Clique covers of H-free graphs

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Abstract

It takes $n^2/4$ cliques to cover all the edges of a complete bipartite graph $K_{n/2,n/2}$, but how many cliques does it take to cover all the edges of a graph G if G has no $K_{t,t}$ induced subgraph? We prove that $O(n^{2-1/(2t)})$ cliques suffice for every n-vertex graph; and also prove that, even for graphs with no stable set of size four, we may need more than linearly many cliques. This settles two questions discussed at a recent conference in Lyon.

1 Introduction

A clique X of a graph G *covers* an edge uv of G if $u, v \in X$, and a *clique cover* of G is a collection of cliques of G that together cover all the edges. The *size* of a clique cover is the number of cliques in the collection. What can we say about the sizes of clique covers?

The complete bipartite graph $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ shows that, for an n-vertex graph, we may need as many as $\lfloor \frac{n^2}{4} \rfloor$ cliques for a clique cover. In fact every graph G has a clique cover of size at most $\lfloor \frac{|G|^2}{4} \rfloor$, where $|G|$ denotes the number of vertices of G. (To see this, note that if x, y are adjacent, we can cover all edges incident with x or y with at most $|G| - 1$ cliques, so we may delete x, y and use induction on $|G|$). But what if we restrict to H-free graphs? (A graph is *H-free* if it does not contain an induced copy of H.) To make a difference, H must be complete bipartite, or else $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$ is H-free; but what happens when H is complete bipartite?

Indeed, what happens if $H = K_{s,0}$? Thus a graph G is H-free if and only if $\alpha(G) < s$. (The sizes of the largest stable set and the largest clique in G are denoted by $\alpha(G), \omega(G)$ respectively.) The minimum size of clique covers in graphs G with $\alpha(G)$ bounded already involves interesting questions. For example, there is a long-standing conjecture that:

1.1 Conjecture. *If $\alpha(G) \le 2$, there is a clique cover of size at most $|G|$.*

("Long-standing", but we do not know the source. Seymour recalls working on it many years ago, possibly in the 1980's.) Which other graphs H have the property that every H-free graph has a clique cover of size at most $|G|$? It turns out that H must be an induced subgraph of $K_{1,3}$. (To see this, observe that every such graph H must be an induced subgraph of a complete bipartite graph, and of the graph obtained from $K_{2,3}$ by subdividing two disjoint edges.) This leads us to the case when $H = K_{1,3}$, and for that there is a remarkable result of Javadi and Hajebi [4]:

1.2 Theorem. *If G is connected and $K_{1,3}$-free, and has a stable set of cardinality three, then G admits a clique cover of size at most $|G|$.*

Thus, if 1.1 is true, then every H-free graph has a clique cover of size at most $|G|$ if and only if H is an induced subgraph of $K_{1,3}$.

We have nothing to contribute to 1.1 itself, but what if we increase the bound on $\alpha(G)$? Javadi and Hajebi [4] asked whether all graphs G with $\alpha(G)$ at most a constant admit clique covers of size $O(|G|)$, but we will disprove this. We will show that:

1.3 Theorem. *There exists $C > 0$ such that for infinitely many n, there is a graph G on n vertices with $\alpha(G) \le 3$ that requires $\frac{Cn^{6/5}}{(\log n)^2}$ cliques in any clique cover.*

And as an upper bound, we will show:

1.4 Theorem. *For every integer $s \ge 3$, if G is a graph with no stable set of size s, then G admits a clique cover of size at most $O(|G|^{2-\frac{1}{s-1}})$.*

At the other extreme, what happens if we exclude $K_{t,t}$? Sepehr Hajebi [3] recently proposed the following:

1.5 Conjecture. *For every integer $t \ge 1$ there exists $\varepsilon > 0$ such that every $K_{t,t}$-free G has a clique cover of size $O(|G|^{2-\varepsilon})$.*

Our main result is a proof of 1.5. We will show that:

1.6 Theorem.

- *For all integers s, t with $s \geq 3$ and $t \geq 2$, every $K_{s,t}$-free graph G with sufficiently many vertices has a clique cover of size at most $\frac{3}{2}|G|^{2-1/(s+t)}$.*
- *For all integers s with $s \geq 3$, every $K_{s,1}$-free graph G (and* a fortiori*, every $K_{s,0}$-free graph) has a clique cover of size at most*

$$O\left(\left(\frac{|G|}{\log |G|}\right)^{\frac{s-2}{s-1}} |G|\right).$$

- *Every $K_{2,2}$-free graph G has a clique cover of size $O(|G|^{3/2})$;*
- *Every $K_{2,3}$-free graph G has a clique cover of size $O(|G|^{3/2}(\log |G|)^{1/2})$.*

The second bullet implies 1.4. We observe that the third bullet here is asymptotically sharp, since there are bipartite $K_{2,2}$-free graphs G with $\Omega(|G|^{3/2})$ edges.

2 Subquadratic clique covers

In this section we will prove 1.6. We begin with some lemmas. Ajtai, Komlós and Szemerédi [1] showed (logarithms in this paper are to base two):

2.1 Lemma. *For every integer $s \geq 2$ there exists $d > 0$ such that, for all integers $a \geq 2$, the Ramsey number*

$$R(a, s) \leq \frac{da^{s-1}}{(\log a)^{s-2}};$$

that is, every graph with at least $\frac{da^{s-1}}{(\log a)^{s-2}}$ vertices has either a clique of size a or a stable set of size s.

Let us rewrite 2.1 in a form more convenient for us:

2.2 Lemma. *For every integer $s \geq 2$ there exists $c > 0$ such that if $w > 1$ is some real number, and G is a graph with $\alpha(G) < s$ and $\omega(G) \leq w$, then*

$$|G| < \frac{cw^{s-1}}{(\log w)^{s-2}}.$$

Proof. Choose d satisfying 2.1, and let $c = 2^{s-1}d$; we claim that c satisfies 2.2. Let $w \geq 1$, and let G be a graph with $\alpha(G) < s$ and $\omega(G) \leq w$. Let $a = \lfloor w \rfloor + 1$. Then $a \geq 2$ is an integer, and $\omega(G) < a$. By 2.1,

$$|G| < \frac{da^{s-1}}{(\log a)^{s-2}}.$$

But $a \leq 2w$ since $w \geq 1$, and so $da^{s-1} \leq cw^{s-1}$, and since $(\log a)^{s-2} \geq (\log w)^{s-2}$, this proves 2.2. ∎

A theorem of Erdős and Hajnal [2], in support of their well-known conjecture, implies that for all s, t there exists $c > 0$ such that if G is $K_{s,t}$-free then G has a clique or stable set of cardinality at least $|G|^c$. But we want to make the result as sharp as we can, so we give a different proof.

2.3 Lemma. *Let s, t be integers with $s \geq t \geq 2$, and let $c \geq s$ satisfy 2.2. If G is $K_{s,t}$-free and $w > 1$ is a real number with $w \geq \omega(G)$, then*

$$|G| \leq \frac{c\alpha(G)^t w^{s-1}}{(\log w)^{s-2}}.$$

Proof. We may assume that $\alpha(G) \geq 2$, because otherwise

$$|G| = \omega(G) \leq \frac{c\alpha(G)^t w^{s-1}}{(\log w)^{s-2}}$$

(since $c \geq 1$ and $s \geq 2$, and $\omega(G) \leq w$, and $w \geq \log w$), and the theorem holds. We first prove the following:

(1) *$V(G)$ is the union of at most $\alpha(G)^t$ sets each including no stable set of cardinality s.*

If $\alpha(G) < s$, the claim holds, so we may assume that $\alpha(G) \geq s$. Let S be a stable set of cardinality $\alpha(G) \geq s$. For $i \in \{t-1, t\}$, let $\mathcal{A}_i$ be the set of all subsets of S of cardinality i. For each $X \in \mathcal{A}_{t-1}$, let R_X be the set of all $v \in V(G)$ such that all neighbours of v in S belong to X (thus, $X \subseteq R_X$). Since S is a largest stable set of G, it follows that $\alpha(G[R_X]) \leq t-1 \leq s-1$, because if there were a larger stable set in R_X, its union with $S \setminus X$ would be a stable set larger than S. For each $X \in \mathcal{A}_t$, let R_X be the set of all $v \in V(G) \setminus S$ that are adjacent to every vertex in X. Then $\alpha(G[R_X]) \leq s-1$ since G is $K_{s,t}$-free. But every vertex with at most $t-1$ neighbours in S belongs to R_X for some $X \in \mathcal{A}_{t-1}$ (here we use that $|S| \geq t-1 \geq 1$), and every vertex with at least t neighbours in S belongs to R_X for some $X \in \mathcal{A}_t$, and so $V(G)$ is the union of the sets R_X ($X \in \mathcal{A}_{t-1} \cup \mathcal{A}_t$). Moreover,

$$|\mathcal{A}_{t-1}| + |\mathcal{A}_t| = \binom{\alpha(G)}{t-1} + \binom{\alpha(G)}{t} = \binom{\alpha(G)+1}{t} \leq \alpha(G)^t.$$

This proves (1).

From the choice of c, if $R \subseteq V(G)$ includes no stable set of size s, then

$$|R| \leq \frac{cw^{s-1}}{(\log w)^{s-2}}.$$

By (1), it follows that

$$|G| \leq \frac{c\alpha(G)^t w^{s-1}}{(\log w)^{s-2}}.$$

This proves 2.3. ∎

We remark that the hypothesis $t \geq 2$ is not necessary. The same statement is true for $t = 0, 1$, but needs a slightly modified proof, which we omit since we only need the result for $t \geq 2$.

2.3 implies that if G is $K_{s,t}$-free then $\max(\alpha(G), \omega(G)) \geq O\big(|G|^{1/(s+t-1)}\big)$. A similar proof shows that for every complete multipartite graph H, there exists ε such that if G is H-free then

$$\max(\alpha(G), \omega(G)) \geq \varepsilon |G|^{1/(|H|-1)}.$$

(We omit the proof, since we shall not use the result.)

Let us prove the first statement of 1.6, the following:

2.4 Theorem. *Let $s \geq 3$ and $t \geq 2$ be integers. There exists N such that every $K_{s,t}$-free graph G with at least N vertices admits a clique cover of size at most $\frac{3}{2}|G|^{2-1/(s+t)}$.*

Proof. We may assume that $s \geq t$, by exchanging them if necessary. Let c satisfy 2.2. Since $s \geq 3$ we may choose N such that

$$(\log N)^{s-2} \geq c(2t)^t (s+t)^{s-2}$$

(this is the only place in the proof that we need $s \geq 3$). Let $d = 1/(s+t)$, and let G be a $K_{s,t}$-free graph with $n \geq N$ vertices. We must show that G has a clique cover of size at most $\frac{3}{2}n^{2-d}$, and so we may assume that $\omega(G) \geq 2$. We begin by choosing a maximal sequence of cliques in G, such that each clique covers at least n^d edges not covered by previous cliques. Thus, so far we have used at most $\frac{1}{2}n^{2-d}$ cliques.

Next, if there is any vertex v that is incident with at most n^{1-d} edges that have not yet been covered, we take copies of K_2 to cover all the uncovered edges incident with v. Repeat this process until no such vertices remain. Note that this step uses at most n^{2-d} cliques in total, so altogether we have used at most $\frac{3}{2}n^{2-d}$ cliques.

We claim that all edges of G have now been covered; so, for a contradiction, suppose not. Call a vertex x *happy* if all edges incident with x have been covered and *unhappy* otherwise; thus there is at least one unhappy vertex. Let H be the subgraph of G with vertex set the unhappy vertices and edge set the uncovered edges. Then H has minimum degree at least n^{1-d}. Furthermore, no clique of G covers at least n^d edges of H, or we could have added it to our maximal sequence at the first step.

Fix an unhappy vertex v, and let D be the set of its neighbours in H, so $|D| \geq n^{1-d}$. There is no clique K of $G[D]$ with size at least n^d, since adding v to K would give a clique of G that covers n^d edges of H (all the edges from v to K). So by 2.3, taking $w = n^d$, it follows that $G[D]$ contains a stable set S where

$$\frac{c|S|^t (n^d)^{s-1}}{(\log n^d)^{s-2}} \geq |D| \geq n^{1-d},$$

that is,

$$|S| \geq d^{(s-2)/t} c^{-1/t} n^{(1-ds)/t} (\log n)^{(s-2)/t} = d^{(s-2)/t} c^{-1/t} n^d (\log n)^{(s-2)/t}$$

(since $d = (1-ds)/t$).

By a *copy* of $K_{1,t}$ we mean an induced subgraph of G isomorphic to $K_{1,t}$, and a *leaf* of a graph means a vertex with degree one. We count copies of $K_{1,t}$ in H with all their leaves in S. Let $L \subseteq S$ with $|L| = t$, and let M be the set of vertices in $V(H) \setminus S$ that are adjacent in H to every vertex in L. Since L is stable in G (as it is a subset of S), and G is $K_{s,t}$-free, it follows that M does not contain a stable set (of G) of size s. Moreover, M contains no clique (of G) of size at least n^d, since adding any vertex of L to such a clique would give a clique in G covering at least n^d edges from H.

From 2.2, $|M| \leq \frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}}$. Since this holds for each choice of L, and there are only $\binom{|S|}{t}$ choices of L, it follows that there are at most

$$\frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}}\binom{|S|}{t} \leq \frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}}\frac{|S|^t}{t!}$$

copies of $K_{1,t}$ with all leaves in S.

On the other hand, there are at least $n^{1-d}|S|$ edges of H with an end in S. For each $y \in V(H)$, let $r(y)$ be the set of vertices in S adjacent in H to y, and let r be the average of the $r(y)$ over $y \in V(H)$. Thus

$$r \geq n^{1-d}|S|/|H| \geq n^{-d}|S| \geq d^{(s-2)/t}c^{-1/t}(\log n)^{(s-2)/t}.$$

Moreover, since $n \geq N$, it follows that

$$(\log n)^{s-2} \geq c(2t)^t d^{2-s}$$

and so

$$r \geq d^{(s-2)/t}c^{-1/t}(\log n)^{(s-2)/t} \geq 2t.$$

The number of copies of $K_{1,t}$ with all leaves in S is at least

$$\sum_{y\in V(H)} \binom{r(y)}{t}$$

(taking $\binom{a}{b} = 0$ when $a < b$); and hence at least

$$|H|\binom{r}{t} \geq |H|\frac{(r-t)^t}{t!} \geq |H|\frac{(r/2)^t}{t!}$$

by convexity and since $r \geq 2t \geq t$. Consequently

$$|H|\frac{(r/2)^t}{t!} \leq \frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}}\frac{|S|^t}{t!},$$

that is,

$$|H|(r/2)^t \leq \frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}}|S|^t.$$

Since $|S| \leq r|H|n^{d-1}$ and $d(s+t) = 1$, the right side of the above is at most

$$\frac{c(n^d)^{s-1}}{(\log n^d)^{s-2}}r^t|H|^t n^{t(d-1)} = \frac{c}{d^{s-2}(\log n)^{s-2}}r^t|H|^t n^{1-d-t} \leq \frac{c}{d^{s-2}(\log n)^{s-2}}r^t|H|n^{-d}.$$

Consequently

$$|H|(r/2)^t \leq \frac{c}{d^{s-2}(\log n)^{s-2}}r^t|H|n^{-d}.$$

that is,

$$(\log n)^{s-2}n^d d^{s-2} \leq c2^t,$$

contradicting that $n \geq N$. This proves 2.4. ∎

A simplified version of the same argument yields a weakened form of the second statement of 1.6:

2.5 Proposition. *For every integer $s \geq 3$ and $t \in \{0, 1\}$, if G is a $K_{s,t}$-free graph, then G admits a clique cover of size at most $O(|G|^{2-1/s})$.*

Proof. Let c be as in the proof of 2.4, let $d = 1/s$, and choose N such that $(\log N)^{s-2} > cs^{s-2}$. We will show that if $|G| = n \geq N$ and G is $K_{s,t}$-free then G admits a clique cover of size at most $\frac{3}{2}|G|^{2-1/s}$, which implies the result. As in the proof of 2.4, we may assume that G has a nonnull subgraph H with minimum degree at least n^{1-d}, such that no clique of G covers n^d edges of H. Choose v, D as before; then $G[D]$ has no clique of size n^d, and no stable set of size s (because G is $K_{s,t}$-free and $t \leq 1$), and so by 2.2,

$$|D| < \frac{cn^{d(s-1)}}{(d \log n)^{s-2}}.$$

But $|D| \geq n^{1-d}$, and so

$$n^{1-d} \leq \frac{cn^{d(s-1)}}{(d \log n)^{s-2}}.$$

Since $1 - d = d(s-1)$, it follows that $(\log n)^{s-2} \leq cs^{s-2}$, contradicting that $n \geq N$. This proves 2.5. ∎

But we can do a little better. To prove the second statement of 1.6 as stated, we use a consequence of 2.2:

2.6 Lemma. *For all integers $s \geq 2$ there exists $c > 0$ such that if G is a graph with $\alpha(G) < s$ and $|G| \geq 2$, then*

$$|G| < \frac{c\omega(G)^{s-1}}{(\log |G|)^{s-2}}.$$

Proof. Choose d such that 2.2 holds with $c = d$. We may assume that $d \geq s^{1/2}$ by increasing d. Choose c such that $c \geq d^2(2 \log d)^{s-2}$ and $c > d(2(s-1))^{s-2}$; we will show that c satisfies the lemma.

Let G be a graph with $\alpha(G) < s$ and $|G| \geq 2$. If $\log |G| \leq 2 \log d$, then $|G| \leq d^2$, and so

$$|G|(\log |G|)^{s-2} \leq d^2(2 \log d)^{s-2} \leq c \leq c\omega(G)^{s-1}$$

as required. Thus we may assume that $\log |G| > 2 \log d$. In particular $|G| \geq d^2 \geq s$ and so G has an edge. By 2.2,

$$|G| < \frac{d\omega(G)^{s-1}}{(\log \omega(G))^{s-2}} \leq d\omega(G)^{s-1}$$

and so $\log |G| \leq \log(d) + (s-1)\log(\omega(G))$. Since $\log |G| > 2\log(d)$, it follows that $\log |G| < 2(s-1)\log(\omega(G))$. Hence

$$|G| < \frac{d\omega(G)^{s-1}}{(\log \omega(G))^{s-2}} \leq \frac{d(2(s-1))^{s-2}\omega(G)^{s-1}}{(\log |G|)^{s-2}} \leq \frac{c\omega(G)^{s-1}}{(\log |G|)^{s-2}}$$

as required. This proves 2.6. ∎

We deduce:

2.7 Lemma. *For all integers $s \geq 3$ there exists $c > 0$ such that for every graph G with $\alpha(G) < s$ and $|G| \geq 2$, $V(G)$ is the union of at most*

$$c\left(\frac{|G|}{\log|G|}\right)^{\frac{s-2}{s-1}}$$

cliques.

Proof. Choose d such that 2.6 holds (with c replaced by d). Choose f such that $2df^{s-1} \geq (1/2)^{s-2}$. Choose $N \geq 4$ such that $\log N > (1-2^{-\frac{1}{s-2}})^{-1}$, and choose c such that $c \geq 2/f$, and $c(n/\log n)^{\frac{s-2}{s-1}} \geq n$ for all nonzero integers $n \leq N$. We will show that c satisfies 2.7. Let G be a graph with $\alpha(G) < s$. We prove that the statement of the theorem is true for G, by induction on $|G|$. If $|G| \leq N$, then $V(G)$ is the union of $|G| \leq c(|G|/\log|G|)^{\frac{s-2}{s-1}}$ cliques and the theorem holds, so we may assume that $|G| > N$.

Choose as many pairwise disjoint cliques as possible that each have cardinality at least

$$f|G|^{\frac{1}{s-1}}(\log|G|)^{\frac{s-2}{s-1}},$$

say $A_1 \dots A_k$. Let $G' = G \setminus (A_1 \cup \dots \cup A_k)$. Thus

$$\omega(G') < f|G|^{\frac{1}{s-1}}(\log|G|)^{\frac{s-2}{s-1}}.$$

We claim:

(1) $|G'| \leq |G|/2$.

Suppose not; then by 2.6,

$$|G'| < \frac{d\omega(G')^{s-1}}{(\log|G'|)^{s-2}},$$

and so

$$(|G|/2)(\log(|G|/2))^{s-2} \leq |G'|(\log|G'|)^{s-2} < d\omega(G')^{s-1} \leq df^{s-1}|G|(\log|G|)^{s-2}.$$

Thus

$$(\log|G|-1)^{s-2} \leq 2df^{s-1}(\log|G|)^{s-2}.$$

But $\log|G| - 1 \geq \frac{1}{2}\log|G|$ (because $|G| \geq N \geq 4$), and so $(1/2)^{s-2} \leq 2df^{s-1}$, a contradiction. This proves (1).

Since $A_1 \dots A_k$ all have cardinality at least $f|G|^{\frac{1}{s-1}}(\log|G|)^{\frac{s-2}{s-1}}$, it follows that

$$k \leq \frac{f^{-1}|G|^{\frac{s-2}{s-1}}}{(\log|G|)^{\frac{s-2}{s-1}}} = f^{-1}\left(\frac{|G|}{\log|G|}\right)^{\frac{s-2}{s-1}} \leq (c/2)\left(\frac{|G|}{\log|G|}\right)^{\frac{s-2}{s-1}}.$$

From the inductive hypothesis, if $|G'| \geq 2$ then $V(G')$ is the union of

$$c\left(\frac{|G'|}{\log|G'|}\right)^{\frac{s-2}{s-1}}$$

cliques. Hence, $V(G')$ is the union of at most

$$c\left(\frac{|G|/2}{\log|G|-1}\right)^{\frac{s-2}{s-1}}$$

cliques, by (1), even if $|G'| \le 1$. But

$$c\left(\frac{|G|/2}{\log|G|-1}\right)^{\frac{s-2}{s-1}} \le (c/2)\left(\frac{|G|}{\log|G|}\right)^{\frac{s-2}{s-1}}$$

since

$$2^{\frac{1}{s-2}} \ge \frac{\log|G|}{\log|G|-1}.$$

Thus, both $A_1 \cup \cdots \cup A_k$ and $V(G')$ are the union of at most

$$(c/2)\left(\frac{|G|}{\log|G|}\right)^{\frac{s-2}{s-1}}$$

cliques. Adding, this proves 2.7. ∎

We use this to show the second statement of 1.6, the following:

2.8 Theorem. *For every integer $s \ge 3$, let c be as in 2.7. If G is a $K_{s,1}$-free graph with $|G| \ge 2$, then G admits a clique cover of size at most*

$$\frac{c|G|^{2-\frac{1}{s-1}}}{(\log|G|)^{\frac{s-2}{s-1}}}.$$

Proof. By 2.7, there is a set $\mathcal{A}$ of cliques of G with union $V(G)$ and with

$$|\mathcal{A}| \le c\left(\frac{|G|}{\log|G|}\right)^{\frac{s-2}{s-1}}.$$

For each $v \in V(G)$ and $A \in \mathcal{A}$, let A_v be the clique consisting of v and the set of neighbours of v that belong to A. Then the set of all the cliques A_v is a clique cover satisfying the theorem. This proves 2.8. ∎

Now we prove the third and fourth statements of 1.6. We will need the following, which is implied by 2.2 with $s = 3$:

2.9 Lemma. *There exists $k > 0$ such that every graph G with no stable set of size three has a clique of size at least $k|G|^{1/2}\sqrt{\log|G|}$.*

We will show:

2.10 Theorem.

- *Every $K_{2,2}$-free graph G has a clique cover of size $O(|G|^{3/2})$.*

- *Every $K_{2,3}$-free graph G has a clique cover of size $O(|G|^{3/2}(\log|G|)^{1/2})$.*

Proof. The proofs for both statements are much the same, and we will do them at the same time. Let G be either $K_{2,2}$-free or $K_{2,3}$-free, let v be a vertex of minimum degree, and let D be the set of its neighbours. We will show that D is the union of a small number of cliques. Adding v to each of these cliques, we see that the edges incident with v can be covered by the same small number of cliques; thus we may delete v and argue by induction. It remains to show that D is the union of an appropriately small number of cliques.

First we need:

(1) *Let $M \subseteq D$. Then either:*

- *there is a set $J_1 \subseteq M$ with $|J_1| \geq |M|^2/(4n)$, and two nonadjacent vertices $x, y \in V(G) \setminus J_1$, both adjacent to every vertex in J_1; or*
- *there is a clique $J_2 \subseteq M$ with $|J_2| \geq |M|/4$.*

Let $A \subseteq M$ be the set of vertices in M with at least $|M|/3$ neighbours outside $D \cup \{v\}$, and let $B = M \setminus A$. Suppose first that $|A| \geq 3|M|/4$. Then the number of edges from A to $V(G) \setminus (D \cup \{v\})$ is at least $|M|^2/4$, and so some vertex $x \in V(G) \setminus (D \cup \{v\})$ has a set J of at least $|M|^2/(4n)$ neighbours in M, and the first bullet of (1) holds (taking $y = v$).

Otherwise $|B| \geq |M|/4$. If B is a clique then the second bullet holds. Otherwise there are nonadjacent vertices $x, y \in B$; and as x, y each have at most $|M|/3$ non-neighbours in D (because v was chosen with minimum degree, and $x, y \in B$), there are at most $2|M|/3$ vertices in M nonadjacent to one of x, y (counting x, y themselves); and so x, y have at least $|M|/3$ common neighbours in M, and the first bullet holds. This proves (1).

We deduce:

(2) *If G is $K_{2,2}$-free, then for every $M \subseteq D$, there is a clique in M with size at least $|M|^2/(4n)$.*

This is immediate from (1), because the set J_1 in (1) must be a clique, since G is $K_{2,2}$-free, and the set J_2 satisfies $|J_2| \geq |M|/4 \geq |M|^2/(4n)$. This proves (2).

(3) *Let k satisfy 2.9, and let $\beta = \min(k/2, 1/4)$. If G is $K_{2,3}$-free, then for every $M \subseteq D$ with $|M|^2 \geq 4n$, there is a clique in M with size at least*

$$\frac{\beta|M|}{\sqrt{n}}\sqrt{\log(|M|^2/(4n))}.$$

By (1), one of the sets J_1, J_2 of (1) exist. If J_1 exists, then it contains no stable triple of vertices, and so by 2.9, it contains a clique of size at least

$$k(|J_1|\log(|J_1|))^{1/2} \geq \frac{k|M|}{2\sqrt{n}}(\log(|M|^2/(4n)))^{1/2},$$

and the claim holds. If J_2 exists then again the claim holds since

$$|M|/4 \geq \beta|M| \geq \frac{\beta|M|}{\sqrt{n}}\sqrt{\log(|M|^2/(4n))}.$$

This proves (3).

Now we will use (2) or (3) to show that the vertices in D can be covered by an appropriately small collection $\mathcal{C}$ of cliques. We choose $\mathcal{C}$ by choosing greedily a largest clique among the uncovered vertices of D until at most $4\sqrt{n}$ vertices remain, and then covering the remaining vertices by singletons. To bound the total number of cliques, we track the process, writing M for the set of uncovered vertices at each stage. We divide the values of $|M|$ into ranges $[1, 4\sqrt{n})$ and $[2^i\sqrt{n}, 2^{i+1}\sqrt{n})$ for $i \geq 2$.

We assume first that G is $K_{2,2}$-free. Thus by (2), if $|M|$ is in the range $[2^i\sqrt{n}, 2^{i+1}\sqrt{n})$, then the size of the clique we obtain is at least $|M|^2/(4n) \geq 2^{2i-2}$, and so there will be at most

$$\frac{2^{i+1}\sqrt{n}}{2^{2i-2}} = \frac{8\sqrt{n}}{2^i}$$

cliques chosen for $|M|$ in this range. The total number of cliques in $\mathcal{C}$ is therefore at most

$$\sum_{i\geq 2} \frac{8\sqrt{n}}{2^i} + 4\sqrt{n} = O(\sqrt{n}).$$

Consequently the first bullet of the theorem follows by induction.

Now we assume that G is $K_{2,3}$-free, and use (3) in place of (2). If $|M|$ is in the range $[2^i\sqrt{n}, 2^{i+1}\sqrt{n})$ where $i \geq 2$, then the size of the clique we obtain is at least

$$\frac{\beta|M|}{\sqrt{n}}\sqrt{\log(|M|^2/(4n))} \geq \beta 2^i\sqrt{\log(2^{2i-2})} = \beta 2^i\sqrt{2i-2}.$$

Consequently, at most

$$\frac{2^{i+1}\sqrt{n}}{\beta 2^i\sqrt{2i-2}} = \frac{2\sqrt{n}}{\beta\sqrt{2i-2}}$$

cliques will be chosen during this range. Thus the total number of cliques is at most

$$\sum_{i=2}^{\log n} \frac{2\sqrt{n}}{\beta\sqrt{2i-2}} + 4\sqrt{n} = O(\sqrt{n\log n}).$$

Hence the second bullet of the theorem follows by induction. This proves 2.10. ∎

3 Lower bounds

What can we say from the other side? For $K_{s,0}$-free graphs, the result of this section, with 1.6, shows that (roughly speaking) the answer is somewhere between $n^{2-4/(s+1)}$ and $n^{2-\frac{1}{s-1}}$. We need the following result of Spencer (theorem 2.2 of [5]):

3.1 Lemma. *For all integers $s \geq 3$, there exists $c > 0$ such that for all integers $t \geq 3$, the Ramsey number $R(s,t)$ is at least $c(t/\log t)^{\frac{s+1}{2}}$. Consequently, for all $s \geq 3$ there exists $C > 0$ such that for infinitely many n, there is a graph J with n vertices such that $\omega(G) < Cn^{\frac{2}{s+1}} \log n$ and $\alpha(G) < s$.*

3.2 Theorem. *For all $s \geq 3$, there exists $c > 0$ such that for infinitely many n, there is a graph with n vertices and with no stable set of size s, such that every clique cover has size at least $cn^{2-4/(s+1)}/(\log n)^2$.*

Proof. Choose C as in the second statement of 3.1, and let c satisfy $c^{-1} = C^2 2^{2-4/(s+1)}$. Now choose $m > 0$ such that there is a graph J with m vertices, and with $\omega(J) < Cn^{\frac{2}{s+1}} \log m$ and $\alpha(J) < s$. Let $n = 2m$. Take two vertex-disjoint copies J_1, J_2 of J, and make every vertex of J_1 adjacent to every vertex of J_2, forming G; thus $|G| = n$. Then G has no stable set of size s; and every clique of G covers at most $C^2 m^{\frac{4}{s+1}} (\log m)^2$ of the edges between $V(J_1)$ and $V(J_2)$. Since there are m^2 such edges, every clique cover of G has size at least

$$C^{-2} m^{2-\frac{4}{s+1}}/(\log m)^2 \geq cn^{2-\frac{4}{s+1}}/(\log n)^2.$$

This proves 3.2. ∎

Taking $s = 4$, this proves 1.3.

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