Finding minimum clique capacity

Maria Chudnovsky[1]
Columbia University, New York, NY 10027

Sang-Il Oum[2]
KAIST, Daejeon, Republic of Korea

Paul Seymour[3]
Princeton University, Princeton, NJ 08544

July 21, 2009; revised July 21, 2009

[1]Supported by NSF grant DMS-0758364.
[2]Supported by
[3]Supported by ONR grant N00014-01-1-0608.

Abstract

Let C be a clique of a graph G. The *capacity* of C is defined to be $(|V(G) \setminus C| + |D|)/2$, where D is the set of vertices in $V(G) \setminus C$ that have both a neighbour and a non-neighbour in C. We give a polynomial-time algorithm to find the minimum clique capacity in a graph G. This problem arose as an open question in a study [1] of packing vertex-disjoint induced three-vertex paths in a graph with no stable set of size three.

1 Introduction

In this paper, all graphs are finite and have no loops or multiple edges. A *clique* is a subset of $V(G)$ of vertices that are pairwise adjacent. A subset X of $V(G)$ is *stable* if all members of X are pairwise nonadjacent, and $\alpha(G)$ denotes the cardinality of the largest stable subset of $V(G)$. If $C \subseteq V(G)$, a vertex $v \in V(G) \setminus C$ is *complete* to C if v is adjacent to every member of C, and *anticomplete* to C if it has no neighbour in C.

Let C be a clique of a graph G. Let $A, B, D \subseteq V(G) \setminus C$ be respectively the sets of all vertices $v \in V(G) \setminus C$ such that

- v is complete to C
- v is anticomplete to C
- v has both a neighbour and a non-neighbour in C.

Thus $A \cup B \cup D = V(G) \setminus C$, and if $C \neq \emptyset$ then A, B, D are pairwise disjoint.

The problem of choosing C with $|C|$ maximum is NP-hard. On the other hand, it is easy to find a clique C with $|C| + |A|/2$ maximum in polynomial time. (To see this, take two copies $p(v), q(v)$ of each vertex v of G, and for distinct $u, v \in V(G)$, make $p(u), q(v)$ adjacent if u, v are nonadjacent in G, forming a bipartite graph H. Find the maximum stable set X in H, and let C be the set of all $v \in V(G)$ such that $p(v), q(v)$ are both in X. It is easy to check that C is the clique of G with $|C| + |A|/2$ maximum.) We define the *capacity* $cap(C)$ of the clique C to be $(|A| + |B|)/2 + |D|$, and in this paper we study finding a clique C with minimum capacity (that is, with $|C| + (|A| + |B|)/2$ maximum). It turns out that we can modify the simple algorithm just given to solve the capacity problem.

A *seagull* in G is an induced three-vertex path in G. In [1] the problem of packing vertex-disjoint seagulls was studied, and a min-max formula was given for the maximum seagull packing in graphs with $\alpha(G) \leq 2$, the following (an *antimatching* means a matching in the complement graph, and the five-wheel is the graph with six vertices in which one vertex is complete to the vertex set of a cycle of length five):

1.1 *Let G be a graph with $\alpha(G) \leq 2$, and let $k \geq 0$ be an integer, such that if $k = 2$ then G is not a five-wheel. Then G has k pairwise disjoint seagulls if and only if*

- $|V(G)| \geq 3k$
- *G is k-connected,*
- *every clique of G has capacity at least k, and*
- *G admits an antimatching of cardinality k.*

This did not directly yield a polynomial-time algorithm to compute the size of the optimum seagull packing, however, because we did not know how to compute in polynomial time whether every clique has capacity at least k, and we had to resort to the ellipsoid method. In this paper we give a polynomial-time algorithm for the missing step. We show

1.2 *There is an algorithm, with running time $O(n^{3.5})$, which with input an n-vertex graph G, finds a clique C in G with minimum capacity.*

We begin with the following; then 1.2 follows by running 1.3 for every vertex c in turn.

1.3 *There is an algorithm, with running time $O(n^{2.5})$, which with input an n-vertex graph G and a vertex $c \in V(G)$, outputs a clique C containing c, with $cap(C)$ minimum over all cliques that contain c.*

Proof. Here is the algorithm. Let N be the set of neighbours of c and M the set of vertices different from c that are nonadjacent to C. Take two copies $p(v), q(v)$ of each vertex $v \in V(G)$, and make a graph H with vertex set

$$\{p(v) : v \in N\} \cup \{q(v) : v \in N \cup M\},$$

with edges as follows:

- $\{p(v) : v \in N\}$ and $\{q(v) : v \in N \cup M\}$ are stable sets
- for all distinct $u, v \in N$, $p(u)$ and $q(v)$ are adjacent if and only if u, v are nonadjacent in G
- for all $u \in N$ and $v \in M$, $p(u)$ and $q(v)$ are adjacent in H if and only if u, v are adjacent in G
- for all $u \in N$, $p(u)$ and $q(u)$ are nonadjacent in H.

Thus H is bipartite. Find the maximum stable subset X of $V(H)$. (This takes time $O(n^{2.5})$.) Then output

$$\{c\} \cup \{v \in N : p(v) \in X \text{ and } q(v) \in X\}.$$

That completes the description of the algorithm. The running time is $O(n^{2.5})$, using the algorithm of Hopcroft and Karp [2]; now we discuss its correctness. Let X be the stable set of H chosen by the algorithm.

(1) *Let k be minimum such that some clique containing c has capacity $k/2$. Then $|X| \geq 2n - k - 2$.*

For let C be a clique of G with $c \in C$ and $cap(C) = k/2$. Let A, B, D be as usual. Thus $A, C \setminus \{c\} \subseteq N$ and $B \subseteq M$. The set

$$\{p(v) : v \in C \setminus \{c\}\} \cup \{q(v) : v \in A \cup B \cup (C \setminus \{c\})\}$$

is a stable set of H, with cardinality

$$|A| + |B| + 2|C| - 2 = 2(|A| + |B| + |C| + |D|) - 2cap(C) - 2 = 2n - k - 2.$$

Since X is a maximum stable set of H, it follows that $|X| \geq 2n - k - 2$. This proves (1).

Let $C = \{c\} \cup \{v \in N : p(v), q(v) \in X\}$. Thus C is the set returned by the algorithm, and $C \subseteq \{c\} \cup N$. Moreover, if $u, v \in C \setminus \{c\}$ are distinct then $p(u), q(v) \in X$, and since X is stable in H, we deduce that $p(u), q(v)$ are nonadjacent in H, and so u, v are adjacent in G. Consequently C is a clique of G.

(2) $cap(C) \leq k/2$.

For let

$$A = \{v \in N \setminus C : \; p(v) \in X \text{ or } q(v) \in X\},$$

$B = \{v \in M : \; q(v) \in X\}$, and $D = V(G) \setminus (A \cup B \cup C)$. Thus $|X| = 2(|C| - 1) + |A| + |B|$, and since $|X| \geq 2n - k - 2$ it follows that $2(|C| - 1) + |A| + |B| \geq 2n - k - 2$, that is,

$$2|C| + |A| + |B| \geq 2(|A| + |B| + |C| + |D|) - k.$$

Consequently $|A|+|B|+2|D| \leq k$. Now since X is stable in H, we deduce that for all $u \in C \setminus \{c\}$ and $v \in B$, $p(u), q(v)$ are nonadjacent in H, and so u, v are nonadjacent in G. Since $B \subseteq M$, it follows that every vertex in B is anticomplete to C. We claim that every vertex in A is complete to C. For let $u \in C \setminus \{c\}$ and $v \in A$. Then $v \in N \setminus C$, and one of $p(v), q(v) \in X$; and so since $p(u), q(u) \in X$ and X is stable in H, it follows that either $p(u), q(v)$ are nonadjacent in H (if $q(v) \in X$) or $q(u), p(v)$ are nonadjacent in H (if $p(v) \in X$). In either case it follows that u, v are adjacent in G, and so v is complete to C, as claimed. Consequently $cap(C) \leq |A| + |B| + 2|D| \leq k/2$. This proves (2).

From (2), and the choice of k, it follows that $cap(C) = k/2$, and so the clique returned by the algorithm is indeed a clique containing c with minimum capacity. This proves 1.3. ▮

References

[1] Maria Chudnovsky and Paul Seymour, "Packing seagulls", submitted for publication.

[2] J. E. Hopcroft and R. M. Karp, "An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs", *SIAM Journal on Computing* 2 (1973), 225–231.