Claw-free Graphs VI. Colouring

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April 18, 2011

[1]This research was partially conducted during the period the author served as a Clay Mathematics Institute Research Fellow at Princeton Univeristy. Partially supportted by NSF grant DMS-0758364.
[2]Partially supported by ONR grant N00014-04-1-0062, and NSF grant DMS03-54465.

Abstract

In this paper we prove that if G is a connected claw-free graph with three pairwise non-adjacent vertices, with chromatic number χ and clique number ω, then $\chi \leq 2\omega$ and the same for the complement of G. We also prove that the choice number of G is at most 2ω, except possibly in the case when G can be obtained from a subgraph of the Schläfli graph by replicating vertices. Finally, we show that the constant 2 is best possible in all cases.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. For a subset X of $V(G)$ we denote by $G|X$ the subgraph of G induced on X. We say that $X \subseteq V(G)$ is a *claw* if $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$, and G is *claw-free* if no subset of $V(G)$ is a claw. Line graphs are a well-known class of claw-free graphs, but there are others, such as *circular interval graphs* and subgraphs of the *Schläfli graph* (a circular interval graph is obtained from a collection of points and intervals of a circle by making two points adjacent if they belong to the same interval). In [4] we prove a theorem that explicitly describes the structure of all claw-free graphs.

Claw-free graphs being a generalization of line graphs, it is natural to ask what properties of line graphs can be extended to all claw-free graphs. A *clique* in a graph is a set of vertices all pairwise adjacent. A *stable set* is a set of vertices all pairwise non-adjacent. A *triangle* is a clique of size three, and a *triad* is a stable set of size three. For a graph G, we denote by $\omega(G)$ the size of the largest clique in G, and by $\chi(G)$ the chromatic number of G. Vizing's theorem [8] gives a bound on $\chi(G)$ in terms of $\omega(G)$ if G is the line graph of a simple graph, namely $\chi \leq \omega + 1$. But what about other claw-free graphs? Does there exist a function f such that if G is a claw-free graph then $\chi(G) \leq f(\omega(G))$? It is easy to see that such f exists, and in fact $\chi(G) \leq \omega(G)^2$ (the neighbourhood of a vertex in a clique of size $\omega(G)$ is the union of at most $\omega(G)$ cliques).

One might hope to get closer to Vizing's bound, asking whether f is a linear function. Unfortunately the answer to this question is negative (in fact, the power two is best possible). If G is a triad-free graph, then $\chi(G) \geq \frac{|V(G)|}{2}$, and yet $\omega(G)$ may be of order $\sqrt{|V(G)| \log |V(G)|}$ [7]. However, if we insist that G contains a triad, and is connected (to prevent counterexamples obtained by taking disjoint unions with large triad-free graphs), then a much stronger result is true. The main result of this paper is the following:

1.1 *Let G be a connected, claw-free graph that contains a triad. Then $\chi(G) \leq 2\omega(G)$.*

This bound is best possible, in the sense that the constant 2 cannot be replaced by any smaller constant (in Section 4, we construct an infinite family of claw-free graphs, satisfying the hypotheses of 1.1, with the ratio of the chromatic number and the clique number arbitrarily close to 2.)

Let us say that a graph G is *tame* if there exists a connected claw-free graph H with a triad, such that G is an induced subgraph of H. We prove a slight strengthening of 1.1, the following:

1.2 *Let G be tame. Then $\chi(G) \leq 2\omega(G)$.*

As we will show in Section 4, the bound of 1.2 is best possible (not only asymptotically). The proof of 1.2 uses the structure theorem mentioned above.

There is a slightly worse, but still linear bound on χ in terms of ω, that has a short proof, without using the structure theorem, and we include it here. We prove:

1.3 *Let G be tame. Then $\chi(G) \leq 4\omega(G)$.*

In fact, we prove the following stronger statement that clearly implies 1.3.

1.4 *Let G be tame. Then every vertex of G has degree at most $4\omega(G) - 1$.*

For $v \in V(G)$ we denote by $N_G(v)$ (or $N(v)$ when there is no ambiguity) the set of neighbours of v in G. Let $X \subseteq V(G)$. We denote by $G \setminus X$ the graph $G|(V(G) \setminus X)$. For $v \in V(G)$ we denote by $G \setminus v$ the graph $G \setminus \{v\}$. We start with two lemmas:

1.5 *Let G be a claw-free graph, let X, Y be disjoint subsets of $V(G)$ with $X \neq \emptyset$, and assume that for every two non-adjacent vertices of Y, every vertex of X is adjacent to exactly one of them. Then Y is the union of two cliques.*

Proof. Since for every two non-adjacent vertices $a, b \in Y$, $N(a) \cap X$ and $N(b) \cap X$ partition X, it follows that $G|Y$ contains no complement of an odd cycle, so $G|Y$ is the complement of a bipartite graph; and in particular Y is the union of two cliques. ∎

1.6 *Let G be a claw-free graph that contains a triad, and assume that there is a vertex $v \in V(G)$, with a neighbour in G, and such that $G \setminus v$ contains no triad. Then $V(G)$ is the union of four cliques, and in particular $\omega(G) \geq \frac{|V(G)|}{4}$.*

Proof. Let X be the set of neighbours of v in G, and let $Y = V(G) \setminus (X \cup \{v\})$. Since G contains a triad, and $G \setminus v$ does not, it follows that there exist two non-adjacent vertices y_1, y_2 in Y. Since v has a neighbour in G, it follows that X is non-empty. For $i = 1, 2$ let N_{y_i} be the set of neighbours of y_i in X. Since $\{x, y_1, y_2, v\}$ is a claw in G for every $x \in N_{y_1} \cap N_{y_2}$, it follows that $N_{y_1} \cap N_{y_2} = \emptyset$. Since $G \setminus v$ contains no triad, $X \setminus N_{y_i}$ is a clique for $i = 1, 2$, and therefore $X \cup \{v\}$ is the union of two cliques. Also since $G \setminus v$ contains no triad, $N_{y_1} \cup N_{y_2} = X$. So for every two non-adjacent vertices in Y every vertex of X is adjacent to exactly one of them. By 1.5 it follows that Y is the union of two cliques. But now $V(G)$ is the union of four cliques, and in particular $\omega(G) \geq \frac{|V(G)|}{4}$, and the theorem holds. ∎

1.6 has the following useful corollary:

1.7 *Let G be tame. Then either G contains a triad, or $V(G)$ is the union of four cliques.*

Proof. Suppose that G contains no triad. Let H be a connected claw-free graph with a triad such that G is an induced subgraph of H. Since H is connected, we can number the vertices of $V(H) \setminus V(G)$ as $\{v_1, \ldots, v_k\}$ such that for $1 \leq i \leq k$, v_i has a neighbour in $V(G) \cup \{v_1, \ldots, v_{i-1}\}$. Choose i minimum such that $V(G) \cup \{v_1, \ldots, v_i\}$ includes a triad, and let G' be the subgraph of H induced on $V(G) \cup \{v_1, \ldots, v_i\}$. Since $G' \setminus v_i$ has no triad, 1.6 implies that $V(G')$ (and hence $V(G)$) is the union of four cliques. This proves 1.7. ∎

Proof of 1.4. Let v be a vertex of maximum degree in G and let N be the set of neighbours of v. Since G is claw-free, $G|(N \cup \{v\})$ contains no triad. Now the result follows from 1.7. This proves 1.4. ∎

We also prove a variant of 1.1 with chromatic number replaced by *choice number*. Let G be a graph, and for every $v \in V(G)$, let L_v be a list of colours. We say that G is *colourable from the lists* $\{L_v\}_{v \in V(G)}$ if there exists a proper colouring of G such that every vertex v is coloured with a colour from L_v. The *choice number* of G is the smallest integer k such that for every set of lists $\{L_v\}$, if $|L_v| \geq k$ for every $v \in V(G)$, then G is colourable from the lists $\{L_v\}$. We denote the choice number of G by $ch(G)$. Clearly, $\chi(G) \leq ch(G)$. In Section 5 we prove that if G tame and G does not belong to a special restricted class of claw-free graphs (that we will define later), then $ch(G) \leq 2\omega(G)$.

Let $\overline{G}$ denote the complement of the graph G (that is, the graph on the same vertex set as G, such that two vertices are adjacent in $\overline{G}$ if and only if they are non-adjacent in G). It turns out that one can also bound the chromatic number of a graph whose complement is claw-free in terms of the size of its maximum clique. We prove:

1.8 *Let G be tame. Then $\chi(\overline{G}) \leq 2\omega(\overline{G})$.*

The remainder of the paper is organized as follows. In Section 2 we develop some tools that will be used in the proof of 1.2. In Section 3 we state the structure theorem from [4], and deduce from it that every connected claw-free graph with a triad either can be handled by the methods developed in Section 2, or is obtained by replicating vertices from an induced subgraph of the *Schläfli* graph (we will define the Schläfli graph and make this precise later). In Section 4 we prove that the conclusion of 1.2 holds for the latter class of claw-free graphs, and thus complete the proof of 1.2. We also show that the constant in 1.1 is best possible, and that the bound of 1.2 is best possible, not only asymptotically. In Section 5 we prove the bound on $ch(G)$. Finally, in Section 6 we prove 1.8.

2 Tools

We start with some definitions. Let G be a graph. A non-empty subset X of $V(G)$ is said to be *connected* if the graph $G|X$ is connected. A *component* of G is a maximal connected subgraph of G. For a vertex v and a set $A \subseteq V(G)$ not containing v, we say that v is *complete (anticomplete)* to A if v is adjacent to every (no) vertex of A, respectively. Two disjoint sets $A, B \subseteq V(G)$ are *complete (anticomplete)* to each other if every vertex of A is complete (anticomplete) to B.

Let G be a connected claw-free graph with a triad. It turns out that in many cases we can prove that G has one of the following properties: either the set of neighbours of some vertex v of G is the union of two cliques (in this case we say that v is *bisimplicial*), or $\omega(G) \geq \frac{|V(G)|}{4}$. In 2.1 and 2.2 we show that both these properties are useful in proving that the conclusion of 1.2 holds for G.

2.1 *Let G be tame. If $\omega(G) \geq \frac{|V(G)|}{4}$, then $\chi(G) \leq 2\omega(G)$.*

Proof. Let $|V(G)| = n$ and let k be the maximum size of a matching in $\overline{G}$. Then $\chi(G) \leq n - k$.

(1) *If $k > \frac{n}{2} - 1$ then the theorem holds.*

If $k > \frac{n}{2} - 1$ then

$$\chi(G) \leq n - k < \frac{n}{2} + 1 \leq 2\omega(G) + 1,$$

and the theorem holds. This proves (1).

From (1) we may assume that $k \leq \frac{n}{2} - 1$. By the Tutte-Berge formula [9], there exists a set $X \subseteq V(G)$ such that $\overline{G \setminus X}$ has $t = |X| + n - 2k$ components, all with an odd number of vertices. Let the components be $Y_1, \ldots, Y_t$. Thus these are induced subgraph of $\overline{G}$.

(2) *For $1 \leq i \leq t$, Y_i contains no triangle and therefore $\chi(Y_i) \leq 4$.*

For G is claw-free, and therefore in $\overline{G}$, for every triangle T and every vertex $v \notin T$, v has a neighbour in T. Since $Y_1, \ldots, Y_t$ are components of $\overline{G} \setminus X$, if some Y_i contains a triangle then $i = t = 1$. But $t = |X| + n - 2k \geq |X| + 2 \geq 2$, a contradiction. This proves the first assertion of (2). The second assertion follows from the first by 1.7. This proves (2).

From (2) and since $|V(Y_i)|$ is odd for all $1 \leq i \leq t$, it follows that each Y_i contains a stable set of size strictly greater than $\frac{|V(Y_i)|}{4}$. In G this means that $\omega(G|(V(Y_i))) > \frac{|V(Y_i)|}{4}$. Now, since Y_i are components of $\overline{G \setminus X}$, it follows that $V(Y_i)$ is complete to $V(Y_j)$ in G for all $1 \leq i < j \leq t$, and hence

$$4\omega(G) \geq \sum_{i=1}^{t} 4\omega(G|(V(Y_i))) \geq \sum_{i=1}^{t}(|V(Y_i)| + 1) \geq n - |X| + t = n - |X| + |X| + n - 2k = 2n - 2k.$$

Thus $\chi(G) \leq n - k \leq 2\omega(G)$ and the theorem holds. This proves 2.1. ∎

2.2 *Let G be a claw-free graph and let $v \in V(G)$ be bisimplicial. If $\chi(G \setminus v) \leq 2\omega(G \setminus v)$, then $\chi(G) \leq 2\omega(G)$, and if $ch(G \setminus v) \leq 2\omega(G \setminus v)$, then $ch(G) \leq 2\omega(G)$.*

Proof. First we prove the first statement of 2.2. Let c be a colouring of $G \setminus v$ with at most $2\omega(G)$ colours. Since v is bisimplicial, at most $2(\omega(G) - 1)$ colours appear in $N(v)$, and so there is a colour that does not appear in $N(v)$. Therefore, the colouring of $G \setminus v$ can be extended to a colouring of G, and $\chi(G) \leq 2\omega(G)$. This proves the first assertion of 2.2.

Let us now prove the second assertion. Let $\{L_u\}_{u \in V(G)}$ be a set of lists such that $|L_u| \geq 2\omega(G)$ for every $u \in V(G)$. Then $G \setminus v$ can be coloured from these lists. Since v is bisimplicial, at most $2(\omega(G) - 1)$ colours appear in $N(v)$, and so there is a colour in L_v that does not appear in $N(v)$. Therefore, the colouring of $G \setminus v$ can be extended to a colouring of G, and $ch(G) \leq 2\omega(G)$. This completes the proof of 2.2. ∎

3 The structure of claw-free graphs

The goal of this section is to state and prove a structural lemma about claw-free graphs that we will later use to prove our main result. The proof of the lemma relies on (an immediate corollary of) the main result of [4], and we start with definitions necessary to state it.

Let G be a graph, and let F be a set of unordered pairs of distinct vertices of G such that every vertex belongs to at most one member of F. Then H is a *thickening* of (G, F) if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$ satisfying the following:

- for each $v \in V(G)$, X_v is a clique of H
- if $u, v \in V(G)$ are adjacent in G and $\{u, v\} \notin F$, then X_u is complete to X_v in H
- if $u, v \in V(G)$ are nonadjacent in G and $\{u, v\} \notin F$, then X_u is anticomplete to X_v in H
- if $\{u, v\} \in F$ then X_u is neither complete nor anticomplete to X_v in H.

First we list some classes of claw-free graphs that are needed for the statement of the structure theorem from [4].

- **Graphs from the icosahedron.** The *icosahedron* is the unique planar graph with twelve vertices all of degree five. Let it have vertices $v_0, v_1, \ldots, v_{11}$, where for $1 \leq i \leq 10$, v_i is adjacent to v_{i+1}, v_{i+2} (reading subscripts modulo 10), and v_0 is adjacent to v_1, v_3, v_5, v_7, v_9, and v_{11} is adjacent to $v_2, v_4, v_6, v_8, v_{10}$. Let this graph be G_0. Let G_1 be obtained from G_0 by deleting v_{11} and let G_2 be obtained from G_1 by deleting v_{10}. Furthermore, let $F' = \{\{v_1, v_4\}, \{v_6, v_9\}\}$.

  Let $G \in \mathcal{T}_1$ if G is a thickening of $(G_0, \emptyset)$, $(G_1, \emptyset)$, or (G_2, F) for some $F \subseteq F'$.

- **Fuzzy long circular interval graphs.** Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \ldots, F_k$ share an endpoint, and no three of them have union Σ. Now let $V \subseteq \Sigma$ be finite, and let H be a graph with vertex set V in which distinct $u, v \in V$ are adjacent precisely if $u, v \in F_i$ for some i.

  Let F' be the set of pairs $\{u, v\}$ such that $u, v \in V$ are distinct endpoints of F_i for some i. Let $F \subseteq F'$. Then G is a *fuzzy long circular interval graph* if for some such H and F, G is a thickening of (H, F).

  Let $G \in \mathcal{T}_2$ if G is a fuzzy long circular interval graph.

- **Fuzzy antiprismatic graphs.** A graph H is called *antiprismatic* if for every triad T and every vertex $v \in V(H) \setminus T$, v has exactly two neighbours in T. Let u, v be two vertices of an antiprismatic graph H. We say that the pair $\{u, v\}$ is *changeable* if u is non-adjacent to v, and the graph obtained from G by adding the edge uv is also antiprismatic. Let H be an antiprismatic graph and let F be a set of changeable pairs of H such that every vertex of H belongs to at most one member of H. We say that a graph G is a *fuzzy antiprismatic graph* if G is a thickening of (H, F).

  Let $G \in \mathcal{T}_3$ if G is a fuzzy antiprismatic graph.

Next, we define what it means for a claw-free graph to admit a "strip-structure". A *hypergraph* H consists of a finite set $V(H)$, a finite set $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (that is, a subset of $V(H) \times E(H)$). For the statement of the structure theorem, we only need hypergraphs such that every member of $E(H)$ is incident with either one or two members of $V(H)$ (thus, these hypergraphs are graphs if we allow "graphs" to have loops and parallel edges). For $F \in E(H)$, let us denote by $\overline{F}$ the set of elements of $V(H)$ incident with F.

Let G be a graph. A *strip-structure* (H, η) of G consists of a hypergraph H with $E(H) \neq \emptyset$, and a function η mapping each $F \in E(H)$ to a subset $\eta(F)$ of $V(G)$, and mapping each pair (F, h) with $F \in E(H)$ and $h \in \overline{F}$ to a subset $\eta(F, h)$ of $\eta(F)$, satisfying the following conditions.

- **(S1)** The sets $\eta(F)$ $(F \in E(H))$ are nonempty and pairwise disjoint and have union $V(G)$.

- **(S2)** For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \overline{F}$ is a clique of G.

- **(S3)** For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in G, then there exists $h \in \overline{F_1} \cap \overline{F_2}$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.

- **(S4)** For every $F \in E(H)$, $h \in \overline{F}$ and $v \in \eta(F, h)$ the set of neighbours of v in $\eta(F) \setminus \eta(F, h)$ is a clique.
- **(S5)** Let $F \in E(H)$ with $|F| = 2$, say $F = \{h_1, h_2\}$. If $\eta(F, h_1) \cap \eta(F, h_2) \neq \emptyset$, then $\eta(F, h_1) \cap \eta(F, h_2) = \eta(F)$.

We say that a strip-structure is *non-trivial* if $|E(H)| \geq 2$.

The following is a corollary of the main theorem of [4].

3.1 *Let G be a connected claw-free graph. Then either*

- *$V(G)$ is the union of three cliques, or*
- *G admits a non-trivial strip-structure, or*
- *$G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.*

We also need a few definitions from [3]. ([3] deals with a class of graphs whose complements are claw-free, so for our purposes in this paper, we need to reformulate the definitions and results of [3] in terms of claw-free graphs.)

Let G be an antiprismatic graph. The *core* of G is the union of all triads of G. Let W be the core of G. For $v \in V(G) \setminus W$, *replicating* v means replacing v by several vertices, all pairwise adjacent, and otherwise with the same neighbours as v. Please note that the graph produced in this manner is still antiprismatic.

Let G have 27 vertices $\{r^i_j, s^i_j, t^i_j : 1 \leq i, j \leq 3\}$, with adjacency as follows. Let $1 \leq i, i', j, j' \leq 3$.

- If $i = i'$ or $j = j'$ then r^i_j is adjacent to $r^{i'}_{j'}$, and s^i_j is adjacent to $s^{i'}_{j'}$, and t^i_j is adjacent to $t^{i'}_{j'}$; while if $i \neq i'$ and $j \neq j'$ then the same three pairs are nonadjacent.
- If $j \neq i'$ then r^i_j is adjacent to $s^{i'}_{j'}$, and s^i_j is adjacent to $t^{i'}_{j'}$, and t^i_j is adjacent to $r^{i'}_{j'}$; while if $j = i'$ then the same three pairs are nonadjacent.

This is the *Schläfli* graph. All induced subgraphs of G are antiprismatic, and we call any such graph *Schläfli-antiprismatic.*

We need the following theorem from [3]:

3.2 *Let G be antiprismatic, with at least one triad. Then one of the following holds:*

- *there is a Schläfli-antiprismatic graph G_0 with no changeable pairs, such that G can be obtained from G_0 by replicating vertices not in the core, or*
- *for some k with $1 \leq k \leq 3$, there is a list of $4k$ cliques of G such that every vertex belongs to exactly k of them.*

We can now state the main result of this section.

3.3 *Let G be an induced subgraph of a connected claw-free graph H such that H contains a triad. Then either*

- *there exists a Schläfli-antiprismatic graph H_0 such that G is a thickening of $(H_0, \emptyset)$, or*

- *G has at least two bisimplicial vertices, or*
- $\omega(G) \geq \frac{|V(G)|}{4}$.

Proof. By 3.1, either

- $V(G)$ is the union of three cliques, or
- G admits a non-trivial strip-structure, or
- $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.

If $V(G)$ is the union of three cliques, then $\omega(G) \geq \frac{|V(G)|}{3} \geq \frac{|V(G)|}{4}$ and the theorem holds, so we may assume that one of the other outcomes holds.

Assume that G admits a non-trivial strip-structure, and let H and η be as in the definition of a strip-structure. For $h \in V(H)$ we denote by $\eta(h)$ the set $\bigcup_{F\,:\,h \in \overline{F}} \eta(F,h)$.

(1) For every $F \in E(H)$ and $h \in \overline{F}$, every vertex of $\eta(F,h)$ is bisimplicial in G.

Let $F \in E(H)$, $h \in \overline{F}$ and $v \in \eta(F,h)$. Suppose first that $\overline{F} = \{h,h'\}$ and $\eta(F,h) \cap \eta(F,h') \neq \emptyset$. By **(S5)**, $\eta(F,h) = \eta(F,h') = \eta(F)$. Consequently, by **(S3)** $N_G(v) = \eta(h) \cup \eta(h')$. But by **(S2)** each of the sets $\eta(h)$ and $\eta(h')$ is a clique, and therefore v is a bisimplicial vertex of G and (1) holds. Thus we may assume that either

- $\overline{F} = \{h\}$, or
- $\overline{F} = \{h,h'\}$ for some $h' \in V(H) \setminus \{h\}$, and $\eta(F,h) \cap \eta(F,h') = \emptyset$.

In both cases, by **(S3)**, $N_G(v) \subseteq \eta(h) \cup \eta(F)$. But, by **(S2)**, $\eta(h)$ is a clique, and by **(S4)** the set of neighbours of v in $\eta(F) \setminus \eta(h)$ is a clique. Consequently, $N_G(v)$ is the union of two cliques, and so v is a bisimplicial vertex of G. This proves (1).

By (1), and since $|E(H)| \geq 2$, it follows that G has at least two bisimplicial vertices, and 3.3 holds.

Thus we may assume that $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. Suppose $G \in \mathcal{T}_1$. Let G_0, G_1, G_2 be as in the definition of $\mathcal{T}_1$. For $0 \leq i \leq 11$, let X_{v_i} be as in the definition of a thickening, except if G is a thickening of G_1, let $X_{v_{11}} = \emptyset$, and if G is a thickening of G_2, let $X_{v_{10}} = X_{v_{11}} = \emptyset$. Let

$$C_1 = X_{v_1} \cup X_{v_9} \cup X_{v_{10}}$$

$$C_2 = X_{v_2} \cup X_{v_3} \cup X_{v_4}$$

$$C_3 = X_{v_6} \cup X_{v_8} \cup X_{v_{11}}$$

$$C_4 = X_{v_0} \cup X_{v_5} \cup X_{v_7}.$$

Then each of C_1, C_2, C_3, C_4 is a clique, and $V(G) = C_1 \cup C_2 \cup C_3 \cup C_4$, and therefore $\omega(G) \geq \frac{|V(G)|}{4}$ and the theorem holds.

If $G \in \mathcal{T}_2$, then every vertex of G is bisimplicial and again the theorem holds. Thus we may assume that $G \in \mathcal{T}_3$, and so there exists an antiprismatic graph H and a set F of changeable pairs of H such that every vertex of H is in at most one member of F, and G is a thickening of (H, F). In particular, if $\{u, v\} \in F$, then u is non-adjacent to v in H. If G contains no triad, then by 1.7, $V(G)$ is the union of four cliques, and therefore $\omega(G) \geq \frac{|V(G)|}{4}$. Thus we may assume that G contains a triad, and consequently so does H. For $v \in V(H)$, let X_v be as in the definition of a thickening. By 3.2, either

- there is a Schläfli-antiprismatic graph H_0 with no changeable pairs, such that H can be obtained from H_0 by replicating vertices not in the core, or
- for some k with $1 \leq k \leq 3$, there is a list of $4k$ cliques of H such that every vertex belongs to exactly k of them.

Suppose that there exists a Schläfli-antiprismatic graph H_0 with no changeable pairs, such that H can be obtained from H_0 by replicating vertices not in the core. Since H_0 has no changeable edges, it follows that neither does H, and so $F = \emptyset$. But now G is a thickening of $(H_0, \emptyset)$, and the theorem holds.

So we may assume that for some k with $1 \leq k \leq 3$, $C_1, \ldots, C_{4k}$ are cliques of H such that every vertex of H belongs to exactly k of them. For $i \in \{1, \ldots, 4k\}$, let $C'_i = \bigcup_{v \in C_i} X_v$. Then, since every vertex pair in F is a non-adjacent pair of H, it follows that each of the sets C'_i is a clique of G, and every vertex of G is in exactly k of them. Thus

$$\sum_{i=1}^{4k} |C'_i| = k|V(G)|,$$

and so for some i, $|C'_i| \geq \frac{|V(G)|}{4}$. Consequently, $\omega(G) \geq \frac{|V(G)|}{4}$, and the theorem holds. This completes the proof of 3.3. ∎

4 The proof of 1.2

The goal of this section is to prove 1.2. We start with a lemma.

4.1 *Let H_0 be a Schläfli-antiprismatic graph, and let G be a thickening of $(H_0, \emptyset)$. Then either $\omega(G) \geq \frac{|V(G)|}{4}$, or $\chi(G) \leq 2\omega(G)$.*

Proof. The proof is by induction on $|V(G)|$. Let $i, j \in \{1, 2, 3\}$ and let r^i_j, s^i_j, t^i_j be as in the definition of the Schläfli graph. Then $V(H_0) \subseteq \{r^i_j, s^i_j, t^i_j \ : \ 1 \leq i, j \leq 3\}$. For $v \in V(H_0)$, let X_v be as in the definition of a thickening. Let $S^i_j = X_{s^i_j}$, $R^i_j = X_{r^i_j}$, $T^i_j = X_{t^i_j}$, where $S^i_j = \emptyset$ if and only if $s^i_j \notin V(H_0)$, and the same for R^i_j and T^i_j. We may assume that $\omega(G) < \frac{|V(G)|}{4}$, and therefore $|V(G)|$ is not the union of four cliques. For $Y \subseteq V(G)$, let the *width* of Y, denoted by *width*(Y), be the number of vertices v of H_0 such that $Y \cap X_v$ is non-empty.

(1) Let K be a maximal clique of G. Then width$(K) > 3$.

Suppose not. From the symmetry of the Schläfli graph [1], we may assume that $K \subseteq S^1_1 \cup S^2_1 \cup S^3_1$.

Since K is a maximum clique in G, it follows that no vertex of G is complete to $S_1^1 \cup S_1^2 \cup S_1^3$, and so $\bigcup_{j=1}^3 (T_j^2 \cup T_j^3) = \emptyset$. But now, for $i \in \{1,2,3\}$, with addition mod 3, let

$$C_i = \bigcup_{j=1}^{3} (R_i^j \cup S_j^{i+1})$$

and let $C_4 = \bigcup_{j=1}^3 T_j^1$. Then each of C_1, C_2, C_3, C_4 is a clique, and $V(G) = \bigcup_{i=1}^4 C_i$, a contradiction. This proves (1).

(2) If either

1. *no clique of G of size $\omega(G)$ has width four, or*
2. *there exists $V_1 \subseteq V(H_0)$ such that*
   - $|V(H_0)| - 1 \leq |V_1|$,
   - $\bigcup_{v \in V_1} X_v$ *includes every clique of size $\omega(G)$ and width four in G, and*
   - $\chi(H_0|V_1) \leq 8$,

then 4.1 holds.

Suppose first that there is no clique of size $\omega(G)$ and width four in G. For every $v \in V(H_0)$, let $x_v \in X_v$ and let $Y = \{x_v\}_{v \in V(H_0)}$ and let $G_1 = G \setminus Y$. Then, by (1), every clique of size $\omega(G)$ in G has width at least five, and so every maximum clique of G meets Y in at least five vertices. Also by (1), every maximal clique of G of size $\omega(G) - 1$ meets Y in at least four vertices. It follows that $\omega(G_1) \leq \omega(G) - 5$. Inductively, $\chi(G_1) \leq 2\omega(G_1)$. Since the Schläfli graph is 9-colourable, it follows that Y is the union of at most nine stable sets. But now

$$\chi(G) \leq \chi(G_1) + 9 \leq 2\omega(G_1) + 9 \leq 2(\omega(G) - 5) + 9 < 2\omega(G)$$

and 4.1 holds.

Next suppose that there exists V_1 as in the second alternative hypothesis of (2). For every $v \in V_1$, let $x_v \in X_v$ and let $Z = \{x_v\}_{v \in V_1}$. Let $G_2 = G \setminus Z$.

By (1), since $|Z| \geq |V(H_0)| - 1$, and since every maximum clique of width four in G is contained in $\bigcup_{v \in V_1} X_v$, it follows that every maximum clique of G meets Z in at least four vertices. Also by (1), every maximal clique of G of size $\omega(G) - 1$ meets Y in at least three vertices. Consequently, $\omega(G_2) \leq \omega(G) - 4$. Inductively, $\chi(G_2) \leq 2\omega(G_2)$. Since $\chi(H_0|V_1) \leq 8$, it follows that Z is the union of at most eight stable sets. But now

$$\chi(G) \leq \chi(G_2) + 8 \leq 2\omega(G_2) + 8 \leq 2(\omega(G) - 4) + 8 \leq 2\omega(G)$$

and 4.1 holds. This proves (2).

We observe that, since G is a thickening of $(H_0, \emptyset)$, if for some $v \in V(H_0)$, X_v meets a maximum clique K of G, then $X_v \subseteq K$. Let $v_0 \in V(H_0)$ be such that X_{v_0} is a subset of some clique of size $\omega(G)$ and width four in G, and subject to that with $|X_{v_0}|$ minimum (by (2), we may assume that there exists a clique of size $\omega(G)$ and width four in G). Let K_0 be a clique of size $\omega(G)$ and width four in G with $X_{v_0} \subseteq K_0$. From the symmetry of the Schläfli graph [1], we may assume that $K_0 = (\bigcup_{i=1}^3 S_3^i) \cup T_3^2$ and $v_0 = t_3^2$. By the maximality of K_0, it follows that $T_1^2 \cup T_2^2 \cup T_3^1 = \emptyset$.

(3) If for some $i \in \{1,2,3\}$, either $S_2^i = \emptyset$ or no clique of size $\omega(G)$ and width four in G includes S_2^1, then 4.1 holds.

From the symmetry, we may assume that either $S_2^1 = \emptyset$, or no clique of size $\omega(G)$ and width four in G includes S_2^1. By an earlier remark, in both cases, $K \cap S_2^1 = \emptyset$ for every clique K of size $\omega(G)$ and width four in G. Let $V_1 = V(H_0) \setminus \{s_2^1\}$. Now

$$\{r_1^1, s_1^1, t_1^1\}, \{r_3^2, s_1^3, t_2^1\}, \{r_2^3, s_2^2, t_3^2\},$$

$$\{r_2^1, s_3^2, t_1^3\}, \{r_1^2, s_3^1, t_2^3\}, \{r_3^3, s_3^3, t_3^3\},$$

$$\{r_3^1, r_2^2, r_1^3\}, \{s_1^2, s_2^3\}$$

are eight stable sets in the Schläfli graph, and their union includes V_1. Thus $\chi(H_0|V_1) \leq 8$, and 4.1 follows from (2). This proves (3).

In view of (3), we may assume that for every $i \in \{1,2,3\}$, $S_2^i \neq \emptyset$, and there exists a clique K_i of size $\omega(G)$ and width four in G, such that $S_2^i \subseteq K_i$. By the choice of K_0 and v_0, it follows that $|S_2^i| \geq |T_3^2|$ for $i \in \{1,2,3\}$. Also, since K_0 is a clique of size $\omega(G)$, and since $(K_0 \setminus T_3^2) \cup T_1^1 \cup T_2^1$ is a clique in G, it follow that $|T_1^1| + |T_2^1| \leq |T_3^2| \leq |S_2^i|$ for $i \in \{1,2,3\}$. In particular,

$$|S_2^1| \geq |T_3^2|$$

and

$$|S_2^2| \geq |T_1^1| + |T_2^1|.$$

For $i \in \{1,2,3\}$, with addition mod 3, let

$$C_i = \bigcup_{j=1}^3 (R_i^j \cup S_j^{i+1})$$

and let

$$C_4 = \bigcup_{j=1}^3 (T_j^3 \cup S_2^j).$$

Then $C_1, \ldots, C_4$ are cliques, and so, for $i \in \{1,2,3,4\}$, $|C_i| \leq \omega(G) < \frac{|V(G)|}{4}$. Thus

$$|V(G)| > \sum_{i=1}^4 |C_i| = |V(G)| + \sum_{j=1}^3 |S_2^j| - |T_1^1| - |T_2^1| - |T_3^2|$$

$$= |V(G)| + (|S_2^1| - |T_3^2|) + (|S_2^2| - |T_1^1| - |T_2^1|) + |S_2^3| \geq |V(G)|,$$

a contradiction. This proves 4.1. ∎

We are now ready to prove 1.2.

Proof of 1.2. The proof is by induction on $|V(G)|$, and so we may assume that if $G' \neq G$ is a proper induced subgraph of G, then $\chi(G') \leq 2\omega(G')$. By 3.3, either

- there exists a Schläfli-antiprismatic graph H_0 such that G is a thickening of $(H_0, \emptyset)$, or
- G has at least two bisimplicial vertices, or
- $\omega(G) \geq \frac{|V(G)|}{4}$.

If $\omega(G) \geq \frac{|V(G)|}{4}$, then 1.2 follows from 2.1, and thus we may assume that $\omega(G) < \frac{|V(G)|}{4}$.

Suppose G contains a bisimplicial vertex v. Inductively it follows that $\chi(G \setminus v) \leq 2\omega(G \setminus v)$, and 1.2 follows from 2.2.

So we may assume that there exists a Schläfli-antiprismatic graph H_0 such that G is a thickening of $(H_0, \emptyset)$. But now, since $\omega(G) < \frac{|V(G)|}{4}$, 1.2 follows from 4.1. This proves 1.2. ∎

Clearly, 1.2 implies 1.1. We remark that 1.1 is tight, in the sense that the constant 2 cannot be replaced with a smaller one. Let n be a positive integer, and let us define the graph G_n as follows:

- $V(G_n)$ is the disjoint union of the sets $\{x, y, z, w\}$, A, B, C, D
- $A = \{a_1, \dots a_n\}$, $B = \{b_1, \dots b_n\}$, $C = \{c_1, \dots c_n\}$ and $D = \{d_1, \dots d_n\}$,
- A, B, C, D are cliques.
- x is complete to $B \cup C$ and anticomplete to $A \cup D$, y is complete to $B \cup D$ and anticomplete to $A \cup C$, z is complete to $A \cup C$ and anticomplete to $B \cup D$, and w is complete to $A \cup D$ and anticomplete to $B \cup C$,
- the pairs xy, xz, wy, wz are adjacent and the pairs xw, yz are non-adjacent,
- for $i, j \in \{1, \dots, n\}$, a_i is adjacent to b_j if and only if $i = j$,
- for $i, j \in \{1, \dots, n\}$, c_i is adjacent to d_j if and only if $i = j$
- for $i, j \in \{1, \dots, n\}$, a_i is adjacent to c_j if and only if $i \neq j$
- for $i, j \in \{1, \dots, n\}$, a_i is adjacent to d_j if and only if $i \neq j$
- for $i, j \in \{1, \dots, n\}$, b_i is adjacent to c_j if and only if $i \neq j$
- for $i, j \in \{1, \dots, n\}$, b_i is adjacent to d_j if and only if $i \neq j$

Then $\overline{G_n}$ are graphs of parallel-square type defined in [3], and therefore the graphs G_n are claw-free. For every n, $|V(G_n)| = 4n + 4$. Since $\{a_1, d_1, x\}$ is a triad, each G_n contains a triad. It is easy to see that all G_n are connected. We also observe that $G_n \setminus \{x, y, z, w\}$ contains no triad, so $\chi(G_n) \geq \frac{|V(G_n)|-4}{2} = 2n$. On the other hand, $\omega(G_n) = n + 2$ (we leave checking this to the reader), and so $\chi(G_n) \geq (2 - \frac{4}{n+2})\omega(G_n)$. Thus $\{G_n\}$ is an infinite family of graphs satisfying the hypotheses of 1.1, with the ratio between the chromatic number and the clique number arbitrarily close to 2.

Finally, we show that 1.2 is tight. Let $G'_n = G_n \setminus \{x, y, z, w\}$. Then G'_n is an induced subgraph of G_n, and G'_n contains no triad. Since $|V(G'_n)| = 4n$, it follows that $\chi(G'_n) \geq 2n$. It is easy to see that $\omega(G'_n) = n$, and therefore $\chi(G'_n) = 2\omega(G'_n)$.

5 Choosability

The goal of this section is to prove the following:

5.1 *Let G be tame, and assume that G is not a thickening of $(H, \emptyset)$ for any Schläfli-antiprismatic graph H. Then $ch(G) \leq 2\omega(G)$.*

Unfortunately, we do not know what the correct bound on $ch(G)$ is if G is a thickening of $(H, \emptyset)$ for some Schläfli-antiprismatic graph H. It may be true that the bound of 5.1 holds for all tame graphs, but we do not know how to prove it. We start with a lemma (we thank Bruce Reed for helping us with the proof).

5.2 *Let G be a claw-free graph. Then $ch(G) \leq \max(\chi(G), \frac{|V(G)|}{2})$.*

Proof. Let $p = \max(\chi(G), \frac{|V(G)|}{2})$. Then G can be coloured with p colours. For a p-colouring c of G, let the *index* of c be the number of colour classes of size two in c. Let c be a colouring of G with maximum index, and let $X_1, \ldots, X_p$ be the colour classes of c.

(1) $|X_i| \leq 2$ for $i \in \{1, \ldots, p\}$.

Suppose $|X_1| \geq 3$. Since $p \geq \frac{|V(G)|}{2}$, it follows that $|X_i| \leq 1$ for some $i \in \{2, \ldots, p\}$, and we may assume that $i = 2$. Since G is claw-free, at most two vertices of X_1 have neighbours in X_2, and so some vertex $y \in X_1$ is anticomplete to X_2. But now

$$X_1 \setminus \{y\}, X_2 \cup \{y\}, X_3, \ldots, X_p$$

is a p-colouring of G with index bigger than that of c, a contradiction. This proves (1).

It follows from (1) that G is a subgraph (not necessarily induced) of the complete p-partite graph $K(2, \ldots, 2)$. By a theorem from [5], the choice number of the p-partite graph $K(2, \ldots, 2)$ is p, and therefore $ch(G) \leq p$. This proves 5.2. ∎

5.2 has the following easy corollary:

5.3 *Let G be tame. If $\omega(G) \geq \frac{|V(G)|}{4}$, then $ch(G) \leq 2\omega(G)$.*

Proof. By 1.1 $\max(\chi(G), \frac{|V(G)|}{2}) \leq 2\omega$. Since by 5.2, $ch(G) \leq \max(\chi(G), \frac{|V(G)|}{2})$, 5.3 follows. ∎

We also need the following:

5.4 *Let H be a Schläfli-antiprismatic graph and let G be a thickening of $(H, \emptyset)$. Then the set of non-neighbours of every vertex of G is the union of two cliques.*

Proof. Let $u \in V(G)$. Let $i, j \in \{1, 2, 3\}$ and let r^i_j, s^i_j, t^i_j be as in the definition of the Schläfli graph. Then $V(H) \subseteq \{r^i_j, s^i_j, t^i_j : 1 \leq i, j \leq 3\}$. For $v \in V(H)$, let X_v be as in the definition of a thickening. Let $S^i_j = X_{s^i_j}$, $R^i_j = X_{r^i_j}$, $T^i_j = X_{t^i_j}$, where $S^i_j = \emptyset$ if and only if $s^i_j \notin V(H)$, and the same for R^i_j and T^i_j. From the symmetry of the Schläfli graph [1], we may assume that $u \in S^1_1$. Let M be the set of non-neighbours of u in G. Then $M = (\bigcup_{i=1}^3 (T^1_i \cup R^i_1)) \cup S^2_2 \cup S^2_3 \cup S^3_2 \cup S^3_3$, and $R^1_1 \cup R^2_1 \cup T^1_3 \cup S^2_2 \cup S^2_3$ and $R^3_1 \cup T^1_1 \cup T^1_2 \cup S^3_2 \cup S^3_3$ are two cliques with union M. This proves 5.4. ∎

We can now prove 5.1.

Proof of 5.1. Let G be tame and not a thickening of $(H, \emptyset)$ for any Schläfli-antiprismatic graph H. The proof is by induction on $|V(G)|$, and so we may assume that if $G' \neq G$ is a proper induced subgraph of G, such that G' is not a thickening of $(H', \emptyset)$ for any Schläfli-antiprismatic graph H', then $ch(G') \leq 2\omega(G')$. Since G is not a thickening of $(H, \emptyset)$ for any Schläfli-antiprismatic graph H, it follows from 3.3 that either

- G has two bisimplicial vertices, or
- $\omega(G) \geq \frac{|V(G)|}{4}$.

If $\omega(G) \geq \frac{|V(G)|}{4}$, then 5.1 follows from 5.3. Thus we may assume that $\omega(G) < \frac{|V(G)|}{4}$, and therefore G contains two bisimplicial vertices, say u and v. Let $G' = G \setminus v$.

(1) If there exists a Schläfli-antiprismatic graph H' such that G' is a thickening of $(H', \emptyset)$, then $ch(G') \leq 2\omega(G')$.

By 5.4, the set of non-neighbours of u in G is the union of two cliques. Since u is bisimplicial, it follows that $N_{G'}(u) \cup \{u\}$ is the union of two cliques. Consequently, $V(G')$ is the union of four cliques, $\omega(G') \geq \frac{|V(G')|}{4}$. Since by 1.2, $\chi(G') \leq 2\omega(G')$, (1) follows from 5.2. This proves (1).

Now, it follows from (1) and the inductive hypothesis that $ch(G') \leq 2\omega(G')$, and 2.2 implies that $ch(G) \leq 2\omega(G)$. This proves 5.1. ∎

6 Colouring the complement

In this section we prove 1.8.

Proof of 1.8. We may assume that G is connected. Let k be the maximum size of a stable set in G. To prove 1.1 we need to show that $V(G)$ is the union of $2k$ cliques. If $k = 2$, then 1.8 follows from 1.7, and so we may assume that $k \geq 3$. Let X be a stable set of size k in G. Since G is claw-free, every vertex of $V(G) \setminus X$ has one or two neighbours in X. Define a new graph H_X with vertex set X and such that vertices $h_1, h_2 \in X$ are adjacent in H_X if in G they have a common neighbour in $V(G) \setminus X$.

(1) *X can be chosen so that H_X is connected.*

Choose a pair (X, C), where C is a component of H_X, with $|V(C)|$ maximum over all such pairs. We claim that H_X is connected. Since C is a component of H, it follows that in G, no vertex $v \in V(G) \setminus X$ has both a neighbour in $V(C)$ and a neighbour in $X \setminus V(C)$. Since G is connected, there exist two adjacent vertices $a, b \in V(G) \setminus X$, such that a has a neighbour in $V(C)$ and b has a neighbour in $X \setminus V(C)$. Since $\{a, b, c_1, c_2\}$ is not a claw in G for distinct $c_1, c_2 \in N(a) \cap V(C)$, we deduce that $|N(a) \cap V(C)| = 1$ and similarly $|N(b) \cap (X \setminus V(C))| = 1$. Let a' be the neighbour of a

in $V(C)$ and let b' be the neighbour of b in $X \setminus V(C)$. Now $X' = X \cup \{b\} \setminus \{b'\}$ is a stable set of size k in G, and the set $V(C) \cup b$ is connected in $H_{X'}$, contrary to the choice of (X, C). This proves (1).

Let X be a stable set of size k in G such that $H = H_X$ is connected. For a vertex $x \in X$ denote by $A(x)$ the set of vertices in $V(G) \setminus X$ adjacent to x and to no other vertex of X. For an edge xy of H denote by $A(xy)$ the set of vertices in $V(G) \setminus X$ adjacent to both x and y. Let $A[x] = A(x) \cup (\bigcup_{xy \in E(H)} A(xy))$.

(2) *Let x be a vertex of H and let $y_1, \dots, y_n$ be the neighbours of x in H with $n \geq 1$. Then $\{x\} \cup (A[x] \setminus A(xy_1))$ is the union of two cliques.*

Since $xy_1 \in E(H)$, it follows that $A(xy_1) \neq \emptyset$. Suppose $u, v \in A[x] \setminus A(xy_1)$ are non-adjacent. Since $\{x, u, v, a\}$ is a claw in G for every $a \in A(xy_1) \setminus (N(u) \cup N(v))$, it follows that $A(xy_1) \subseteq of N(u) \cup N(v)$. Since $\{a, u, v, y_1\}$ is a claw for every $a \in A(xy_1) \cap N(u) \cap N(v)$, it follows that $A(xy_1) \cap N(u) \cap N(v) = \emptyset$. So every vertex in $A(xy_1)$ is adjacent to exactly one of u, v and $A[x] \setminus A(xy_1)$ is the union of two cliques by 1.5. Since every vertex of $A[x] \setminus A(xy_1)$ is adjacent to x, it follows that $\{x\} \cup A[x] \setminus A(xy_1)$ is the union of two cliques. This proves (2).

(3) *If H is not a tree then the theorem holds.*

Let C be a cycle in H. Let H' be a maximal subgraph of H with $V(H) = V(H')$, such that C is the unique cycle in H'. Direct the edges of H' so that C is a directed cycle and every path with an end in $V(C)$ and no other vertex in $V(C)$ is directed away from C. By (2) for every edge ab of H' directed from b to a, the set of vertices of $V(G) \setminus X$ adjacent to a and not to b, together with $\{a\}$, is the union of two cliques. Also, since every vertex of $V(G) \setminus X$ has a neighbour in X, and no vertex of $V(G) \setminus X$ is complete to $V(C)$ (since $V(C)$ is a stable set of size at least three in G), it follows that for every vertex $v \in V(G) \setminus X$, there exists an edge ab of H' directed from b to a such that v is adjacent to a and not b. But now, by (2), $V(G)$ is the union of $2|E(H')| \leq 2|V(H)| = 2k$ cliques and the theorem holds. This proves (3).

By (3) we may assume that H is a tree. Since $|X| \geq 3$, it follows that some vertex of H has degree at least 2. Let r be a vertex of degree at least two in H. Direct the edges of H so that r has in-degree zero, and every path of H starting at r is a directed path. Then every vertex $x \in X \setminus \{r\}$ has exactly one in-neighbour. Denote this in-neighbour by $i(x)$. Since X is a maximum stable set in G, and $|X| \geq 3$, it follows that every vertex of $V(G) \setminus X$ has both a neighbour and a non-neighbour in X. Therefore, for every vertex $v \in V(G) \setminus X$, either v is adjacent to r, or there exists $x \in X \setminus \{r\}$, such that v is adjacent to x and non-adjacent to $i(x)$.

Now

$$V(G) = (\bigcup_{x \in X \setminus \{r\}} (\{x\} \cup (A[x] \setminus A(xi(x))))) \cup \{r\} \cup A[r].$$

Applying (2) twice we deduce that $A[r] \cup \{r\}$ is the union of four cliques. Let L be the set of leaves of H. Then $|L| \geq 2$. By the definition of H no vertex of $V(G) \setminus X$ is adjacent to two members of L, and by the maximality of X, for all $x \in L$ the set $(\{x\} \cup A[x]) \setminus A(xi(x)) = \{x\} \cup A(x)$ is a clique. By (2) applied to each $x \in X \setminus (L \cup \{r\})$, it follows that $\{x\} \cup (A[x] \setminus A(xi(x)))$ is the union of two cliques. So $V(G)$ is the union of $2(k-1-|L|) + 4 + |L| = 2k + 2 - |L| \leq 2k$ cliques. This completes the proof of 1.8. ∎

We remark that the constant 2 in 1.8 is best possible. For every positive integer n, let G_n be the line graph of the compete graph on $2n+1$ vertices. Then G_n is claw-free, the size of the maximum stable set in G_n is n and $\chi(\overline{G_n}) = 2n-1$. This suggests that $\chi(\overline{G})$ may be bounded above by $2\omega(\overline{G}) - 1$ for every tame graph G. However, this is false, since if G is the Schläfli graph, then $\omega(\overline{G}) = 3$ and $\chi(\overline{G}) = 6$.

There remains an obvious question: can we bound the choice number of complements of tame graphs by some function of their clique number? Next we construct a family of graphs that shows that no such function exists, and so there is no analogue of 5.1 for complements of tame graphs. Let G_n be defined as follows. Let $V(G_n) = A_n \cup B_n \cup \{v_n\}$ where A_n and B_n are disjoint cliques and $v_n \notin A_n \cup B_n$. Moreover, there exist $x_n \in A_n$ and $y_n, z_n \in B$ such that $x_n y_n, z_n v_n \in E(G_n)$, and there are no other edges in G_n. Then G_n is a tame graph, $\omega(\overline{G_n}) = 3$, and, since $\overline{G_n}$ contains the complete bipartite graph $K_{n-1,n-1}$, it follows that $ch(\overline{G_n})$ tends to infinity with n.

References

[1] P. Cameron, "Strongly regular graphs", in *Topics in Algebraic Graph Theory* (ed. L. W. Beineke and R. J. Wilson), Cambridge Univ. Press, Cambridge, 2004, 203-221.

[2] M. Chudnovsky and P. Seymour, "Claw-free graphs I. Orientable prismatic graphs", *J. Combin. Theory B* 97 (2007), 866-901.

[3] M. Chudnovsky and P. Seymour, "Claw-free graphs II. Non-orientable prismatic graphs", *J. Combin. Theory B* 98 (2008), 812-834.

[4] M. Chudnovsky and P. Seymour, "Claw-free graphs V. Global structure", *J. Combin. Theory B* 98 (2008), 1373-1410.

[5] P. Erdös, A.L. Rubin, H. Taylor, "Choosability in graphs", *Congr. Numer.* 26 (1980), 125-157.

[6] P. Hall, "On representation of subsets", *J. Lond. Math. Sc.* 10 (1935), 26-30.

[7] J.H. Kim, "The Ramsey number $R(3,t)$ has order of magnitude $\frac{t^2}{\log t}$" *Random Structures and Algorithms* 7 (1995), 173-207.

[8] V. Vizing, "On an estimate of the chromatic class of a p-graph", *Diskret. Analiz* 3(1964), 25-30.

[9] C. Berge, "Sur le couplage maximum d'un graphe", *C.R. Acad. Sci. Paris*, 247 (1958), 258–259.