Claw-free Graphs. III. Circular interval graphs

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Abstract

Construct a graph as follows. Take a circle, and a collection of intervals from it, no three of which have union the entire circle; take a finite set of points V from the circle; and make a graph with vertex set V in which two vertices are adjacent if they both belong to one of the intervals. Such graphs are "long circular interval graphs", and they form an important subclass of the class of all claw-free graphs. In this paper we characterize them by excluded induced subgraphs. This is a step towards the main goal of this series, to find a structural characterization of all claw-free graphs.

This paper also gives an analysis of the claw-free graphs G with a clique the deletion of which disconnects G into two parts both with at least two vertices.

1 Introduction

Let G be a graph. (All graphs in this paper are finite and simple.) If $X \subseteq V(G)$, the subgraph $G|X$ *induced on* X is the subgraph with vertex set X and edge-set all edges of G with both ends in X. ($V(G)$ and $E(G)$ denote the vertex- and edge-sets of G respectively.) We say that $X \subseteq V(G)$ is a *claw in* G if $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$. We say G is *claw-free* if no $X \subseteq V(G)$ is a claw in G. Our objective in this series of papers is to show that every claw-free graph can be built starting from some basic classes by means of some simple constructions. One important subclass of claw-free graphs are the "long circular interval graphs", the main topic of this paper.

Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be "intervals", that is, each is homeomorphic to the interval $[0, 1]$. Now let $V \subseteq \Sigma$ be finite, and let G be the graph with vertex set V in which two vertices are adjacent if and only if some F_i contains both of them. Such a graph G is called a *circular interval graph*. If in addition no three of $F_1, \ldots, F_k$ have union Σ, we call G a *long circular interval graph*. (They are not to be confused with what are called "circular arc graphs"; one is a proper subclass of the other.)

Long circular interval graphs are claw-free, and these together with line graphs turn out to be the two "principal" basic classes of claw-free graphs. Our (lengthy) proof of that fact includes a characterization of long circular interval graphs by excluded induced subgraphs, and it is convenient to spin that off into a separate paper, the present paper. Incidentally, one might ask why we insist that no three of the intervals $F_1, \ldots, F_k$ mentioned above can have union the circle. If we omit this condition, the graphs we produce are still claw-free; but we do not know how to characterize this larger class by excluded induced subgraphs, and for our application we do not need to do so.

One form of our main result is the following (the terms used are defined later in terms of "trigraphs", but here have their conventional graph-theory meaning).

1.1 *Let G be a graph. Then G is a long circular interval graph if and only if no induced subgraph is a claw, net, antinet or $(1, 1, 1)$-prism, and every hole is dominating, and has no centre.*

However, for our application in [1], we need to prove something stronger, connecting the size of the largest hole in G with the type of excluded subgraph that we use. This needs a number of further definitions, and we postpone the precise statement until 3.3.

It is advantageous for the application to work, not just with graphs, but with slightly more general objects that we call "trigraphs". In a graph, every pair of vertices are either adjacent or nonadjacent, but in a trigraph, some pairs may be "undecided". For our purposes, we may assume that this set of undecided pairs is a matching. Thus, let us say a *trigraph* G consists of a finite set $V(G)$ of vertices, and a map $\theta_G : V(G)^2 \to \{1, 0, -1\}$, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v), \theta_G(u, w) = 0$.

We call θ_G the *adjacency function* of G. For distinct u, v in $V(G)$, we say that u, v are *strongly adjacent* if $\theta_G(u, v) = 1$, *strongly antiadjacent* if $\theta_G(u, v) = -1$, and *semiadjacent* if $\theta_G(u, v) = 0$. We say that u, v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. Also, we say u is adjacent to v and u is a *neighbour* of v if u, v are adjacent; u is antiadjacent to v and u is an *antineighbour* of v if u, v are antiadjacent. We denote by $F(G)$ the set of all pairs $\{u, v\}$ such that $u, v \in V(G)$ are distinct and semiadjacent. Thus a trigraph G is a graph if $F(G) = \emptyset$.

For a vertex a and a set $B \subseteq V(G) \setminus \{a\}$ we say that a is *complete* to B or B*-complete* if a is adjacent to every vertex in B; and that a is *anticomplete* to B or B*-anticomplete* if a has no neighbour in B. For two disjoint subsets A and B of $V(G)$ we say that A is *complete*, respectively *anticomplete*, to B if every vertex in A is complete, respectively anticomplete, to B. Similarly, we say that a is *strongly complete* to B if a is strongly adjacent to every member of B, and so on.

Let G be a trigraph. A *clique* in G is a subset $X \subseteq V(G)$ such that every two members of X are adjacent, and a *strong clique* is a subset such that every two of its members are strongly adjacent. A set $X \subseteq V(G)$ is *stable* if every two of its members are antiadjacent, and *strongly stable* if every two of its members are strongly antiadjacent.

If $X \subseteq V(G)$, we define the trigraph $G|X$ *induced on* X as follows. Its vertex set is X, and its incidence function is the restriction of θ_G to X^2. Isomorphism for trigraphs is defined in the natural way, and if G, H are trigraphs, we say that G *contains* H if there exists $X \subseteq V(G)$ such that H is isomorphic to $G|X$.

An induced subtrigraph $G|X$ of G is said to be a *path from u to v* if $|X| = n$ for some $n \geq 1$, and X can be ordered as $\{p_1, \ldots, p_n\}$, satisfying

- $p_1 = u$ and $p_n = v$
- p_i is adjacent to p_{i+1} for $1 \leq i < n$, and
- p_i is antiadjacent to p_j for $1 \leq i, j \leq n$ with $i + 2 \leq j$.

We say it has *length* $n - 1$. (Thus it has length 0 if and only if $u = v$.) It is often convenient to describe such a path by the sequence p_1-p_2-$\cdots$-p_n. Note that the sequence is uniquely determined by the set $\{p_1, \ldots, p_n\}$ and the vertices u, v, because $F(G)$ is a matching.

A *hole* in G is an induced subtrigraph C with n vertices for some $n \geq 4$, whose vertex set can be ordered as $\{c_1, \ldots, c_n\}$, satisfying

- c_i is adjacent to c_{i+1} for $1 \leq i < n$, and also c_n is adjacent to c_1, and
- c_i is antiadjacent to c_j for $1 \leq i, j \leq n$ with $i + 2 \leq j$ and $(i, j) \neq (1, n)$.

Again, it is often convenient to describe C by the sequence c_1-c_2-$\cdots$-c_n-c_1, and we say it has *length* n. The sequence is uniquely determined by a knowledge of $V(C)$, up to choice of the first term and up to reversal. An n*-hole* means a hole of length n. A *centre* for a hole C is a vertex in $V(G) \setminus V(C)$ that is adjacent to every vertex of the hole. A hole C is *dominating* in G if every vertex in $V(G) \setminus V(C)$ has a neighbour in C.

A *claw* is a trigraph with four vertices a_0, a_1, a_2, a_3, such that $\{a_1, a_2, a_3\}$ is stable and a_0 is complete to $\{a_1, a_2, a_3\}$. (Thus, for example, if $a_0, a_1, a_2, a_3 \in V(G)$, and a_0a_3 and a_1a_2 are semi-adjacent pairs, and a_0a_1, a_0a_2 are strongly adjacent, and a_1a_3, a_2a_3 are strongly antiadjacent, then $G|\{a_0, a_1, a_2, a_3\}$ is a claw.) If $X \subseteq V(G)$ and $G|X$ is a claw, we often loosely say that X is a claw; and if no induced subtrigraph of G is a claw, we say that G is *claw-free.*

A *net* is a trigraph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$, such that $\{a_1, a_2, a_3\}$ is a clique and a_i, b_i are adjacent for $i = 1, 2, 3$, and all other pairs are antiadjacent. An *antinet* is the "complement trigraph" of a net; that is, a trigraph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$, such that $\{a_1, a_2, a_3\}$ is stable and a_i, b_i are antiadjacent for $i = 1, 2, 3$, and all other pairs are adjacent. A $(1,1,1)$-*prism* is a trigraph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$, such that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are cliques, and a_i, b_i are adjacent for $i = 1, 2, 3$, and all other pairs are antiadjacent.

Let Σ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$. Assume that no three of $F_1, \ldots, F_k$ have union Σ, and no two of $F_1, \ldots, F_k$ share an end-point. Now let $V \subseteq \Sigma$ be finite, and let G be a trigraph with vertex set V in which, for distinct $u, v \in V$,

- if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent
- if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

Such a trigraph G is called a *long circular interval trigraph.*

Let G be a trigraph, and let D be a directed graph. We say that D is a *direction* of G if $V(D) = V(G)$, and distinct u, v are adjacent in D if and only if they are adjacent in G. If $u, v \in V(G)$, we write $uv \in E(D)$ to mean that there is an edge of D between u, v, and u is its tail and v its head in D.

Let Σ be a circle, and assign an orientation to it called "clockwise". Let G be a trigraph, and let ϕ be a map from $V(G)$ into Σ; then for $\sigma \in \Sigma$, $\phi^{-1}(\sigma)$ denotes the set $\{v \in V(G) : \phi(v) = \sigma\}$. Let D be a direction of G, and suppose that the following conditions are satisfied:

- For each $\sigma \in \Sigma$, $\phi^{-1}(\sigma)$ is a strong clique of G.
- Every directed cycle in D has length at least four.
- If $\sigma_1, \sigma_2 \in \Sigma$ are distinct, there do not exist $u_i, v_i \in \phi^{-1}(\sigma_i)$ $(i = 1, 2)$ such that $u_1v_2, u_2v_1 \in E(D)$.
- If $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ are distinct and in clockwise order, let $X_i = \phi^{-1}(\sigma_i)$ $(i = 1, 2, 3)$, and suppose that there exist $u \in X_1$ and $v \in X_3$ such that $uv \in E(D)$. Then X_2 is strongly complete to $X_1 \cup X_3$ in G. Moreover, in D the edges between X_1, X_2 all have tail in X_1, and the edges between X_2, X_3 all have tail in X_2.
- For each $\sigma_1 \in \Sigma$, if $|\phi^{-1}(\sigma_1)| > 1$ then there exists $\sigma_2 \in \Sigma$ with $\sigma_2 \neq \sigma_1$ such that $\phi^{-1}(\sigma_1)$ is neither strongly complete nor strongly anticomplete to $\phi^{-1}(\sigma_2)$. Moreover, for any such σ_2, every vertex in $\phi^{-1}(\sigma_1)$ has both a neighbour and an antineighbour in $\phi^{-1}(\sigma_2)$.
- For each $\sigma_1 \in \Sigma$ there is at most one $\sigma_2 \in \Sigma$ with $\sigma_2 \neq \sigma_1$ such that $\phi^{-1}(\sigma_1)$ is neither strongly complete nor strongly anticomplete to $\phi^{-1}(\sigma_2)$.

We call such a pair (ϕ, D) a *metacircular arrangement* of G; and if G admits such a pair (ϕ, D) we say that G is *metacircular.* Note that ϕ can be chosen to be an injection if and only if G is a long circular interval trigraph; so metacircular trigraphs are just a slight variant of long circular interval trigraphs. If (ϕ, D) is a metacircular arrangement of G, and C_0 is a hole in G, we say that C_0 is a *basis* for the arrangement if the points $\phi(v)$ $(v \in V(C_0))$ are all distinct.

2 Circle arrangements

Being metacircular implies being claw-free, but it is convenient to relax the definition of "metacircular" further, to a condition which is much easier to recognize, but which no longer implies being claw-free, as follows.

As before, let Σ be a circle, and assign an orientation to it called "clockwise". If ϕ is a map from $V(G)$ into Σ, we say a triple (u, v, w) of vertices of G is *clockwise* if $\phi(u), \phi(v), \phi(w)$ are distinct and in clockwise order in Σ. By a *circle arrangement* of a trigraph G we mean a pair (ϕ, D), where ϕ is a map from $V(G)$ into Σ, and D is a direction of G, satisfying the following conditions:

- for all $\sigma \in \Sigma$, $\phi^{-1}(\sigma)$ is a strong clique of G
- every directed cycle in D has length at least four
- for every clockwise triple (u, v, w), if $uw \in E(D)$ then v is strongly adjacent to u, w in G, and $uv, vw \in E(D)$
- if $u, v, w \in V(G)$ and $uv, vw \in E(D)$, and $\phi(u) = \phi(w)$, then $\phi(u) = \phi(v)$.

We call the first condition the *first axiom* and so on. If C_0 is a hole in G, we say it is a *basis* for the circle arrangement if the points $\phi(v)$ $(v \in V(C_0))$ are all distinct. The next result implies that to show that a claw-free trigraph is metacircular, it suffices to exhibit a circle arrangement.

2.1 *Every claw-free trigraph G that admits a circle arrangement is metacircular; and indeed, for every circle arrangement (ϕ_0, D_0) of G, there is a metacircular arrangement (ϕ, D) of G such that every basis for (ϕ_0, D_0) is a basis for (ϕ, D).*

Proof. If (ϕ, D) is a circle arrangement of G, we define $\Sigma(\phi)$ to be the set of all $\sigma \in \Sigma$ such that $\phi^{-1}(\sigma) \neq \emptyset$. Let (ϕ_0, D_0) be a circle arrangement of G; and choose a circle arrangement (ϕ, D) such that every basis for (ϕ_0, D_0) is a basis for (ϕ, D), and $|\Sigma(\phi)|$ is maximum with this property. (We call this property the "maximality" of (ϕ, D).) We shall prove that (ϕ, D) is in fact a metacircular arrangement. For a vertex u, let $A(u), B(u)$ be the sets of all $v \in V(G)$ such that $\phi(u) \neq \phi(v)$ and $uv \in E(D)$ (respectively, $vu \in E(D)$). Let $A^*(u)$ be the set of all $v \in A(u)$ such that u, v are strongly adjacent in G, and define $B^*(u)$ similarly. (Thus $A^*(u)$ includes all of $A(u)$ except possibly one vertex.)

(1) *For all $s \in \Sigma(\phi)$, and all $X \subseteq \phi^{-1}(s)$, let $Y = \phi^{-1}(s) \setminus X$; if X, Y are both nonempty, then there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A^*(y)$, or $B(y) \not\subseteq B^*(x)$.*

For suppose not. Choose two points r, t of $\Sigma \setminus \Sigma(\phi)$, immediately before and after s; more precisely, such that r, s, t are in clockwise order, and t, s', r are in clockwise order for all $s' \in \Sigma(\phi) \setminus \{s\}$. For $v \in V(G)$, define $\phi'(v)$ as follows: if $v \in X$, $\phi'(v) = r$; if $v \in Y$, $\phi'(v) = t$; and otherwise $\phi'(v) = \phi(v)$. Define a new direction D' of G as follows: for each edge $e = yx$ of D with $x \in X$ and $y \in Y$, redirect e so that x is its tail. For all other edges e let the tail of e in D' be the same as its tail in D. We claim that the pair (ϕ', D') is a circle arrangement of G. Let us check the axioms. The first axiom clearly holds. For the second, suppose that $uv, vw, wu \in E(D')$. Then one of them does not belong to $E(D)$, say wu, and so $w \in X$ and $u \in Y$. Since $uv \in E(D')$ it follows that $v \notin X$, and similarly $v \notin Y$; and so $v \notin \phi^{-1}(s)$. Consequently $uv, vw \in E(D)$; but then (ϕ, D) fails to satisfy the fourth axiom, a contradiction. For the third axiom, suppose that (u, v, w) is clockwise (with respect to ϕ'), and $uw \in E(D')$. If at most one of u, v, w belong to $\phi^{-1}(s)$, then (u, v, w) is also clockwise with respect to ϕ, and $uw \in E(D)$, and hence $uv, vw \in E(D)$ since (ϕ, D) satisfies the third axiom; but then $uv, vw \in E(D')$ as required. We may therefore assume that at least two of u, v, w belong to $\phi^{-1}(s)$. Since (u, v, w) is clockwise with respect to ϕ', at most one of u, v, w is in X and at most one in Y. There are therefore three possibilities: $u \in X$, $v \in Y$ and $w \notin \phi^{-1}(s)$; or $v \in X$, $w \in Y$ and $u \notin \phi^{-1}(s)$; or $w \in X, u \in Y$ and $v \notin \phi^{-1}(s)$. In the first case, u, v are strongly adjacent, since $X \cup Y$ is a strong clique; $uv \in E(D')$ from the definition of D'; also, $w \in A(u)$, and since the claim of (1) is false, $w \notin A(u) \setminus A^*(v)$, and consequently $vw \in E(D)$ and v, w are strongly adjacent; and therefore the axiom is satisfied. The second case is similar. The third case cannot occur since $uw \in E(D')$. Thus (ϕ', D') satisfies the third axiom. Finally, for the fourth axiom, suppose that $u, v, w \in V(G)$ and $uv, vw \in E(D')$, and $\phi'(u) = \phi'(w)$. Consequently $\phi(u) = \phi(w)$. We claim that $\phi(u) = \phi(v)$. For if $uv, vw \in E(D)$ then $\phi(u) = \phi(v)$, since (ϕ, D) satisfies the fourth axiom; and otherwise $X \cup Y$ contains v and at least one of u, w, and therefore contains both of u, w since $\phi(u) = \phi(w)$, and so again $\phi(u) = \phi(v)$. Thus in either case $\phi(u) = \phi(v) = \phi(w)$, and we may therefore assume that $u, v, w \in X \cup Y$. Since $\phi'(u) = \phi'(w)$, both u, w belong to X or both to Y; and since $uv, vw \in E(D')$, it follows that v belongs to the same set, and therefore $\phi'(u) = \phi'(v)$. This verifies the fourth axiom, and therefore proves that (ϕ', D') is a circle arrangement of G. But $|\Sigma(\phi)| < |\Sigma(\phi')|$, since X, Y are nonempty, and every basis for (ϕ, D) is a basis for (ϕ', D'), contrary to the maximality of (ϕ, D). This proves (1).

(2) *If $u, v \in V(G)$ and $\phi(u) = \phi(v)$ then either $A(u) \subseteq A^*(v)$ or $B(u) \subseteq B^*(v)$.*

Suppose not. Then there exist a vertex $w \in A(u) \setminus A^*(v)$ and a vertex $x \in B(u) \setminus B^*(v)$. Consequently, $uw, xu \in E(D)$, and $\phi(w), \phi(x) \neq \phi(u) = \phi(v)$. The fourth axiom therefore implies that $wv, vx \notin E(D)$. If v, w are strongly adjacent, it follows that $vw \in E(D)$ and so $w \in A^*(v)$, a contradiction; so v, w are antiadjacent, and similarly so are v, x. Since $\phi(u) = \phi(v)$, the first axiom implies that u, v are adjacent in G. Since $\{u, v, w, x\}$ is not a claw, it follows that w, x are strongly adjacent in G. By the second axiom, $wx \notin E(D)$, and so $xw \in E(D)$. Now there are three subcases: (x, v, w) is clockwise; (v, x, w) is clockwise; and $\phi(w) = \phi(x)$. If (x, v, w) is clockwise, the third axiom implies that v, w are strongly adjacent, a contradiction; if (v, x, w) is clockwise, then (u, x, w) is clockwise, and the third axiom applied to uw implies that $ux \in E(D)$, which is false since $xu \in E(D)$; and if $\phi(x) = \phi(w)$, the fourth axiom is violated. This proves (2).

(3) *Let $s \in \Sigma(\phi)$. Then either $A(u) = A^*(v)$ for all $u, v \in \phi^{-1}(s)$, or $B(u) = B^*(v)$ for all $u, v \in \phi^{-1}(s)$.*

For let H be the digraph with vertex set $\phi^{-1}(s)$, and edge set all pairs (u, v) with $u, v \in \phi^{-1}(s)$ (possibly with $u = v$) such that $B(v) \not\subseteq B^*(u)$. Suppose first that some $w \in V(H)$ has outdegree zero in H (that is, no edge of H has tail w.) In particular, $(w, w) \notin E(H)$, and so $B(w) = B^*(w)$. Let X be the set of all vertices $x \in V(H)$ such that $B(x) = B^*(x) = B(w)$. If $X = V(H)$ then the claim holds, so we may assume that $V(H) \neq X$. By (1), there exist $x \in X$ and $y \in V(G) \setminus X$ such that either $A(x) \not\subseteq A^*(y)$, or $B(y) \not\subseteq B^*(x)$. Since $(w, y) \notin E(H)$, it follows that $B(y) \subseteq B^*(w) = B^*(x)$, and so $A(x) \not\subseteq A^*(y)$. From (2), $B(x) \subseteq B^*(y)$, and therefore $B(y) = B^*(y) = B(x) = B(w)$, contradicting that $y \notin X$.

We may therefore assume that every vertex of H has positive outdegree in H. Suppose that H is not strongly connected. Then there is a subset $X \subseteq V(H)$ such that $X, V(H) \setminus X \neq \emptyset$, and the restriction (denoted $H|X$) of H to X is strongly connected, and there is no $(u,v) \in E(H)$ with $u \in X$ and $v \in V(G) \setminus X$. By (1), there exist $x \in X$ and $y \in V(G) \setminus X$ such that either $A(x) \not\subseteq A^*(y)$, or $B(y) \not\subseteq B^*(x)$. Since $(x,y) \notin E(H)$, it follows that $B(y) \subseteq B^*(x)$, and so $A(x) \not\subseteq A^*(y)$. By (2), $B(x) \subseteq B^*(y)$, and therefore $B(x) = B^*(x) = B(y) = B^*(y)$. If $x' \in X$ and $(x',x) \in E(H)$, then $B(y) = B(x) \not\subseteq B^*(x')$, and so $(x',y) \in E(H)$, a contradiction. Hence x has indegree zero in $H|X$. Since $H|X$ is strongly connected, it follows that $|X| = 1$, and in particular x has outdegree zero in $H|X$; but this contradicts that x has positive outdegree in H.

Consequently H is strongly connected. We claim that $A(v) \subseteq A^*(u)$ for all distinct $u,v \in V(H)$. To see this, choose a directed path $u = h_1\text{-}\cdots\text{-}h_k = v$ of H. For $1 \le i < k$, since $(h_i, h_{i+1}) \in E(H)$, it follows that $B(h_{i+1}) \not\subseteq B^*(h_i)$, and therefore $A(h_{i+1}) \subseteq A^*(h_i)$ by (2). Consequently $A(v) \subseteq A^*(u)$, as claimed. Since also $A(u) \subseteq A^*(v)$, it follows that $A(u) = A^*(u) = A(v) = A^*(v)$ for all distinct $u,v \in V(H)$. Consequently, if $|V(H)| \ge 2$ then (3) holds; and if $|V(H)| \le 1$ then again (3) holds, since $F(G)$ is a matching. This proves (3).

Let $s,t \in \Sigma(\phi)$ be distinct. We write $s \Rightarrow t$ if all edges between $\phi^{-1}(s)$ and $\phi^{-1}(t)$ are directed in D from $\phi^{-1}(s)$ to $\phi^{-1}(t)$.

(4) *Let $s,t \in \Sigma(\phi)$ be different. Then either $s \Rightarrow t$ or $t \Rightarrow s$.*

For suppose there exist $u,u' \in \phi^{-1}(s)$ and $v,v' \in \phi^{-1}(t)$ such that $uv, v'u' \in E(D)$. By (3) and the symmetry, we may assume that $A(u) = A(u')$. Since $v \in A(u)$ it follows that $v \in A(u')$ and so $u'v \in E(D)$. But $v'u' \in E(D)$, contrary to the fourth axiom. This proves (4).

(5) *Let $r,s,t \in \Sigma(\phi)$ be clockwise, and suppose that $r \Rightarrow t$. Then $r \Rightarrow s$ and $s \Rightarrow t$. Moreover, either $\phi^{-1}(r)$ is strongly anticomplete to $\phi^{-1}(t)$, or $\phi^{-1}(s)$ is strongly complete to both $\phi^{-1}(r)$ and $\phi^{-1}(t)$.*

For the first claim, we may assume from the symmetry that $r \not\Rightarrow s$. Choose $u \in \phi^{-1}(r)$ and $v \in \phi^{-1}(s)$, such that $vu \in E(D)$. Choose $w \in \phi^{-1}(t)$. Since (s,t,r) is clockwise, the third axiom implies that $wu \in E(D)$, contradicting that $r \Rightarrow t$. This proves the first claim.

For the second, we may assume that $\phi^{-1}(r)$ is not strongly anticomplete to $\phi^{-1}(t)$, and (from the symmetry) not strongly complete to $\phi^{-1}(s)$. Let X be the set of all $u \in \phi^{-1}(r)$ with an antineighbour in $\phi^{-1}(s)$, and let $Y = \phi^{-1}(r) \setminus X$. By the third axiom, every vertex of $\phi^{-1}(r)$ with a neighbour in $\phi^{-1}(t)$ belongs to Y, and so X, Y are both nonempty. In particular, not all $x,y \in \phi^{-1}(r)$ have $A^*(x) = A(y)$, and so by (3), $B^*(x) = B(y)$ for all $x,y \in \phi^{-1}(r)$. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A^*(y)$, or $B(y) \not\subseteq B^*(x)$. The first holds, since we just saw that $B(y) = B^*(x)$. Choose $z \in A(x) \setminus A^*(y)$, and let $\phi(z) = t'$ say. Now $t' \neq s$ since y is strongly complete to $\phi^{-1}(s)$ and $r \Rightarrow s$; if (r,s,t') is clockwise, then the third axiom implies that x is strongly complete to $\phi^{-1}(s)$, contradicting that $x \in X$; and if (r,t',s) is clockwise, then since y has a neighbour in $\phi^{-1}(s)$ and $yz \notin E(D)$, the third axiom is violated. This proves (5).

Let $s,t \in \Sigma(\phi)$ be distinct. We say s,t is a *mixed pair* if $\phi^{-1}(s)$ is neither strongly complete nor strongly anticomplete to $\phi^{-1}(t)$.

(6) *Let s,t be a mixed pair. Then there exists $u \in \phi^{-1}(s)$ such that u has a neighbour and an antineighbour in $\phi^{-1}(t)$.*

For suppose not. So every vertex in $\phi^{-1}(s)$ is either strongly complete or strongly anticomplete to $\phi^{-1}(t)$. Let X be the set of members of $\phi^{-1}(s)$ that are strongly $\phi^{-1}(t)$-complete, and Y the set of those that are strongly $\phi^{-1}(t)$-anticomplete. Since s,t is mixed, X,Y are both nonempty. By (4) and the symmetry, we may assume that $t \Rightarrow s$. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A^*(y)$, or $B(y) \not\subseteq B^*(x)$. But $B(x) \not\subseteq B^*(y)$, since $\phi^{-1}(t)$ is included in the first set and not the second; and so from (2), $A(x) \subseteq A^*(y)$. Consequently, $B(y) \not\subseteq B^*(x)$. Choose $v \in \phi^{-1}(t)$ and $w \in B(y) \setminus B^*(x)$. Since $B(y) \cap \phi^{-1}(t) = \emptyset$, $\phi(w) \neq t = \phi(v)$, and therefore one of $(v,w,y),(w,v,y)$ is clockwise. If (v,w,y) is clockwise then (v,w,x) is clockwise, and yet $vx \in E(D)$ and $wx \notin E(D)$, a contradiction to the third axiom. If (w,v,y) is clockwise then $wy \in E(D), vy \notin E(D)$ again contradicts the third axiom. This proves (6).

(7) *For every $s \in \Sigma(\phi)$ there is at most one $t \in \Sigma(\phi)$ such that s,t is a mixed pair.*

For let $s,t_1,t_2 \in \Sigma(\phi)$ be distinct, and suppose that the pairs s,t_1 and s,t_2 are both mixed. By (3) and the symmetry, we may assume that $B^*(x) = B(y)$ for all $x,y \in \phi^{-1}(s)$. By (6) for $i = 1,2$ there exists $v_i \in \phi^{-1}(t_i)$ with a neighbour u_i and an antineighbour u_i' in $\phi^{-1}(s)$. Since $v_i \notin B^*(u_i') = B(u_i)$ it follows that $v_iu_i \notin E(D)$, and so $u_iv_i \in E(D)$. By (4), $s \Rightarrow t_i$ for $i = 1,2$. From the symmetry between t_1,t_2, we may assume that (s,t_1,t_2) is clockwise; but then (5) implies that either $\phi^{-1}(s)$ is strongly anticomplete to $\phi^{-1}(t_2)$, or $\phi^{-1}(s)$ is strongly complete to $\phi^{-1}(t_1)$, contradicting that the pairs s,t_1 and s,t_2 are both mixed. This proves (7).

(8) *If s,t is a mixed pair, then every $v \in \phi^{-1}(s)$ has both a neighbour and an antineighbour in $\phi^{-1}(t)$.*

For we may assume that $s \Rightarrow t$. Let $x,y \in \phi^{-1}(s)$, and suppose that there exists $u \in \phi^{-1}(t)$ such that x,u are antiadjacent and y,u are adjacent, and there is no $u' \in \phi^{-1}(t)$ such that x,u' are adjacent and y,u' are antiadjacent. We claim that $A(x) \subseteq A^*(y)$ and $B(y) \subseteq B^*(x)$. For suppose that there exists $v \in A(x) \setminus A^*(y)$. Now $\phi(v) \neq s$ since $v \in A(x)$, and $\phi(v) \neq t$ since v is adjacent to x and antiadjacent to y; and so $s,t,\phi(v)$ are all distinct. If $s,\phi(v),t$ are in clockwise order, then (y,v,u) is clockwise, and the third axiom implies that $v \in A^*(y)$, a contradiction. If $s,t,\phi(v)$ are in clockwise order, then (x,u,v) is clockwise, and the third axiom implies that x,u are strongly adjacent, again a contradiction. This proves that $A(x) \subseteq A^*(y)$. Moreover, $A(y) \not\subseteq A^*(x)$, since $u \in A(y) \setminus A^*(x)$; so by (2), $B(y) \subseteq B^*(x)$. This proves our claim that $A(x) \subseteq A^*(y)$ and $B(y) \subseteq B^*(x)$.

Let X be the set of all vertices in $\phi^{-1}(s)$ that are strongly anticomplete to $\phi^{-1}(t)$, and suppose that X is nonempty. Let $Y = \phi^{-1}(s) \setminus X$. Since s,t is a mixed pair, $Y \neq \emptyset$. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A^*(y)$, or $B(y) \not\subseteq B^*(x)$. But since x is strongly anticomplete to $\phi^{-1}(t)$ and y is not, it follows that there exists $u \in \phi^{-1}(t)$ such that x,u are antiadjacent and y,u are adjacent, and there is no $u' \in \phi^{-1}(t)$ such that x,u' are adjacent and y,u' are antiadjacent. This contradicts the claim above. Hence X is empty, and so every vertex in $\phi^{-1}(s)$ has a neighbour in

$\phi^{-1}(t)$.

Now let X be the set of all vertices in $\phi^{-1}(s)$ that are not strongly complete to $\phi^{-1}(t)$, and let $Y = \phi^{-1}(s) \setminus X$. Since s, t is a mixed pair, $X \neq \emptyset$. Suppose that Y is nonempty. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A^*(y)$, or $B(y) \not\subseteq B^*(x)$. But since y is strongly complete to $\phi^{-1}(t)$ and x is not, it follows that that there exists $u \in \phi^{-1}(t)$ such that x, u are antiadjacent and y, u are adjacent, and there is no $u' \in \phi^{-1}(t)$ such that x, u' are adjacent and y, u' are antiadjacent, again contrary to the claim above. Thus $Y = \emptyset$, and so every vertex in $\phi^{-1}(s)$ has an antineighbour in $\phi^{-1}(t)$. This proves (8).

(9) *If $s \in \Sigma$ and $|\phi^{-1}(s)| > 1$, then there exists $t \in \Sigma$ with $t \neq s$ such that s, t is a mixed pair.*

For suppose not; and partition $\phi^{-1}(s)$ into two nonempty subsets X, Y. By (1), there exist $x \in X$ and $y \in Y$ such that either $A(x) \not\subseteq A^*(y)$, or $B(y) \not\subseteq B^*(x)$. But $A(x) = A^*(y)$ and $B^*(x) = B(y)$ since there is no t satisfying the claim, a contradiction. This proves (9).

By (4), (5), (7), (8), and (9), it follows that (ϕ, D) is a metacircular arrangement of G. This proves 2.1. ■

3 The main theorem

We begin with some more definitions. Let C be a hole with vertices $c_1\text{-}\cdots\text{-}c_n\text{-}c_1$ in order. We call the sequence $c_1\text{-}\cdots\text{-}c_n\text{-}c_1$ an *n-numbering* of C. Let $v \in V(G) \setminus V(C)$, and let $N \subseteq V(C)$, such that v is adjacent to every member of N and antiadjacent to every member of $V(C) \setminus N$. We say that (relative to C):

- v is a *hat* if $N = \{c_i, c_{i+1}\}$ for some i
- v is a *clone* if $N = \{c_{i-1}, c_i, c_{i+1}\}$ for some i
- v is a *star* if $n \geq 5$ and $N = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ for some i
- v is a *hub* if $n \geq 6$ and there exist i, j with $1 \leq i, j \leq n$ such that $i-1, i, i+1, j-1, j, j+1$ are all distinct (modulo n) and $N = \{c_i, c_{i+1}, c_j, c_{j+1}\}$
- v is a *centre* if $n \leq 5$ and $N = V(C)$.

If in addition v is strongly adjacent to every member of N and strongly antiadjacent to every member of $V(C) \setminus N$, then we say v is a *strong* hat, clone, star, hub or centre respectively.

Here are two lemmas. The proofs of both are clear, and we omit them.

3.1 *Let C be a hole in a claw-free trigraph G, and let $v \in V(G) \setminus V(C)$. Then either:*

- *v has no neighbours in $V(C)$, or*
- *v is a hat, clone, star or hub relative to C, or*
- *C has length ≤ 5 and v is a centre relative to C.*

Moreover, if C has vertices c_1-c_2-$\cdots$-c_n-c_1 in order, and v is semiadjacent to say c_1, then either v is strongly antiadjacent to one of c_2, c_n and v is a clone or star relative to C, or $n \leq 5$ and v is a centre relative to C.

3.2 *Let G be a claw-free trigraph, and let C be a hole in G. Let $v_1, v_2 \in V(G) \setminus V(C)$, and for $i = 1, 2$, let N_i, A_i be respectively the sets of neighbours and antineighbours of v_i in $V(C)$.*

- *If there exist $x \in N_1 \cap N_2$ and $y \in A_1 \cup A_2$, consecutive in C, then v_1, v_2 are strongly adjacent.*
- *If there exist $x, y \in N_1 \cap A_2$ that are antiadjacent, then v_1, v_2 are strongly antiadjacent.*

We need some further definitions. Let C be a hole in a trigraph G, with vertices c_1-c_2-$\cdots$-c_n-c_1 in order. Let $v_1, \dots, v_k \in V(G) \setminus V(C)$, and for $1 \leq i \leq k$ let $N_i \subseteq V(C)$ such that v_i is complete to N_i and anticomplete to $V(C) \setminus N_i$.

- If $k = 2$ and $N_1 = \{c_i, c_{i+1}\}$ and $N_2 = \{c_j, c_{j+1}\}$ for some i, j, and $N_1 \cap N_2 = \emptyset$, and v_1, v_2 are adjacent, we call $\{v_1, v_2\}$ a *hat-diagonal* for C.
- If $n \geq 5$ and $k = 2$ and $N_1 = \{c_i, c_{i+1}\}$ and $N_2 = \{c_{i-1}, c_i, c_{i+1}, c_{i+2}\}$ for some i, we call $\{v_1, v_2\}$ a *coronet* for C.
- If $n \geq 5$ and $k = 2$ and $N_1 = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ and $N_2 = \{c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}\}$ for some i, and v_1, v_2 are antiadjacent, we call $\{v_1, v_2\}$ a *crown* for C.
- If $n = 5$ or 6 and $k = 2$ and $N_1 = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ and $N_2 = \{c_{i+3}, c_{i+4}, c_{i+5}, c_{i+6}\}$ and v_1, v_2 are adjacent, we call $\{v_1, v_2\}$ a *star-diagonal* for C.
- If $n = 6$ and $k = 3$ and $N_1 = \{c_i, c_{i+1}, c_{i+2}, c_{i+3}\}$ and $N_2 = \{c_{i+2}, c_{i+3}, c_{i+4}, c_{i+5}\}$ and $N_3 = c_{i-2}, c_{i-1}, c_i, c_{i+1}\}$ for some i, and $\{v_1, v_2, v_3\}$ is a clique, we call $\{v_1, v_2, v_3\}$ a *star-triangle* for C.

Now we prove the main result of this paper, the following. (Please note that for some values of n, certain hypotheses of the theorem are vacuous. For instance, if $n > 5$ then no n-hole has a centre; star-triangles are only possible if $n = 6$; and so on.)

3.3 *Let G be a claw-free trigraph with a hole, and let n be the maximum length of holes in G. Suppose that every n-hole is dominating, and has no hub, coronet, crown, hat-diagonal, star-diagonal, star-triangle or centre. Then G is metacircular. Indeed, for every n-hole C, there is a metacircular arrangement (ϕ, D) of G with basis C.*

Proof. Let C be an n-hole in G, with a numbering c_2-c_4-c_6-$\cdots$-c_{2n}-c_2 say. Since C is dominating and has no hub or centre, it follows from 3.1 that, relative to C, every vertex of G not in $V(C)$ is either a strong hat, a strong clone, a strong star, or both a hat and a clone, or both a clone and a star.

For every $v \in V(G)$, we define $\pi(v)$ as follows. Let N be the set of neighbours of v in $V(C)$, together with v if $v \in V(C)$. Since either $v \in V(C)$ or v is a hat, clone or star, there is a unique choice of $i \in \{1, \dots, 2n\}$ such that N is one of the three sets $\{c_{i-1}, c_{i+1}\}$, $\{c_{i-2}, c_i, c_{i+2}\}$, and $\{c_{i-3}, c_{i-1}, c_{i+1}, c_{i+3}\}$. We define $\pi(v) = i$.

Let Σ be a circle, with circumference $2n$, and let $\sigma_1, \dots, \sigma_{2n}$ be equally spaced points of it, in order. For each vertex v of G, we define $\phi(v) = \sigma_{\pi(v)}$. If $u, v \in V(G)$, we define $d(u, v)$ to be the length of the shorter arc of Σ joining $\phi(u), \phi(v)$ (or 0, if $\phi(u) = \phi(v)$); thus $d(u, v) \leq n$. (For example, $d(c_2, c_4) = 2$.)

For $i = 1, \dots, 2n$ with i even, let C_i be the set of all clones v with $\pi(v) = i$. (Thus every member of C_i is either a strong clone, or both a clone and a hat.) For i odd, let H_i and S_i be respectively the set of all hats and stars v with $\pi(v) = i$. (Every member of H_i is a strong hat; and every member of S_i is either a strong star, or both a star and a clone.) If $n = 4$ then the sets S_i are all empty, by definition. We read all these subscripts modulo $2n$; so for instance H_{2n+1} means H_1 and so on.

(1) *If $u, v \in V(G)$ are adjacent and $u \in V(C)$ then $d(u, v) \leq 3$.*

The proof is clear.

(2) *If $u, v \in V(G)$ are adjacent then $d(u, v) \leq 4$.*

For since $d(u, v) \leq n$, we may assume that $n \geq 5$; and by (1) we may assume that $u, v \notin V(C)$. If either of u, v is a strong hat, then $d(u, v) \leq 4$ if u, v have a common neighbour in C, and otherwise the claim follows from 3.2 since there is no hat-diagonal for C; so we assume that u, v both have at least three neighbours in C. Suppose first that u is a star, say $u \in S_i$ where i is odd. If v is adjacent to c_{i+1}, then

$$v \in S_{i-2} \cup C_{i-1} \cup S_i \cup C_{i+1} \cup S_{i+2} \cup C_{i+3} \cup S_{i+4},$$

and consequently $d(u, v) \leq 4$ as required. We assume then that v is strongly antiadjacent to c_{i+1}, and similarly to c_{i-1}. Since $\{u, v, c_{i-3}, c_{i+1}\}$ is not a claw, it follows that v is strongly adjacent to c_{i-3}, and similarly to c_{i+3}. Since v is not a hub, it follows that $n \leq 6$. If $n = 6$, then $v \in S_{i+6}$, and $\{u, v\}$ is a star-diagonal for C, a contradiction; while if $n = 5$, then $u \in C_{i+5}$, and u is a centre for the 5-hole v-c_{i-3}-c_{i-1}-c_{i+1}-c_{i+3}-v, a contradiction. We may therefore assume that u is not a star, and similarly v is not a star. They are therefore both clones; but then $d(u, v) \leq 4$ by 3.2. This proves (2).

(3) *If $u, v \in V(G)$ and $d(u, v) \leq 1$ then u, v are strongly adjacent.*

For if one of $u, v \in V(C)$, this is clear, so we assume both are in $V(G) \setminus V(C)$. If $d(u, v) = 0$, then either u, v are both hats, or both clones, or both stars, or one is a star and one is a hat. In the first three cases, u, v are strongly adjacent by 3.2, and in the last case, $n \geq 5$ and $\{u, v\}$ is a coronet for C, a contradiction. Thus if $d(u, v) = 0$ then the claim holds. We assume then that $u \in C_i$ and $v \in H_{i+1} \cup S_{i+1}$. But then u, v are strongly adjacent by 3.2. This proves (3).

If $u, v, w \in V(G)$, let us say that v is *between* u, w if $d(u, w) < n$, and $d(u, v), d(v, w) > 0$, and $d(u, v) + d(v, w) = d(u, w)$ (and so $\phi(v)$ lies on the shorter circle arc between $\phi(u), \phi(w)$).

(4) *If $u, v, w \in V(G)$ and v is between u, w, and $d(u, w) \geq 4$, and u, w are adjacent, then v is strongly adjacent to both u, w.*

For then $u, w \notin V(C)$ by (1), and $d(u, w) = 4$ by (2). Since $d(u, w) < n$ it follows that $n \geq 5$. It suffices from the symmetry to prove that u, v are strongly adjacent, and we therefore assume they are antiadjacent, for a contradiction. Let $\pi(u) = i$. We may assume that $\pi(w) = i + 4$, and $\pi(v) \in \{i + 1, i + 2, i + 3\}$. By (3), $\pi(v) \neq i + 1$, and so $\pi(v) = i + 2$ or $i + 3$. There are two cases, depending whether i is odd or even.

Suppose first that i is even. If $\pi(v) = i + 2$, then v, w are antiadjacent, since $\{w, u, v, c_{i+6}\}$ is not a claw; but then $\{u, w\}$ is a hat-diagonal for the n-hole obtained from C by replacing c_{i+2} by v, a contradiction. If $\pi(v) = i + 3$, then by (3), v, w are adjacent; since $\{w, u, v, c_{i+6}\}$ is not a claw, v is adjacent to c_{i+6} and hence to c_i; and then $\{u, v\}$ is a crown for the n-hole obtained from C by replacing c_{i+4} by w, a contradiction.

Now suppose that i is odd. Since u, w are adjacent and $n \geq 5$, and there is no hat-diagonal, it follows from 3.2 that u, w are both stars, and so $u \in S_i, w \in S_{i+4}$. If $n = 5$, then $\{u, w\}$ is a star-diagonal for C, a contradiction, and so $n \geq 6$. If $\pi(v) = i + 2$, then by 3.2, $v \notin H_{i+2}$, and so $v \in S_{i+2}$; and then $\{u, v\}$ is a crown for C, a contradiction. If $\pi(v) = i + 3$, then v, w are adjacent by (3), but then $\{w, u, v, c_{i+7}\}$ is a claw, a contradiction. This proves (4).

(5) *If $u, v, w \in V(G)$ and v is between u, w, and u, w are adjacent, then v is strongly adjacent to both u, w.*

For by (4) we may assume that $d(u, w) \leq 3$, and by (3) the claim is true if $d(u, w) \leq 2$, so we may assume that $d(u, w) = 3$. Thus exactly one of $\pi(u), \pi(w)$ is even; and so from the symmetry we may assume that $\pi(u)$ is even, $\pi(u) = i$ say, and $\pi(w) = i + 3$, and $\pi(v)$ is one of $i + 1, i + 2$. Thus $u \in C_i \cup \{c_i\}$ and $w \in H_{i+3} \cup S_{i+3}$, and $v \in H_{i+1} \cup S_{i+1} \cup C_{i+2} \cup \{c_{i+2}\}$. Suppose first that $v \in H_{i+1} \cup S_{i+1}$. Then by (3), u, v are strongly adjacent, and so we may assume that v, w are antiadjacent. Now w is not adjacent to c_{i-2}, since $w \in H_{i+3} \cup S_{i+3}$ (and if $n = 4$ then $w \in H_{i+3}$, by definition of a star). Since $\{u, v, w, c_{i-2}\}$ is not a claw, v is adjacent to c_{i-2}, and therefore $v \in S_{i+1}$. Since v, w are antiadjacent, 3.2 implies that w is not a hat, and so $w \in S_{i+3}$; and then $\{v, w\}$ is a crown for C, a contradiction. Suppose now that $v \in C_{i+2} \cup \{c_{i+2}\}$. By (1), v, w are strongly adjacent, and so we may assume that u, v are antiadjacent. If $w \in S_{i+3}$ then $n \geq 5$ and $\{w, u, v, c_{i+6}\}$ is a claw, a contradiction; and if $w \in H_{i+3}$ then w, c_i are strongly antiadjacent, and so $u \neq c_i$, and $\{u, w\}$ is a hat-diagonal for the n-hole induced on $(V(C) \setminus \{c_{i+2}\}) \cup \{v\}$, a contradiction. This proves (5).

(6) *If $u, v \in V(G)$ are adjacent, then $d(u, v) < n$.*

By (2) we assume that $d(u, v) = n = 4$. From (1), $u, v \notin V(C)$, and since $n = 4$, there are no stars relative to C. Thus u, v are both hats or both clones. If they are both hats, then $\{u, v\}$ is a hat-diagonal for C, a contradiction. If they are both clones, say $u \in C_2, v \in C_6$, then v is a centre for the 4-hole u-c_4-c_6-c_8-u, a contradiction. This proves (6).

Fix an orientation of Σ, called clockwise. Define a direction D of G as follows. For all distinct $u, v \in V(G)$ that are adjacent in G, if $\phi(u) \neq \phi(v)$ then by (6), one of the arcs of Σ between $\phi(u), \phi(v)$ has length $< n$ (and therefore the other has length $> n$). We call this shorter arc the *route* between $\phi(u), \phi(v)$. Let e be the edge of D with ends u, v. Direct e from u to v if moving clockwise along Σ from $\phi(u)$ to $\phi(v)$ traverses the route between $\phi(u), \phi(v)$; and otherwise direct e from v to u. For each $\sigma \in \Sigma$, direct the edges of D with both ends in $\phi^{-1}(\sigma)$ in such a way that there is no directed cycle with vertex set in $\phi^{-1}(\sigma)$.

(7) *Every directed cycle in D has length at least four.*

For certainly there is no directed cycle of length at most two. Suppose that a_1-a_2-a_3-a_1 is a directed cycle of length three. Not all of $\phi(a_1), \phi(a_2), \phi(a_3)$ are equal, and so we may assume that $\phi(a_2), \phi(a_3) \neq \phi(a_1)$. Thus moving clockwise along Σ from $\phi(a_1)$ to $\phi(a_2)$, and then moving clockwise from $\phi(a_2)$ to $\phi(a_3)$, traverses the route between $\phi(a_1)$, $\phi(a_2)$ and then traverses the route between $\phi(a_2)$, $\phi(a_3)$. Since both these arcs have length $< n$ by (6), it follows that $\phi(a_2), \phi(a_3)$ are different, and $\phi(a_1), \phi(a_2), \phi(a_3)$ are in clockwise order, and the three routes joining pairs of them have union Σ. Since a_1, a_2 are adjacent, it follows that $d(a_1, a_2) \leq 4$ by (2); and similarly $d(a_2, a_3), d(a_3, a_1) \leq 4$. Since the three routes have union the whole circle of length $2n$, it follows $2n \leq 12$, and so $n \leq 6$.

Suppose first that $n = 6$. Then equality holds throughout. Hence we may assume that either $\pi(a_1) = 1$, $\pi(a_2) = 5$ and $\pi(a_3) = 9$, or $\pi(a_1) = 2$, $\pi(a_2) = 6$ and $\pi(a_3) = 10$. In the first case, at most one of a_1, a_2, a_3 is a hat, since there is no hat-diagonal; and then 3.2 implies that none of a_1, a_2, a_3 is a hat, and so $\{a_1, a_2, a_3\}$ is a star-triangle for C, a contradiction. In the second case, a_3-c_{12}-c_2-c_4-c_6-c_8-a_3 is a 6-hole, and $\{a_1, a_2\}$ is a star-diagonal for it, a contradiction. Thus $n < 6$.

Suppose next that $n = 5$. Then the sum of $d(a_1, a_2)$, $d(a_2, a_3)$ and $d(a_3, a_1)$ is at least 10, and so we may assume that $d(a_1, a_2) = 4$. Suppose that a_1 is a star, say $a_1 \in S_1$; then we may assume that $a_2 \in H_5 \cup S_5$. By 3.2, $a_2 \notin H_5$, and if $a_2 \in S_5$ then $\{a_1, a_2\}$ is a star-diagonal for C, a contradiction. Thus a_1 is not a star, and similarly a_2 is not a star. Not both a_1, a_2 are hats, since $\{a_1, a_2\}$ is not a hat-diagonal for C. Thus we may assume that $a_1 \in C_2 \cup \{c_2\}$, and $a_2 \in C_6 \cup \{c_6\}$, and therefore

$$c_3 \in C_8 \cup \{c_8\} \cup H_9 \cup S_9 \cup C_{10} \cup \{c_{10}\}.$$

Since a_1, a_2 are adjacent, $a_1 \neq c_2$ and $a_2 \neq c_6$. Now $a_3 \neq c_8$ since c_8 is not adjacent to a_1, and similarly $a_3 \neq c_{10}$. If a_3 is antiadjacent to c_2, then a_1 is a centre for the 5-hole a_2-a_3-c_{10}-c_2-c_4-a_2, a contradiction. Thus a_3 is adjacent to c_2 and similarly to c_6, and so $a_3 \in S_9$. But then $\{a_2, a_3\}$ is a star-diagonal for the 5-hole a_1-c_4-c_6-c_8-c_{10}-a_1, a contradiction. Thus $n \neq 5$.

Consequently $n = 4$. By (6), we may assume that $a_1 \in C_2 \cup \{c_2\}$, $a_2 \in H_5$, and $a_3 \in H_7 \cup C_8 \cup \{c_8\}$. Then a_2 is a centre for the 4-hole a_1-c_4-c_6-a_3-a_1, a contradiction. This proves (7).

We claim that (ϕ, D) is a circle arrangement of G. The first axiom holds by (3), the second by (7), the third by (5), and the fourth by (6). Hence (ϕ, D) is a circle arrangement of G. Moreover, C is a basis for (ϕ, D), and the result follows from 2.1. This proves 3.3. ■

Next we prove a corollary of 3.3 that is a little more convenient to apply. We need another definition. Let $A, B \subseteq V(G)$. We call (A, B) a *W-join* if

- A, B are disjoint nonempty strong cliques of G, and at least one of A, B has at least two members
- every member of $V(G) \setminus (A \cup B)$ is either strongly A-complete or strongly A-anticomplete and either strongly B-complete or strongly B-anticomplete, and
- A is neither strongly complete to B nor strongly anticomplete to B.

Indeed, in this paper we find that our W-joins have a stronger property, that

- no member of A is strongly B-complete or strongly B-anticomplete, and no member of B is strongly A-complete or strongly A-anticomplete.

If this property also holds, we call the W-join *proper*. A W-join (A, B) is *coherent* if the set of all strongly $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a strong clique.

3.4 *Let G be a claw-free trigraph, with a hole, and let n be the maximum length of holes in G. Suppose that every n-hole is dominating, and has no hub, coronet, crown, hat-diagonal, star-diagonal, star-triangle or centre. Then either G admits a coherent proper W-join, or G is a long circular interval trigraph.*

Proof. By 3.3, G is metacircular; choose a metacircular arrangement (ϕ, D) of G. If $|\phi^{-1}(\sigma)| \leq 1$ for each $\sigma \in \Sigma$, then G is a long circular interval trigraph. Thus we may assume that $|\phi^{-1}(\sigma_1)| > 1$ for some $\sigma_1 \in \Sigma$. By the fifth and sixth conditions in the definition of a metacircular arrangement, there is a unique $\sigma_2 \in \Sigma$ with $\sigma_2 \neq \sigma_1$ such that $\phi^{-1}(\sigma_1)$ is neither strongly complete nor strongly anticomplete to $\phi^{-1}(\sigma_2)$; and every vertex in $\phi^{-1}(\sigma_1)$ has both a neighbour and an antineighbour in $\phi^{-1}(\sigma_2)$, and vice versa. Thus, $(\phi^{-1}(\sigma_1), \phi^{-1}(\sigma_2))$ is a proper W-join, and we claim that it is coherent. Let $X_i = \phi^{-1}(\sigma_i)$ $(i = 1, 2)$, and choose $v_i \in X_i$ $(i = 1, 2)$, such that v_1, v_2 are adjacent in G. From the symmetry, we may assume that $v_1v_2 \in E(D)$. Let S be the set of all $\sigma \in \Sigma$ such that $\sigma_2, \sigma_1, \sigma$ are in clockwise order; and let N be the union of all the sets $\phi^{-1}(\sigma)$ $(\sigma \in S)$. Then N is a strong clique, by the fourth condition in the definition of a metacircular arrangement. Let $v \in V(G) \setminus (X_1 \cup X_2)$ be complete to $X_1 \cup X_2$. To prove that (X_1, X_2) is coherent, it suffices to show that $v \in N$. Now $v \in \phi^{-1}(\sigma_3)$ for some $\sigma_3 \in \Sigma$, and $\sigma_1, \sigma_2, \sigma_3$ are all different. Suppose first that $\sigma_1, \sigma_2, \sigma_3$ are in clockwise order. Since X_1 is not strongly complete to X_2, it follows that $v_1v \notin E(D)$ and $vv_2 \notin E(D)$ from the fourth condition in the definition of a metacircular arrangement. Hence v-v_1-v_2-v is a directed cycle of D, a contradiction. Thus $\sigma_2, \sigma_1, \sigma_3$ are in clockwise order, and so $\sigma_3 \in S$, and therefore $v \in N$ as required. This proves 3.4. ■

4 Circular interval graphs

In this section we derive 1.1 from 3.3, but we need several more definitions. We say that a trigraph G is a *linear interval trigraph* if the vertices of G can be numbered $v_1, \dots, v_n$ such that for all i, j with $1 \leq i < j \leq n$, if v_i is adjacent to v_j then $\{v_i, v_{i+1}, \dots, v_{j-1}\}$ and $\{v_{i+1}, v_{i+2}, \dots, v_j\}$ are strong cliques. (Linear interval trigraphs are a subclass of long circular interval trigraphs, as may easily be checked.)

Two adjacent vertices of a trigraph G are called *twins* if (apart from each other) they have the same neighbours in G, and the same antineighbours, and if there are two such vertices, we say "G admits twins". Let (V_1, V_2) be a partition of $V(G)$ such that V_1, V_2 are nonempty and V_1 is strongly anticomplete to V_2. We call the pair (V_1, V_2) a *0-join* in G.

Let (V_1, V_2) be a partition of $V(G)$, such that for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- $A_1 \cup A_2$ is a strong clique, and for $i = 1, 2$, $A_i, V_i \setminus A_i$ are both nonempty, and
- V_1 is strongly anticomplete to $V_2 \setminus A_2$, and V_2 is strongly anticomplete to $V_1 \setminus A_1$.

In these circumstances, we say that (V_1, V_2) is a 1-*join*. We start with the following lemma.

4.1 *Let G be a claw-free trigraph with no hole, that is not a linear interval trigraph. Then G contains a net or antinet. Moreover, either G is an antinet, or G admits a 0-join, a 1-join, or twins.*

Proof. We proceed by induction on $|V(G)|$. If there exist $u, v \in V(G)$ that are twins, then the trigraph obtained from G by deleting v is also not a linear interval graph, and the result follows from the inductive hypothesis. Thus we may assume that G does not admit twins. If (V_1, V_2) is a 0-join, then one of $G|V_1, G|V_2$ is not a linear interval graph, and again the result follows from the inductive hypothesis. So we may assume that G does not admit a 0-join.

Since G has no hole, the adjacency relation of G defines a chordal graph, and therefore (by a well-known folklore theorem) there is a tree T and a family $(T_v : v \in V(G))$ of subtrees of T such that for all distinct $u, v \in V(G)$, $V(T_u \cap T_v) \neq \emptyset$ if and only if u, v are adjacent in G. Choose such a tree T with $|V(T)|$ minimum.

(1) *For every edge st of T, there exist $u, v \in V(G)$, strongly antiadjacent, such that T_u contains s and not t, and T_v contains t and not s.*

For let e be the edge st. Let T' be the tree obtained from T by contracting the edge st. For each $v \in V(G)$, let $T'_v = T_v$ if $s, t \notin V(T_v)$, and otherwise let S be the subtree of T with edge-set $E(T_v) \cup \{e\}$, and let T'_v be the subtree of T' obtained from S by contracting e. For all distinct $u, v \in V(G)$, if u, v are adjacent then $V(T_u \cap T_v) \neq \emptyset$ and so $V(T'_u \cap T'_v) \neq \emptyset$; and hence from the minimality of $|V(T)|$, the converse is false. In other words, there exist strongly antiadjacent $u, v \in V(G)$ such that $V(T_u \cap T_v) = \emptyset$ and $V(T'_u \cap T'_v) \neq \emptyset$. This proves (1).

(2) *For all distinct $u, v \in V(G)$, if u, v are semiadjacent then $|V(T_u \cap T_v)| = 1$ and the unique vertex in $T_u \cap T_v$ is incident with at most one edge in T_u, and with at most one edge in T_v.*

For suppose first that $|V(T_u \cap T_v)| > 1$; then we may choose distinct $s, t \in V(T_u \cap T_v)$, adjacent in T. By (1), there exist $w, x \in V(G)$, strongly antiadjacent, with $s \in V(T_w) \setminus V(T_x)$ and $t \in V(T_x) \setminus V(T_w)$. Since the trees T_u, T_w, T_x are distinct, it follows that u, w, x are all distinct, and similarly so are v, w, x; and so u, v, w, x are all different. Since $s \in V(T_u \cap T_w)$, it follows that u, w are adjacent, and similarly the pairs ux, vw, vx are adjacent. Since w-u-x-v-w is not a 4-hole, we deduce that u, v are strongly adjacent. This proves that there is a vertex $s \in V(T)$ with $V(T_u \cap T_v) = \{s\}$. To prove the second assertion, suppose that, say, s is incident with two edges st_1 and st_2 of T_u. For $i = 1, 2$, there is by (1) a vertex $v_i \in V(G)$ such that $t_i \in V(T_{v_i})$ and $s \notin V(T_{v_i})$. But then T_{v_1}, T_{v_2}, T_v are pairwise vertex-disjoint, and so v_1, v_2, v are pairwise strongly antiadjacent, and therefore $\{u, v, v_1, v_2\}$ is a claw, a contradiction. This proves (2).

(3) *If T is a path then the theorem holds.*

For let T be a path, with vertices $t_1, \ldots, t_m$ in order. For each $v \in V(G)$, T_v is a nonempty subpath of T, say from $t_{a(v)}$ to $t_{b(v)}$, where $1 \leq a(v) \leq b(v) \leq m$. For distinct $u, v \in V(G)$, write $u < v$ if $a(u) \leq a(v)$ and $b(u) \leq b(v)$, and if $T_u = T_v$ then there exists $w \in V(G) \setminus \{u, v\}$ with $T_w \neq T_u (= T_v)$ such that either

- w is semiadjacent to v with $a(w) \leq a(v)$ and $b(w) \leq b(v)$, or
- w is semiadjacent to u with $a(u) \leq a(w)$ and $b(u) \leq b(w)$.

We claim that for all distinct $u, v \in V(G)$, either $u < v$ or $v < u$, and not both. For let $u, v \in V(G)$ be distinct, and suppose first that $a(u), a(v)$ are distinct, say $a(u) < a(v)$. If also $b(u) \leq b(v)$ then $u < v$ and $v \not< u$ as claimed, so we may assume that $b(u) > b(v)$. By (1), there exist $x, y \in V(G)$ such that T_x contains $t_{a(u)}$ and not $t_{a(u)+1}$, and T_y contains $t_{b(u)}$ and not $t_{b(u)-1}$. But then $\{u, v, x, y\}$ is a claw in G, a contradiction. This proves our claim when $a(u) \neq a(v)$, and similarly it holds when $b(u) \neq b(v)$. Now we assume that $a(u) = a(v)$ and $b(u) = b(v)$. Hence u, v have the same neighbours, except for each other. Since G does not admit twins, there is a vertex $w \neq u, v$ that is semiadjacent to one of u, v. Suppose first that $a(u) = b(u)$, and let Z be the set of all $z \in V(G)$ with $t_{a(u)} \in V(T_z)$. Then Z is a clique in G, and since G has no 4-hole, it follows that at most one pair of vertices in Z is semiadjacent. In particular, since $u, v, w \in Z$, it follows that w is the only vertex that is semiadjacent to one of u, v. If $a(w) < a(u)$, then $w < u$ as we already saw, and in particular $b(w) = a(u)$, and just one of the statements $u < v$ and $v < u$ is true, as required. So we may assume that $a(w) \geq a(u)$, and since $t_{a(u)} \in V_w$, it follows that $a(w) = a(u)$. Similarly $b(w) = b(u)$; but then w and one of u, v are twins, a contradiction. This proves the claim if $a(u) = b(u)$. We may therefore assume that $a(u) < b(u)$, and hence from (2) we may assume that $a(w) \leq b(w) = a(u) < b(u)$. If no other vertex is semiadjacent to one of u, v then the claim holds, so we may assume that there exists $x \neq u, v, w$, semiadjacent to one of u, v. As before, let Z be the set of all $z \in V(G)$ with $t_{a(u)} \in V(T_z)$; then at most one pair of vertices in Z is semiadjacent. Since $u, v, w \in Z$, it follows that $x \notin Z$, and so $b(u) = a(x) \leq b(x)$ from (2). Since one of w, x is semiadjacent to u and the other to v (because $F(G)$ is a matching), we deduce that again the claim holds. This proves our claim that for all distinct u, v, either $u < v$ or $v < u$ and not both.

Next we claim that this relation is transitive. For suppose not; then there exist distinct $u, v, w \in V(G)$ such that $u < v < w < u$. Thus $a(u) \leq a(v) \leq a(w) \leq a(u)$, and so $a(u) = a(v) = a(w)$, and similarly $b(u) = b(v) = b(w)$. Since $w < u$, we may assume that there exists $x \in V(G) \setminus \{u, w\}$ semiadjacent to w with $a(w) \leq a(x)$ and $b(w) \leq b(x)$ and with $T_x \neq T_w$. But then $w < v$, a contradiction. This proves that the relation is transitive, and therefore is a total order. Hence we may number the vertices of G as $v_1, \ldots, v_n$ such that $v_i < v_j$ for $1 \leq i < j \leq n$.

Now suppose that $1 \leq h < k \leq n$ and v_h, v_k are adjacent. To prove that G is a linear interval trigraph, we must show that $\{v_h, \ldots, v_{k-1}\}$ and $\{v_{h+1}, \ldots, v_k\}$ are strong cliques, and from the symmetry it suffices to show the former. Let $h \leq i < j \leq k-1$; we shall prove that v_i, v_j are strongly adjacent. Since $h < k$ and therefore $v_h < v_k$, we deduce that $a(v_h) \leq a(v_k)$ and $b(v_h) \leq b(v_k)$; and since v_h, v_k are adjacent, it follows that $a(v_k) \leq b(v_h)$. Thus

$$a(v_h) \leq a(v_k) \leq b(v_h) \leq b(v_k).$$

Now $h \leq i < j \leq k-1$, and so

$$a(v_h) \leq a(v_i) \leq a(v_j) \leq a(v_{k-1}) \leq a(v_k)$$

and

$$b(v_h) \leq b(v_i) \leq b(v_j) \leq b(v_{k-1}) \leq b(v_k).$$

Consequently, $a(v_i) \leq a(v_k) \leq b(v_h) \leq b(v_i)$, and similarly $a(v_j) \leq a(v_k) \leq b(v_j)$. It follows that $a(v_k)$ belongs to both T_{v_i} and T_{v_j}, and therefore v_i, v_j are weakly adjacent. Suppose they are not strongly adjacent. Then by (2), $a(v_k)$ is the unique vertex in both T_{v_i} and T_{v_j}, and since $a(v_i) \le a(v_j)$ and $b(v_i) \le b(v_j)$, (2) implies that $b(v_i) = a(v_j)$. Now

$$a(v_j) \le a(v_k) \le b(v_h) \le b(v_i) = a(v_j),$$

and so $a(v_j) = a(v_k) = b(v_h)$. Suppose that $b(v_j) < b(v_k)$. Then by (1), there exists $u \in V(G)$ such that $t_{b(v_j)} \notin V(T_u)$ and $t_{b(v_k)} \in V(T_u)$. Hence $b(v_j) < a(u)$, and so v_i, u are antiadjacent; but then v_i, v_j, u are pairwise antiadjacent (since v_i, v_j are semiadjacent) and all three are adjacent to v_k (because $a(v_k)$ belongs to T_{v_i} and to T_{v_j}, and $b(v_k)$ belongs to T_u), and therefore $\{v_k, u, v_i, v_j\}$ is a claw, a contradiction. Thus $b(v_j) \ge b(v_k)$, and since $j < k$ it follows that $b(v_j) = b(v_k)$, and so $T_{v_j} = T_{v_k}$. But there exists $w \in V(G) \setminus \{v_j, v_k\}$ with $T_w \ne T_{v_j}$ and with $a(w) \le a(v_j)$ and $b(w) \le b(v_j)$ that is semiadjacent to v_j, namely $w = v_i$, and so $v_k < v_j$, a contradiction. This proves that v_i, v_j are strongly adjacent, and so G is a linear interval trigraph. This proves (3).

In view of (3), we may assume that there is a vertex t_0 of T with d neighbours $t_1, \dots, t_d$ in T, where $d \ge 3$. Let Y be the set of all $v \in V(G)$ such that $t_0 \in V(T_v)$; thus, Y is a clique of G. For $1 \le i \le d$, let X_i be the set of all $v \in V(G)$ such that $V(T_v)$ contains t_i and not t_0, and let Y_i be the set of all $v \in Y$ with $t_0, t_i \in V(T_v)$. By (1), $X_i \ne \emptyset$, for $1 \le i \le d$; choose $x_i \in X_i$. Since $t_i \in V(T_{x_i})$, it follows that every member of Y_i is adjacent to x_i.

(4) *For $1 \le i \le d$, $\emptyset \ne Y_i \ne Y$.*

For suppose that $Y_1 = \emptyset$ say. Let S_1, S_2 be the two components of $T \setminus e$, where e is the edge t_0t_1. For $j = 1, 2$, let A_j be the set of all $v \in V(G)$ such that $V(T_v) \subseteq V(S_j)$; then since $Y_1 = \emptyset$, $A_1 \cup A_2 = V(G)$. Since each T_v has at least one vertex, $A_1 \cap A_2 = \emptyset$. Moreover, by (1) $A_1, A_2 \ne \emptyset$; and if $v_1 \in A_1$ and $v_2 \in A_2$ then $V(T_{v_1} \cap T_{v_2}) = \emptyset$, and so v_1, v_2 are strongly antiadjacent. It follows that G admits a 0-join, a contradiction. This proves that $Y_i \ne \emptyset$, for $1 \le i \le d$. Moreover, by (1) there exists $v \in V(G)$ such that T_v contains t_0 and not t_i, and therefore $v \in Y \setminus Y_i$; and consequently $Y_i \ne Y$ for $1 \le i \le d$. This proves (4).

(5) *For $1 \le i < j \le d$, either $Y_i \cap Y_j = \emptyset$ or $Y_i \cup Y_j = Y$; and for $1 \le i < j < k \le d$, $Y_i \cap Y_j \cap Y_k = \emptyset$.*

For if $y \in Y_i \cap Y_j$ and $y' \in Y \setminus (Y_i \cup Y_j)$, then $\{y, y', x_i, x_j\}$ is a claw, a contradiction; and if $y \in Y_i \cap Y_j \cap Y_k$, then $\{y, x_i, x_j, x_k\}$ is a claw, a contradiction. This proves (5).

(6) *Either $Y_1, \dots, Y_d$ are pairwise disjoint, or $d = 3$ and every vertex of Y is in exactly two of the sets Y_1, Y_2, Y_3.*

For if $i, j \in \{1, \dots, d\}$ are distinct, let us say that the pair (i, j) is *big* if $Y_i \cap Y_j \ne \emptyset$. We claim that if (i, j) is big, then (i, k) is big for all $k \in \{1, \dots, d\}$ with $k \ne i$. For suppose not; then $j \ne k$, and $Y_k \cap Y_i = \emptyset$. Since $Y_i \cap Y_j \ne \emptyset$, (5) implies that $Y_i \cup Y_j = Y$, and therefore $Y_k \subseteq Y_j$, contrary to (4) and (5) (applied to the pair j, k). This proves our claim. Hence either no pair is big, or every pair is big. In the first case the claim holds, so we assume the second. By the final statement of (5), no vertex of Y belongs to three of the sets $Y_1, \dots, Y_d$; and yet by the first assertion of (5), the sets $Y \setminus Y_i$ ($1 \le i \le d$) are pairwise disjoint, and so every vertex in Y is in at least

$d-1$ of the set $Y_1, \dots, Y_d$. Consequently $d = 3$ and the second statement of (6) holds. This proves (6).

(7) *Y is a strong clique.*

For suppose that $u, v \in Y$ are antiadjacent. Suppose first that for some i, $u, v \notin Y_i$. Choose $y_i \in Y_i$; then $\{y_i, x_i, u, v\}$ is a claw, a contradiction. So each of $Y_1, \dots, Y_d$ contains one of u, v, and in particular $Y_1, \dots, Y_d$ are not pairwise disjoint, since $d \geq 3$; and so by (6), $d = 3$ and every vertex of Y is in exactly two of the sets Y_1, Y_2, Y_3. We may therefore assume that $u \in Y_1 \cap Y_3$ and $v \in Y_2 \cap Y_3$, say. By (1) there exists $y \in V(G)$ such that $V(T_y)$ contains t_0 and not t_3; but then u-y-v-x_3-u is a 4-hole, a contradiction. This proves (7).

For $1 \leq i \leq d$, let S_i be the component of $T \setminus \{t_0\}$ that contains t_i; and let V_i be the set of all $v \in V(G)$ with T_v a subtree of $S_i \setminus \{t_i\}$. It follows that Y and the sets X_i, V_i $(1 \leq i \leq d)$ are pairwise disjoint and have union $V(G)$.

(8) *If $Y_1, \dots, Y_d$ are pairwise disjoint, then G contains a net and G admits a 1-join.*

For suppose that $Y_1, \dots, Y_d$ are pairwise disjoint. Choose $y_i \in Y_i$ for $i = 1, 2, 3$; then the trigraph induced on $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ is a net, so G contains a net. Moreover, Y is a strong clique, and we claim that

$$(V_1 \cup X_1 \cup Y_1, V(G) \setminus (V_1 \cup X_1 \cup Y_1))$$

is a 1-join. To show this, it suffices to check that if $u \in V_1 \cup X_1 \cup Y_1$ and $v \in V(G) \setminus (V_1 \cup X_1 \cup Y_1)$ then u, v are strongly anticomplete unless $u \in Y_1$ and $v \in Y$. Since $v \notin V_1 \cup X_1 \cup Y_1$, it follows that $T_v \not\subseteq S_1$ and $t_1 \notin V(T_v)$; and so T_v is disjoint from S_1. If $T_u \subseteq S_1$ then $T_u \cap T_v$ is empty and therefore u, v are strongly antiadjacent as required, so we may assume that $T_u \not\subseteq S_1$. Consequently $u \in Y_1$, and $t_0, t_1 \in V(T_u)$. Since $Y_1, Y_2, \dots, Y_d$ are pairwise disjoint, it follows that u does not belong to $Y_2 \cup \dots \cup Y_d$, and therefore $t_2, \dots, t_d \notin V(T_u)$. If $t_0 \notin V(T_v)$, then again T_u, T_v are disjoint and the claim follows; and if $t_0 \in V(T_v)$, then $v \in Y$ and again the claim holds. Thus in this case G admits a 1-join. This proves (8).

In view of (6) and (8), we may therefore assume that $d = 3$ and every vertex in Y is in exactly two of Y_1, Y_2, Y_3. Choose $y_i \in Y_i$ for $i = 1, 2, 3$; then the trigraph induced on $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ is an antinet, so G contains an antinet. Moreover, for every $v \in V(G)$, if $t_0 \in V(T_v)$ then t_0 is incident with exactly two edges of T_v. We may also assume that the same is true for every vertex of T with degree at least three. We deduce that

- every vertex of T has degree at most three
- for every $v \in V(G)$, T_v is a path in T, and both its ends (or its end, if it is a path of length zero) have degree at most two in T
- for every two edges of T that have a common end of degree three in T, there exists $v \in V(G)$ such that both these edges belong to T_v (this follows from (4)).

(9) *If $u, v \in V(G)$ and $V(T_u \cap T_v) \neq \emptyset$, then T_u contains an end of T_v and vice versa.*

For suppose that no end of T_u belongs to T_v. Then $u \neq v$, and since some vertex of T_u belongs to T_v, it follows that T_u has length at least two; let its ends be s, t. By (1), there exists $w \in V(G)$ such that $V(T_u \cap T_w) = \{s\}$, and there exists $x \in V(G)$ such that $V(T_u \cap T_x) = \{t\}$. In particular, T_v, T_w, T_x are pairwise disjoint, and so $\{u, v, w, x\}$ is a claw, a contradiction. This proves (9).

(10) *If $v \in V(T)$ and some vertex of T_v has degree three in T, then T_v is a path of length two.*

For let $t_0 \in V(T_v)$ have degree three in T. Then t_0 is an internal vertex of the path T_v; let T_v have ends s_1, s_2, and let P_1, P_2 be the paths of T between t_0 and s_1, s_2 respectively. Thus $T_v = P_1 \cup P_2$. Let t_1, t_2, t_3 be the neighbours of t_0 in T, where $t_i \in V(P_i)$ for $i = 1, 2$. Certainly P_v has length at least two; suppose it has length at least three. Then from the symmetry we may assume that P_1 has length at least two. Thus $t_1 \neq s_1$; let t_1' be the neighbour of t_1 in P_1 different from t_0. From (1), there exists $u \in V(G)$ such that T_u contains t_1 and not t_1'. By (9), one of s_1, s_2 belongs to T_u, and therefore $s_2 \in V(T_u)$, and so P_2 is a path in T_u. Choose $w \in V(G)$ such that T_w contains t_1, t_0, t_3 (such a vertex exists, by the displayed statements above). Then by (9), some end of T_v belongs to T_w, and so $s_1 \in V(T_w)$. Also, some end of T_w belongs to T_v, and therefore s_1 is an end of T_w. By (9) again, some end of T_w belongs to T_u, and this is impossible, since $s_1 \notin V(T_u)$ (because $t_1' \notin V(T_u)$), and the other end of T_w is separated from T_u by the edge t_0t_3. This proves (10).

Now choose $t_0 \in V(T)$ with degree three, and define Y and t_i, Y_i, X_i, V_i ($i = 1, 2, 3$) as in (4)–(8). We see that if $u \in V_1$ and $v \in Y_1$ then T_u, T_v are disjoint (by (10)); and so V_1 is strongly anticomplete to Y_1 and hence to $V(G) \setminus (V_1 \cup X_1)$. Moreover, X_1 is strongly anticomplete to $V(G) \setminus (V_1 \cup X_1 \cup Y_1)$ (for if $u \in X_1$ and $v \in V(G) \setminus (V_1 \cup X_1 \cup Y_1)$ then T_u, T_v are separated in T by the edge t_0t_1). Third, we claim that $X_1 \cup Y_1$ is a strong clique. For suppose that $u, v \in X_1 \cup Y_1$ are antiadjacent. By (7), not both $u, v \in Y$, and so we may assume that $u \in X_1$, and $v \notin Y_2$. Choose $y_2 \in Y_1 \cap Y_2$, and $x_2 \in X_2$ (these exist, by (4) and (1)); then $\{y_2, x_2, u, v\}$ is a claw, a contradiction. This proves that $X_1 \cup Y_1$ is a strong clique. Since $V_1 \cup X_1 \cup Y_1 \neq V(G)$, it follows that if $V_1 \neq \emptyset$ then $(V_1 \cup X_1, V(G) \setminus (V_1 \cup X_1))$ is a 1-join as required; so we may assume that $V_1 = \emptyset$, and similarly $V_2 = V_3 = \emptyset$. Since $X_1 \cup Y_1$ is a strong clique, all members of X_1 are twins, and so $|X_1| = 1$, and similarly $|X_2| = |X_3| = 1$; and since also Y is a strong clique, all members of Y_i are twins for $i = 1, 2, 3$, and so $|Y_i| = 1$ ($i = 1, 2, 3$). But then G is an antinet, as required. This proves 4.1. ■

Now we prove 1.1, which we restate (note that this is a theorem about graphs, not trigraphs):

4.2 *Let G be a graph. Then G is a long circular interval graph if and only if no induced subgraph is a claw, net, antinet or $(1,1,1)$-prism, and every hole is dominating and has no centre.*

Proof. The "only if" statement is easy, and we omit it. For the "if" part, let G be a graph such that no induced subgraph is a claw, net, antinet or $(1,1,1)$-prism, and such that every hole in G is dominating and has no centre. By 4.1, we may assume that G has a hole. Let the longest hole in G have length n, and let C_0 be an n-hole.

(1) *There is a metacircular arrangement (ϕ, D) of G with basis C_0.*

For by 3.3, it suffices to show that every n-hole is dominating, and has no hub, coronet, crown, hat-diagonal, star-diagonal, star-triangle or centre. Let C be an n-hole, with vertices c_1-$\cdots$-c_n-c_1

say. By hypothesis, C is dominating and has no centre. If it has a hub v, then there are two holes with vertex set in $V(C) \cup \{v\}$ different from C, and both are nondominating, a contradiction. Similarly C has no coronet.

Suppose that there is a crown for C; then $n \geq 5$ and there are nonadjacent stars s_1, s_2, adjacent say to $c_1, \ldots, c_4$ and to $c_2, \ldots, c_5$ respectively. If $n \geq 6$, the hole s_1-c_2-s_2-c_4-s_1 is not dominating, since c_6 has no neighbour in it. If $n = 5$, then the subgraph induced on $\{c_1, c_2, c_4, c_5, s_1, s_2\}$ is a $(1,1,1)$-prism, a contradiction. Thus there is no crown for C.

Suppose that there is a hat-diagonal for C; that is, there are adjacent hats $h_1, h_2 \in V(G) \setminus V(C)$, where h_1 is adjacent to c_1, c_2 say, and h_2 is adjacent to c_i, c_{i+1}, for some i with $3 \leq i < n$. If $i \geq 4$ then the hole h_1-h_2-c_{i+1}-c_{i+2}-$\cdots$-c_1-h_1 is not dominating, since c_3 has no neighbour in it, a contradiction. Thus $i = 3$ and similarly $i + 1 = n$, and so $n = 4$. But then the subgraph induced on $V(C) \cup \{h_1, h_2\}$ is a $(1,1,1)$-prism, a contradiction. Thus there is no hat-diagonal for C.

Suppose that there is a star-diagonal for C; that is, $n = 5$ or 6, and there are adjacent stars s_1, s_2 adjacent respectively to c_{n-1}, c_n, c_1, c_2 and c_2, c_3, c_4, c_5 say. Then s_2 is a centre for the $(n-1)$-hole s_1-c_2-c_3-$\cdots$-c_{n-1}-s_1, a contradiction. Finally, suppose that there is a star-triangle for C; that is, $n = 6$ and there are three pairwise adjacent stars s_1, s_2, s_3 adjacent respectively to $c_1, \ldots, c_4$, to $c_3, \ldots, c_6$ and to c_5, c_6, c_1, c_2. Then s_2 is a centre for the 4-hole s_1-c_4-c_5-s_3-s_1, a contradiction. Thus the hypotheses of 3.3 are satisfied, and consequently this proves (1).

Let (ϕ, D) be as in (1). Suppose that $|\phi^{-1}(\sigma_1)| > 1$ for some $\sigma_1 \in \Sigma$. From the fifth axiom for a metacircular arrangement, there exists $\sigma_2 \in \Sigma$ with $\sigma_2 \neq \sigma_1$, such that X_1 is neither complete nor anticomplete to X_2, and every vertex in X_1 has both a neighbour and a nonneighbour in X_2, and vice versa, where $X_i = \phi^{-1}(\sigma_i)$ $(i = 1, 2)$. Choose $u_1 \in X_1$ with as many neighbours in X_2 as possible. Choose $v_2 \in X_2$ nonadjacent to u_1. Let $v_1 \in X_1$ be a neighbour of v_2. Since v_1 does not have more neighbours in X_2 than u_1, and v_1 is adjacent to v_2 and u_1 is not, it follows that there is a neighbour $u_2 \in X_2$ of u_1 that is nonadjacent to v_1. Since X_1, X_2 are cliques by the first axiom, it follows that u_1-u_2-v_2-v_1-u_1 is a hole, C say. We may assume that in D, all edges between X_1 and X_2 have tail in X_1. By hypothesis, C has no centre, and so there does not exist $\sigma_3 \in \Sigma$ different from σ_1, σ_2 with $\phi^{-1}(\sigma_3)$ nonempty such that $\sigma_1, \sigma_3, \sigma_2$ are in clockwise order. Since σ_2 is the only point $\sigma \in \Sigma$ such that $\phi^{-1}(\sigma_1)$ is neither complete nor anticomplete to $\phi^{-1}(\sigma)$, it follows that every $v \in V(G) \setminus (X_1 \cup X_2)$ is either complete or anticomplete to X_1, and similarly is either complete or anticomplete to X_2. But no such v is complete to both X_1, X_2, since C has no centre, and no v is anticomplete to both X_1, X_2 since C is dominating. For $i = 1, 2$, let A_i be the set of all $v \in V(G) \setminus (X_1 \cup X_2)$ that are complete to X_i; then A_1, A_2, X_1, X_2 are pairwise disjoint and have union $V(G)$, and A_1 is complete to X_1 and anticomplete to X_2, and A_2 is complete to X_2 and anticomplete to X_1. Since G contains no $(1,1,1)$-prism, it follows that A_1 is anticomplete to A_2. Since G is claw-free, and A_1 is complete to $\{u_1\}$ and anticomplete to $\{u_2\}$, it follows that A_1 is a clique, and similarly so is A_2. Hence for $i = 1, 2$, and for every vertex $v \in A_i$, all neighbours of v are pairwise adjacent, and therefore v belongs to no hole. We recall that C_0 is a basis for (ϕ, D). It follows that $V(C_0) \subseteq X_1 \cup X_2$, which is impossible since $\phi(v)$ $(v \in V(C_0))$ are all different. This proves that there is no such σ_1.

We have therefore proved that $|\phi^{-1}(\sigma_1)| \leq 1$ for all $\sigma_1 \in \Sigma$. Consequently G is a long circular interval graph. This proves 4.2. ■

5 Internal clique cutsets

Now we come to the second main result of the paper. We recall that the goal of this series of papers is to give a construction for all claw-free graphs. A *clique cutset* in a trigraph G is a strong clique C such that $G \setminus C$ admits a 0-join, that is, $V(G) \setminus C$ can be partitioned into two nonempty sets V_1, V_2 such that V_1 is strongly anticomplete to V_2. In this situation, G may be regarded as being constructed from the trigraphs $G|(V_i \cup C)$ $(i = 1, 2)$ in the natural way, but this construction is not useful for our purposes, since it does not preserve the property of being claw-free (that is, G may not be claw-free even if both the graphs $G|(V_i \cup C)$ $(i = 1, 2)$ are claw-free).

This difficulty can be partly obviated. Let us say the clique cutset C is *internal* if the sets V_1, V_2 can be chosen with $|V_1|, |V_2| > 1$. The main result of this section asserts that if G is claw-free and admits an internal clique cutset, then either G admits a decomposition that *is* useful for us, or it belongs to a special class of trigraphs that we can characterize completely.

For a trigraph G and a vertex $v \in V(G)$ we denote by $N(x)$ the set of neighbours of x in G, and $N^*(v)$ denotes the set of strong neighbours of v. We need the following lemma.

5.1 *Let G be a claw-free trigraph and let C be a minimal clique cutset in G. Let V_1, V_2 be a partition of $V(G) \setminus C$ such that $V_1, V_2 \neq \emptyset$ and V_1 is anticomplete to V_2. Then*

- *for all $u \in C$, both $N(u) \cap V_1$ and $N(u) \cap V_2$ are strong cliques, and*
- *for all $u, v \in C$, either $N(u) \cap V_1 \subseteq N^*(v) \cap V_1$ or $N(u) \cap V_2 \subseteq N^*(v) \cap V_2$.*

Proof. Suppose that for some vertex $u \in C$ there exist two antiadjacent vertices x, y in $N(u) \cap V_1$. By the minimality of C, there exists a vertex $z \in N(u) \cap V_2$. But now $\{u, x, y, z\}$ is a claw in G, a contradiction. This proves the first assertion of the theorem.

For the second, assume that there exist $v_1 \in (N(u) \setminus N^*(v)) \cap V_1$ and $v_2 \in (N(u) \setminus N^*(v)) \cap V_2$. Since C is a clique, u is adjacent to v. But then $\{u, v, v_1, v_2\}$ is a claw, a contradiction. This proves the second assertion of the theorem and completes the proof of 5.1. ■

The following is the main result of this section.

5.2 *Let G be a claw-free trigraph with an internal clique cutset, such that G does not admit twins, a 0-join, or a 1-join. Then every hole in G has length four; if there is a 4-hole then G admits a coherent proper W-join, and otherwise G is a linear interval trigraph.*

Proof. Let G be a claw-free trigraph admitting a internal clique cutset, such that G does not admit twins, a 0-join, or a 1-join. In particular, G is not an antinet. If G has no hole, the theorem therefore follows from 4.1. Hence we may assume that there is a hole H in G. Choose H with length at least five if possible.

Let us say a *clique-separation* in G is a triple (C, V_1, V_2), such that

- C is a strong clique of G, and (V_1, V_2) is a partition of $V(G) \setminus C$,
- V_1 is strongly anticomplete to V_2, and
- $V(H) \cap V_2 = \emptyset$.

(1) *There is a clique-separation (C, V_1, V_2) in G with $|V_2| \geq 2$ and with $|V_2|$ maximum; and it can be chosen such that $C \neq \emptyset$, and every vertex in C has a neighbour in V_1 and a neighbour in V_2.*

For since G admits an internal clique cutset, there is a triple (C, V_1, V_2) satisfying the first and second conditions in the definition of a clique-separation, with $|V_1|, |V_2| \geq 2$; and since C is a strong clique, it follows that $V(H)$ has empty intersection with one of V_1, V_2. Hence (possibly after exchanging V_1, V_2), it follows that G contains a clique-separation (C, V_1, V_2) with $|V_2| \geq 2$. Choose a clique-separation (C, V_1, V_2) with $|V_2|$ maximum (and therefore with $|V_2| \geq 2$), and subject to that, with C minimal. Since G does not admit a 0-join, it follows that $C \neq \emptyset$. Let $c \in C$. If c has no neighbour in V_2, then $(C \setminus \{c\}, V_1 \cup \{c\}, V_2)$ is also a clique-separation with $|V_2|$ maximum, contradicting the minimality of C; and if c has no neighbour in V_1, then $c \notin V(H)$ (since every vertex in $V(H) \cap C$ has a neighbour in $V(H) \setminus C \subseteq V_1$, because C is a strong clique), and therefore $(C \setminus \{c\}, V_1, V_2 \cup \{c\})$ is a clique-separation contradicting the maximality of $|V_2|$. This proves (1).

For a vertex $c \in C$ and for $i = 1, 2$, let $N_i(c)$ be the set of neighbours of c in V_i, and let $N_i^*(c)$ be the set of strong neighbours of c in V_i. Let J be the digraph with $V(J) = C$ and edge set all pairs (u, v) with $u, v \in C$ (possibly equal), such that $N_1(v) \not\subseteq N_1^*(u)$. Since C is nonempty, there is a strong component of J that is a "sink component"; that is, there exists $X \subseteq C$ such that

- X is nonempty and $J|X$ is strongly connected
- there is no edge $(u, v) \in E(J)$ with $u \in X$ and $v \notin X$.

(2) *For all distinct $u, v \in X$, $N_2(u) = N_2^*(u) = N_2(v) = N_2^*(v)$.*

For since X is strongly connected, there is a directed path of J from u to v, say $u = v_1\text{-}\cdots\text{-}v_k = v$. For $1 \leq i < k$, since $(v_i, v_{i+1}) \in E(J)$, it follows that $N_1(v_{i+1}) \not\subseteq N_1^*(v_i)$, and therefore $N_2(v_{i+1}) \subseteq N_2^*(v_i)$ by the second statement of 5.1. Consequently $N_2(v) \subseteq N_2^*(u)$. Similarly $N_2(u) \subseteq N_2^*(v)$. This proves (2).

Let $Z = \bigcap_{x \in X} N_1^*(x)$.

(3) *$X \neq C$.*

For suppose that $X = C$. Choose $c \in C$, and let $Y = N_2(c)$. By 5.1, Y is a strong clique. There are two cases, depending whether $N_2(c') = N_2^*(c') = Y$ for all $c' \in C$ or not. Suppose first that $N_2(c') = N_2^*(c') = Y$ for all $c' \in C$. Then $C \cup Y$ is a strong clique. If $Y = V_2$ then since $|V_2| > 1$ it follows that G admits twins, a contradiction; and if $Y \neq V_2$ then $(V_1 \cup C, V_2)$ is a 1-join, again a contradiction. Thus we may assume that there exists $c' \in C$ with one of $N_2(c'), N_2^*(c')$ different from Y. By (2), $|C| = 1$, and so $c' = c$ and $N_2(c) \neq N_2^*(c)$. Hence $N_1(c) = N_1^*(c) = Z$ (since c is semiadjacent to a member of V_2 and $F(G)$ is a matching); and by 5.1, Z is a strong clique, and therefore so is $Z \cup C$. But $Z \neq V_1$, because $G|V_1$ contains a hole and therefore V_1 is not a strong clique; and so $(V_1, V_2 \cup C)$ is a 1-join, a contradiction. This proves (3).

(4) *$X \cup Z$ is a strong clique, and $N_1(c) \subseteq Z$ for every vertex $c \in C \setminus X$, and H is a 4-hole, and $V(H)$ consists of two vertices of $C \setminus X$ and two vertices of Z.*

For the first statement of 5.1 implies that Z is a strong clique, and therefore $X \cup Z$ is a strong clique. Let $c \in C \setminus X$, and $x \in X$. Since $(x,c) \notin E(J)$, it follows that $N_1(c) \subseteq N_1^*(x)$. Since this holds for all $x \in X$, we deduce that $N_1(c) \subseteq Z$. From (3), $(X \cup Z, V_1 \setminus Z, V_2 \cup (C \setminus X))$ is not a clique-separation of G, and so $V(H) \cap (C \setminus X) \neq \emptyset$. Let H have vertices h_1-$\cdots$-h_n-h_1 in order, where $h_1 \in C \setminus X$. Then $h_2, h_n \in C \cup N_1(h_1) \subseteq C \cup Z$, and since C, Z are both strong cliques and h_2, h_n are antiadjacent, we may assume that $h_2 \in C$ and $h_n \in Z$. Since h_2, h_n are antiadjacent, and $X \cup Z$ is a strong clique, it follows that $h_2 \notin X$, and so $h_2 \in C \setminus X$. Thus by the same argument $h_3 \in Z$. Since $h_3, h_n \in Z$ and Z is a strong clique, it follows that $n = 4$, and so H is a 4-hole. This proves (4).

Let us say a *step* is a 4-hole consisting of two vertices of $C \setminus X$ and two vertices of Z. We have seen that H is a step. We say a pair (A, B) is a *step-connected strip* if $A \subseteq Z$ and $B \subseteq C \setminus X$, and for every partition (P, Q) of A or of B with P, Q nonempty, there is a step S with $V(S) \subseteq A \cup B$ and with $P \cap V(S), Q \cap V(S) \neq \emptyset$. Certainly the pair $(V(H) \cap Z, V(H) \cap (C \setminus X))$ is a step-connected strip; so we may choose a step-connected strip (A, B) with $V(H) \subseteq A \cap B$ and with $A \cup B$ maximal.

(5) *Every vertex in $V(G) \setminus (A \cup B)$ is either strongly complete or strongly anticomplete to A, and either strongly complete or strongly anticomplete to B. Moreover, the set of vertices $V(G) \setminus (A \cup B)$ that are complete to $A \cup B$ is a strong clique.*

For let $v \in V(G) \setminus (A \cup B)$, and let A_1, B_1 be the set of members of A, B respectively that are adjacent to v. Let A_2, B_2 be the set of members of A, B respectively that are antiadjacent to v. Suppose first that A_1, A_2 are both nonempty. Then $v \notin V_2$, since V_2 is strongly anticomplete to A, and so $v \in C \cup V_1$. Since $|A| \geq 2$, there is a partition (P, Q) of A with P, Q nonempty and with $P \subseteq A_1$ and $Q \subseteq A_2$. Hence, since (A, B) is step-connected, there is a step a_1-a_2-b_2-b_1-a_1 with $a_1 \in A_1$, $a_2 \in A_2$, and $b_1, b_2 \in B$. Since $\{a_1, b_1, a_2, v\}$ is not a claw, v is strongly adjacent to b_1. Since $X \cup Z$ is a strong clique containing a_2, and v is antiadjacent to a_2, it follows that $v \notin X \cup Z$; and therefore $v \notin N_1(b_1)$ by (4). Consequently $v \in C \setminus X$, and therefore v is adjacent to b_2. But then a_1-v-b_2-a_2-a_1 is a step, and so $(A, B \cup \{v\})$ is a step-connected strip, contrary to the maximality of $A \cup B$. Hence not both A_1, A_2 are nonempty. This proves the first assertion of (5).

Now assume that B_1, B_2 are both nonempty, and choose a step a_1-a_2-b_2-b_1-a_1 with $a_1, a_2 \in A$, $b_1 \in B_1$ and $b_2 \in B_2$. Since $\{b_1, a_1, b_2, v\}$ is not a claw in G, it follows that v is strongly adjacent to a_1. Since not both A_1, A_2 are nonempty, it follows that $a_2 \in A_1$. Also, v is not strongly anticomplete to A, and so $v \notin V_2$; v is antiadjacent to b_2, and so $v \notin C$; and therefore $v \in V_1$. Consequently $v \in N_1(b_1) \subseteq Z$. Hence v-a_2-b_2-b_1-v is a step, and so $(A \cup \{v\}, B)$ is a step-connected strip, contrary to the maximality of $A \cup B$. This proves the second assertion of (5).

Now suppose that $a, b \in V(G) \setminus (A \cup B)$ are both complete to $A \cup B$, and are antiadjacent. In particular, a, b are not strongly anticomplete to A, and so $a, b \notin V_2$; and they are both adjacent to a vertex of $C \setminus X$, and therefore both belong to $C \cup Z$. Since C, Z are strong cliques, we may assume that $a \in Z$ and $b \in C$. Since $X \cup Z$ is a strong clique, it follows that $b \in C \setminus X$. Choose $a' \in V(H) \cap Z$ and $b' \in V(H) \cap C$, antiadjacent. Then a-a'-b-b'-a is a step, and so $(A \cup \{a\}, B \cup \{b\})$ is a step-connected strip, contrary to the maximality of $A \cup B$. Thus there are no such a, b. This proves (5).

From (4), H has length four, and so no hole of G has length more than four; and from (5), G admits a coherent proper W-join. This proves 5.2. ■

References

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