Claw-free Graphs. I. Orientable prismatic graphs

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Abstract

A graph is *prismatic* if for every triangle T, every vertex not in T has exactly one neighbour in T. In this paper and the next in this series, we prove a structure theorem describing all prismatic graphs. This breaks into two cases depending whether the graph is 3-colourable or not, and in this paper we handle the 3-colourable case. (Indeed we handle a slight generalization of being 3-colourable, called being "orientable".)

Since complements of prismatic graphs are claw-free, this is a step towards the main goal of this series of papers, providing a structural description of all claw-free graphs (a graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours).

1 Introduction

Let G be a graph. (All graphs in this paper are finite and simple.) A *clique* in G is a set of pairwise adjacent vertices, and a *triangle* is a clique with cardinality three. We say G is *prismatic* if for every triangle T, every vertex not in T has exactly one neighbour in T. Our objective, in this paper and the next [1] of this series, is to describe all prismatic graphs.

A graph is *claw-free* if no vertex has three pairwise nonadjacent neighbours. The main goal of this series of papers is to give a structure theorem describing all claw-free graphs. Complements of prismatic graphs are claw-free, and we find it best to handle such graphs separately from the general case, since they seem to require completely different methods.

A 3-*colouring* of a graph G is a triple (A, B, C) such that A, B, C are pairwise disjoint stable subsets of $V(G)$ with union $V(G)$; and we call the quadruple (G, A, B, C) a 3-*coloured graph*. One way to make a (3-colourable) prismatic graph is to take several smaller prismatic graphs, each with a 3-colouring, and piece them together in a "chain". (We explain the details later.) This kind of chain construction is only needed in the 3-colourable case, and for this reason and others, it seems best to treat 3-colourable prismatic graphs separately, and that is one of our goals in this paper.

The graph G we construct by this chaining process depends not only on the graphs that are the building blocks, but also on the 3-colouring selected for each; so for this to count as a "construction" for G, we need constructions for all these smaller 3-coloured graphs. For this reason, our aim in this paper is to construct not only all 3-colourable prismatic graphs, but all 3-colourings of such graphs. But it turns out that, with a few small exceptions, a prismatic graph that admits none of our decompositions has at most one 3-colouring (up to exchanging the colour classes), so enumerating its 3-colourings is not a problem.

Let $T = \{a, b, c\}$ be a set with a, b, c distinct. There are two cyclic permutations of T, and we use the notation $a \to b \to c \to a$ to denote the cyclic permutation mapping a to b, b to c and c to a. (Thus $a \to b \to c \to a$ and $b \to c \to a \to b$ mean the same permutation.)

Let G be a prismatic graph. If S, T are triangles of G with $S \cap T = \emptyset$, then since every vertex of S has a unique neighbour in T and vice versa, it follows that there are precisely three edges of G between S and T, forming a 3-edge matching. An *orientation* $\mathcal{O}$ of G is a choice of a cyclic permutation $\mathcal{O}(T)$ for every triangle T of G, such that if $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are triangles with $S \cap T = \emptyset$, and $s_i t_i$ is an edge for $1 \le i \le 3$, then $\mathcal{O}(S)$ is $s_1 \to s_2 \to s_3 \to s_1$ if and only if $\mathcal{O}(T)$ is $t_1 \to t_2 \to t_3 \to t_1$. We say that G is *orientable* if it admits an orientation. Every 3-colourable prismatic graph is orientable, as we shall see later. It turns out that orientable prismatic graphs are not much more general than 3-colourable ones, and it is convenient to handle them at the same time.

In order to state our main results (a construction for all 3-colourable prismatic graphs, and a construction for all orientable prismatic graphs), we need a number of further definitions, and it is convenient to postpone the full statement of these theorems until section 11.

2 A construction

First we give a construction for a subclass of prismatic graphs. We present this in the hope of aiding the reader's understanding for what will come later; the truth of the claims in this section is not crucial, and we leave the proofs to the reader. (Our main result is that every orientable prismatic graph can be built from the graphs presented in this section and one other class, by certain composition operations.)

There are four stages in the construction. First, we need what we call "linear vines" and "circular vines".

- Start with a directed path or directed cycle S with vertices $s_1, \ldots, s_n$ in order with $n \geq 1$, such that if S is a cycle then $n \geq 5$ and $n = 2$ modulo 3.
- Choose a stable subset $W \subseteq V(S)$ (with $s_1, s_n \notin W$ if S is a path).
- For each $s_i \in W$, duplicate s_i arbitrarily often (that is, add a set of new vertices to the digraph, each incident with the same in-neighbours and out-neighbours as s_i). Let $\hat{X}_{2i}$ be the set consisting of s_i and these copies, and for $1 \leq i \leq n$ with $s_i \notin W$, let $\hat{X}_{2i} = \{s_i\}$. Let the digraph just constructed be J_1.
- For every edge uv of J_1, add a new vertex w to J_1, adjacent only to u and v, in such a way that the cycle with vertex set $\{u, v, w\}$ is a directed cycle. For $1 \leq i < n$, let M_{2i+1} be the set of all such w where $u \in \hat{X}_{2i}$ and $v \in \hat{X}_{2i+2}$. (If S is a path, let $M_1 = M_{2n+1} = \emptyset$.) Let this form a digraph J_2.
- For each $s_i \notin W$, add arbitrarily many adjacent pairs of new vertices x, y to J_2, such that x, y are adjacent only to s_i and to each other, and the cycle with vertex set $\{x, y, s_i\}$ is directed. Let R_{2i-1}, L_{2i+1} be the set of new out-neighbours and new in-neighbours of s_i, respectively. (Ensure that if S is a path then R_1, L_{2n+1} are large enough that in the digraph we construct, s_1, s_n are both in at least two triangles.) Define $R_{2i-1} = L_{2i+1} = \emptyset$ for $1 \leq i \leq n$ with $s_i \notin W$ (and if S is a path let $L_1 = R_{2n+1} = \emptyset$).

If S is a path we call the digraph we construct a *linear vine*, and if S is a cycle we call it a *circular vine*. (We give a more formal definition later.) In the remainder of the construction, we assume that H is a linear vine; the modifications when H is circular are easy, and we leave them to the reader. For $1 \leq i \leq n+1$ let $X_{2i-1} = L_{2i-1} \cup M_{2i-1} \cup R_{2i-1}$.

The second step of the construction is, we take the undirected graph underlying H, and add some new vertices to it. For $1 \leq i \leq n$ let X_{2i} be a set including $\hat{X}_{2i}$, such that the members of $X_{2i} \setminus \hat{X}_{2i}$ are new vertices, and in particular the sets $X_2, \ldots, X_{2n}$ are pairwise disjoint. For each new vertex $w \in X_{2i} \setminus \hat{X}_{2i}$, all its neighbours belong to $R_{2i-1} \cup L_{2i+1}$, and w is adjacent to exactly one end of every edge of H' between R_{2i-1} and L_{2i+1}. Let the graph we obtain be H'.

Third, now we add more new edges to H'. We add the edge uv for each choice of vertices $u, v \in V(H')$ satisfying the following: $u \in X_i$ and $v \in X_j$, where $1 \leq i < j \leq 2n+1$ and $j \geq i+2$, and either

- $j \geq i+3$ and $j - i = 2$ modulo 3;
- $j = i+2$ and i is even;
- $j = i+2$ and i is odd, and either $u \notin R_i$ or $v \notin L_{i+2}$, and u, v have no common neighbour in $\hat{X}_{i+1}$.

Let the graph just constructed be G'.

The fourth and final step of the construction is, for all even i, j with $2 \leq i < j \leq 2n$, we may arbitrarily delete any of the edges between $X_i \setminus \hat{X}_i$ and $X_j \setminus \hat{X}_j$. Let the graph we produce be G.

We leave the reader to check that G is prismatic and orientable (and indeed, the edges of G in cycles of length 3 are precisely the edges of H, and their directions in H define an orientation of G in the natural way). We call such a graph G a *path of triangles graph.* (Again, we give a formal definition later.) There is a similar construction starting from a circular vine, and again the graphs that result are prismatic and orientable; we call them *cycle of triangles graphs.*

3 Core structure

Before we begin on the main theorem (or even attempt its statement; the statement of the main theorem will appear in section 11) we study the question under two simplifying assumptions. We say G is *triangle-covered* if every vertex of G belongs to a triangle; and G is *triangle-connected* if there is no partition A, B of $V(G)$ into two subsets, both including a triangle, such that every triangle of G is included in one of A, B. We shall explain the structure of 3-colourable prismatic graphs that are triangle-covered and triangle-connected.

If $X \subseteq V(G)$, we denote the subgraph of G induced on X by $G|X$. If $Y \subseteq V(G)$ and $x \in V(G) \setminus Y$, we say that x is *complete to* Y or Y*-complete* if x is adjacent to every member of Y; and x is *anticomplete to* Y or Y*-anticomplete* if x is adjacent to no member of Y. If $X, Y \subseteq V(G)$ are disjoint, we say that X is *complete* to Y (or the pair (X, Y) is *complete*) if every vertex of X is adjacent to every vertex of Y. We say that X is *anticomplete* to Y (or (X, Y) is *anticomplete*) if (X, Y) is complete in $\overline{G}$. If $X, Y \subseteq V(G)$, we say that X, Y are *matched* if $X \cap Y = \emptyset$, $|X| = |Y|$, and every vertex in X has a unique neighbour in Y and vice versa.

Let us say that G is a *path of triangles* graph if for some integer $n \geq 1$ there are pairwise disjoint stable subsets $X_1, \ldots, X_{2n+1}$ of $V(G)$ with union $V(G)$, satisfying the following conditions **(P1)**–**(P7)**.

(P1) For $1 \leq i \leq n$, there is a nonempty subset $\hat{X}_{2i} \subseteq X_{2i}$; $|\hat{X}_2| = |\hat{X}_{2n}| = 1$, and for $0 < i < n$, at least one of $\hat{X}_{2i}, \hat{X}_{2i+2}$ has cardinality 1.

(P2) For $1 \leq i < j \leq 2n + 1$

(1) if $j - i = 2$ modulo 3 and there exist $u \in X_i$ and $v \in X_j$, nonadjacent, then either i, j are odd and $j = i + 2$, or i, j are even and $u \notin \hat{X}_i$ and $v \notin \hat{X}_j$;

(2) if $j - i \neq 2$ modulo 3 then either $j = i + 1$ or X_i is anticomplete to X_j.

(P3) For $1 \leq i \leq n + 1$, X_{2i-1} is the union of three pairwise disjoint sets $L_{2i-1}, M_{2i-1}, R_{2i-1}$, where $L_1 = M_1 = M_{2n+1} = R_{2n+1} = \emptyset$.

(P4) If $R_1 = \emptyset$ then $n \geq 2$ and $|\hat{X}_4| > 1$, and if $L_{2n+1} = \emptyset$ then $n \geq 2$ and $|\hat{X}_{2n-2}| > 1$.

(P5) For $1 \leq i \leq n$, X_{2i} is anticomplete to $L_{2i-1} \cup R_{2i+1}$; $X_{2i} \setminus \hat{X}_{2i}$ is anticomplete to $M_{2i-1} \cup M_{2i+1}$; and every vertex in $X_{2i} \setminus \hat{X}_{2i}$ is adjacent to exactly one end of every edge between R_{2i-1} and L_{2i+1}.

(P6) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| = 1$, then

(1) R_{2i-1}, L_{2i+1} are matched, and every edge between $M_{2i-1} \cup R_{2i-1}$ and $L_{2i+1} \cup M_{2i+1}$ is between R_{2i-1} and L_{2i+1};

(2) the vertex in $\hat{X}_{2i}$ is complete to $R_{2i-1} \cup M_{2i-1} \cup L_{2i+1} \cup M_{2i+1}$;

(3) L_{2i-1} is complete to X_{2i+1} and X_{2i-1} is complete to R_{2i+1}

(4) if $i > 1$ then $M_{2i-1}, \hat{X}_{2i-2}$ are matched, and if $i < n$ then $M_{2i+1}, \hat{X}_{2i+2}$ are matched.

(P7) For $1 < i < n$, if $|\hat{X}_{2i}| > 1$ then

(1) $R_{2i-1} = L_{2i+1} = \emptyset$;

(2) if $u \in X_{2i-1}$ and $v \in X_{2i+1}$, then u, v are nonadjacent if and only if they have the same neighbour in $\hat{X}_{2i}$.

We leave the reader to check that this is equivalent to the definition presented in the previous section. It is easy to see a vertex of G is in no triangle of G if and only if it belongs to one of the sets $X_{2i} \setminus \hat{X}_{2i}$. If for each i we have $\hat{X}_{2i} = X_{2i}$, then G is triangle-covered, and G is called a *core path of triangles* graph. The sequence $X_1, \ldots, X_{2n+1}$ is called a *(core) path of triangles decomposition* of G. We shall prove the following.

3.1 *Let G be a non-null 3-colourable prismatic graph that is triangle-covered and triangle-connected. Then either G is isomorphic to $L(K_{3,3})$, or G is a core path of triangles graph.*

($K_{3,3}$ is the complete bipartite graph on two sets of cardinality three, and $L(H)$ denotes the line graph of a graph H.) The proof is contained in the next four sections.

4 Orientable prismatic graphs

We defined what we mean by an orientation in the first section, and it is convenient to prove an extension of 3.1 in which we replace the 3-colourable hypothesis by the weaker assumption that G is orientable. To begin, let us see that this is indeed weaker.

4.1 *Every 3-colourable prismatic graph is orientable.*

Proof. Let (A, B, C) be a 3-colouring of an orientable prismatic graph G. For each triangle T, define $\mathcal{O}(T)$ to be $a \to b \to c \to a$ where $T = \{a, b, c\}$ and $a \in A, b \in B$ and $c \in C$. We claim that $\mathcal{O}$ is an orientation of G. For let $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ be disjoint triangles where s_1t_1, s_2t_2, s_3t_3 are edges. Let $\mathcal{O}(S)$ be $s_1 \to s_2 \to s_3 \to s_1$; thus we may assume that $s_1 \in A, s_2 \in B$ and $s_3 \in C$. We must show that $\mathcal{O}(T)$ is $t_1 \to t_2 \to t_3 \to t_1$. Certainly $t_1 \notin A$, since s_1, t_1 are adjacent, and so either $t_1 \in B$ or $t_1 \in C$. If $t_1 \in B$, then since t_3 is adjacent to both s_3 and t_1, it follows that $t_3 \in A$ and therefore $t_2 \in C$ and the claim follows; and if $t_1 \in C$, then $t_2 \in A$ and $t_3 \in B$ and again the claim follows. This proves 4.1. ■

The converse to this is false; there are orientable prismatic graphs that are not 3-colourable. For instance, let G have vertex set $\{v_0, \ldots, v_9\}$, with edges v_iv_{i+1} and v_iv_{i+5} (for all i), and v_iv_{i+2} (for i even), reading subscripts modulo 10. (We call this graph the *core ring of five.*) Nevertheless, orientable prismatic graphs are not much more general than 3-colourable prismatic graphs, as we shall see. We need a slight modification of an earlier definition, as follows.

Let us say that G is a *cycle of triangles* graph if for some integer $n \geq 5$ with $n = 2$ modulo 3, there are pairwise disjoint stable subsets $X_1, \ldots, X_{2n}$ of $V(G)$ with union $V(G)$, satisfying the following conditions **(C1)–(C6)** (reading subscripts modulo $2n$):

(C1) For $1 \leq i \leq n$, there is a nonempty subset $\hat{X}_{2i} \subseteq X_{2i}$, and at least one of $\hat{X}_{2i}, \hat{X}_{2i+2}$ has cardinality 1.

(C2) For $i \in \{1, \ldots, 2n\}$ and all k with $2 \leq k \leq 2n-2$, let $j \in \{1, \ldots, 2n\}$ with $j = i + k$ modulo $2n$:

(1) if $k = 2$ modulo 3 and there exist $u \in X_i$ and $v \in X_j$, nonadjacent, then either i, j are odd and $k \in \{2, 2n-2\}$, or i, j are even and $u \notin \hat{X}_i$ and $v \notin \hat{X}_j$;

(2) if $k \neq 2$ modulo 3 then X_i is anticomplete to X_j.

(Note that $k = 2$ modulo 3 if and only if $2n - k = 2$ modulo 3, so these statements are symmetric between i and j.)

(C3) For $1 \leq i \leq n+1$, X_{2i-1} is the union of three pairwise disjoint sets $L_{2i-1}, M_{2i-1}, R_{2i-1}$.

(C4) For $1 \leq i \leq n$, X_{2i} is anticomplete to $L_{2i-1} \cup R_{2i+1}$; $X_{2i} \setminus \hat{X}_{2i}$ is anticomplete to $M_{2i-1} \cup M_{2i+1}$; and every vertex in $X_{2i} \setminus \hat{X}_{2i}$ is adjacent to exactly one end of every edge between R_{2i-1} and L_{2i+1}.

(C5) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| = 1$, then

(1) R_{2i-1}, L_{2i+1} are matched, and every edge between $M_{2i-1} \cup R_{2i-1}$ and $L_{2i+1} \cup M_{2i+1}$ is between R_{2i-1} and L_{2i+1};

(2) the vertex in $\hat{X}_{2i}$ is complete to $R_{2i-1} \cup M_{2i-1} \cup L_{2i+1} \cup M_{2i+1}$;

(3) L_{2i-1} is complete to X_{2i+1} and X_{2i-1} is complete to R_{2i+1}

(4) $M_{2i-1}, \hat{X}_{2i-2}$ are matched and $M_{2i+1}, \hat{X}_{2i+2}$ are matched.

(C6) For $1 \leq i \leq n$, if $|\hat{X}_{2i}| > 1$ then

(1) $R_{2i-1} = L_{2i+1} = \emptyset$;

(2) if $u \in X_{2i-1}$ and $v \in X_{2i+1}$, then u, v are nonadjacent if and only if they have the same neighbour in $\hat{X}_{2i}$.

Again, if $\hat{X}_{2i} = X_{2i}$ for $1 \leq i \leq n$ we call G a *core cycle of triangles graph.* We call the sequence $X_1, \ldots, X_{2n}$ a *(core) cycle of triangles decomposition* of G. We shall prove the following.

4.2 *Let G be a non-null orientable prismatic graph that is triangle-covered and triangle-connected. Then either G is isomorphic to $L(K_{3,3})$, or G is a core cycle of triangles graph, or G is a core path of triangles graph.*

To show that this implies 3.1, we need the second statement of the following lemma.

4.3 *Every core path of triangles graph is 3-colourable, and no core cycle of triangles graph is 3-colourable.*

Proof. Let $X_1, \ldots, X_{2n+1}$ be a core path of triangles decomposition of G. Then

$$(X_1 \cup X_4 \cup X_7 \cup \cdots, X_2 \cup X_5 \cup X_8 \cup \cdots, X_3 \cup X_6 \cup X_9 \cup \cdots)$$

is a 3-colouring of G. This proves the first assertion.

For the second, let $X_1, \ldots, X_{2n}$ be a core cycle of triangles decomposition of G, and for each i choose $x_i \in X_i$, so that x_i, x_{i+1} are adjacent for all i. Let (A, B, C) be a 3-colouring of G. Since n is not divisible by 3, it is not the case that for all i, the vertices $x_{2i}, x_{2i+2}, x_{2i+4}$ all have different colours. Since x_{2i+2} is adjacent to both x_{2i} and x_{2i+4}, we may therefore assume that (say) $x_2, x_6 \in A$ and $x_4 \in B$, and therefore $x_3, x_5 \in C$. Since x_8 is adjacent to $x_3 \in C$ and to $x_6 \in A$, it follows that $x_8 \in B$; and since x_{10} is adjacent to $x_2 \in A, x_5 \in C$ and to $x_8 \in B$, this is impossible. This proves 4.3. ■

5 Vines and their structure

In this section we prove a lemma that will be needed for the proof of 4.2. If u, v are adjacent vertices of a digraph H, we write $u \to v$ to denote that the edge uv has tail u and head v. (We only use this notation in digraphs with no directed cycle of length 2.)

We regard a digraph as a graph with additional structure; and in particular, we define the triangles, paths, cycles etc. of a digraph to mean the corresponding object in the undirected graph. When we mean a *directed* cycle or similar, we shall say so explicitly. We say a *thorn* of a digraph H is a vertex belonging to only one triangle of H. An edge uv of H is a *twig* if there is a unique vertex w such that $\{u, v, w\}$ is a triangle, and this vertex w is a thorn of H. A path P of H is called a *twig path* if all its edges are twigs. We say that a digraph H is a *vine* if it satisfies the following conditions **(V1)**–**(V7)**.

(V1) H has at least one edge, and H is connected (as a graph), and every cycle of H has length at least three.

(V2) Every edge of H is in a unique cycle of length 3.

(V3) Every cycle of H of length 3 is a directed cycle.

(V4) Every triangle of H contains a thorn of H.

(V5) If h_1-h_2-h_3-h_4-h_5 are the vertices in order of a 4-edge twig path of H (not necessarily an induced subgraph), then either $h_2 \to h_3 \to h_4$ or $h_4 \to h_3 \to h_2$.

(V6) If h_1-h_2-h_3-h_4-h_1 are the vertices in order of a 4-vertex cycle of H and $h_1 \to h_2$, then $h_4 \to h_3$.

(V7) If C is a cycle of H with length at least five, and no vertex of C is a thorn of H, then C has length 2 modulo 3.

Here is a useful lemma.

5.1 *Let uv be an edge of a vine H. If neither of u, v is a thorn then uv is a twig.*

Proof. There is a triangle T containing u, v; let $T = \{u, v, w\}$ say. Since some vertex of T is a thorn, it follows that w is a thorn, and so uv is a twig. ■

In section 2 we introduced linear and circular vines. It is easy to check that they are indeed vines. What follows is a more formal definition of the same thing. A vine H is said to be *linear* (respectively, *circular*) if there is a directed path (respectively, directed cycle) S of H, with vertices $s_1 \to s_2 \to \cdots \to s_n$ for some $n \geq 1$, such that, denoting by $N_S(v)$ the set of neighbours in $V(S)$ of $v \in V(H) \setminus V(S)$, the following conditions **(LV1)**–**(LV4)** are satisfied.

(LV1) S is an induced subgraph of H, and none of its vertices are thorns.

(LV2) If S is a cycle then $n \geq 5$ and $n = 2$ modulo 3 (and if so then in what follows subscripts are to be read modulo n).

(LV3) Every vertex in $V(H) \setminus V(S)$ has a neighbour in $V(S)$.

(LV4) For every $v \in V(H) \setminus V(S)$, if v is not a thorn then for some $i \in \{1, \dots, n\}$, where $1 < i < n$ if S is a path

- $N_S(v) = \{s_{i-1}, s_{i+1}\}$
- every neighbour of s_i or of v in $V(H) \setminus V(S)$ is a thorn adjacent to one of s_{i-1}, s_{i+1}
- $s_{i-1} \to v \to s_{i+1}$.

In this case we call S a *stem* of the vine. We will show the following.

5.2 *Every vine with at least two triangles is either linear or circular.*

Proof. Let H be a vine with at least two triangles. If C is a cycle of H of length at least five, and no vertex of C is a thorn, then all its edges are twigs by 5.1, and any five consecutive vertices of C form a five-vertex twig path, in which the two middle edges form a directed path, from **(V5)**. Consequently every two consecutive edges of C form a directed path, that is, C is a directed cycle. If H has a cycle of length at least five of which no vertex is a thorn, let S be such a cycle. Otherwise, since H has at least two triangles and is connected, there is a vertex that is not a thorn, and consequently we may choose S to be a directed path as long as possible such that no vertex of S is a thorn of H.

Let the vertices of S be $s_1, \dots, s_n$ in order, where $s_1 \to s_2 \to \cdots \to s_n$, and if S is a cycle then $s_n \to s_1$. Thus $n \geq 1$.

(1) *S is an induced subgraph of H.*

For suppose that there exist $i, j \in \{1, \dots, n\}$ such that $s_i s_j$ is an edge of H and not of S. Let P be a subpath of S between s_i, s_j; then P is a directed path. Let C be the cycle obtained by adding the edge $s_i s_j$ to P. Then C has length at least four, since no vertex of S is a thorn and every triangle contains a thorn. Since P is a directed path, **(V6)** implies that C has length at least five. Consequently H has a cycle of length at least five in which no vertex is a thorn, and therefore S is a directed cycle; and so there are two choices in S for the path P. For one of these two choices the cycle C is not a directed cycle, contrary to **(V5)**. This proves (1).

(2) *If $u, v \in V(H) \setminus V(S)$ are adjacent, and u has a neighbour in $V(S)$, then u, v have a common neighbour in $V(S)$.*

For suppose first that for some $i \in \{1, \dots, n\}$, u is adjacent to s_i and v is not. From the symmetry we may assume that $u \to s_i$. Since u has two nonadjacent neighbours, u is not a thorn, and so us_i is a twig by 5.1; and certainly all edges of S are twigs. Let $v' \in V(H)$ such that $\{u, v, v'\}$ is a triangle. Since s_i has a unique neighbour in this triangle, it follows that s_i, v' are nonadjacent. If $v' \in V(S)$, then u, v have a common neighbour in $V(S)$ as claimed, so we may assume that $v' \notin V(S)$. Since one of v, v' is a thorn, and neither of them has a common neighbour with u in $V(S)$, we may assume that uv is a twig, by exchanging v, v' if necessary.

If either $i \geq 3$ or S is a cycle, then the two middle edges of the path s_{i-2}-s_{i-1}-s_i-u-v both have the same head, namely s_i, a contradiction to **(V5)**. So $i \leq 2$ and S is a path. Let S' be the directed path u-s_i-s_{i+1}-$\cdots$-s_n. Its length is at least that of S, and u is not a thorn of H; so from the maximality of the length of S, it follows that $i = 2$. Since u is not a thorn, no member of $\{s_1, s_2, u\}$ is a thorn, and so this set is not a triangle, that is, u is not adjacent to s_1. Since s_1 is not a thorn of H, it follows from **(V2)** that s_1 has a neighbour $x \neq s_2$ with x, s_2 nonadjacent. From (1), $x \notin V(S)$, and $x \neq u$ since u, s_1 are nonadjacent. We claim that we may choose x so that xs_1 is a twig. For if xs_1 is not a twig, then x is a thorn; choose w so that $\{w, x, s_1\}$ is a triangle, and so ws_1 is a twig. Then $w \neq s_2$ since x, s_2 are nonadjacent, and so $w \notin V(S)$, and w, s_2 are nonadjacent since s_2 has only one neighbour in this triangle; and hence (by exchanging w, x if necessary) we may assume that xs_1 is a twig. If $x \neq v$, then the two middle edges of the path x-s_1-s_2-u-v have the same head, contrary to **(V5)**; and so $x = v$. But then v-s_1-s_2-u-v is a cycle of length four, and since $u \to s_2$ it follows that $v \to s_1$. Since u, s_1 are nonadjacent it follows that v is not a thorn. Also v-s_1-$\cdots$-s_n is a directed path, contrary to the maximality of the length of S. This proves that there is no such i, and so $N_S(u) \subseteq N_S(v)$. From the symmetry between u, v we deduce that $N_S(u) = N_S(v)$; and since $N_S(u) \neq \emptyset$ and at most one triangle contains both u, v, it follows that $|N_S(u)| = 1$, $N_S(u) = N_S(v) = \{s_i\}$ say. Suppose that u is not a thorn; then it has a neighbour w different from v, s_i. Since $N_S(u) = \{s_i\}$, it follows that $w \notin V(S)$, and so by what we already proved, $N_S(u) = N_S(w)$; but then w has two neighbours in the triangle $\{u, v, s_i\}$, a contradiction. Hence u, and similarly v, is a thorn. This proves (2).

(3) *If $v \in V(H) \setminus V(S)$, then $1 \leq |N_S(v)| \leq 2$. If $|N_S(v)| = 2$, then either*

- *$N_S(v) = \{s_{i-1}, s_{i+1}\}$ for some $i \in \{1, \dots, n\}$ (where $1 < i < n$ if S is a path), and $s_{i-1} \to v \to s_{i+1}$, or*
- *$N_S(v) = \{s_i, s_{i+1}\}$ for some $i \in \{1, \dots, n\}$ (where $i < n$ if S is a path), and v is a thorn, and $s_{i+1} \to v \to s_i$.*

For if v has no neighbour in $V(S)$, then since H is connected, there is an induced path w-x-y of H where $w \in V(S)$ and $x, y \notin V(S)$, contrary to (2). Thus v has a neighbour in $V(S)$. If every two neighbours of v in S are adjacent, then the claim holds, so we may assume that v is adjacent to s_i, s_j where $i < j$ and s_i, s_j are nonadjacent. Hence v is not a thorn. If every path of S between s_i, s_j has length at least three, then H has a cycle of length at least five no vertex of which is a thorn of H, and so S is a directed cycle, and there are two paths in S between s_i, s_j; and for both of them, their union with the path s_i-v-s_j makes a directed cycle, which is impossible. Thus there is a path of length two in S between s_i, s_j, and we may assume that $1 \leq i \leq n-2$ and $j = i+2$. From the cycle v-s_i-s_{i+1}-s_{i+2}-v, it follows that $s_i \to v \to s_{i+2}$. If v has another neighbour in S, say s_k, then $k \neq i, i+1, i+2$, and we may assume that $k \neq i-1$ from the symmetry. By the same argument applied to s_i, s_k, it follows that $k = i - 2$ (and so $i \geq 3$ if S is a path), and that $v \to s_i$, a contradiction. Thus $N_S(v) = \{s_i, s_{i+2}\}$. This proves (3).

(4) *If $v \in V(H) \setminus V(S)$ is not a thorn then*

- *$N_S(v) = \{s_{i-1}, s_{i+1}\}$ for some $i \in \{1, \dots, n\}$, where $1 < i < n$ if S is a path*
- *every neighbour of s_i or of v in $V(H) \setminus V(S)$ is a thorn adjacent to one of s_{i-1}, s_{i+1}*
- *$s_{i-1} \to v \to s_{i+1}$.*

For the first and third assertions follow from (3). For the second, suppose that $u \in V(H) \setminus V(S)$ is adjacent to one of v, s_i, and either it is not a thorn or it is nonadjacent to both s_{i-1}, s_{i+1}. Let $\{v, s_i\} = \{x, y\}$, where u is adjacent to x. We claim that we may choose u so that ux is a twig. For suppose it is not; then u is a thorn, and therefore u is nonadjacent to s_{i-1}, s_{i+1}. Let $\{w, u, x\}$ be a triangle; then $w \neq s_{i-1}, s_{i+1}$ since u is nonadjacent to them. Since s_{i-1} has only one neighbour in this triangle, it follows that w, s_{i-1} are nonadjacent, and similarly w, s_{i+1} are nonadjacent, and so we may replace u by w. This proves that we may assume that ux is a twig. But there is a five-vertex path u-x-s_{i-1}-y-s_{i+1}, and all its edges are twigs, and its two middle edges both have tail s_{i-1}, contrary to **(V5)**. This proves (4).

From (1)–(4), it follows that S is a stem and H is either a linear or circular vine. This proves 5.2. ■

6 The triangular digraph

In this section we make another step in the proof of 4.2. We show that, if G satisfies the hypotheses of that claim, then (provided that $G \neq L(K_{3,3})$) we can associate a vine with G.

Let G be prismatic with an orientation $\mathcal{O}$. Let H be the subgraph of G with $V(H) = V(G)$, and with edges the edges of G that belong to cycles of length 3. Let us direct the edges of H, so that H is a digraph, as follows. For every triangle $T = \{a, b, c\}$ where $\mathcal{O}(T)$ is $a \to b \to c \to a$, direct the edges ab, bc, ca of H so that $a \to b, b \to c, c \to a$. Since every edge of H belongs to exactly one triangle (since G is prismatic), this gives a well-defined digraph H. We call H the *triangular digraph* of G.

6.1 *Let G be prismatic, triangle-covered and triangle-connected, and not isomorphic to $L(K_{3,3})$, and let $\mathcal{O}$ be an orientation. Let H be the corresponding triangular digraph. Then for every triangle T, some vertex of T is a thorn of H.*

Proof. Let $T = \{t_1, t_2, t_3\}$ and suppose that for $i = 1, 2, 3$ there is a triangle $T_i \neq T$ containing t_i. Any vertex in $T_1 \cap T_2$ would be adjacent in G to both t_1, t_2, which is impossible since G is prismatic, and so $T_1 \cap T_2 = \emptyset$; and similarly T_1, T_2, T_3 are pairwise disjoint. Let $T_i = \{r_i, s_i, t_i\}$ say, where $\mathcal{O}(T_i)$ is $t_i \to r_i \to s_i \to t_i$ for $i = 1, 2, 3$. Since t_1, t_2 are adjacent, it follows that r_1r_2 and s_1s_2 are edges, and similarly that $r_1r_3, r_2r_3, s_1s_3, s_2s_3$ are edges. Let $W = T_1 \cup T_2 \cup T_3$. Thus $G|W$ is isomorphic to $L(K_{3,3})$.

Since G is not isomorphic to $L(K_{3,3})$, it follows that $V(G) \neq W$. Since G is triangle-connected and triangle-covered, there is a triangle Q that has nonempty intersection with W and with $V(G)\backslash W$. Since every two adjacent vertices in W belong to a triangle included in W, and belong to only one triangle, it follows that $|Q \cap W| = 1$; and we may assume that $Q \cap W = \{t_1\}$ from the symmetry. Let $Q = \{q_1, q_2, t_1\}$, where $\mathcal{O}(Q)$ is $t_1 \to q_1 \to q_2 \to t_1$. For $i = 2, 3$, since t_1, t_i are adjacent and $\mathcal{O}(T_i)$ is $t_i \to r_i \to s_i \to t_i$, it follows that q_1 is adjacent to r_i. In particular, q_1 has two neighbours in the triangle $\{r_1, r_2, r_3\}$, a contradiction. Thus not all of T_1, T_2, T_3 exist. This proves 6.1. ■

6.2 *Let G be prismatic, triangle-connected, triangle-covered, and not isomorphic to $L(K_{3,3})$. Let $\mathcal{O}$ be an orientation, and let H be the corresponding triangular digraph. Then H is a vine.*

Proof. We must verify the seven conditions **(V1)**–**(V7)** in the definition of a vine. Since G is triangle-covered and triangle-connected, it follows that H is connected. Every cycle of H is a cycle of G, and therefore has length at least three. Thus **(V1)** holds. Conditions **(V2)** and **(V3)** are clear, and **(V4)** follows from 6.1.

For **(V5)**, let h_1-h_2-h_3-h_4-h_5 be the vertices of a 4-edge twig path P of H. If h_1, h_3 are adjacent in H, then since h_1h_2 is a twig it follows that h_3 is a thorn, a contradiction since h_3 has three neighbours. So h_1, h_3 are nonadjacent, and similarly h_3, h_5 are nonadjacent. Let $m_1, m_2, m_3, m_4 \in V(H)$ such that for $i = 1, \ldots, 4$, $T_i = \{h_i, h_{i+1}, m_i\}$ is a triangle. Thus m_1, m_2, m_3, m_4 are thorns; and since $m_1, \ldots, m_4$ all have different sets of neighbours, it follows that $m_1, \ldots, m_4$ are all different. Since m_1 has only two neighbours h_1, h_2, it follows that $m_1 \neq h_3, h_4, h_5$ and so $m_1 \notin V(P)$. Since m_2 only has two neighbours h_2, h_3, it follows that $m_2 \neq h_4, h_5$; and $m_2 \neq h_1$ since h_1, h_3 are nonadjacent. So $m_2 \notin V(P)$. Similarly $m_3, m_4 \notin V(P)$.

Suppose that h_3 is the head of the edge h_2h_3. Thus $\mathcal{O}(T_2)$ is $m_2 \to h_2 \to h_3 \to m_2$. Let $\mathcal{O}(T_1)$ be $x_1 \to y_1 \to h_2 \to x_1$ say, where $\{x_1, y_1\} = \{h_1, m_1\}$; and similarly let $\mathcal{O}(T_4)$ be $x_2 \to y_2 \to h_4 \to x_2$. From the pair T_2, T_4, since h_3, h_4 are adjacent it follows that y_2, h_2 are adjacent. From the pair T_1, T_4, since y_2, h_2 are adjacent, it follows that x_1, h_4 are adjacent. From the pair T_1, T_3, since x_1h_4 and h_2h_3 are edges, it follows that $\mathcal{O}(T_3)$ is $m_3 \to h_3 \to h_4 \to m_3$, and so $h_3 \to h_4$ in H. Thus in this case h_3 is the head of exactly one of the two edges. The argument when h_3 is the tail of h_2h_3 is similar (and indeed can be reduced to the case we already did by reversing the orientation of every triangle). This proves **(V5)**.

For **(V6)**, let h_1-h_2-h_3-h_4-h_1 be the vertices in order of a cycle of length 4, where $h_1 \to h_2$. Let $m_1, m_2 \in V(G)$ such that $\{h_1, h_2, m_1\} = T_1$ and $\{h_3, h_4, m_2\} = T_2$ are triangles. Since no edge is in two triangles, $m_1, m_2, h_1, h_2, h_3, h_4$ are all different. Since $h_1 \to h_2$, it follows that $\mathcal{O}(T_1)$ is $m_1 \to h_1 \to h_2 \to m_1$. Since h_2h_3 and h_1h_4 are edges, and m_2 has a neighbour in T_1, it follows that m_1, m_2 are adjacent in G, and so $\mathcal{O}(T_2)$ is m_2-h_4-h_3-m_2. Hence $h_3 \to h_4$ in H. This proves **(V6)**.

For **(V7)**, let h_1-$\cdots$-h_n-h_1 be the vertices of a cycle C of H, in order, with $n \geq 5$, such that none of them are thorns of H. We may assume that $h_1 \to h_2$. By **(V5)**, $h_2 \to h_3$, and so on; in general (reading subscripts modulo n), $h_i \to h_{i+1}$. For $1 \leq i \leq n$, let $m_i \in V(H)$ such that $\{m_i, h_i, h_{i+1}\}$ is a triangle T_i. Since T_i contains a thorn, it follows that m_i is a thorn, and therefore $m_i \notin V(C)$. Now for $2 \leq i \leq n-2$, the triangles T_i, T_n are disjoint, and so if h_i is adjacent in G to some $x \in T_n$, then h_{i+1} is adjacent (in G) to the image of x under the permutation $\mathcal{O}(T_n)$. Since h_2 is adjacent to h_1, we deduce that h_i is adjacent (in G) to h_1 if $i = 2$ modulo 3, to m_n if $i = 0$ modulo 3, and to h_n if $i = 1$ modulo 3. Since h_{n-1} is adjacent to h_n and therefore nonadjacent to h_1, m_n, we deduce that

$n-1 = 1$ modulo 3, that is, $n = 2$ modulo 3. This proves **(V7)**, and therefore completes the proof of 6.2. ■

The next result allows us to reconstruct G from a knowledge of its triangular digraph. If H is the triangular digraph as usual, and P is a twig path of H of length at least three, we define the *signed length* $sl(P)$ of P as follows. Let P have vertices $p_1, \dots, p_k$ in order. Since H is a vine and P is a twig path, the path obtained from P by deleting p_1, p_k is a directed path Q_0; let Q be the unique maximal directed subpath of P that contains Q_0. An edge of P is called a *forward* edge if it belongs to Q, and any other edge of P is a *backward* edge. Thus, all edges of P are forward edges except possibly for the first and last. We define the signed length $sl(P)$ of P to be $d_1 - d_2$, where d_1, d_2 are the numbers of forward edges and backward edges in P, respectively.

6.3 *Let G be prismatic, triangle-connected, triangle-covered, and not isomorphic to $L(K_{3,3})$. Let $\mathcal{O}$ be an orientation of G, and let H be the corresponding triangular digraph. Let P be a twig path of H of length at least* 3*. Then the ends of P are adjacent in G if and only if $sl(P) = 1$ modulo* 3*.*

Proof. Let P have vertices $p_1, \dots, p_k$ in order, where $k \geq 4$. From 6.2, it follows that by exchanging p_1, p_k if necessary, we may assume that $p_2 \to p_3 \to \cdots \to p_{k-1}$. We claim that for $1 \leq i \leq k-2$, p_i and p_{i+2} are nonadjacent. For suppose they are adjacent; then since p_ip_{i+1} and $p_{i+1}p_{i+2}$ are both twigs, it follows that p_i, p_{i+2} are both thorns. In particular, since p_i has degree 2 it follows that $i = 1$, and since p_{i+2} has degree 2 it follows that $i+2 = k$, and so $k = 3$, a contradiction. This proves our claim that p_i and p_{i+2} are nonadjacent. It follows that $p_2, \dots, p_{k-1}$ are not thorns.

For each i with $1 \leq i < k$, choose a thorn $m_i \in V(H)$ such that $\{p_i, p_{i+1}, m_i\}$ is a triangle T_i say. If $p_1 = m_i$ for some i, then $2 \leq i < k$; $i \neq 2$ since p_1, p_3 are nonadjacent, and yet $p_2 \in \{p_i, p_{i+1}\}$ since p_i, p_{i+1} are the only neighbours of m_i, which is impossible. Thus $m_1, \dots, m_{k-1} \neq p_1$, and similarly they are different from p_k, and therefore they do not belong to $V(P)$. Moreover, they are all distinct.

Let π be the permutation $\mathcal{O}(T_1)$. For $i \in \{3, \dots, k\}$, let x_i be the unique vertex of T_1 that is adjacent in G to p_i; thus $x_3 = p_2$. For $3 \leq j \leq k-2$, since p_j is mapped to p_{j+1} by the permutation $\mathcal{O}(T_j)$, it follows that $x_{j+1} = \pi(x_j)$. Consequently $x_{k-1} = \pi^{k-4}(p_2)$. Let $n = k-3$ if $p_{k-1}p_k$ has tail p_{k-1}, and $n = k-5$ if it has tail p_k. In the first case $x_k = \pi(x_{k-1})$, and in the second $x_k = \pi^{-1}(x_{k-1})$, and so in both cases $x_k = \pi^n(p_2)$. We claim that $x_k = \pi^{sl(P)-1}(p_1)$. For if p_1p_2 has tail p_1, then $sl(P) = n+2$, and $p_2 = \pi(p_1)$, and so $x_k = \pi^{sl(P)-1}(p_1)$; and if p_1p_2 has tail p_2, then $sl(P) = n$, and $p_2 = \pi^{-1}(p_1)$, and so again $x_k = \pi^{sl(P)-1}(p_1)$. Consequently $x_k = p_1$ if and only if $sl(P) = 1$ modulo 3. This proves 6.3. ■

6.3 can be viewed another way. We are trying to make a "construction" of all orientable triangle-connected triangle-covered prismatic graphs. We showed so far that such a graph gives rise to a vine, and it can be reconstructed from a knowledge of the vine. But as we explained in section 2, *every* vine can be converted to an orientable triangle-connected triangle-covered prismatic graph, by following the rule for adjacency described in 6.3, and so we can regard this as a construction for all orientable triangle-connected triangle-covered prismatic graphs.

7 The proof of 4.2

Now we come to put the pieces of the last few sections together.

Proof of 4.2. Let G be a non-null orientable prismatic graph that is triangle-covered and triangle-connected. Let $\mathcal{O}$ be an orientation, and let H be the corresponding triangular digraph. We may assume that G is not isomorphic to $L(K_{3,3})$, for otherwise the theorem holds. Hence by 6.1, each triangle contains a thorn of H. By 6.2, H is a vine. We may assume that G has at least two triangles, for otherwise G is a core path of triangles graph. Consequently by 5.2, H is either a linear or circular vine. Let $s_1, \dots, s_n$ be the vertices in order of some stem S of H. For each vertex $v \in V(H) \setminus V(S)$, let $N_S(v)$ be the set of vertices of S adjacent to v in H.

We will show that if S is a cycle then G is a core cycle of triangles graph, and if S is a path then G is a core path of triangles graph. The two proofs are almost identical, so we only give the second (the first is a little easier since we do not have to worry about "end effects"). Thus, henceforth S is a path. (The reader is warned that there is a difference between adjacency in H and adjacency in G in what follows.)

Let $X_2 = \{s_1\}$ and $X_{2n} = \{s_n\}$. For $1 < i < n$, let X_{2i} be the union of $\{s_i\}$ and the set of all vertices $v \in V(H) \setminus V(S)$ such that $N_S(v) = \{s_{i-1}, s_{i+1}\}$. Let

$$Z = X_2 \cup X_4 \cup \cdots \cup X_{2n}.$$

No member of Z is a thorn, since every member of Z either belongs to $V(S)$ or is adjacent in H to two nonadjacent vertices of S. Let $M_1 = M_{2n+1} = \emptyset$. For $1 \le i < n$, let M_{2i+1} be the set of all vertices in $V(G) \setminus Z$ adjacent in H to a member of X_{2i} and to a member of X_{2i+2}. Let $R_{2n+1} = \emptyset$, and for $1 \le i \le n$, let R_{2i-1} be the set of all thorns $v \in V(H) \setminus Z$ such that s_i is the unique vertex of Z adjacent in H to v, and $s_i \to v$ in H. Similarly, let $L_1 = \emptyset$, and for $1 \le i \le n$, let L_{2i+1} be the set of all thorns $v \in V(H) \setminus Z$ such that s_i is the unique vertex of Z adjacent in H to v, and $v \to s_i$ in H. It follows that the sets $X_2, X_4, \dots, X_{2n}$ and all the sets $L_{2i+1}, M_{2i+1}, R_{2i+1}$ $(0 \le i \le n)$ are pairwise disjoint (we shall show below that they have union $V(G)$). For $1 \le i \le n+1$ let $X_{2i-1} = L_{2i-1} \cup M_{2i-1} \cup R_{2i-1}$. We will show that $X_1, \dots, X_{2n+1}$ is a core path of triangles decomposition.

(1) *For every triangle T of G, either there exists i with $1 \le i < n$ such that $X_{2i}, M_{2i+1}, X_{2i+2}$ each contain a vertex of T, or there exists i with $1 \le i \le n$ such that $R_{2i-1}, X_{2i}, L_{2i+1}$ each contain a vertex of T.*

For let $T = \{u, v, w\}$. At least one of u, v, w is a thorn, say w, and so $w \notin V(S)$ (and indeed, $w \notin Z$); and since by **(LV3)** w has a neighbour in $V(S)$, we may assume that $u = s_i$ where $1 \le i \le n$. Thus $u \in X_{2i}$. If $v \in V(S)$, then since S is induced in H, we may assume that say $v = s_{i+1}$; and so $v \in X_{2i+2}$ and $w \in M_{2i+1}$ and the claim holds. So we may assume that $v \notin V(S)$. Since w is a thorn, it follows that $N_S(w) = \{u\}$. Suppose that $|N_S(v)| \ge 2$. Then since v is adjacent in H to a vertex not in $V(S)$ (namely w) and hence has at least three neighbours in H, it follows that v is not a thorn; and from **(LV4)**, we may assume that $N_S(v) = \{s_i, s_{i+2}\}$; and so $v \in X_{2i+2}$, and again $w \in M_{2i+1}$ and the claim holds. So we may assume that $N_S(v) = \{u\}$. From **(LV4)**, it follows that v is a thorn, and so $v \notin Z$ and v, w are adjacent in H to no members of Z except s_i (since they both have degree two in H). In particular, the symmetry between v, w is restored. From this symmetry, we may assume that uv has tail v. But then $v \in L_{2i+1}$ and $w \in R_{2i-1}$. This proves (1).

It follows from (1) that the sets $X_1, \dots, X_{2n+1}$ have union $V(G)$, since G is triangle-covered.

(2) *For $1 \le i < n$, the following hold:*

- *one of X_{2i}, X_{2i+2} has cardinality* 1
- *X_{2i}, X_{2i+2} are complete to each other*
- *every edge between X_{2i} and X_{2i+2} has tail in X_{2i}*
- *every edge between X_{2i} and M_{2i+1} has tail in M_{2i+1}, and*
- *every edge between M_{2i+1} and X_{2i+2} has tail in X_{2i+2}.*

For suppose that $|X_{2i}|, |X_{2i+2}| > 1$. Since $|X_2| = |X_{2n}| = 1$, it follows that $1 < i \le n-2$. Choose $u \in X_{2i}$ and $v \in X_{2i+2}$ with $u \ne s_{2i}$ and $v \ne s_{2i+2}$. From the definition of X_{2i}, it follows that $N_S(u) = \{s_{i-1}, s_{i+1}\}$, and similarly $N_S(v) = \{s_i, s_{i+2}\}$. In particular, u, v are not thorns. From **(LV4)**, since $N_S(u) = \{s_{i-1}, s_{i+1}\}$ it follows that every vertex in $V(H) \setminus V(S)$ adjacent in H to s_i is a thorn, and yet v is adjacent in H to s_i, a contradiction. This proves that one of X_{2i}, X_{2i+2} has cardinality 1, and so the first assertion holds. The second holds since we may assume from the symmetry that $X_{2i+2} = \{s_{i+1}\}$, and every member of X_{2i} is adjacent to s_{i+1} from the definition of X_{2i}. We prove the final three assertions together. By (1), every edge between two of the three sets $X_{2i}, M_{2i+1}, X_{2i+2}$ is in a triangle included in the union of these three sets; so let $T = \{u, v, w\}$ be a triangle with $u \in X_{2i}, w \in M_{2i+1}$ and $v \in X_{2i+2}$. It suffices to show that $\mathcal{O}(T)$ is $w \to u \to v \to w$. If $u = s_i$ and $v = s_{i+1}$, the claim holds since $s_i s_{i+1}$ has tail s_i. Thus we may assume from the symmetry that $v \ne s_{i+1}$. Consequently $|X_{2i+2}| > 1$, and so $i \le n-2$. Choose x so that $\{s_{i+1}, s_{i+2}, x\}$ is a triangle T'. From (1), $x \in M_{2i+3}$, and so T, T' are disjoint. Also $\mathcal{O}(T')$ is $x \to s_{i+1} \to s_{i+2} \to x$, as we saw already. From the pair T, T', since us_{i+1} and vs_{i+2} are edges, it follows that $\mathcal{O}(T)$ is $w \to u \to v \to w$. This proves the final three assertions and so proves (2).

(3) *For $1 \le i \le n$, R_{2i-1}, L_{2i+1} are matched in G, and if $R_{2i-1} \cup L_{2i+1} \ne \emptyset$ then $X_{2i} = \{s_i\}$. Moreover, if $u \in R_{2i-1}$ and $v \in L_{2i+1}$ are adjacent, and T is the triangle $\{u, v, s_i\}$, then $\mathcal{O}(T)$ is $s_i \to u \to v \to s_i$.*

For every member of $R_{2i-1} \cup L_{2i+1}$ is adjacent in H to s_i. Let $u \in R_{2i-1}$; then $u \in V(H) \setminus Z$, $N_S(u) = \{s_i\}$ and the edge us_i has tail s_i. Choose $v \in V(H)$ so that $\{u, v, s_i\}$ is a triangle. From (1), $v \in L_{2i+1}$. Consequently every member of R_{2i-1} is adjacent in H to a member of L_{2i+1} and vice versa. Since no edge of H belongs to two triangles, and every edge of G between R_{2i-1} and L_{2i+1} is an edge of H, it follows that R_{2i-1}, L_{2i+1} are matched in H and in G. This proves the first claim. For the second, suppose that $u \in R_{2i-1} \cup L_{2i+1} \ne \emptyset$. Then u is a thorn. Since u is adjacent in H to s_i and to neither of s_{i-1}, s_{i+1}, it follows from **(LV4)** that there is no vertex $w \in V(H) \setminus V(S)$ with $N_S(w) = \{s_{i-1}, s_{i+1}\}$; and therefore $X_{2i} = \{s_i\}$. This proves the second claim. For the third, let $u \in R_{2i-1}$ and $v \in L_{2i+1}$ be adjacent, and let $T = \{u, v, s_i\}$. Since $v \in L_{i+1}$ it follows that vs_i has tail v in H; that is, $\mathcal{O}(T)$ is $s_i \to u \to v \to s_i$. This proves (3).

(4) *For $1 \le i \le 2n+1$, X_i is stable in G.*

For suppose that $u, v \in X_i$ are adjacent in G. If i is even, then since $|X_2| = 1$, it follows that

$i > 2$, and from (2) $s_{(i/2)-1}$ is adjacent to both u, v, contrary to (1). Thus i is odd, say $i = 2j + 1$. If $u \in R_{2j+1}$, then $j < n$, and since u is a thorn adjacent in H to s_{j+1} and to v, it follows that $\{u, v, s_{j+1}\}$ is a triangle, contrary to (1). Thus $u \notin R_{2j+1}$, and similarly $u, v \notin R_{2j+1} \cup L_{2j+1}$. Hence $u, v \in M_{2j+1}$. By (2), one of X_{2j}, X_{2j+2} has only one member say r, and so $\{r, u, v\}$ is a triangle, contrary to (1). This proves (4).

(5) *For $1 \le i, j \le 2n + 1$ with $j \ge i + 3$, if $j - i = 2$ modulo 3 then X_i is complete in G to X_j, and otherwise X_i is anticomplete in G to X_j.*

For let $u \in X_i$ and $v \in X_j$. We must show that u, v are adjacent in G if and only if $j - i = 2$ modulo 3. In most cases we will choose a twig path P of H between u, v, and prove that $sl(P) = 1$ modulo 3 if and only if $j - i = 2$ modulo 3, and then the claim will follow from 6.3. First suppose that i, j are even; say $i = 2s, j = 2t$, where $1 \le s < t \le n$. Let P be the path with vertices u-s_{s+1}-s_{s+2}-$\cdots$-s_{t-1}-v in order; then P is directed by (2), it has length > 2 (since $j \ge i+3$ by hypothesis), all its edges are twigs (by 5.1, since none of its vertices are thorns) and $sl(P) = t - s = (j-i)/2$. Hence $sl(P) = 1$ modulo 3 if and only if $j - i = 2$ modulo 3, as claimed.

Next suppose that i is odd and j is even; say $i = 2s - 1$ and $j = 2t$, where $1 \le s < t \le n$ (since $j \ge i + 3$). Then $u \in L_{2s-1} \cup M_{2s-1} \cup R_{2s-1}$ and v is adjacent in H to s_{t-1}. Suppose that $u \in L_{2s-1}$, and let P have vertices u-s_{s-1}-s_s-$\cdots$-s_{t-1}-v in order; then P is a directed path by (2), all its edges are twigs, and $sl(P) = t - s + 2 = (j - i + 3)/2$, and so $sl(P) = 1$ modulo 3 if and only if $j - i = 2$ modulo 3 as required. Next suppose that $u \in R_{2s-1}$. If $t = s + 1$, then u, v are nonadjacent by (1), since they are both adjacent to s_s, and the claim holds; so we may assume that $t \ge s + 2$. Let P be the path with vertices u-s_s-$\cdots$-s_{t-1}-v in order. Then P has length at least 3, all its edges are twigs, and $sl(P) = t - s - 1 = (j - i - 3)/2$, and so again $sl(P) = 1$ modulo 3 if and only if $j - i = 2$ modulo 3 as required. Thus we may assume that $u \in M_{2s-1}$, and therefore $\{u, x_{s-1}, x_s\}$ is a triangle for some $x_{s-1} \in X_{2s-2}$ and $x_s \in X_{2s}$. The edges ux_s and ux_{s-1} are not twigs, so in this case we cannot construct P. Let $i_1 = i - 1$, $i_2 = i + 1$. Then i_1, i_2 are even, and $x_{s-1} \in X_{i_1}$ and $x_s \in X_{i_2}$. From what we already proved, x_{s-1} is adjacent to v if and only if $j - i_1 = 2$ modulo 3, and x_s is adjacent to v if and only if $j - i_2 = 2$ modulo 3 (this follows from (2) if $j - i_2 = 2$, and from what we already proved if $j - i_2 \ge 3$). But $j - i = 2$ modulo 3 if and only if $j - i_1, j - i_2 \ne 2$ modulo 3, and v is adjacent to u if and only if v is nonadjacent to both x_{s-1}, x_s, since $\{u, x_{s-1}, x_s\}$ is a triangle. Thus again u, v are adjacent in G if and only if $j - i = 2$ modulo 3. The proof is similar if j is odd and we omit the details. This proves (5).

So far we have verified conditions **(P1)**, **(P2)** and **(P3)** in the definition of a core path of triangles decomposition. For **(P4)** note that s_1 is in at least two triangles from the definition of a stem, and so if $R_1 = \emptyset$ then from (1), $n \ge 2$ and $|X_4| > 1$. This proves **(P4)**. Condition **(P5)** holds since if $u \in L_{2i-1}$ and $v \in X_{2i}$ are adjacent in G then $\{s_{i-1}, u, v\}$ is a triangle, contrary to (1). Condition **(P6)** follows from the next assertion.

(6) *For $1 \le i \le n$, if $|X_{2i}| = 1$, then*

- *R_{2i-1}, L_{2i+1} are matched in G, and every edge of G between $M_{2i-1} \cup R_{2i-1}$ and $L_{2i+1} \cup M_{2i+1}$ is between R_{2i-1} and L_{2i+1};*
- *the vertex in X_{2i} is complete in H to $R_{2i-1} \cup M_{2i-1} \cup L_{2i+1} \cup M_{2i+1}$;*

- *if $u \in X_{2i-1}$ and $v \in X_{2i+1}$ are nonadjacent in G then $u \in M_{2i-1} \cup R_{2i-1}$ and $v \in L_{2i+1} \cup M_{2i+1}$*
- *if $i > 1$ then M_{2i-1}, X_{2i-2} are matched in G, and if $i < n$ then M_{2i+1}, X_{2i+2} are matched in G.*

For let $|X_{2i}| = 1$; then $X_{2i} = \{s_i\}$. From (3), R_{2i-1}, L_{2i+1} are matched in G. If $u \in M_{2i-1} \cup R_{2i-1}$ and $v \in L_{2i+1} \cup M_{2i+1}$ are adjacent in G, then since they are both adjacent in H to s_i, it follows from (1) that $u \in R_{2i-1}$ and $v \in L_{2i+1}$, and so the first claim of (6) holds. The second is clear. For the third, suppose that $u \in X_{2i-1}$ and $v \in X_{2i+1}$ are nonadjacent in G, and $u \in L_{2i-1}$. Choose $x \in V(H)$ so that $\{u, s_{i-1}, x\}$ is a triangle; then $x \in R_{2i-3}$ by (1). By (5), v is nonadjacent in G to x, and therefore is adjacent in G to no member of this triangle, a contradiction. Thus $u \notin L_{2i-1}$, and similarly $v \notin R_{2i+1}$. This proves the third claim. For the fourth, suppose that $i > 1$. From the definition of M_{2i-1}, every vertex in X_{2i-2} is adjacent in H to a member of M_{2i-1} and vice versa; and since no edge is in two triangles and s_i is complete to $X_{2i-2} \cup M_{2i-1}$, it follows that X_{2i-2}, M_{2i-1} are matched in G. Similarly if $i < n$ then X_{2i+2}, M_{2i+1} are matched in G. This proves the fourth assertion of (6), and so completes the proof of (6).

Finally, condition **(P7)** follows from the next assertion.

(7) *For $1 < i < n$, if $|X_{2i}| > 1$ then*

- *$R_{2i-1} = L_{2i+1} = \emptyset$;*
- *if $u \in X_{2i-1}$ and $v \in X_{2i+1}$, then u, v are nonadjacent in G if and only if there is a vertex in X_{2i} adjacent in G to both u, v.*

For let $|X_{2i}| > 1$. The first assertion of (7) follows from (3). For the second, let $u \in X_{2i-1}$ and $v \in X_{2i+1}$. If in G, u, v have a common neighbour in X_{2i}, then they are nonadjacent in G by (1), so it remains to prove the converse. Suppose then that u, v are nonadjacent in G. Since $|X_{2i}| > 1$, (2) implies that $X_{2i-2} = \{s_{i-1}\}$. Since $R_{2i-1} = \emptyset$, it follows that $u \in L_{2i-1} \cup M_{2i-1}$, and therefore is adjacent in H to s_{i-1}. Choose $x \in V(H)$ so that $\{u, x, s_{i-1}\}$ is a triangle T. By (1), either $x \in R_{2i-3}$ and $u \in L_{2i-1}$, or $x \in X_{2i}$ and $u \in M_{2i-1}$. Now v is not adjacent in G to s_{i-1} by (5). Since v is adjacent in G to a member of T and v is not adjacent in G to u, s_{i-1}, it follows that v, x are adjacent in G. Since X_{2i+1}, X_{2i-3} are anticomplete in G by (5), it follows that $x \in X_{2i}$, and x is adjacent in G to both u, v. This proves the second assertion, and therefore proves (7).

Consequently the sequence $X_1, \ldots, X_{2n+1}$ is indeed a core path of triangles decomposition. This proves 4.2. ■

8 A stable neighbourhood

Let G be prismatic and triangle-covered. We say $N \subseteq V(G)$ is a *crosscut* if N is stable and $|N \cap T| = 1$ for every triangle T. Our next objective is to study crosscuts. The reason for this is, we need to investigate the structure of prismatic graphs H that are not triangle-covered. The *core* of H is the union of all triangles of G. Let H be prismatic with core W, let $G = H|W$, let $v \in V(H) \setminus W$, and let N be the set of members of W that are adjacent to v. Then N is a crosscut in G, since v is in no triangles and G is prismatic. Thus an understanding of crosscuts will tell us all possible ways to add one vertex not in the core to a triangle-covered prismatic graph. (The core ring of five was defined in section 4.)

8.1 *Let $X_1, \ldots, X_{2n}$ be a core cycle of triangles decomposition of G, and let the sets $L_{2i+1}, M_{2i+1}, R_{2i+1}$ $(1 \leq i \leq n)$ be as in the definition of a core cycle of triangles graph. Let $N \subseteq V(G)$ be a crosscut. Then either:*

- *G is the core ring of five, or*
- *there exists $i \in \{1, \ldots, n\}$ such that N contains exactly one end of every edge between R_{2i-1} and L_{2i+1}, and (reading subscripts modulo $2n$)*

$$N \setminus (R_{2i-1} \cup L_{2i+1}) = \bigcup (X_{2i+2+k} : 0 \leq k \leq 2n-4 \text{ and } k \text{ is divisible by } 3).$$

Proof. Since $X_1, \ldots, X_{2n}$ is a core cycle of triangles decomposition of G, it follows that $n \geq 5$ and $n = 2$ modulo 3; and we read the subscripts of X_i modulo $2n$. Let

$$P = \{i : 1 \leq i \leq n \text{ and } N \cap X_{2i} \neq \emptyset\}.$$

(1) *We may assume that $P \neq \emptyset$.*

For suppose that $P = \emptyset$. For each $i \in \{1, \ldots, n\}$, one of X_{2i}, X_{2i+2} has cardinality 1 and M_{2i+1} is matched with the other, and in particular, $M_{2i+1} \neq \emptyset$ and every vertex of M_{2i+1} is in a triangle included in $X_{2i} \cup M_{2i+1} \cup X_{2i+2}$. Since N meets all these triangles it follows that $\emptyset \neq M_{2i+1} \subseteq N$. If $n > 5$ then this is impossible since M_1 is complete to M_{11} and yet N is stable. Thus $n = 5$. If $|X_2| > 1$ then M_1, M_3 are both matched with X_2, and so there exist $u \in M_1$ and $v \in M_3$ with no common neighbour in X_2; then u, v are adjacent from **(C6)**. But $u, v \in N$ and N is stable, which is impossible. This proves that $|X_2| = 1$, and similarly $|X_{2i}| = 1$ for $i = 1, \ldots, 5$. Hence $|M_{2i+1}| = 1$ for $i = 1, \ldots, 5$. Suppose that $|V(G)| > 10$. Then one of the sets $R_1, R_3, \ldots, R_9, L_1, L_3, \ldots, L_9$ is nonempty, say R_1. Choose $u \in R_1$. Then there exists $v \in L_3$ such that $\{u, v, s\}$ is a triangle, where $X_2 = \{s\}$. Since N meets this triangle we may assume that $v \in N$. But v is complete to M_5, by **(C6)**, a contradiction since N is stable. Hence $|V(G)| = 10$ and the first outcome of the theorem holds. This proves (1).

(2) *If $i \in P$ then $i + 1 \notin P$ and one of $i + 2, i + 3 \in P$.*

For let $1 \in P$ say; thus $N \cap X_2 \neq \emptyset$. Since X_2 is complete to X_4 it follows that $N \cap X_4 = \emptyset$, and so $2 \notin P$. Suppose that $3, 4 \notin P$. Since there is a triangle included in $X_6 \cup M_7 \cup X_8$, it follows that $N \cap M_7 \neq \emptyset$; and yet X_2 is complete to X_7, a contradiction. This proves (2).

Since n is not divisible by 3 and $P \neq \emptyset$, it follows from (2) that there exists $i \in P$ such that $i + 2 \in P$, and we may assume that $1, 3 \in P$. Since X_2 is complete to X_i for $i = 4, 7, 10, 13, \ldots, 2n$ and X_6 is complete to X_i for $i = 8, 11, 14, 17, \ldots, 2n-2, 1, 4$, we deduce that

$$N \subseteq X_2 \cup X_3 \cup X_5 \cup X_6 \cup \bigcup (X_i : 9 \leq i \leq 2n-1 \text{ and } i \text{ is divisible by 3.})$$

Let $9 \leq i \leq 2n-1$ with i divisible by 3. If i is even then every vertex of X_i belongs to a triangle included in $X_{i-2} \cup X_{i-1} \cup X_i$, and so $X_i \subseteq N$. If i is odd then every vertex in X_i belongs to a triangle included in one of $X_{i-2} \cup X_{i-1} \cup X_i$ (for a vertex in L_i), $X_{i-1} \cup X_i \cup X_{i+1}$ (for a vertex in M_i), $X_i \cup X_{i+1} \cup X_{i+2}$ (for a vertex in R_i). Since N meets these triangles it follows again that $X_i \subseteq N$. Moreover, every vertex in X_6 belongs to a triangle included in $X_6 \cup X_7 \cup X_8$, so $X_6 \subseteq N$, and similarly $X_2 \subseteq N$. Since every member of $L_3 \cup M_3$ has a neighbour in X_2, it follows that $N \cap X_3 \subseteq R_3$, and similarly $N \cap X_5 \subseteq L_5$. If $|X_4| > 1$, then the second outcome of the theorem holds, because $R_3 = L_5 = \emptyset$; so we assume that $X_4 = \{w\}$ say. If $u \in R_3, v \in L_5$ are adjacent, then since $|N \cap \{u,v,w\}| = 1$, it follows that N contains exactly one of u, v, and so the second outcome of the theorem holds. This proves 8.1. ■

Let us say a prismatic graph G is *k-substantial* if for every $S \subseteq V(G)$ with $|S| < k$ there is a triangle T with $S \cap T = \emptyset$. We need an analogue of 8.1 for paths of triangles, and it is helpful to assume that the graph is 3-substantial to eliminate some degenerate cases.

8.2 *Let G be 3-substantial, let $X_1, \ldots, X_{2n+1}$ be a core path of triangles decomposition of G, and let the sets $L_{2i+1}, M_{2i+1}, R_{2i+1}$ $(1 \leq i \leq n)$ be as usual. Let $N \subseteq V(G)$ be a crosscut. Then either:*

- *there exists $i \in \{1, \ldots, n\}$ such that N contains exactly one end of every edge between R_{2i-1} and L_{2i+1} and*

$$N \setminus (R_{2i-1} \cup L_{2i+1}) = \bigcup (X_h : 1 \leq h \leq 2n+1 \text{ and } |h - 2i| = 2 \text{ modulo } 3)$$

or

- *there exists $k \in \{0,1,2\}$ such that $N = \bigcup (X_i : 1 \leq i \leq 2n+1 \text{ and } i = k \text{ modulo } 3)$.*

Proof. If $n \leq 2$ then $X_2 \cup X_{2n}$ meets all triangles, contradicting that G is 3-substantial. Thus $n \geq 3$. It is convenient to define $X_i = \emptyset$ for all integers $i \notin \{1, \ldots, 2n+1\}$. Once again, let $P = \{i : 1 \leq i \leq n \text{ and } N \cap X_{2i} \neq \emptyset\}$.

(1) *$P \neq \emptyset$.*

For suppose that P is empty. Then as in the proof of 8.1, $\emptyset \neq M_{2i+1} \subseteq N$ for $1 \leq i < n$. We claim that $R_{2i-1} \subseteq N$ for $i = 1, \ldots, n-2$. For let $u \in R_{2i-1}$, and choose $v \in L_{2i+1}$ so that $\{u, v, w\}$ is a triangle, where $X_{2i} = \{w\}$. Since v is complete to M_{2i+3}, it follows that $v \notin N$, and so $u \in N$. Hence $R_{2i-1} \subseteq N$ as claimed. Similarly $L_{2i+1} \subseteq N$ for $i = 3, \ldots, n$.

We claim that $|X_{2i}| = 1$ for $i = 1, \ldots, n$. For if $i = 1$ or $i = n$ the claim holds by **(P1)**, so we assume that $2 \leq i \leq n-1$. Suppose that $v_1, v_2 \in X_i$ are distinct. Then X_{2i} is matched with both M_{2i-1}, M_{2i+1} and so there exist $u \in M_{2i-1}$ and $w \in M_{2i+1}$ such that uv_1, v_2w are edges. Then u, w are adjacent from **(P7)**, a contradiction since they both belong to N. This proves that $|X_{2i}| = 1$ for $1 \leq i \leq n$. Since $|X_4| = 1$, it follows from **(P4)** that $R_1 \neq \emptyset$, and similarly $L_{2n+1} \neq \emptyset$. Thus R_1 is a nonempty subset of N. If $n \geq 4$, then R_1 is complete to $L_9 \cup M_9$, and $L_9 \cup M_9$ is also a nonempty subset of N (because $M_9 \neq \emptyset$ if $n \geq 5$, and $L_9 \neq \emptyset$ if $n = 4$), a contradiction. Hence $n = 3$. Since R_1 is complete to R_3, and L_7 is complete to L_5, it follows that $R_3 \cup L_5$ is disjoint from N, and since R_3, L_5 are matched, it follows that $R_3 = L_5 = \emptyset$. But then $X_2 \cup X_6$ meets every triangle of G, contradicting that G is 3-substantial. This proves (1).

(2) *If $i \in P$ and $i < n$ then $i+1 \notin P$; and if $i \leq n-3$ then one of $i+2, i+3 \in P$.*

The proof is just as in 8.1.

(3) *We may assume that there does not exist i with $2 \leq i \leq n-1$ such that $i-1, i+1 \in P$.*

For suppose that $i-1, i+1 \in P$. Thus N meets both X_{2i-2}, X_{2i+2}. For $1 \leq h < 2i-2$ we claim that $N \cap X_h = \emptyset$ if $2i-2 \neq h$ modulo 3, and $X_h \subseteq N$ if $2i-2 = h$ modulo 3. For if $2i-2 \neq h$ modulo 3, then $2i-h = 0$ or 1 modulo 3. If $2i-h = 0$ modulo 3, then $(2i+2)-h = 2$ modulo 3 and so X_h is complete to X_{2i+2}; and consequently $N \cap X_h = \emptyset$. If $2i-h = 1$ modulo 3, then X_h is complete to X_{2i-2} and again $N \cap X_h = \emptyset$. Now let $2i-2 = h$ modulo 3. Then N is disjoint from the four sets $X_{h-2}, X_{h-1}, X_{h+1}, X_{h+2}$, because all the numbers $h-2, h-1, h+1, h+2$ are less than $2i-2$ and are different from $2i-2$ modulo 3. But if $v \in X_h$, there is a triangle T containing v with

$$T \setminus \{v\} \subseteq X_{h-2} \cup X_{h-1} \cup X_{h+1} \cup X_{h+2},$$

and since $N \cap T \neq \emptyset$, it follows that $v \in N$. Hence $X_h \subseteq N$. This proves our claim. Similarly, for $h > 2i+2$, if $h \neq 2i+2$ modulo 3 then $N \cap X_h = \emptyset$, and if $h = 2i+2$ modulo 3 then $X_h \subseteq N$. Since X_{2i} is complete to X_{2i-2}, it follows that $N \cap X_{2i} = \emptyset$. We claim that $X_{2i-2} \subseteq N$. For suppose not; then since $N \cap X_{2i-2} \neq \emptyset$, it follows that $|X_{2i-2}| > 1$, and therefore $i > 2$. Let $v \in X_{2i-2} \setminus N$. Then there is a triangle T containing v with $T \setminus \{v\} \subseteq M_{2i-3} \cup X_{2i-4}$, and therefore $N \cap T = \emptyset$, a contradiction. This proves that $X_{2i-2} \subseteq N$, and similarly $X_{2i+2} \subseteq N$. It remains to examine $N \cap X_{2i-1}$ and $N \cap X_{2i+1}$. Since every vertex of $L_{2i-1} \cup M_{2i-1}$ has a neighbour in $X_{2i-2} \subseteq N$, it follows that $N \cap X_{2i-1} \subseteq R_{2i-1}$, and similarly $N \cap X_{2i+1} \subseteq L_{2i+1}$. For every edge uv between R_{2i-1} and L_{2i+1}, exactly one end of this edge belongs to N since $|X_{2i}| = 1$, say $X_{2i} = \{w\}$, and $|N \cap \{u, v, w\}| = 1$. Hence the first outcome of the theorem holds. This proves (3).

(4) *We may assume that for $1 \leq i \leq n$, if $N \cap X_{2i} \neq \emptyset$ then $X_{2i} \subseteq N$.*

For suppose that $v, v' \in X_{2i}$ with $v \notin N$ and $v' \in N$. Since $|X_{2i}| > 1$, it follows that $i > 1$ and $|X_{2i-2}| = 1$, and similarly $i < n$ and $|X_{2i+2}| = 1$. Let $X_{2i-2} = \{s_{2i-2}\}$ and $X_{2i+2} = \{s_{2i+2}\}$. Since X_{2i} is matched with M_{2i-1}, there exists $u \in M_{2i-1}$ such that $\{s_{2i-2}, u, v\}$ is a triangle, and similarly there exists $w \in M_{2i+1}$ such that $\{v, w, s_{2i+2}\}$ is a triangle. Since N meets these triangles and is disjoint from X_{2i-2}, X_{2i+2}, it follows that $u, w \in N$. If $i \leq n-3$, then by (2) and (3), $N \cap X_{2i+6} \neq \emptyset$, and yet $w \in X_{2i+1}$ is complete to X_{2i+6}, a contradiction. Thus $i \geq n-2$, and similarly $i \leq 3$. If $n = 3$, then $X_2 \cup X_6$ meets all triangles, contradicting that G is 3-substantial; so $n \geq 4$, and from the symmetry we may therefore assume that $i = 3$. Since $|X_4| = 1$, it follows that $R_1 \neq \emptyset$, and so there exist $a \in R_1, b \in L_3$ such that $\{a, b, s_2\}$ is a triangle, where $X_2 = \{s_2\}$. By (3), $s_2 \notin N$, and so one of $a, b \in N$; yet $a \in X_1$ is adjacent to $v' \in X_6$, because X_1 is complete to X_6, and b is adjacent to u by **(P6)**, a contradiction. This proves (4).

From (1)–(4), there exists $k \in \{0, 1, 2\}$ such that for all even i with $1 \leq i \leq 2n+1$, if $i = k$ modulo 3 then $X_i \subseteq N$, and otherwise $N \cap X_i = \emptyset$.

(5) *For* $1 \leq i \leq 2n+1$ *with* i *odd and* $i = k$ *modulo* 3*, if* $N \cap X_{i-2} = N \cap X_{i+2} = \emptyset$*, then* $X_i \subseteq N$*.*

For let $v \in X_i$. There is a triangle T containing v with $T \setminus \{v\} \subseteq X_{i-2} \cup X_{i-1} \cup X_{i+1} \cup X_{i+2}$. Now $N \cap X_{i-1} = N \cap X_{i+1} = \emptyset$ from the choice of k since $i = k$ modulo 3 and $i-1, i+1$ are even, and $N \cap X_{i-2} = N \cap X_{i+2} = \emptyset$ by hypothesis. Since $N \cap T \neq \emptyset$, it follows that $v \in N$, and so $X_i \subseteq N$. This proves (5).

Now if there does not exist $i \in \{1, \ldots, 2n+1\}$, odd, such that $i \neq k$ modulo 3 and $N \cap X_i \neq \emptyset$, then by (5), $X_i \subseteq N$ for all odd i with $i = k$ modulo 3, and so the second outcome of the theorem holds. Thus we may assume that $N \cap X_i \neq \emptyset$ for some odd $i \in \{1, \ldots, 2n+1\}$, such that $i \neq k$ modulo 3. Let $v \in N \cap X_i$. By reversing the sequence $X_1, \ldots, X_{2n+1}$ if necessary, we may assume that $i = k+2$ modulo 3. Since $X_{i+1} \subseteq N$, it follows that v has no neighbour in X_{i+1}, and so $v \in L_i$. Consequently $i \geq 3$, and $|X_{i-1}| = 1$. If $i \geq 7$, then $X_{i-5} \subseteq N$ is complete to X_i, a contradiction, and so $i \leq 5$. Suppose that $i = 5$. Then since $|X_4| = 1$, it follows that $R_1 \neq \emptyset$, and so there exist $a \in R_1$ and $b \in L_3$ such that $\{a, b, s_2\}$ is a triangle, where $X_2 = \{s_2\}$. But $a \in X_1$ is complete to X_6, and $b \in X_3$ is complete to X_5, and $N \cap X_2 = \emptyset$ by the choice of k. Hence N is disjoint from the triangle $\{a, b, s_2\}$, a contradiction. Thus $i \neq 5$, and so $i = 3$. Since $i = k+2$ modulo 3, it follows that $k = 1$. Suppose that there exists $i' \neq i$ such that $1 \leq i' \leq 2n+1$, $i' \neq k$ modulo 3 and $N \cap X_{i'} \neq \emptyset$. We assumed that $i = k+2$ modulo 3 and deduced that $i = 3$, and since $i' \neq 3$, it follows that $i' \neq k+2$ modulo 3. Thus $i' = k+1$ modulo 3. By reversing the sequence $X_1, \ldots, X_{2n+1}$, we deduce that $i' = 2n-1$. Since $k = 1$ and $i' = k+1$ modulo 3, it follows that n is divisible by 3. But L_3 is complete to X_{2n-1} (since X_3 is complete to X_{2n-1} if $n > 3$, and L_3 is complete to X_5 from **(P6)**), a contradiction. We deduce that for all j with $4 \leq j \leq 2n+1$, if $j \neq 1$ modulo 3 then $N \cap X_j = \emptyset$. From (5), it follows that for all j with $4 \leq j \leq 2n+1$, if $j = 1$ modulo 3 then $X_j \subseteq N$. But then the first outcome of the theorem holds, taking $i = 1$. This proves 8.2. ■

9 Vertices not in the core

We can use 8.1 and 8.2 to analyze the structure of vertices not in the core. We begin with the following.

9.1 *Let G be prismatic, with core W, such that $G|W$ is a core cycle of triangles graph. Then either G is a cycle of triangles graph, or $G|W$ is the core ring of five.*

Proof. Let $X_1, \ldots, X_{2n}$ be a core cycle of triangles decomposition of $G|W$, and let the sets L_i, M_i, R_i be defined as usual; and we read these subscripts modulo $2n$ as usual. For each $v \in V(G) \setminus W$, let N_v be the set of vertices in W adjacent to v. Thus for each such v, N_v is a crosscut in $G|W$. For $1 \leq i \leq n$, let Y_{2i} be the set of all $v \in V(G) \setminus W$ such that N_v contains exactly one end of every edge between R_{2i-1} and L_{2i+1} and

$$N_v \setminus (R_{2i-1} \cup L_{2i+1}) = \bigcup (X_{2i+2+k} : 0 \leq k \leq 2n-4 \text{ and } k \text{ is divisible by } 3).$$

We may assume that $G|W$ is not the core ring of five, and so by 8.1, the sets Y_{2i} $(1 \leq i \leq n)$ have union $V(G) \setminus W$.

We propose to construct a cycle of triangles decomposition $X'_1, \ldots, X'_{2n}$ of G, where $X'_i = X_i$ for i odd, and $X'_i = X_i \cup Y_i$ for i even (and then defining $\hat{X}'_{2i} = X_{2i}$). It remains to verify the six conditions **(C1)**–**(C6)**. Since $X_1, \ldots, X_{2n}$ is a core cycle of triangles decomposition, we need only to prove the following:

- for $1 \le i \le n$, $X_{2i} \cup Y_{2i}$ is stable;
- for all $i \in \{1, \ldots, n\}$ and all k with $2 \le k \le 2n-2$, let $j \in \{1, \ldots, 2n\}$ with $j = 2i + k$ modulo $2n$:
  (1) if $k = 2$ modulo 3 and there exist $u \in Y_{2i}$ and $v \in X_j \cup Y_j$, nonadjacent, then j is even, and $v \in Y_j$;
  (2) if $k \neq 2$ modulo 3 then Y_{2i} is anticomplete to $X_j \cup Y_j$;
- for $1 \le i \le n$, Y_{2i} is anticomplete to $L_{2i-1} \cup M_{2i-1} \cup M_{2i+1} \cup R_{2i+1}$, and every vertex in Y_{2i} is adjacent to exactly one end of every edge between R_{2i-1} and L_{2i+1}.

Since $X_{2i} \cup Y_{2i}$ is complete to X_{2i+2}, and no vertex in Y_{2i} is in a triangle, and X_{2i} is stable, the first assertion follows. The third follows from the definition of Y_{2i}, and it remains to check the second. Thus, let $i \in \{1, \ldots, n\}$, let $2 \le k \le 2n-2$, and let $j \in \{1, \ldots, 2n\}$ with $j = 2i + k$ modulo $2n$. Suppose first that $k = 2$ modulo 3 and there exist $u \in Y_{2i}$ and $v \in X_j \cup Y_j$, nonadjacent. Since $X_j = X_{2i+2+(k-2)}$, and $0 \le k-2 \le 2n-4$ and $k-2$ is divisible by 3, it follows from the definition of Y_{2i} that $X_j \subseteq N_u$, and so $v \notin X_j$. Consequently j is even, and $v \in Y_j$. Finally, for the second half of the second assertion, suppose that $k \neq 2$ modulo 3, and that $u \in Y_{2i}$ is adjacent to $v \in X_j \cup Y_j$. Again from the definition of Y_{2i} it follows that j is even and $v \in Y_j$. Let $h = j/2$. Since u, v are adjacent and they do not belong to triangles, it follows that $N_u \cap N_v = \emptyset$. Let $k' = 2n - k$; then $2 \le k' \le 2n-2$, and $2i = 2h + k'$ modulo $2n$, and $k' \neq 2$ modulo 3 (since $n = 2$ modulo 3). Thus there is symmetry between h and i, and from this symmetry we may assume that $1 \le h \le i \le n$ and so $2i = 2h + k'$. If $i = h+1$ modulo 3, then $k' = 2$ modulo 3; if $i = h$ modulo 3, then N_u, N_v both include X_{2i+2}; and if $i = h+2$ modulo 3 then they both include X_{2i-2}, in each case a contradiction. This completes the proof of 9.1. ■

Again, we need an analogous result for paths of triangles, as follows.

9.2 *Let G be a prismatic graph, with core W, such that $G|W$ is a 3-substantial core path of triangles graph. Let $X_1, \ldots, X_{2n+1}$ be a core path of triangles decomposition of $G|W$, and for $k = 0, 1, 2$, let $A_k = \bigcup(X_i : 1 \le i \le 2n+1$ and $i = k$ modulo $3)$. Then either*

- *there exists $v \in V(G) \setminus W$ such that the set of neighbours of v in W is one of A_1, A_2, A_3, or*
- *G is a path of triangles graph.*

Proof. Since $G|W$ is 3-substantial, it follows that $n \ge 3$. For each $v \in V(G) \setminus W$, let N_v be the set of vertices in W adjacent to v. For $1 \le i \le n$, let Y_{2i} be the set of all $v \in V(G) \setminus W$ such that N_v contains exactly one end of every edge between R_{2i-1} and L_{2i+1}, and

$$N_v \setminus (R_{2i-1} \cup L_{2i+1}) = \bigcup(X_h : 1 \le h \le 2n+1 \text{ and } |2i-h| = 2 \text{ modulo } 3).$$

We may assume that the first outcome of the theorem does not hold, and so by 8.2, the sets Y_{2i} ($1 \le i \le n$) have union $V(G) \setminus W$. Again, we add Y_{2i} to X_{2i} to produce a path of triangles decomposition. The proof is exactly like that in 9.1, except in one step, when we need to prove the following.

(1) *Let $1 \le i \le j \le n$, and let $u \in Y_{2i}$ and $v \in Y_{2j}$. If u, v are adjacent then $2j - 2i = 2$ modulo 3.*

For $N_u \cap N_v = \emptyset$. If $j = i + 2$ modulo 3 then N_u, N_v both include X_{2i+2}, a contradiction, so we may assume that $j = i$ modulo 3. If $i > 1$ then N_u, N_v both include X_{2i-2}, so $i = 1$, and similarly $j = n$. Consequently $n = 1$ modulo 3. But $L_3 \subseteq X_3$ is a subset of N_v, since $3 \le 2n-2$ and $3 = 2n-2$ modulo 3, and since $N_u \cap N_v = \emptyset$ it follows that $N_u \cap L_3 = \emptyset$. Since $u \in Y_2$, and every member of R_1 has a neighbour in L_3, it follows that $X_1 = R_1 \subseteq N_u$. But also since $u \in Y_2$,

$$N_u \setminus (R_1 \cup L_3) = A_1 \setminus (R_1 \cup L_3)$$

and so $N_u = A_1$ and the first outcome of the theorem holds. This proves (1).

All the other steps of the verification of **(P1)**–**(P7)** are obvious modifications of the verification in the proof of 9.1, and we omit them. This proves 9.2. ■

10 The degenerate cases

We are almost ready to begin on the general characterization of orientable prismatic graphs, but first we need to examine the various degenerate cases that were exceptions to the theorems of the last section.

It is possible to give explicit constructions for all orientable triangle-connected prismatic graphs that are not 3-substantial. For instance, let $k \ge 1$; let K be the set of all subsets of $\{1, \ldots, k\}$; and let G be a graph with vertex set the disjoint union of a set $W = \{a_1, \ldots, a_k, b_1, \ldots, b_k, c\}$, a set U, and for each $I \in K$ a set V_I. The adjacency in G is as follows. The sets $\{a_i, b_i, c\}$ are triangles for $i = 1, \ldots, k$, and there are no other edges with both ends in W; c is complete to U, and has no other neighbours outside of W; for $I \in K$ and $1 \le i \le k$, if $i \in I$ then a_i is complete to V_I and b_i is anticomplete to V_I, and vice versa if $i \notin I$; each of the sets V_I ($I \in K$) is stable, and so is U; and if $I, I' \in K$ and $I' \neq \{1, \ldots, k\} \setminus I$ then $V_{I'}$ is anticomplete to V_I. For $I \in K$, let $I' = \{1, \ldots, k\} \setminus I$; the adjacency between members of distinct sets $U, V_I, V_{I'}$ is arbitrary except that there is no triangle with vertices in U, V_I and $V_{I'}$. Such a graph G is prismatic, and we call the class of all such graphs (for all k) $\mathcal{P}_1$.

10.1 *If G is a prismatic graph with a triangle, such that for some vertex c every triangle contains c, then $G \in \mathcal{P}_1$.*

Proof. Let the list of all triangles be $\{a_i, b_i, c\}$ ($1 \le i \le k$); thus the core W of G is

$$\{a_1, \ldots, a_k, b_1, \ldots, b_k, c\}.$$

Let U be the set of neighbours of c not in W. For each $v \in V(G) \setminus (W \cup U)$, let

$$I(v) = \{i : 1 \leq i \leq k \text{ and } a_i \text{ is adjacent to } v\}.$$

Since v has a unique neighbour in $\{a_i, b_i, c\}$, it follows that v is adjacent to b_i if and only if $i \notin I(v)$. Let K be the set of all subsets of $\{1, \ldots, k\}$, and for each $I \in K$ let $V_I = \{v \in V(G) \setminus (W \cup U) : I(v) = I\}$. If $v, v' \in V(G) \setminus (W \cup U)$ are adjacent, then they have no common neighbour in $W \cup U$, and therefore $I(v), I(v')$ are complementary subsets of $\{1, \ldots, k\}$. It follows that $G \in \mathcal{P}_1$. This proves 10.1. ■

It is possible to give similar, more complicated constructions for the orientable, triangle-connected prismatic graphs in which the smallest set of vertices meeting all triangles has cardinality 2; but they are rather messy, and yet easy for the reader to work out independently. We therefore omit these "constructions".

We need two more, when the core is the core ring of five, and when the core is $L(K_{3,3})$. Thus, let G be a graph with $V(G)$ the union of the disjoint sets $W = \{a_1, \ldots, a_5, b_1, \ldots, b_5\}$ and $V_0, V_1, \ldots, V_5$. Let adjacency be as follows (reading subscripts modulo 5). For $1 \leq i \leq 5$, $\{a_i, a_{i+1}, b_{i+3}\}$ is a triangle, and a_i is adjacent to b_i; V_0 is complete to $\{b_1, \ldots, b_5\}$ and anticomplete to $\{a_1, \ldots, a_5\}$; $V_0, V_1, \ldots, V_5$ are all stable; for $i = 1, \ldots, 5$, V_i is complete to $\{a_{i-1}, b_i, a_{i+1}\}$ and anticomplete to the remainder of W; V_0 is anticomplete to $V_1 \cup \cdots \cup V_5$; for $1 \leq i \leq 5$ V_i is anticomplete to V_{i+2}; and the adjacency between V_i, V_{i+1} is arbitrary. We call such a graph a *ring of five*.

10.2 *If G is prismatic and its core is the core ring of five then G is a ring of five.*

The proof is straightforward and we omit it.

Finally, let G be a graph with $V(G)$ the union of seven sets

$$W = \{a^i_j : 1 \leq i, j \leq 3\}, V^1, V^2, V^3, V_1, V_2, V_3,$$

with adjacency as follows. For $1 \leq i, j, i', j' \leq 3$, a^i_j and $a^{i'}_{j'}$ are adjacent if and only if $i' \neq i$ and $j' \neq j$. For $i = 1, 2, 3$, V^i, V_i are stable; V^i is complete to $\{a^i_1, a^i_2, a^i_3\}$, and anticomplete to the remainder of W; and V_i is complete to $\{a^1_i, a^2_i, a^3_i\}$ and anticomplete to the remainder of W. Moreover, $V^1 \cup V^2 \cup V^3$ is anticomplete to $V_1 \cup V_2 \cup V_3$, and there is no triangle included in $V^1 \cup V^2 \cup V^3$ or in $V_1 \cup V_2 \cup V_3$. We call such a graph G a *mantled* $L(K_{3,3})$.

10.3 *If G is prismatic with core W, and $G|W$ is isomorphic to $L(K_{3,3})$, then G is a mantled $L(K_{3,3})$.*

Again, the proof is easy and we omit it.

11 Statement of the theorem

Our next goal is to state precisely the main theorem, the structure theorem for 3-coloured prismatic graphs and for orientable prismatic graphs. Before we can do so we need to introduce a composition operation for 3-coloured prismatic graphs. Let $n \geq 0$, and for $1 \leq i \leq n$, let (G_i, A_i, B_i, C_i) be a 3-coloured prismatic graph, where $V(G_1), \ldots, V(G_n)$ are all nonempty and pairwise vertex-disjoint. Let $A = A_1 \cup \cdots \cup A_n$, $B = B_1 \cup \cdots \cup B_n$, and $C = C_1 \cup \cdots \cup C_n$, and let G be the graph with vertex set $V(G_1) \cup \cdots \cup V(G_n)$ and with adjacency as follows:

- For $1 \leq i \leq n$, $G|V(G_i) = G_i$;
- for $1 \leq i < j \leq n$, A_i is anticomplete to $V(G_j) \setminus B_j$; B_i is anticomplete to $V(G_j) \setminus C_j$; and C_i is anticomplete to $V(G_j) \setminus A_j$; and
- for $1 \leq i < j \leq n$, if $u \in A_i$ and $v \in B_j$ are nonadjacent then u, v are both in no triangles; and the same applies if $u \in B_i$ and $v \in C_j$, and if $u \in C_i$ and $v \in A_j$.

In particular, A, B, C are stable, and so (G, A, B, C) is a 3-coloured graph; we call the sequence (G_i, A_i, B_i, C_i) $(i = 1, \dots, n)$ a *worn chain decomposition* or *worn n-chain* for (G, A, B, C). Note also that every triangle of G is a triangle of one of $G_1, \dots, G_n$, and G is prismatic. If we replace the third condition above by the strengthening

- for $1 \leq i < j \leq n$, the pairs $(A_i, B_j), (B_i, C_j)$ and (C_i, A_j) are complete

we call the sequence a *chain decomposition* or *n-chain* for (G, A, B, C). (Thus a worn chain decomposition is not in general a chain decomposition.)

If $X_1, \dots, X_{2n+1}$ is a path of triangles decomposition of G, let

$$A_k = \bigcup (X_i : 1 \leq i \leq 2n+1 \text{ and } i = k \text{ modulo } 3) \; (k = 0, 1, 2).$$

We have already seen that (G, A_1, A_2, A_3) is a 3-coloured graph. For any 3-coloured graph (G, A, B, C), if there is a path of triangles decomposition $X_1, \dots, X_{2n+1}$ of G and sets A_1, A_2, A_3 as above, with $\{A_1, A_2, A_3\} = \{A, B, C\}$, we call (G, A, B, C) a *canonically-coloured path of triangles graph.*

Let $\mathcal{Q}_0$ be the class of all 3-coloured graphs (G, A, B, C) such that G has no triangle; let $\mathcal{Q}_1$ be the class of all 3-coloured graphs (G, A, B, C) where G is isomorphic to the line graph of $K_{3,3}$; and let $\mathcal{Q}_2$ be the class of all canonically-coloured path of triangles graphs. Now we can state the main theorem.

11.1 *Every 3-coloured prismatic graph admits a worn chain decomposition with all terms in $\mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$.*

For general orientable prismatic graphs the analogous result is the following.

11.2 *Every orientable prismatic graph that is not 3-colourable is either not 3-substantial, or a cycle of triangles graph, or a ring of five graph, or a mantled $L(K_{3,3})$.*

12 Chains of 3-coloured prismatic graphs

Our objective in this section is to develop some useful ways to recognize that our graph admits a worn chain decomposition. We begin with the following. Let us say that a 3-coloured graph (G, A, B, C) is *prime* if $V(G) \neq \emptyset$ and (G, A, B, C) cannot be expressed as a worn 2-chain.

12.1 *Every 3-coloured prismatic graph admits a worn chain decomposition each term of which is prime.*

Proof. Let (G, A, B, C) be a 3-coloured prismatic graph. We proceed by induction on $|V(G)|$. If $V(G) = \emptyset$ we may take the null sequence, and if (G, A, B, C) is prime then we may take the sequence with only one term (G, A, B, C). Hence we may assume that (G, A, B, C) admits a worn 2-chain $(G_1, A_1, B_1, C_1), (G_2, A_2, B_2, C_2)$. Consequently G_1, G_2 both have fewer vertices than G, and so from the inductive hypothesis, each of them admits a worn chain decomposition into prime terms. The sequence obtained by concatenating these two sequences appropriately is a worn chain decomposition of (G, A, B, C) into prime terms. This proves 12.1. ■

In view of 12.1, to construct all 3-coloured prismatic graphs it suffices to construct all prime 3-coloured prismatic graphs, and now we turn to that.

In this paper, a *hypergraph* H consists of a finite set $V(H)$ of *vertices* and a finite set $E(H)$ of *edges*, where each edge is a nonempty subset of $V(H)$. If H is a hypergraph, we say that $X \subseteq V(H)$ is *connected* if $X \neq \emptyset$ and there is no partition A, B of X into two nonempty subsets such that every edge of H included in X is included in one of A, B. We say H is *connected* if $V(H)$ is connected. A *component* of H is a connected subset of $V(H)$ that is maximal under inclusion.

Let G be prismatic. The *hypergraph of triangles* of G is the hypergraph with vertex set the core of G and edges the triangles of G. Thus if G has a triangle, then G is triangle-connected if and only if its hypergraph of triangles is connected.

12.2 *Let G be prismatic, and suppose that $G|(V_1 \cup V_2)$ admits a 3-colouring for some two components V_1, V_2 of the hypergraph of triangles of G. Then:*

- *G admits a 3-colouring, and*
- *for every 3-colouring (A, B, C) of G, (G, A, B, C) is not prime.*

Proof. Let $V_1, \ldots, V_n$ be the components of the hypergraph of triangles, and for $1 \leq i \leq n$ let $G_i = G|V_i$. By hypothesis, $G|(V_1 \cup V_2)$ admits a 3-colouring; and so for $i = 1, 2$ there is a 3-colouring (A_i, B_i, C_i) of G_i, such that $A_1 \cup A_2, B_1 \cup B_2$ and $C_1 \cup C_2$ are stable.

(1) *A_1 is complete to one of B_2, C_2 and anticomplete to the other.*

For let $a_1 \in A_1$. We prove first that a_1 is complete to one of B_2, C_2 and anticomplete to the other. For since $a_1 \in V_1$, there is a triangle $\{a_1, b_1, c_1\}$ of G, where $b_1 \in B_1$ and $c_1 \in C_1$. For every triangle $\{a_2, b_2, c_2\}$ of G_2 with $a_2 \in A_2, b_2 \in B_2$ and $c_2 \in C_2$, since a_1 has a unique neighbour in this triangle and a_1, a_2 are nonadjacent (since $A_1 \cup A_2$ is stable), it follows that a_1 is adjacent to exactly one of b_2, c_2. Similarly b_1 is adjacent to exactly one of c_2, a_2, and c_1 to exactly one of a_2, b_2. Thus the three edges between $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$ are either a_1b_2, b_1c_2, c_1a_2 or a_1c_2, b_1a_2, c_1b_2. We say $\{a_2, b_2, c_2\}$ is *white* in the first case and *black* in the second. Suppose there is both a white triangle and a black triangle in G_2. Since G_2 is triangle-connected, and every triangle in G_2 is either white or black, it follows that there is a white triangle and a black triangle in G_2 that share a vertex. From the symmetry we may assume that $\{a_2, b_2, c_2\}$ is a white triangle, and $\{a_2, b_2', c_2'\}$ is a black triangle, where $a_2 \in A_2, b_2, b_2' \in B_2$ and $c_2, c_2' \in C_2$. Since $\{a_2, b_2, c_2\}$ is white, we deduce that a_1b_2, b_1c_2, c_1a_2 are edges, and similarly a_1c_2', b_1a_2, c_1b_2' are edges; but then a_2 has two neighbours in $\{a_1, b_1, c_1\}$, a contradiction. Thus either all triangles in G_2 are white, or they are all black, and from the symmetry we may assume that they are all white. Hence a_1 is complete to B_2 and anticomplete to C_2, as claimed. Choose $b_2 \in B_2$. Similarly b_2 is complete to one of A_1, C_1 and anticomplete to the other. Since b_2 is adjacent to a_1, it is not anticomplete to A_1, and so b_2 is complete to A_1. Since this holds for all $b_2 \in B_2$, it follows that A_1 is complete to B_2. Every vertex in A_1 is anticomplete to one of B_2, C_2, and therefore A_1 is anticomplete to C_2. This proves (1).

(2) *G admits a 3-colouring.*

For from (1) we may assume that the pairs $(A_1, B_2), (B_1, C_2), (C_1, A_2)$ are complete, and the other three pairs $(A_1, C_2), (B_1, A_2), (C_1, B_2)$ are anticomplete. (Note also that the pairs $(A_1, A_2), (B_1, B_2), (C_1, C_2)$ are anticomplete.) Define A_3, B_3, C_3 to be the sets of all B_2-complete, C_2-complete, and A_2-complete vertices in $V(G) \setminus (V_1 \cup V_2)$ respectively. Define A_4, B_4, C_4 to be the sets of all C_1-complete, A_1-complete, and B_1-complete vertices in $V(G) \setminus (V_1 \cup V_2 \cup A_3 \cup B_3 \cup C_3)$ respectively. Let $A = A_1 \cup A_2 \cup A_3 \cup A_4$, and define B, C similarly. We claim that (A, B, C) is a 3-colouring of G. For A, B, C are pairwise disjoint, from their definition. We must check that they are stable and have union $V(G)$.

To show that A is stable, let $a_3 \in A_3$. Then a_3 is complete to B_2, and has only one neighbour in each triangle of G_2, and therefore a_3 is anticomplete to A_2. Moreover, any two members of $A_1 \cup A_3$ have a common neighbour in B_2, and therefore are nonadjacent (since V_1, V_2 are components of the hypergraph of triangles of G). We deduce that $A_1 \cup A_2 \cup A_3$ is stable, and similarly $A_1 \cup A_2 \cup A_4$ is stable. Suppose that $a_3 \in A_3$ and $a_4 \in A_4$ are adjacent. Since $a_4 \in A_4$, it is not complete to C_2; choose $c_2 \in C_2$ nonadjacent to a_4. Choose a triangle $\{a_2, b_2, c_2\}$ with $a_2 \in A_2$ and $b_2 \in B_2$. Since a_4 has a neighbour in this triangle, and we have already seen that a_4 is anticomplete to A_2, it follows that a_4 is adjacent to b_2; but then $\{a_3, a_4, b_2\}$ is a triangle, a contradiction (since V_2 is a component of the hypergraph of triangles). This proves that A_3 is anticomplete to A_4, and so A is stable, and similarly B, C are stable.

To show that $A \cup B \cup C = V(G)$, let $v \in V(G)$. If $v \in V_1 \cup V_2$ then $v \in A \cup B \cup C$, so we may assume that $v \notin V_1 \cup V_2$. Since A_1 is complete to B_2, and no triangle meets both A_1 and B_2, it follows that v is anticomplete to at least one of A_1, B_2. Similarly v is anticomplete to at least one of B_1, C_2, and to at least one of C_1, A_2. Hence v is either anticomplete to at least two of A_1, B_1, C_1, or to at least two of A_2, B_2, C_2. In the first case, since v has a neighbour in every triangle of G_1, it follows that v is complete to one of A_1, B_1, C_1, and therefore belongs to $A \cup B \cup C$, a contradiction. The second case is similar. This proves that $A \cup B \cup C = V(G)$, and therefore proves (2).

From (2), the first assertion of the theorem follows. To prove the second assertion, let (A, B, C) be a 3-colouring of G. Let W be the core of G.

(3) *The 3-coloured graph $(G|W, A \cap W, B \cap W, C \cap W)$ is not prime.*

To see this, for $1 \le i \le n$, let $A_i = A \cap V(G_i)$, and define B_i, C_i similarly. For $1 \le i, j \le n$ with $i \neq j$, we write $i \to j$ if the pairs $(A_i, B_j), (B_i, C_j)$ and (C_i, A_j) are complete, and the pairs $(A_i, C_j), (B_i, A_j)$ and (C_i, B_j) are anticomplete. By (1) (with V_1, V_2 replaced by V_i, V_j) it follows that either $i \to j$ or $j \to i$, and not both. We claim that this relation is transitive. For let $i, j, k \in \{1, \dots, n\}$ be distinct, and suppose that $i \to j$ and $j \to k$. If $k \to i$, then $A_i \cup B_j \cup C_k$ includes a triangle, which is impossible. Thus $i \to k$, and so the relation is transitive. Hence we may renumber $V_1, \dots, V_n$ so that $i \to j$ if and only if $j > i$. But then

$$(G|V_1, A_1, B_1, C_1), (G|(W \setminus V_1), A_2 \cup \dots \cup A_n, B_2 \cup \dots \cup B_n, C_2 \cup \dots \cup C_n)$$

is a 2-chain for $(G|W, A \cap W, B \cap W, C \cap W)$, and consequently the latter is not prime. This proves (3).

In view of (3) and since $G|W$ is triangle-covered, we may choose a 2-chain for $(G|W, A \cap W, B \cap W, C \cap W)$, say (F_i, A_i, B_i, C_i) $(i = 1, 2)$. Define sets $A_3, B_3, C_3, A_4, B_4, C_4 \subseteq V(G) \setminus W$ as follows.

- A_3 is the set of all B_2-complete vertices in $A \setminus W$
- B_3 is the set of all C_2-complete vertices in $B \setminus W$
- C_3 is the set of all A_2-complete vertices in $C \setminus W$
- A_4 is the set of all C_1-complete vertices in $A \setminus (W \cup A_3)$
- B_4 is the set of all A_1-complete vertices in $B \setminus (W \cup B_3)$
- C_4 is the set of all B_1-complete vertices in $C \setminus (W \cup C_3)$.

(4) *$A = A_1 \cup A_2 \cup A_3 \cup A_4$, and analogous statements hold for B, C.*

For let $v \in A$, and suppose that $v \notin A_1 \cup A_2 \cup A_3$. Thus $v \notin W$. Since $v \notin A_3$, v has a non-neighbour in B_2, and since it has no neighbours in A_2 (because A is stable), it follows that v has a neighbour in C_2. Since B_1 is complete to C_2 and no triangle meets both B_1 and C_2, it follows that v is anticomplete to B_1. Since it is also anticomplete to A_1, we deduce that v is complete to C_1, and so $v \in A_4$. This proves (4).

Let $G_3 = G|(V(F_1) \cup A_3 \cup B_3 \cup C_3)$, and $G_4 = G|(V(F_2) \cup A_4 \cup B_4 \cup C_4)$. Then $(A_1 \cup A_3, B_1 \cup B_3, C_1 \cup C_3)$ is a 3-colouring of G_3, by (4), and the analogous statement holds for G_4. We claim that

$$(G_3, A_1 \cup A_3, B_1 \cup B_3, C_1 \cup C_3), (G_4, A_2 \cup A_4, B_2 \cup B_4, C_2 \cup C_4)$$

is a worn 2-chain for (G, A, B, C). To see this, it suffices from the symmetry to check that

- if $a \in A_1 \cup A_3$ and $c \in C_2 \cup C_4$, then a, c are nonadjacent, and
- if $a \in A_1 \cup A_3$ and $b \in B_2 \cup B_4$, and at least one of $a, b \in W$, then a, b are adjacent.

For the first statement, let $a \in A_1 \cup A_3$ and $c \in C_2 \cup C_4$, and suppose a, c are adjacent. Since a is complete to B_2, it follows that c is anticomplete to B_2, and in particular $c \notin C_2$ (since F_2 is triangle-covered). Since c is anticomplete to C_2 (because C is stable), it follows that c is A_2-complete. But then $c \in C_3$, a contradiction. For the second statement, suppose that $a \in A_1 \cup A_3$ and $b \in B_2 \cup B_4$, and at least one of $a, b \in W$, and a, b are nonadjacent. Since $a \in A_1 \cup A_3$, a is B_2-complete, and so $b \notin B_2$, and similarly $a \notin A_1$; but then $a, b \notin W$, a contradiction. This proves our claim that (G, A, B, C) admits a worn 2-chain, and consequently is not prime; and therefore completes the proof of 12.2. ■

We deduce the following corollary.

12.3 *If (G, A, B, C) is a prime 3-coloured prismatic graph with nonnull core, then G is triangle-connected.*

The proof is clear. The next result is another corollary of 12.2.

12.4 *Let G be prismatic and orientable, with nonnull core. If G is not triangle-connected, then G is 3-colourable.*

Proof. Since G has nonnull core and is not triangle-connected, its hypergraph of triangles has at least two components. Let V_1, V_2 be two such components. For $i = 1, 2$, let $S_i \subseteq V_i$ be a triangle. Let $\mathcal{O}$ be an orientation of G, and let $\mathcal{O}(S_i)$ be $p_i \to q_i \to r_i \to p_i$, where p_1p_2, q_1q_2, r_1r_2 are edges. Every vertex in V_1 is adjacent to exactly one of p_2, q_2, r_2; let A_1, B_1, C_1 be the sets of those $v \in V_1$ adjacent to p_2, q_2, r_2 respectively. Define A_2, B_2, C_2 similarly. Certainly $A_1, B_1, C_1, A_2, B_2, C_2$ are all stable, since no triangle meets both V_1 and V_2. Since $\mathcal{O}(S_2)$ is $p_2 \to q_2 \to r_2 \to p_2$ and a_1p_2, b_1q_2, c_1r_2 are edges, we have

(1) *Let $T_1 \subseteq V_1$ be a triangle, where $T_1 = \{a_1, b_1, c_1\}$ and $a_1 \in A_1, b_1 \in B_1$ and $c_1 \in C_1$; then $\mathcal{O}(T_1)$ is $a_1 \to b_1 \to c_1 \to a_1$. The analogous statement holds for triangles in V_2.*

For $i = 1, 2$, let $T_i = \{a_i, b_i, c_i\}$ be a triangle with $a_i \in A_i, b_i \in B_i$ and $c_i \in C_i$. Each of a_1, b_1, c_1 has a neighbour in T_2; let us say the pair (T_1, T_2) is *good* if every edge between T_1 and T_2 is either between A_1 and A_2, or between B_1 and B_2, or between C_1 and C_2; and *bad* otherwise.

(2) *Every pair (T_1, T_2) is good.*

For since V_1, V_2 are components, it suffices (from the symmetry between V_1, V_2) to show that if T_1 is a triangle in V_1, and T_2, T_2' are triangles in V_2 that share a vertex, and (T_1, T_2) is good, then so is (T_1, T_2'). Let $T_1 = \{a_1, b_1, c_1\}$, $T_2 = \{a_2, b_2, c_2\}$, and $T_2' = \{a_2', b_2', c_2\}$, where $a_1 \in A_1, b_1 \in B_1, c_1 \in C_1$, $\{a_2, a_2'\} \subseteq A_2, \{b_2, b_2'\} \subseteq B_2$ and $c_2 \in C_2$. Since (T_1, T_2) is good, it follows that c_1c_2 is an edge. But from (1), $\mathcal{O}(T_1)$ is $a_1 \to b_1 \to c_1 \to a_1$ and $\mathcal{O}(T_2')$ is $a_2' \to b_2' \to c_2 \to a_2'$. Since c_1c_2 is an edge, we deduce that a_1a_2' and b_1b_2' are edges, and so (T_1, T_2') is good. This proves (2).

Since every vertex of $V_1 \cup V_2$ belongs to a triangle, (2) implies that every edge between V_1 and V_2 is either between A_1 and A_2, or between B_1 and B_2, or between C_1 and C_2. In particular, $A_1 \cup B_2, B_1 \cup C_2, C_1 \cup A_2$ are three stable sets, and so $G|(V_1 \cup V_2)$ is 3-colourable. By 12.2, G is 3-colourable. This proves 12.4. ■

13 Orientable and not 3-colourable

In this section we complete the proof of 11.2. We need two more lemmas. The first is the following. ($K_{3,3} \setminus e$ is the graph obtained from $K_{3,3}$ by deleting one edge.)

13.1 *Let G be prismatic and triangle-connected, with core W. Suppose that $(G|W,A,B,C)$ and $(G|W,A',B',C')$ are 3-coloured graphs with $\{A,B,C\} \neq \{A',B',C'\}$. Then either*

- *$G|W$ is isomorphic to $L(K_{3,3})$ or to $L(K_{3,3} \setminus e)$, or*
- *there is a clique $X \subseteq W$ with $1 \leq |X| \leq 2$ such that every triangle has nonempty intersection with X.*

Proof. For more convenient notation, let $W_1 = A, W_2 = B, W_3 = C$ and $W^1 = A', W^2 = B', W^3 = C'$. For $1 \leq i,j \leq 3$, let $W^i_j = W^i \cap W_j$. Thus W is the union of the nine pairwise disjoint sets W^i_j. Let T be a triangle of G, with $T = \{t_1,t_2,t_3\}$. Let $t_k \in W^{i_k}_{j_k}$ for $k = 1,2,3$. Thus i_1,i_2,i_3 are distinct, and so are j_1,j_2,j_3; and so the map sending i_k to j_k for $k = 1,2,3$ is a permutation of $\{1,2,3\}$, denoted by $\pi(T)$. The sign of this permutation is called the *sign of* T. (Thus, the identity map and the two cyclic permutations have positive sign, and the three involutions have negative sign.)

(1) *If S,T are triangles with opposite sign, then $S \cap T \neq \emptyset$.*

For from the symmetry we may assume that $S = \{s_1,s_2,s_3\}$ where $s_i \in W^i_i$ for $i = 1,2,3$, and $T = \{t_1,t_2,t_3\}$ where $t_1 \in W^1_2, t_2 \in W^2_1$ and $t_3 \in W^3_3$. Suppose that $S \cap T = \emptyset$. Since t_1 has a neighbour in S, and is nonadjacent to s_1,s_2 (because W^1,W_2 are stable), it follows that t_1 is adjacent to s_3. Similarly t_2 is adjacent to s_3, and so s_3 has two neighbours in T, a contradiction. This proves (1).

Let Π be the set of all (six) permutations of $\{1,2,3\}$. For each $\pi \in \Pi$, let $X(\pi)$ be the union of all the triangles T with $\pi(T) = \pi$.

(2) *Not all triangles have the same sign.*

For suppose they do; they all have positive sign say. Let $\pi_1,\pi_2,\pi_3 \in \Pi$ be the permutations with positive sign. Any two triangles S,T with the same sign with $\pi(S) \neq \pi(T)$ are disjoint, and so $X(\pi_1), X(\pi_2), X(\pi_3)$ are pairwise disjoint. Moreover their union is W, and since G is triangle-connected and every triangle is a subset of one of $X(\pi_1), X(\pi_2), X_(\pi_3)$, it follows that two of these sets are empty. We may therefore assume that $\pi(T) = \pi_1$ for all triangles T, where π_1 is the identity permutation say. Since every vertex of W belongs to a triangle, and so belongs to W^k if and only if it belongs to W_k (for $k = 1,2,3$), it follows that $W^k = W_k$ for $k = 1,2,3$, contradicting that $\{A,B,C\} \neq \{A',B',C'\}$. This proves (2).

(3) *If there are two triangles T_1,T_2 with positive sign and with $\pi(T_1) \neq \pi(T_2)$, and two triangles T_1,T_2 with negative sign and with $\pi(T_3) \neq \pi(T_4)$, then $G|W$ is isomorphic to $L(K_{3,3})$ or to $L(K_{3,3} \setminus e)$.*

For in this case, suppose that T,T' are triangles with $\pi(T) = \pi(T')$. From the symmetry we may assume that $\pi(T)$ is the identity permutation. By (1) T,T' both meet T_3 and T_4, and therefore both contain the unique vertex of T_3 that lies in $W^1_1 \cup W^2_2 \cup W^3_3$, and the unique vertex of T_4 that lies in the same set. Hence $|T \cap T'| \geq 2$ and so $T = T'$. Thus G has between four and six triangles, all with $\pi(T)$ different. From this and (1), it follows that $|W^i_j| \leq 1$ for $1 \leq i,j \leq 3$; and so $G|W$ is isomorphic to one of $L(K_{3,3}), L(K_{3,3} \setminus e)$, and the theorem holds. This proves (3).

In view of (3), we may assume that for every triangle T, if T has positive sign then $\pi(T)$ is the identity. From (2), some triangle S has positive sign; say $S = \{s_1^1, s_2^2, s_3^3\}$ where $s_i^i \in W_i^i$ for $i = 1, 2, 3$. Again from (2), there is a triangle T with negative sign, and by (1) we may assume $T = \{t_2^1, t_1^2, s_3^3\}$ where $t_2^1 \in W_2^1$ and $t_1^2 \in W_1^2$. Suppose that some triangle $R \neq S$ also has positive sign; say $R = \{r_1^1, r_2^2, r_3^3\}$ where $r_i^i \in W_i^i$ for $i = 1, 2, 3$. Since R meets T, it follows that $r_3^3 = s_3^3$. We claim that every triangle contains s_3^3. For we have seen this already for triangles of positive sign; and if T' has negative sign then since it meets both R and S, and $r_i^i \neq s_i^i$ for $i = 1, 2$, it follows that T contains s_3^3 as claimed. Thus in this case the second statement of the theorem holds with $X = \{s_3^3\}$.

Consequently we may assume that S is the only triangle that has positive sign. Every triangle with negative sign contains one of s_1^1, s_2^2, s_3^3, and so we may assume that there are three triangles T_1, T_2, T_3, all with negative sign and with $s_i^i \in T_i$ for $i = 1, 2, 3$ (for otherwise the second statement of the theorem holds). Thus there exist $s_j^i \in W_j^i$ for all distinct $i, j \in \{1, 2, 3\}$, such that $\{s_1^1, s_3^2, s_2^3\}$, $\{s_3^1, s_2^2, s_1^3\}$ and $\{s_2^1, s_1^2, s_3^3\}$ are triangles. Since s_2^1 has a neighbour in $\{s_3^1, s_2^2, s_1^3\}$, and is nonadjacent to s_3^1, s_2^2, it follows that s_2^1 is adjacent to s_1^3. Similarly every two of s_2^1, s_3^2, s_1^3 are adjacent; but then they form a second triangle with positive sign, a contradiction. This proves 13.1. ∎

The next lemma is a convenient corollary of 13.1 and 8.2.

13.2 *Let G be prismatic and 3-substantial, with core W. If $G|W$ is a core path of triangles graph, then G is 3-colourable.*

Proof. Let $X_1, \ldots, X_{2n+1}$ be a core path of triangles decomposition of $G|W$. For $k = 1, 2, 3$, let $A_k = \bigcup(X_i : 1 \leq i \leq 2n+1 \text{ and } i = k \text{ modulo } 3)$. For each vertex $v \in V(G) \setminus W$, let N_v be the set of neighbours of v in W. By 8.2, N_v is disjoint from at least one of A_1, A_2, A_3. Let B_1 be the set of all $v \in V(G) \setminus W$ such that $N_v \cap A_2, N_v \cap A_3 \neq \emptyset$, and define B_2, B_3 similarly. For $i = 1, 2, 3$ let C_i be the set of all $v \in V(G) \setminus W$ such that $N_v \subseteq A_i$. (Note that if $v \in C_i$ then $N_v = A_i$, since N_v meets every triangle.) The sets $B_1, B_2, B_3, C_1, C_2, C_3$ are pairwise disjoint and have union $V(G) \setminus W$.

(1) *For $i = 1, 2, 3$, $A_i \cup B_i$ is stable.*

Let $i = 3$ say. Certainly A_3 is stable; and by definition of B_3, B_3 is anticomplete to A_3. Suppose that there exist $u, v \in B_3$, adjacent. For $i = 1, 2$ let U_i, V_i be the set of neighbours in A_i of u, v respectively. Since u is in no triangle, it follows that $U_i \cap V_i = \emptyset$ for $i = 1, 2$. We claim that $U_1 \cup V_1 = A_1$; for suppose that there exists $a_1 \in A_1 \setminus (U_1 \cup V_1)$. Choose a triangle $\{a_1, a_2, a_3\}$ with $a_2 \in A_2$ and $a_3 \in A_3$. Since $U_2 \cap V_2 = \emptyset$, not both u, v are adjacent to a_2, and since neither of them is adjacent to a_1, a_3, not both u, v have a neighbour in this triangle, a contradiction. This proves that $U_1 \cup V_1 = A_1$, and similarly $U_2 \cup V_2 = A_2$. Hence N_u, N_v are disjoint and have union $A_1 \cup A_2$. But N_u, N_v are both stable, and so (N_u, N_v, A_3) is a 3-colouring of $G|W$. Since G is 3-substantial and $L(K_{3,3})$ is not a core path of triangles graph, 13.1 implies that N_u is one of A_1, A_2, a contradiction since $u \in B_3$. This proves (1).

Now for $i = 1, 2, 3$, C_i is stable since its members are not in the core and have a common neighbour. Moreover, $A_2 \cup A_3$ is anticomplete to C_1 by definition, and if $x \in B_2 \cup B_3$ then x has a neighbour (in A_1) which is adjacent to every vertex of C_1, and therefore x is anticomplete to C_1. In particular, $A_2 \cup B_2 \cup C_1$ is stable, and so are $A_3 \cup B_3 \cup C_2$ and $A_1 \cup B_1 \cup C_3$. Since these three sets have union $V(G)$, it follows that G is 3-colourable. This proves 13.2. ∎

Proof of 11.2. Let G be prismatic, orientable and not 3-colourable, and let W be its core. We may assume that G is 3-substantial, for otherwise the theorem holds. By 12.4, it follows that G is triangle-connected. By 4.2, either $G|W$ is isomorphic to $L(K_{3,3})$, or $G|W$ is a core cycle of triangles graph, or $G|W$ is a core path of triangles graph. If $G|W$ is isomorphic to $L(K_{3,3})$, then G is a mantled $L(K_{3,3})$ by 10.3, and the theorem holds. If $G|W$ is a core cycle of triangles graph, then by 9.1 and 10.2, either G is a cycle of triangles graph, or G is a ring of five graph, and in either case the theorem holds. If $G|W$ is a path of triangles graph, then by 13.2 G is 3-colourable, a contradiction. This proves 11.2. ■

14 The 3-colourable case

It remains to prove 11.1; and in view of 12.1, it suffices to show that the following:

14.1 *If (G, A, B, C) is a prime 3-coloured triangle-connected prismatic graph, then $(G, A, B, C) \in \mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$.*

(We recall that $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2$ were defined just before the statement of 11.1.) This therefore is the goal of the remainder of the paper. Here is an immensely useful lemma.

14.2 *Let (G, A, B, C) be a prime 3-coloured prismatic graph, with nonnull core W. Then every vertex in $V(G) \setminus W$ has neighbours in exactly two of $W \cap A, W \cap B, W \cap C$.*

Proof. Certainly no vertex in $V(G) \setminus W$ has neighbours in all three of $W \cap A, W \cap B, W \cap C$, since it belongs to one of A, B, C and these three sets are stable. Since W is nonnull and therefore W includes a triangle, every vertex in $V(G) \setminus W$ has at least one neighbour in W. Let

$$
\begin{aligned}
A_1 &= \{v \in A \setminus W : v \text{ is } C \cap W\text{-anticomplete}\} \\
B_1 &= \{v \in B \setminus W : v \text{ is } A \cap W\text{-anticomplete}\} \\
C_1 &= \{v \in C \setminus W : v \text{ is } B \cap W\text{-anticomplete}\},
\end{aligned}
$$

and define $A_2 = A \setminus A_1, B_2 = B \setminus B_1$ and $C_2 = C \setminus C_1$. Let $V_i = A_i \cup B_i \cup C_i$, and let $G_i = G|V_i$ for $i = 1, 2$. Then $W \subseteq V_2$ and so $V_2 \neq \emptyset$; suppose that also $V_1 \neq \emptyset$. Then (G_i, A_i, B_i, C_i) $(i = 1, 2)$ is a 2-term sequence of 3-coloured prismatic graphs, and we claim it is a worn 2-chain for (G, A, B, C). To show this, it suffices (from the symmetry between A, B, C) to show that if $u \in A_1$ (and hence $u \notin W$) then

- u is anticomplete to $A_2 \cup C_2$, and
- if u is nonadjacent to $v \in B_2$ then $v \notin W$.

Now u has no neighbour in A_2 and hence none in $A \cap W$ since A is stable, and no neighbour in $C \cap W$ from the definition of A_1. On the other hand every vertex in $B \cap W$ is in a triangle T, and u has a neighbour in T; and consequently u is $B \cap W$-complete. This proves the second assertion above. For the first assertion, we already saw that u is A_2-anticomplete, so let $v \in C_2$. We claim that v has a neighbour in $B \cap W$. For if $v \in W$ then v belongs to a triangle with a vertex in $B \cap W$, and if $v \in C \setminus W$ then v has a neighbour in $B \cap W$ since $v \notin C_1$. This proves the claim. Since u is

$B \cap W$-complete, it follows that there is a vertex in $B \cap W$ adjacent to both u, v. Since u is in no triangle, it follows that u, v are nonadjacent. This proves that u is anticomplete to C_2, and therefore proves that (G, A, B, C) admits a worn 2-chain, a contradiction since it is prime. We deduce that $V_1 = \emptyset$. Thus every vertex in $A \setminus W$ has a neighbour in $C \cap W$, and similarly has a neighbour in $B \cap W$ (and evidently has none in $A \cap W$, since A is stable), and the result follows. ■

To complement 13.1, we prove the following.

14.3 *Let (G, A, B, C) be a prime 3-coloured prismatic graph with nonnull core, and let W be the core of G.*

- *If $G|W$ is isomorphic to $L(K_{3,3})$ then $(G, A, B, C) \in \mathcal{Q}_1$.*
- *If G is not 3-substantial then $(G, A, B, C) \in \mathcal{Q}_2$.*

Proof. Suppose first that $G|W$ is isomorphic to $L(K_{3,3})$. Thus $|W| = 9$, and we may number $W = \{w^i_j : 1 \leq i, j \leq 3\}$ such that distinct $w^i_j, w^{i'}_{j'}$ are adjacent if and only if $i \neq i'$ and $j \neq j'$. Since the three sets A, B, C are stable and their union includes W, we may assume that

$$\begin{aligned} A \cap W &= \{w^1_1, w^2_1, w^3_1\} \\ B \cap W &= \{w^1_2, w^2_2, w^3_2\} \\ C \cap W &= \{w^1_3, w^2_3, w^3_3\}. \end{aligned}$$

If there exists $v \in A \setminus W$, let N be the set of neighbours of v in W. Then N satisfies:

- N is stable (since v is in no triangle)
- N is disjoint from $A \cap W$ (since A is stable)
- N meets every triangle (since G is prismatic), and
- N has nonempty intersection with both B and C (by 14.2, since (G, A, B, C) is prime).

But there is no such subset in $L(K_{3,3})$, and so v does not exist. Hence $A \subseteq W$, and similarly $B, C \subseteq W$, and so $W = V(G)$ and $(G, A, B, C) \in \mathcal{Q}_1$ as required.

Next suppose that $G|W$ is isomorphic to $L(K_{3,3} \setminus e)$. Thus $|W| = 8$, and W can be numbered as

$$W = \{w^i_j : 1 \leq i, j, \leq 3 \text{ and } (i, j) \neq (3, 3)\}$$

where distinct $w^i_j, w^{i'}_{j'}$ are adjacent if and only if $i \neq i'$ and $j \neq j'$. From the symmetry we may assume that

$$\begin{aligned} A \cap W &= \{w^1_1, w^2_1, w^3_1\} \\ B \cap W &= \{w^1_2, w^2_2, w^3_2\} \\ C \cap W &= \{w^1_3, w^2_3\}. \end{aligned}$$

As before, it follows that $A, B \subseteq W$, but the argument does not quite work for C. Suppose that there exists $v \in C \setminus W$, and let N be its set of neighbours in W. Then again, N is stable, meets all triangles, is disjoint from C and meets both A and B, but there is one such subset, namely

$\{w_1^3, w_2^3\}$. Thus every vertex not in W belongs to C and its neighbour set in W is $\{w_1^3, w_2^3\}$. But then $(G, A, B, C) \in \mathcal{Q}_2$. To see this let $n = 3$, and let

$$\begin{aligned}
X_1 &= \emptyset \\
\hat{X}_2 = X_2 &= \{w_1^3\} \\
M_3 = X_3 &= \{w_2^1, w_2^2\} \\
\hat{X}_4 &= \{w_3^1, w_3^2\} \\
X_4 &= \{w_3^1, w_3^2\} \cup (V(G) \setminus W) \\
M_5 = X_5 &= \{w_1^1, w_1^2\} \\
\hat{X}_6 = X_6 &= \{w_2^3\} \\
X_7 &= \emptyset,
\end{aligned}$$

with all the sets L_i, R_i empty.

Next, suppose that there is a vertex c that belongs to every triangle of G. We may assume that $c \in C$. Let the triangles containing c be $\{a_i, b_i, c\}$ for $1 \le i \le k$. Let $v \in V(G) \setminus W$. If v is adjacent to c, then it is anticomplete to both $A \cap W$ and $B \cap W$ (since v is in no triangle), contrary to 14.2; so c has no other neighbours. By 14.2, v has a neighbour in $A \cap W$ and a neighbour in $B \cap W$, and therefore $v \in C$. For $1 \le i \le k$, v is adjacent to exactly one of a_i, b_i; and so by setting $n = 1$, $X_1 = A$, $\hat{X}_2 = \{c\}$, $X_2 = C$, $X_3 = B$, we see that $(G, A, B, C) \in \mathcal{Q}_2$.

Next, suppose that there exist adjacent $a, b \in V(G)$ so that every triangle contains one of a, b. We may assume that $a \in A$ and $b \in B$, and that not every triangle contains a, so at least one contains b and not a, and similarly at least one contains a and not b. Every vertex in W is in a triangle containing a or b, and so is adjacent to a or b (or both). Let

$$\begin{aligned}
A_b &= \{v \in (A \cap W) \setminus \{a\} : v \text{ is adjacent to } b\} \\
B_a &= \{v \in (B \cap W) \setminus \{b\} : v \text{ is adjacent to } a\} \\
C_b &= \{v \in C \cap W : v \text{ is adjacent to } b \text{ and not to } a\} \\
C_a &= \{v \in C \cap W : v \text{ is adjacent to } a \text{ and not to } b\} \\
C_0 &= \{v \in C \cap W : v \text{ is adjacent to both } a \text{ and } b.\}
\end{aligned}$$

Thus these five sets are pairwise disjoint and have union $W \setminus \{a, b\}$. Every triangle that contains a and not b is a subset of $\{a\} \cup B_a \cup C_a$, and every triangle containing b and not a is a subset of $\{b\} \cup A_b \cup C_b$. Moreover A_b is matched with C_b, and B_a is matched with C_a. Since by 12.3 G is triangle-connected, it follows that some (necessarily unique) triangle contains both a, b, and so $|C_0| = 1$, say $C_0 = \{c\}$. If $u \in C_a$, then u is anticomplete to $\{b\} \cup C_b$, and since u has a neighbour in every triangle that contains b and not a, it follows that u is A_b-complete. Hence C_a is complete to A_b, and similarly C_b is complete to B_a. Let $v \in V(G) \setminus W$, and let N be the set of neighbours of v in W. If v is adjacent to c, then from the symmetry we may assume that $v \in A$, and since N meets every triangle that contains b and not a, and $N \cap (A_b \cup \{a\}) = \emptyset$, it follows that $C_b \subseteq N$. Since B_a is complete to C_b and $C_b \neq \emptyset$, and v is in no triangle, it follows that v is anticomplete to B_a; but then v is anticomplete to both $A \cap W$ and $B \cap W$, contrary to 14.2. Thus every neighbour of c belongs to W. Now suppose that $v \in V(G) \setminus W$ is adjacent to a. Since a is complete to $B \cap W$, it follows that v has no neighbours in $B \cap W$, and so by 14.2, v has neighbours in both $A \cap W$ and in $C \cap W$.

Consequently $v \in B$. Let B_0 be the set of all such v, that is, all $v \in B \setminus W$ that are adjacent to a. Similarly let A_0 be the set of all $v \in A \setminus W$ that are adjacent to b. Then $V(G) \setminus W = A_0 \cup B_0$. Let $n = 2$, and let

$$\begin{aligned} R_1 = X_1 &= B_a \\ \hat{X}_2 &= \{a\} \\ X_2 &= \{a\} \cup A_0 \\ L_3 &= C_a \\ M_3 &= \{c\} \\ R_3 &= C_b \\ X_3 &= C \\ \hat{X}_4 &= \{b\} \\ X_4 &= \{b\} \cup B_0 \\ L_5 = X_5 &= A_b. \end{aligned}$$

This sequence shows that $(G, A, B, C) \in \mathcal{Q}_2$.

Finally, suppose that there exist nonadjacent $a_0, b_0 \in V(G)$ so that every triangle contains one of a_0, b_0. By what we already proved, we may assume that there is no clique of cardinality at most two meeting all triangles, and $G|W$ is not isomorphic to $L(K_{3,3} \setminus e)$. There is at least one triangle containing a_0 with nonempty intersection with a triangle containing b_0, since G is triangle-connected. Since no clique with cardinality at most two meets every triangle, it follows that a_0 is in at least two triangles, and so is b_0. Define $\hat{X}_4$ to be the set of all vertices v such that some triangle contains v, a_0, and some triangle contains v, b_0. Now there are four kinds of triangles in G; those containing a_0 and a vertex of $\hat{X}_4$; those containing b_0 and a vertex of $\hat{X}_4$; those containing a_0 disjoint from $\hat{X}_4$; and those containing b_0 disjoint from $\hat{X}_4$. We call them *left inner, right inner, left outer* and *right outer* respectively. Let $\hat{X}_2 = \{a_0\}, \hat{X}_6 = \{b_0\}$. Let $X_1 = R_1$ be the set of vertices in left outer triangles that are adjacent to b_0, and let L_3 be the vertices different from a_0 that are in left outer triangles and are not adjacent to b_0 Similarly, let $X_7 = L_7$ be the set of neighbours of a_0 in right outer triangles, and R_5 the set of nonneighbours of a_0 in right outer triangles (different from b_0). Let M_3 be the set of all vertices in left inner triangles and not in $\hat{X}_4 \cup \{a_0\}$, and let M_5 be those in right inner triangles and not in $\hat{X}_4 \cup \{b_0\}$. Let $X_3 = L_3 \cup M_3$, and $X_5 = M_5 \cup R_5$. The sets

$$R_1, \hat{X}_2, L_3, M_3, \hat{X}_4, M_5, R_5, \hat{X}_6, L_7$$

are pairwise disjoint, and have union the core W. It follows that the sequence

$$X_1, \hat{X}_2, X_3, \hat{X}_4, X_5, \hat{X}_6, X_7$$

is a core path of triangles decomposition of $G|W$ (note that since a_0 is in at least two triangles, it follows that if $R_1 = \emptyset$ then $|\hat{X}_4| > 1$, and the same holds for b_0). By 13.1, we may assume that $\hat{X}_2, X_5 \subseteq A$, and $X_3, \hat{X}_6 \subseteq B$, and $X_1, \hat{X}_4, X_7 \subseteq C$.

Let us examine the vertices not in the core. Define X_2, X_4, X_6 as follows:

- let X_2 be the union of $\hat{X}_2$ and the set of all vertices in A that are nonadjacent to b_0 and complete to $\hat{X}_4 \cup L_7$;

- let X_4 be the union of $\hat{X}_4$ and the set of all vertices $v \in C \setminus W$ that are adjacent to both a_0, b_0 and have no other neighbours in W;
- let X_6 be the union of $\hat{X}_6$ and the set of all vertices in B that are nonadjacent to a_0 and complete to $\hat{X}_4 \cup R_1$.

We claim that every vertex not in the core belongs to one of X_2, X_4, X_6. For let $v \in V(G) \setminus W$. If v is adjacent to both a_0, b_0, then it has no other neighbours in the core and $v \in C$, and so $v \in X_4$. Next, suppose that v is adjacent to b_0 and not to a_0. Then v is anticomplete to R_1, M_4, R_5, M_5, L_7 (since these are all complete to b_0), and therefore every neighbour of v in W belongs to B, contrary to 14.2. Similarly every vertex not in $W \cup X_4$ is nonadjacent to both a_0, b_0. Let v be such a vertex. If $v \in A$, then v has no neighbours in $M_5 \cup \{b_0\}$, and so v is complete to $\hat{X}_4$; and v has no neighbours in R_5, and so is complete to L_7, and consequently $v \in X_2$. Similarly if $v \in B$ then $v \in X_6$. We therefore suppose that $v \in C$. Hence v is anticomplete to $\hat{X}_4$, and therefore complete to $M_3 \cup M_5$. We deduce that M_3 is anticomplete to M_5, and so $|\hat{X}_4| = 1$. Also, since M_5 is complete to L_3 and v is complete to L_3, we deduce that $L_3 = \emptyset$, contradicting that a_0 is in at least two triangles. Thus, no such v exists. This proves our claim that every vertex not in the core belongs to one of X_2, X_4, X_6.

Since X_2, X_6 are complete to $\hat{X}_4$, they are anticomplete to each other. It follows that

$$X_1, X_2, X_3, X_4, X_5, X_6, X_7$$

is a path of triangles decomposition of G. But $A = X_2 \cup X_5, B = X_3 \cup X_6$, and $C = X_1 \cup X_4 \cup X_7$, and so $(G, A, B, C) \in \mathcal{Q}_2$. This completes the proof of 14.3. ■

Now we can complete the proof of the characterization for 3-coloured prismatic graphs.

Proof of 14.1. Let (G, A, B, C) be a prime 3-coloured prismatic graph. Let W be the core of G. If $W = \emptyset$ then $(G, A, B, C) \in \mathcal{Q}_0$ as required, so we assume that W is nonnull. By 12.3, G is triangle-connected. By 14.3, we may assume that $G|W$ is 3-substantial and not isomorphic to $L(K_{3,3})$. By 3.1, $G|W$ is a core path of triangles graph. Hence by 13.1 if $G|W$ is not isomorphic to $L(K_{3,3}) \setminus e$, and by inspection if $G|W$ is isomorphic to $L(K_{3,3}) \setminus e$, it follows that $(G|W, A \cap W, B \cap W, C \cap W) \in \mathcal{Q}_2$. Every vertex not in the core has neighbours in exactly two of $A \cap W, B \cap W, C \cap W$, by 14.2. By 9.2, G is a path of triangles graph. Hence there is a 3-colouring (A', B', C') of G with $(G, A', B', C') \in \mathcal{Q}_2$, and by 13.1, we may assume that $A \cap W \subseteq A'$, $B \cap W \subseteq B'$ and $C \cap W \subseteq C'$. Since every vertex not in the core has neighbours in exactly two of $A \cap W, B \cap W, C \cap W$, it follows that $A' = A, B' = B$ and $C' = C$, and so $(G, A, B, C) \in \mathcal{Q}_2$. This proves 14.1, and therefore proves 11.1. ■

As we observed earlier, this also completes the proof of 11.1.

15 Four-colouring

For an application in a future paper, it is convenient now to prove a lemma. This will avoid having to redefine "path of triangles graph" and all the rest in that paper. We wish to prove the following.

15.1 *Let G be an orientable prismatic graph with nonnull core.*

- *If G is a mantled $L(K_{3,3})$, then there are twelve stable sets of G so that every vertex is in three of them.*

- *If not, then G is 4-colourable.*

Proof. Suppose first that G is a mantled $L(K_{3,3})$. Then $V(G)$ is the union of seven sets $W = \{a^i_j : 1 \le i, j \le 3\}, V^1, V^2, V^3, V_1, V_2, V_3$, with adjacency as in the definition of a mantled $L(K_{3,3})$. Reading the subscripts and superscripts modulo 3, we see that the nine sets

$$V^i \cup V_j \cup \{a^{i+1}_k : k \in \{1,2,3\} \setminus \{j\}\} \ (1 \le i, j \le 3)$$

are all stable, and so are the three sets $\{a^i_1, a^i_2, a^i_3\}$ $(1 \le i \le 3)$; and every vertex is in exactly three of these twelve sets. This proves the first claim.

Now we assume that G is not a mantled $L(K_{3,3})$, and let W be its core.

(1) *If there is a stable set $X \subseteq V(G)$ such that $G \setminus X$ has a triangle and the hypergraph of triangles of $G \setminus X$ is not connected, then G is 4-colourable.*

For since $G \setminus X$ is prismatic and orientable, 12.4 implies that $G \setminus X$ is 3-colourable, and therefore G is 4-colourable, as required. This proves (1).

(2) *If G is 3-substantial then G is 4-colourable.*

For suppose that G is 3-substantial. We may assume that G is not 3-colourable, and so by 11.2, G is either a cycle of triangles graph, or a ring of five graph. In either case $G|W$ is a core cycle of triangles graph. Let $X_1, \dots, X_{2n}$ be a core cycle of triangles decomposition of $G|W$. Thus $n \ge 5$. Let $X = X_1 \cup X_5$. Then X is stable, and every triangle of $G \setminus X$ either meets $X_2 \cup X_4$ or meets $X_6 \cup \dots \cup X_{2n}$; there is a triangle of each type, and no triangle of the first kind intersects any triangle of the second kind. Hence the hypergraph of triangles of $G \setminus X$ is disconnected, and the claim follows from (1). This proves (2).

(3) *If some vertex belongs to every triangle of G then G is 4-colourable.*

For suppose that c belongs to every triangle. Choose a triangle $T = \{a, b, c\}$, and let A, B, C be the sets of vertices in $V(G) \setminus T$ adjacent to a, b, c respectively. Thus A, B, C, T are pairwise disjoint and have union $V(G)$. Since every triangle contains c, it follows that A, B are both stable. The subgraph induced on $C \cup \{a, b\}$ is a matching and so is 2-colourable; let X, Y be disjoint stable sets with union $C \cup \{a, b\}$. Then $X, Y, A, B \cup \{c\}$ are four stable sets with union $V(G)$. This proves (3).

(4) *If there exist two adjacent vertices a, b so that every triangle contains one of a, b, then G is 4-colourable.*

For by (3) we may assume that some triangle contains a and not b, and some triangle contains b and not a. Let X be the set of all (at most one) vertices that are adjacent to both a, b. Then X is stable, and the hypergraph of triangles of $G \setminus X$ is not connected, and the claim follows from (2). This proves (4).

(5) *If there exist nonadjacent a_0, b_0 so that every triangle contains one of a_0, b_0, then G is 4-colourable.*

For by (4), we may assume that there is no clique of cardinality at most two meeting all triangles. Define

$$X_1 = R_1, \hat{X}_2, L_3, M_3, X_3, \hat{X}_4, M_5, R_5, \hat{X}_6, X_7 = L_7$$

as in the proof of 14.3. As in that proof, it follows that the sequence

$$X_1, \hat{X}_2, X_3, \hat{X}_4, X_5, \hat{X}_6, X_7$$

is a core path of triangles decomposition of $G|W$. If $R_1 \neq \emptyset$, then the hypergraph of triangles of $G \setminus M_3$ is not connected, and the result follows from (2). We assume that $R_1 = \emptyset$, and consequently $L_3 = \emptyset$. Similarly we may assume that $R_5 = L_7 = \emptyset$. If $|\hat{X}_4| = 1$, then $\hat{X}_4 \cup X_2$ meets all triangles and is a clique of cardinality 2, a contradiction, so $|\hat{X}_4| \geq 2$. For each $x \in \hat{X}_4$, let $r_x \in M_3$ be the vertex such that $\{a_0, x, r_x\}$ is a triangle, and define $s_x \in M_5$ similarly. Let $v \in V(G) \setminus W$, and let N be the set of neighbours of v in W. We say:

- $v \in C$ if $N = \{a_0, b_0\}$
- $v \in A$ if $N = \{a_0\} \cup M_5$
- $v \in B$ if $N = \{b_0\} \cup M_3$
- $c \in D_0$ if $N = \hat{X}_4$
- $c \in D_x$ for $x \in \hat{X}_4$ if $N = (\hat{X}_4 \setminus \{x\}) \cup \{r_x, s_x\}$.

It follows that the sets A, B, C, D_0 and D_x ($x \in \hat{X}_4$) are pairwise disjoint. We claim that they have union $V(G) \setminus W$. For let $v \in V(G) \setminus W$, and define N as before. If $a_0, b_0 \in N$ then since every vertex of W is adjacent to one of a_0, b_0 and N is stable, it follows that $v \in C$. We assume then that $b_0 \notin N$. If $a_0 \in N$, then N is disjoint from $X_3 \cup \hat{X}_4$, and so $M_5 \subseteq N$, and therefore $v \in A$. We assume therefore that $a_0 \notin A$. If $\hat{X}_4 \subseteq N$ then $v \in D_0$, so we assume that $x \notin N$ for some $x \in \hat{X}_4$. Since N meets the triangle $\{a_0, x, r_x\}$, it follows that $r_x \in N$, and similarly $s_x \in N$. Since r_x is adjacent so s_y for all $y \in \hat{X}_4 \setminus \{x\}$, it follows that x is the unique member of $\hat{X}_4$ that is not in N, and so $v \in D_x$. This proves our claim that the sets A, B, C, D_0 and D_x ($x \in \hat{X}_4$) have union $V(G) \setminus W$.

The four sets $X_2 \cup M_5 \cup B$, $X_6 \cup M_3 \cup A$, $\hat{X}_4 \cup C$, and $D_0 \cup \bigcup(D_x : x \in \hat{X}_4)$ have union $V(G)$, and the first three are stable; so we assume the fourth is not stable. Hence there exist $d_1, d_2 \in D_0 \cup \bigcup(D_x : x \in \hat{X}_4)$, adjacent. Since d_1, d_2 are not in triangles, they have no common neighbour; and so $|\hat{X}_4| = 2$, $\hat{X}_4 = \{x_1, x_2\}$ say, and $d_i \in D_{x_i}$ for $i = 1, 2$. But then the sets

$$\{a_0, s_{x_2}\} \cup D_{x_1}, \{b_0, r_{x_1}\} \cup D_0 \cup D_{x_2}, \{x_1, r_{x_2}\} \cup A, \{x_2, s_{x_1}\} \cup B \cup C$$

are stable and have union $V(G)$, and so G is 4-colourable. This proves (5).

From (2)–(5) we deduce that G is 4-colourable. This proves 15.1. ■

16 Changeable edges

Let G be a prismatic graph and let $e \in E(G)$. We say that uv is *changeable* if $G \setminus e$ is also prismatic. For another application in a future paper, it is helpful to study here which edges are changeable, if G is orientable. Let T be a triangle of a prismatic graph H, say $T = \{a, b, c\}$. We say T is a *leaf triangle at* c if a, b both only belong to one triangle of H (namely, T). We observe first that:

16.1 *Let G be a prismatic graph, and let $e \in E(G)$, with ends u, v. Then e is changeable if and only if either u, v are both not in the core of G, or there is a leaf triangle $\{u, v, w\}$ at some vertex w.*

Proof. If there is a triangle of G that contains u and not v, then $G \setminus e$ is not prismatic, and u is in the core, and there is no leaf triangle $\{u, v, w\}$ for any vertex w, and so the claim holds. We may assume then that u, v belong to the same triangles. If neither of them is in the core, then e is changeable and the claim holds; so we may assume that there is a triangle $\{u, v, w\}$ for some w. Since G is prismatic, w is unique, and $\{u, v, w\}$ is a leaf triangle at w; but then e is changeable and the claim holds. This proves 16.1. ■

Now let us examine which triangles are leaf triangles, if G is orientable.

16.2 *Let G be prismatic and orientable, and let $T = \{u, v, w\}$ be a triangle of G. Then T is a leaf triangle at w if and only if either:*

- *G admits a worn chain decomposition, and T is a leaf triangle at w in some term of the chain, or*
- *there exists $S \subseteq V(G)$ with $|S| \le 2$ such that every triangle meets S, and $w \in S$, and u, v belong to no triangle that meets $S \setminus \{w\}$, or*
- *G admits a path of triangles decomposition $X_1, \dots, X_{2n+1}$ or cycle of triangles decomposition $X_1, \dots, X_{2n}$, and for some i, $w \in \hat{X}_{2i}$ and $u \in R_{2i-1}$ and $v \in L_{2i+1}$ (or vice versa), with the usual notation.*

Proof. The "if" part is clear. Suppose then that T is a leaf triangle at w. If G admits a worn chain decomposition, then $\{u, v, w\}$ is a leaf triangle in one of the terms of the chain; so we may assume that G admits no such decomposition. Since G has a leaf triangle, it follows from 11.1 that either G is a path of triangles graph or it is not 3-colourable. We may assume that G has at least two triangles.

Suppose then that G is a path of triangles graph. Let $X_1, \dots, X_{2n+1}$ be a path of triangles decomposition of G, and let $L_{2i+1}, M_{2i+1}, R_{2i+1}$ $(1 \le i \le n)$ be as usual. Then for $1 \le i \le n$, every edge between $u \in R_{2i-1}$ and $v \in L_{2i+1}$ is changeable, since $\{u, v, w\}$ is a leaf triangle where $\hat{X}_{2i} = \{w\}$. We claim that there are no other leaf triangles; for suppose that $T = \{u, v, w\}$ is a leaf triangle at w. As in statement (1) of the proof of 4.2, either there exists i with $1 \le i < n$ such that $X_{2i}, M_{2i+1}, X_{2i+2}$ each contain a vertex of T, or there exists i with $1 \le i \le n$ such that $R_{2i-1}, X_{2i}, L_{2i+1}$ each contain a vertex of T. In the second case T is of the kind we already described, so we assume the first holds. From the symmetry we may assume that $u \in \hat{X}_{2i}$. Suppose that $|\hat{X}_{2i+2}| > 1$. By **(P1)**, $|\hat{X}_{2i}| = 1$, and by **(P6)**, $M_{2i+1}, \hat{X}_{2i+2}$ are matched; but then u belongs to more than one triangle, a contradiction. Thus $|\hat{X}_{2i+2}| = 1$. Suppose that $i > 1$. Then the same argument shows that $|\hat{X}_{2i-2}| = 1$, and by **(P6)**, $\hat{X}_{2i}$ is matched with both M_{2i+1} and M_{2i-1}, and again u is in more than one triangle. Hence $i = 1$, and so $|\hat{X}_4| = 1$. By **(P4)**, $R_1 \neq \emptyset$. But R_1 is matched with L_3, and so again u is in more than one triangle. This proves our claim.

We may therefore assume that G is not 3-colourable. Then G is triangle-connected by 12.4, and it has more than one triangle. Hence every triangle contains a vertex that belongs to another triangle, and so is a leaf triangle at at most one vertex. By 11.2, G is either not 3-substantial, or a cycle of triangles graph, or a ring of five graph, or a mantled $L(K_{3,3})$. Suppose it is not 3-substantial, and let $S \subseteq V(G)$ with $|S| \leq 2$ such that every triangle contains a vertex of S. Choose S minimal with this property. If $|S| = 1$, $S = \{s\}$ say, then every triangle is a leaf triangle at s, so we assume that $S = \{s_1, s_2\}$. Then the leaf triangles are those triangles that contain exactly one member of S, say s_1, and intersect no triangle that contains s_2. (It is easy to list these explicitly if we first formulate an explicit construction for G, which as we mentioned before is left to the reader.) Now suppose that G is a cycle of triangles graph. Then as for the path of triangles case, it follows easily that the changeable edges in leaf triangles are the edges between R_{2i-1} and L_{2i+1} for some i. Finally, if G is either a ring of five graph or a mantled $L(K_{3,3})$, then G has no leaf triangles. This proves 16.2. ■

References

[1] Maria Chudnovsky and Paul Seymour, "Claw-free graphs. II. Non-orientable prismatic graphs", submitted for publication (manuscript February 2004).