Claw-free Graphs. III. Sparse decomposition

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Abstract

A graph is *claw-free* if no induced subgraph is isomorphic to the complete bipartite graph $K_{1,3}$. In this series of papers we give a structural description of all claw-free graphs.

The most well-known class of claw-free graphs is the class of line graphs, but there are also claw-free graphs that are far from being line graphs; and one key distinction turns out to be, are there four vertices such that at most one pair of them are adjacent? The structure of the connected claw-free graphs which have such a quadruple (such as the icosahedron, and most line graphs and circular interval graphs) is quite different from those that do not (such as the Schläfli graph), and they are handled by different methods.

This paper is the first half of the proof; we prove that every claw-free graph which has such a quadruple either belongs to one of a few basic classes, or admits a decomposition in a useful way. The other half of the proof will be in the next paper of this series.

1 Introduction

Let G be a graph. (All graphs in this paper are finite and simple.) If $X \subseteq V(G)$, the subgraph $G|X$ *induced on* X is the subgraph with vertex set X and edge-set all edges of G with both ends in X. ($V(G)$ and $E(G)$ denote the vertex- and edge-sets of G respectively.) We say that $X \subseteq V(G)$ is a *claw in* G if $|X| = 4$ and $G|X$ is isomorphic to the complete bipartite graph $K_{1,3}$. We say G is *claw-free* if no $X \subseteq V(G)$ is a claw in G. Our objective in this series of papers is to show that every claw-free graph can be built starting from some basic classes by means of some simple constructions.

What constructions should we permit? That is not such an easy question, as we shall see. For instance, here is a construction that is natural and acceptable, "duplicating" a vertex: if G is claw-free, and $u \in V(G)$ with neighbour set N, we could add a new vertex v to G, with neighbour set $N \cup \{u\}$. Here is another "construction" – given any claw-free graph, add to it a vertex with any set of neighbours, provided that the enlarged graph has no claw. This second construction is evidently not what we want; if we allow it, constructing claw-free graphs becomes trivial. But it is difficult (indeed, beyond us) to see any formal difference between the two, at least from the viewpoint of complexity theory, because one can easily check in polynomial time whether adding a certain vertex introduces a claw. So it seems that we perhaps have to fall back on nebulous concepts like "naturalness" and "depth" to justify our work.

On the other hand, there is definitely something under here, waiting to be excavated. For instance, one of the first things we shall show is that if G is claw-free, and has an induced subgraph that is a line graph of a (not too small) cyclically 3-connected graph, then either the whole graph G is a line graph, or G admits a decomposition of one of two possible types. That suggests that we should investigate which other claw-free graphs do not admit either of these decompositions; and that turns out to be a good question, because at least when $\alpha(G) \geq 4$ there is a nice answer. (We denote the size of the largest stable set of vertices in G by $\alpha(G)$.) All claw-free graphs G with $\alpha(G) \geq 4$ that do not admit either of these decompositions can be explicitly described, and fall into a few basic classes; and all connected claw-free graphs G with $\alpha(G) \geq 4$ can be built from these basic types by simple constructions. When $\alpha(G) \leq 3$ the situation becomes more complicated; there are both more basic types and more decompositions required, as we shall explain.

Let us say a graph is *prismatic* if for every three pairwise adjacent vertices u, v, w, every vertex different from u, v, w is adjacent to exactly one of them. We say G is *antiprismatic* if its complement is prismatic. Antiprismatic graphs are claw-free, but they seem to need to be treated in a different way from the general case; indeed, the methods that we developed for the general case completely failed to work on antiprismatic graphs, and we had to find quite different techniques. For that reason, and the length of the present paper, we decided to write up the antiprismatic case in a separate paper. Thus, in this paper we just handle claw-free graphs that are not antiprismatic.

There is a difference between a "decomposition theorem" and a "structure theorem", although they are closely related. In this paper we prove a decomposition theorem for claw-free graphs that are not antiprismatic; we show that they all either belong to a few basic classes or admit certain decompositions. But this can be refined into a structure theorem that is more informative; for instance, every connected claw-free graph G with $\alpha(G) \geq 4$ has the same overall "shape" as a line graph, and more or less can be regarded as a line graph with "interval graph strips" substituted for some of the vertices. For reasons of space, that development, and its application to several open questions about claw-free graphs, is postponed to a future paper.

2 The main theorem

In this section we state our main theorem, but first we need a number of definitions. A *hole* in G means an induced subgraph which is a cycle with at least four vertices. A *path* in G means an induced nonnull connected subgraph in which no vertex is adjacent to more than two others. The *length* of a path or hole is the number of edges in it. If $X \subseteq V(G)$, the graph obtained from G by deleting X is denoted by $G \setminus X$. We say $X \subseteq V(G)$ is a *clique* in G if every two members of X are adjacent. A clique with cardinality 3 is a *triangle*. A *triad* in G means a set of three vertices of G, pairwise nonadjacent. Two subsets X, Y of $V(G)$ with $X \cap Y = \emptyset$ are *complete* to each other if every vertex of X is adjacent to every vertex of Y, and *anticomplete* if no vertex in X is adjacent to a member of Y. If $A \subseteq V(G)$, and $v \in V(G) \setminus A$, we say v is *A-complete* if it is adjacent to every vertex in A, and *A-anticomplete* if it has no neighbour in A. Distinct vertices u, v of G are *twins* (in G) if they are adjacent and have exactly the same neighbours in $V(G) \setminus \{u, v\}$.

Next, let us explain the decompositions that we shall use in the main theorem. The first is just that there are two vertices in G that are twins, or briefly, "G admits twins". For the second, let A, B be disjoint subsets of $V(G)$. The pair (A, B) is called a *homogeneous pair* in G if for every vertex $v \in V(G) \setminus (A \cup B)$, v is either A-complete or A-anticomplete and either B-complete or B-anticomplete. Let (A, B) be a homogeneous pair, such that A, B are both cliques, and A is neither complete nor anticomplete to B. In these circumstances we call (A, B) a *W-join*. (Note that there is no requirement that $A \cup B \neq V(G)$. If the complement of G is bipartite, then G admits a W-join except in degenerate cases.) The pair (A, B) is *nondominating* if some vertex of $G \setminus (A \cup B)$ has no neighbour in $A \cup B$; and it is *coherent* if the set of all $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a clique. In some applications, nondominating W-joins and coherent W-joins are easier to handle than general W-joins, and it turns out that throughout the proof in this paper, in every instance where we exhibit a W-join, it is either nondominating or coherent. We might as well take advantage of that convenient fact to save ourselves trouble in the future, so we confine ourselves to W-joins which are either nondominating or coherent.

Next, suppose that V_1, V_2 is a partition of $V(G)$ such that V_1, V_2 are nonempty and there are no edges between V_1 and V_2. We call the pair (V_1, V_2) a *0-join* in G. Thus G admits a 0-join if and only if it is not connected.

Next, suppose that V_1, V_2 partition $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq V_i$ such that:

- for $i = 1, 2$, A_i is a clique, and $A_i, V_i \setminus A_i$ are both nonempty
- A_1 is complete to A_2
- every edge between V_1 and V_2 is between A_1 and A_2.

In these circumstances, we say that (V_1, V_2) is a *1-join*.

Next, suppose that V_0, V_1, V_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of V_i satisfying the following:

- for $i = 1, 2$, A_i, B_i are cliques, $A_i \cap B_i = \emptyset$ and A_i, B_i and $V_i \setminus (A_i \cup B_i)$ are all nonempty
- A_1 is complete to A_2, and B_1 is complete to B_2, and there are no other edges between V_1 and V_2, and
- V_0 is a clique; and for $i = 1, 2$, V_0 is complete to $A_i \cup B_i$ and anticomplete to $V_i \setminus (A_i \cup B_i)$.

We call the triple (V_1, V_0, V_2) a *generalized 2-join*, and if $V_0 = \emptyset$ we call the pair (V_1, V_2) a *2-join*. (This is closely related to, but not the same as, what has been called a 2-join in other papers.)

We use one more decomposition, the following. Let (V_1, V_2) be a partition of $V(G)$, such that for $i = 1, 2$ there are cliques $A_i, B_i, C_i \subseteq V_i$ with the following properties:

- For $i = 1, 2$ the sets A_i, B_i, C_i are pairwise disjoint and have union V_i
- V_1 is complete to V_2 except that there are no edges between A_1 and A_2, between B_1 and B_2, and between C_1 and C_2.
- V_1, V_2 are both nonempty.

In these circumstances we say that G is a *hex-join* of $G|V_1$ and $G|V_2$. Note that if G is expressible as a hex-join as above, then the sets $A_1 \cup B_2$, $B_1 \cup C_2$ and $C_1 \cup A_2$ are three cliques with union $V(G)$, and consequently no graph G with $\alpha(G) > 3$ is expressible as a hex-join.

Next, we list some basic classes of graphs.

- **Line graphs.** If H is a graph, its *line graph* $L(H)$ is the graph with vertex set $E(H)$, in which distinct $e, f \in E(H)$ are adjacent if and only if they have a common end in H. We say $G \in \mathcal{S}_0$ if G is isomorphic to a line graph.
- **The icosahedron.** This is the unique planar graph with twelve vertices all of degree five. For $0 \le k \le 3$, $icosa(-k)$ denotes the graph obtained from the icosahedron by deleting k pairwise adjacent vertices. We say $G \in \mathcal{S}_1$ if G is isomorphic to $icosa(0)$, $icosa(-1)$ or $icosa(-2)$.
- **The graphs $\mathcal{S}_2$.** Let G be the graph with vertex set $\{v_1, \dots, v_{13}\}$, with adjacency as follows. v_1-$\cdots$-v_6 is a hole in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5, and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; v_{12} is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$; and v_{13} is adjacent to $v_1, v_2, v_4, v_5, v_7, v_8$. We say $H \in \mathcal{S}_2$ if H is isomorphic to $G \setminus X$, where $X \subseteq \{v_{11}, v_{12}, v_{13}\}$.
- **Circular interval graphs.** Let Σ be a circle and let $F_1, \dots, F_k$ be subsets of Σ, each homeomorphic to the closed interval $[0, 1]$, and no three with union Σ. Let V be a finite subset of Σ, and let G be the graph with vertex set V in which $v_1, v_2 \in V$ are adjacent if and only $v_1, v_2 \in F_i$ for some i. Such a graph is called a *circular interval graph*. We write $G \in \mathcal{S}_3$ if G is a circular interval graph.
- **An extension of $L(K_6)$.** Let H be the graph with seven vertices $h_0, \dots, h_6$, in which $h_1, \dots, h_6$ are pairwise adjacent and h_0 is adjacent to h_1. Let G be the graph obtained from the line graph $L(H)$ of H by adding one new vertex, adjacent precisely to the members of $V(L(H)) = E(H)$ that are not incident with h_1 in H. Then G is claw-free. Let $\mathcal{S}_4$ be the class of all graphs isomorphic to induced subgraphs of G.
- **The graphs $\mathcal{S}_5$.** Let $n \ge 0$. Let $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ be three cliques, pairwise disjoint. For $1 \le i, j \le n$, let a_i, b_j be adjacent if and only if $i = j$, and let c_i be adjacent to a_j, b_j if and only if $i \ne j$. Let d_1, d_2, d_3, d_4, d_5 be five more vertices, where d_1 is $A \cup B \cup C$-complete; d_2 is complete to $A \cup B \cup \{d_1\}$; d_3 is complete to $A \cup \{d_2\}$; d_4 is complete to $B \cup \{d_2, d_3\}$; d_5 is adjacent to d_3, d_4; and there are no more edges. Let the

graph just constructed be G. We say $H \in \mathcal{S}_5$ if (for some n) H is isomorphic to $G \setminus X$ for some $X \subseteq A \cup B \cup C$.

- **2-simplicial graphs of antihat type.** Let $n \geq 0$. Let $A = \{a_0, a_1, \ldots, a_n\}$, $B = \{b_0, b_1, \ldots, b_n\}$ and $C = \{c_1, \ldots, c_n\}$ be three cliques, pairwise disjoint. For $0 \leq i, j \leq n$, let a_i, b_j be adjacent if and only if $i = j > 0$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let c_i be adjacent to a_j, b_j if and only if $i \neq j \neq 0$. Let the graph just constructed be G. We say $H \in \mathcal{S}_6$ if (for some n) H is isomorphic to $G \setminus X$ for some $X \subseteq A \cup B \cup C$, and then H is said to be 2-*simplicial of antihat type.*

Now we can state the main result of this paper, the following.

2.1 *Let G be claw-free. Then either*

- *$G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_6$, or*
- *G admits either twins, a nondominating W-join, a coherent W-join, a 0-join, a 1-join, a generalized 2-join, or a hex-join, or*
- *G is antiprismatic.*

The proof is given in the final section of the paper. We postpone to future papers the study of antiprismatic graphs, and the problem of converting this decomposition theorem to a structure theorem.

3 More on decompositions

Before we begin the main proof, it is helpful to develop a few tools that will enable us to prove more easily that graphs are decomposable. First, here is another useful decomposition. Suppose that there is a partition (A, B, X) of $V(G)$ such that X is a clique, and $|A|, |B| \geq 2$, and A is anticomplete to B. In these circumstances we say that X is an *internal clique cutset*. This is not one of the decompositions used in the statement of the main theorem (indeed, it is not the inverse of a composition that preserves being claw-free, unlike the other decompositions we mentioned). Nevertheless, we win if we can prove that our graph admits an internal clique cutset, because of the following, proved in [1]. (G is said to be a *linear interval graph* if there is a linear order $v_1, \ldots, v_n$ of its vertex set such that for every edge v_iv_j with $j > i$, the set $\{v_i, v_{i+1}, \ldots, v_j\}$ is a clique. Every such graph is a circular interval graph.)

3.1 *Let G be claw-free. If G admits an internal clique cutset, then either G is a linear interval graph, or G admits either a 1-join, or a 0-join, or a coherent W-join, or twins.*

For brevity, let us say that G is *decomposable* if it admits either a generalized 2-join, or a 1-join, or a 0-join, or a nondominating W-join, or a coherent W-join, or twins, or an internal clique cutset, or a hex-join. There follow four lemmas that will speed up our recognition of decomposable graphs.

3.2 *Let G be claw-free, and let $A, C \subseteq V(G)$ be disjoint, such that*

- *A is a clique*
- *if $C = \emptyset$ then $|A| > 1$*
- *every vertex in $V(G) \setminus (A \cup C)$ is C-anticomplete, and either A-complete or A-anticomplete*
- *$|V(G) \setminus (A \cup C)| \geq 2$.*

Then G is decomposable.

Proof. If C is empty then $|A| > 1$ and any two members of A are twins. So we may assume that C is nonempty. If A is anticomplete to C then G admits a 0-join, so we may assume that $a \in A$ and $c \in C$ are adjacent. Let Y be the set of vertices in $V(G) \setminus (A \cup C)$ that are A-complete, and let $Z = V(G) \setminus (A \cup C \cup Y)$. If $y_1, y_2 \in Y$, then since $\{c, a, y_1, y_2\}$ is not a claw, it follows that y_1, y_2 are adjacent, and so Y is a clique. If Z is nonempty then $(A \cup C, Y \cup Z)$ is a 1-join, so we assume that Z is empty. But then $|Y| \geq 2$ by hypothesis, and all members of Y are twins, and so G is decomposable. This proves 3.2. ■

3.3 *Let G be claw-free, and let (A, B) be a homogeneous pair of cliques in G. Suppose that at least one of A, B has cardinality > 1, and either (A, B) is nondominating or the set of all $(A \cup B)$-complete vertices in $V(G) \setminus (A \cup B)$ is a clique. Then either*

- *(A, B) is a nondominating W-join, or a coherent W-join (respectively), or*
- *one of A, B has cardinality > 1 and all its members are twins.*

Proof. We may assume that $|A| > 1$. If B is either complete or anticomplete to A then the elements of A are twins, and otherwise (A, B) is either a nondominating or coherent W-join. This proves 3.3. ■

We say a triple (A, C, B) is a *breaker* in G if it satisfies:

- A, B, C are disjoint nonempty subsets of $V(G)$, and A, B are cliques
- every vertex in $V(G) \setminus (A \cup B \cup C)$ is either A-complete or A-anticomplete, and either B-complete or B-anticomplete, and C-anticomplete
- there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a neighbour in A and a nonneighbour in B; there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a neighbour in B and a nonneighbour in A; and there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a nonneighbour in A and a nonneighbour in B
- if A is complete to B, then there do not exist adjacent $x, y \in V(G) \setminus (A \cup B \cup C)$ such that x is $A \cup B$-complete and y is $A \cup B$-anticomplete.

The reason for interest in breakers is that they allow us to deduce that our graph admits one of our decompositions, without having to figure out which one, in view of the following theorem.

3.4 *Let G be claw-free. If G admits a breaker, then G admits either a 0-join, a 1-join, or a generalized 2-join.*

Proof. Let (A_1, C_1, B_1) be a breaker; let $V_1 = A_1 \cup B_1 \cup C_1$, let V_0 be the set of all vertices not in V_1 that are $A_1 \cup B_1$-complete, and let $V_2 = V(G) \setminus (V_1 \cup V_0)$. Let A_2 be the set of A_1-complete vertices in V_2, and B_2 the set of B_1-complete vertices in V_2. Let $C_2 = V_2 \setminus (A_2 \cup B_2)$. By hypothesis, A_2, B_2, C_2 are all nonempty. If there are no edges between C_1 and $A_1 \cup B_1$ then G admits a 0-join, so from the symmetry we may assume that there is an edge between C_1 and A_1. Since $A_1 \cup C_1 \cup A_2 \cup V_0$ includes no claw, it follows that $A_2 \cup V_0$ is a clique. Let A' be the set of vertices in A_1 with a neighbour in C_1. Since $B_2 \neq \emptyset$ and we may assume that $(C_1 \cup A', V(G) \setminus (C_1 \cup A'))$ is not a 1-join, it follows that A' is not anticomplete to B_1. Consequently some vertex $a \in A_1$ has a neighbour $b \in B_1$ and a neighbour $c \in C_1$. Since $A_2 \neq \emptyset$ and there is no claw, it follows that b, c are adjacent. Consequently $B_2 \cup V_0$ is a clique. Suppose that there is an edge xy between some $x \in V_0$ and $y \in C_2$. By hypothesis, A_1 is not complete to B_1; choose $a \in A_1$ and $b \in B_1$, nonadjacent. Then $\{x, y, a, b\}$ is a claw, a contradiction. It follows that there is no such edge xy, and so V_0 is anticomplete to C_2, and consequently (V_1, V_0, V_2) is a generalized 2-join. This proves 3.4. ■

Here is another shortcut, this time useful for handling hex-joins.

3.5 *Let G be claw-free, and let A, B, C be disjoint nonempty cliques. Suppose that every vertex in $V(G) \setminus (A \cup B \cup C)$ is complete to two of A, B, C and anticomplete to the third. Suppose also that one of A, B, C has cardinality > 1, and $A \cup B \cup C \neq V(G)$. Then G admits either a hex-join, or a nondominating W-join, or twins.*

Proof. Let $V_1 = A \cup B \cup C$, and $V_2 = V(G) \setminus V_1$. Let A_2 be the set of vertices in V_2 that are anticomplete to A, and define B_2, C_2 similarly. If A_2, B_2, C_2 are cliques, then G is the hex-join of $G|V_1$ and $G|V_2$, so we may assume that there exist nonadjacent $u, v \in A_2$. For $w \in A$ and $x \in B \cup C$, $\{x, w, u, v\}$ is not a claw, and so w, x are nonadjacent; and consequently A is anticomplete to $B \cup C$. Thus (B, C) is a homogeneous pair of cliques, and it is nondominating since A is nonempty; so by 3.3 we may assume that $|B|, |C| = 1$, and therefore $|A| > 1$ by hypothesis, and yet every two members of A are twins. This proves 3.5. ■

4 The icosahedron

Our first main goal is to prove that claw-free graphs that include a "substantial" line graph either are line graphs or are decomposable. To make this theorem as useful as possible, we want to weaken the meaning of "substantial" as far as we can; and on the borderline where the theorem is just about to become false, there are two situations where the theorem is false in a way we can handle. It is convenient to deal with them first before we embark on line graphs in general. We do one in this section and the other in the next, and then start on line graphs proper in the section after that.

The icosahedron is claw-free, and in this section we study claw-free graphs which contain it (or most of it) as an induced subgraph. If H is an induced subgraph of G, and $v \in V(G) \setminus V(H)$, we say that v is a *clone* of $u \in V(H)$ (with respect to H) if v is adjacent to u and u, v have exactly the same neighbours in $V(H) \setminus \{u\}$.

Frequently we assume that our current claw-free graph G has an induced subgraph H that we know, and we wish to enumerate all the possibilities for the neighbours set in $V(H)$ of vertices in $V(G) \setminus V(H)$. And having done so, then we try to figure out the adjacencies between the vertices in $V(G) \setminus V(H)$. To aid with that, here are three trivial facts that are used so often that it is worth stating them explicitly. (All three proofs are obvious and we omit them.)

4.1 *Let G be claw-free, and let H be an induced subgraph of G. Let $v \in V(G) \setminus V(H)$, and let N be the set of neighbours of v in $V(H)$. Then N includes no triad.*

4.2 *Let G be claw-free, and let H be an induced subgraph of G. Let $v \in V(G) \setminus V(H)$, and let N be the set of neighbours of v in $V(H)$. Then there is no path of length 2 in H with middle vertex in N and no other vertex in N.*

4.3 *Let G be claw-free, and let H be an induced subgraph of G. Let $u, v \in V(G) \setminus V(H)$ have a common neighbour $a \in V(H)$ and a common non-neighbour $b \in V(H)$. If a, b are adjacent then u, v are adjacent.*

4.4 *Let G be claw-free, containing $icosa(-1)$ as an induced subgraph. Then either $G \in \mathcal{S}_1$, or two vertices of G are twins, or G admits a 0-join. In particular, either $G \in \mathcal{S}_1$, or G is decomposable.*

Proof. Let $H = icosa(-1)$. Number the vertex set of H as $\{v_1, \ldots, v_{11}\}$, where for $1 \leq i < j \leq 10$, v_i is adjacent to v_j if either $j - i \leq 2$ or $j - i \geq 8$, and v_{11} is adjacent to v_1, v_3, v_5, v_7, v_9.

For $1 \leq i \leq 11$, let N_i be the union of $\{v_i\}$ and the set of neighbours of v_i in H, and let C_i be the union of $\{v_i\}$ and the set of all clones of v (with respect to H) in $V(G) \setminus V(H)$. Let $N_{12} = \{v_2, v_4, v_6, v_8, v_{10}\}$, and let C_{12} be the set of all $v \in V(G) \setminus V(H)$ whose set of neighbours in $V(H)$ is N_{12}.

(1) *Every vertex in $V(G)$ with at least one neighbour in $V(H)$ belongs to one of $C_1, \ldots, C_{12}$.*

For certainly each v_i belongs to C_i, for $1 \leq i \leq 11$; let $v \in V(G) \setminus V(H)$, and let N be the set of neighbours of v in $V(H)$. By hypothesis N is nonempty. Suppose first that $v_{11} \in N$. By 4.2 with v_1-v_{11}-v_5, at least one of $v_1, v_5 \in N$. Similarly N contains at least one of every two nonadjacent members of $\{v_1, v_3, v_5, v_7, v_9\}$, and so we may assume that $v_1, v_3, v_5 \in N$, from the symmetry. If $v_7, v_9 \in N$, then by 4.1, it follows that $|N| = 6$ and v is a clone of v_{11}. So we may assume from the symmetry that $v_9 \notin N$. By 4.2 with v_2-v_1-v_9, $v_2 \in N$. By 4.1, it follows that $v_6, v_8 \notin N$. By 4.2 with v_6-v_7-v_9, $v_7 \notin N$, and by 4.2 with v_4-v_5-v_7, $v_4 \in N$. By 4.1 with $\{v_4, v_{10}, v_{11}\}$, $v_{10} \notin N$. Thus v is a clone of v_3.

We may therefore assume that $v_{11} \notin N$. Suppose next that $v_1 \in N$. By 4.2 with v_{11}-v_1-v_2, $v_2 \in N$, and similarly $v_{10} \in N$. By 4.2 with v_3-v_1-v_9, one of $v_3, v_9 \in N$, and from the symmetry we may assume that $v_3 \in N$. By 4.2 with v_4-v_3-v_{11}, $v_4 \in N$. By 4.1, $v_6, v_7, v_8 \notin N$. By 4.2 with v_6-v_5-v_{11} and with v_8-v_9-v_{11}, $v_5, v_9 \notin N$. But then v is a clone of v_2.

We may therefore assume that $v_1 \notin N$, and by the symmetry that $v_3, v_5, v_7, v_9 \notin N$. By 4.2 with v_1-v_2-v_4 and v_2-v_4-v_5, it follows that N contains both or neither of v_2, v_4, and the same holds for all adjacent pairs of $v_2, v_4, v_6, v_8, v_{10}$. Thus either N consists of all these vertices or none. Since N is nonempty, the first case applies, and so $v \in C_{12}$. This proves (1).

We may assume that G is connected, for otherwise the theorem holds.

(2) *Every vertex of G has a neighbour in $V(H)$.*

For let C_0 be the set of all vertices of G with no neighbour in $V(H)$, and suppose that C_0 is nonempty. Since G is connected, there exist $x \in C_0$ and $y \in V(G) \setminus C_0$, adjacent. Since y has neighbours in $V(H)$, it follows from (1) that y belongs to some C_i. In particular, y has two nonadjacent neighbours in $V(H)$, say a, b. But then $\{y, a, b, x\}$ is a claw, a contradiction. This proves (2).

From (1) and (2), the sets $C_1, \ldots, C_{12}$ are pairwise disjoint and have union $V(G)$.

(3) *Each C_i is a clique, and for $1 \leq i < j \leq 12$, C_i, C_j are either complete or anticomplete to each other.*

The first statement follows from 4.3. For the second, let $1 \leq i < j \leq 12$, and let $u \in C_i$ and $v \in C_j$. We claim that v is adjacent to u if and only if $v_i \in N_j$. This is clear if either of u, v belongs to $V(H)$, so we assume that both belong to $V(G) \setminus V(H)$. Let $H' = G|(\{u\} \cup V(H) \setminus \{v_i\})$; then H' is isomorphic to H. For $1 \leq k \leq 12$, let $N'_k = N_k$ if $v_i \notin N_k$, and $N'_k = \{u\} \cup (N_k \setminus \{v_i\})$ otherwise. From (1) and (2) applied to H', the set of neighbours of v in $V(H')$ is one of $N'_1, \ldots, N'_{12}$, say N'_k. The set of neighbours of v in $V(H)$ is N_j, and since H and H' differ only by the vertices u, v_i, it follows that $N'_k \subseteq N_j \cup \{u\}$. But $N_k \subseteq N'_k \cup \{v_i\}$, and since $u \notin N_k$ we deduce that $N_k \subseteq N_j \cup \{v_i\}$. Consequently $j = k$, and the set of neighbours of v in $V(H')$ is N'_j, and so v is adjacent to u if and only if $u \in N'_j$, that is, if and only if $v_i \in N_j$. This proves our claim. Consequently whether u, v are adjacent depends only on i, j, and this proves (3).

By (3) every two members of each C_i are twins; and so we may assume that each C_i has at most one member, for otherwise the theorem holds. Since $v_i \in C_i$ for $1 \leq i \leq 11$, we deduce that either $G = H$ or G is isomorphic to the icosahedron, depending whether C_{12} has cardinality 0 or 1. This proves 4.4. ■

4.4 handles claw-free graphs that contain $icosa(-1)$; next we need to consider $icosa(-2)$.

4.5 *Let G be claw-free, with an induced subgraph isomorphic to $icosa(-2)$. Then either $G \in \mathcal{S}_1$, or G is decomposable.*

Proof. Since G has an induced subgraph isomorphic to $icosa(-2)$, we may choose ten disjoint nonempty cliques $A_1, B_1, C_1, A_2, B_2, C_2, D_1, D_2, E, F$ in G, satisfying:

- The following pairs are complete: for $i = 1, 2$, $A_iC_i, B_iC_i, A_iD_i, B_iD_i, C_iD_i, A_iE, D_iE, B_iF, D_iF$, also, A_1A_2, B_1B_2, EF.
- The pairs A_1B_1 and A_2B_2 are not complete (but not necessarily anticomplete).
- All remaining pairs are anticomplete.

Let us choose such a set of cliques with maximal union W say. Suppose first that $W = V(G)$. Then (A_1, B_1) is a homogeneous pair of cliques, nondominating since $C_2 \neq \emptyset$, and so by 3.3 we may assume $|A_1| = |B_1| = 1$, and similarly $|A_2| = B_2| = 1$. If one of the other six cliques has cardinality > 1, say X, then the members of X are twins and the theorem holds. If all ten cliques have cardinality 1 then G is isomorphic to $icosa(-2)$, as required. So we may assume that $W \neq V(G)$.

We may assume that G is connected, and so there exists $v \in V(G)\setminus W$ with at least one neighbour in W. Let N be the set of neighbours of v in W.

(1) *If N meets both C_1, C_2 then the theorem holds.*

For in that case, by 4.1 N is disjoint from E, F. Since A_1, B_1 are not complete, 4.1 (with A_1, B_1, C_2) implies that $A_1 \cup B_1 \not\subseteq N$; and so 4.2 (with A_1, D_1, F if $A_1 \not\subseteq N$) implies that $D_1 \cap N = \emptyset$. Similarly $D_2 \cap N = \emptyset$. Since A_1, B_1 are not complete, 4.2 (with A_1, B_1, C_1) implies that N meets at least one of A_1, B_1, say A_1. Then 4.2 (with D_1, A_1, A_2) implies $A_2 \subseteq N$, and by symmetry $A_1 \subseteq N$. Similarly, if B_1 meets N then $B_2 \subseteq N$, contrary to 4.1 (with A_2, B_2, C_1), and so $B_1 \cap N = \emptyset$, and by symmetry $B_2 \cap N = \emptyset$. Then G contains an induced subgraph isomorphic to $icosa(-1)$ (choose one vertex from each of the ten cliques, choosing neighbours of v from C_1, C_2, and such that for $i = 1, 2$ the representatives of A_i, B_i are nonadjacent; and take v as the eleventh vertex). But the the theorem holds by 4.4. This proves (1).

(2) *If N meets $C_1 \cup C_2$ then the theorem holds.*

For by (1) we may assume that N meets C_1 and is disjoint from C_2. Suppose first that N meets A_2. 4.2 (with A_1, A_2, C_2 and with E, A_2, C_2) implies that $A_1, E \subseteq N$. 4.1 (with A_2, C_1, F) implies that $N \cap F = \emptyset$. 4.2, applied in turn to the triples A_2, E, F; C_2, B_2, F; C_2, D_2, F; D_1, E, D_2; C_1, D_1, F implies that $A_2 \subseteq N$; $N \cap B_2 = \emptyset$; $N \cap D_2 = \emptyset$; $D_1 \subseteq N$, and $C_1 \subseteq N$. But then v can be added to A_1, contrary to the maximality of W. This proves that $N \cap A_2 = \emptyset$, and by symmetry $N \cap B_2 = \emptyset$. 4.2 (with A_2, D_2, B_2) implies that $N \cap D_2 = \emptyset$. 4.2 (with A_1, C_1, B_1) implies that N meets one of A_1, B_1. 4.2 (with D_1, A_1, A_2 if N meets A_1)implies that $D_1 \subseteq N$. Suppose first that N is disjoint from both E, F. Then 4.2 (with B_1, D_1, E and A_1, D_1, F) implies that $B_1, A_1 \subseteq N$, and 4.2 (with A_2, A_1, C_1) implies that $C_1 \subseteq N$. But then v can be added to C_1, contradicting the maximality of W. Hence N is not disjoint from both E, F, and from the symmetry we may assume it meets E. 4.2 (with A_2, E, F) implies that $F \subseteq N$, and from symmetry $E \subseteq N$. 4.2 (with A_1, E, D_2) implies $A_1 \subseteq N$, and by symmetry $B_1 \subseteq N$; and 4.2 (with C_1, A_1, A_2) implies that $C_1 \subseteq N$. Then v can be added to D_1, contrary to the maximality of W. This proves (2).

To finish the proof, we assume by (2) that N is disjoint from $C_1 \cup C_2$. Suppose that N meets A_1. 4.2 (with C_1, A_1, A_2 and A_1, A_2, C_2) implies that $A_2, A_1 \subseteq N$. 4.2 (with C_1, A_1, E) implies that $E \subseteq N$. Suppose in addition that N meets $B_1 \cup B_2$. Then from the symmetry, $B_1 \cup B_2 \cup F \subseteq N$; 4.1 (with A_1, B_1, D_2 and A_2, B_2, D_1) implies that N is disjoint from D_1, D_2, contrary to 4.2 (with D_1, E, D_2). So N is disjoint from B_1, B_2. 4.2 (with D_1, E, D_2) implies that N includes one of D_1, D_2, say D_1; 4.2 (with C_1, D_1, F) implies that $F \subseteq N$; 4.2 (with B_1, F, D_2) implies that $D_2 \subseteq N$; but then v can be added to E, contrary to the maximality of W. This proves that N is disjoint from A_1, and by symmetry from B_1, A_2, B_2. 4.2 (with A_1, D_1, B_1) implies that $N \cap D_1 = \emptyset$, and by symmetry $N \cap D_2 = \emptyset$; and then 4.2 (with D_1, E, D_2 and D_1, F, D_2) implies that N is disjoint from E, F. But then $N = \emptyset$, a contradiction. Thus there is no such vertex v. This proves 4.5. ■

Next we need to consider deleting two vertices (distance 2 apart) from the icosahedron. This is the smallest graph in $\mathcal{S}_2$; it is also a case of what we call an XX-configuration. Let G be a graph. An *XX-configuration in G* means an induced subgraph H, consisting of

- eight vertices $b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2$
- edges $a_ib_i, a_ic_i, b_ic_i, b_id_i, c_id_i, b_ib_3, c_ic_3, d_ib_3, d_ic_3$ for $i = 1, 2$, the edge d_1d_2, and possibly the edge a_1a_2.

4.6 *Let G be claw-free, and contain an XX-configuration. Then either $G \in \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

Proof. Let H be an XX-configuration in G, and let $a_1, a_2, \ldots$ be as in the definition of an XX-configuration. We may therefore choose eleven disjoint subsets $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2$, with the following properties:

- the ten sets $A_1, A_2, B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2$ are nonempty cliques
- the pairs $A_iB_i, A_iC_i, B_iC_i, B_iB_3, C_iC_3, B_iD_i, C_iD_i, B_3D_i, C_3D_i$ are complete for $i = 1, 2$, and all the other pairs of the eleven subsets named are anticomplete, with the exception of D_1D_2, A_1A_2, A_1A_3, A_2A_3
- $a_1 \in A_1$, $a_2 \in A_2$ and so on.

(To see this, take $A_1 = \{a_1\}, B_1 = \{b_1\}$ and so on, with $A_3 = \emptyset$.) Consequently we may choose these eleven sets with maximal union. Let J be the subgraph of G induced on their union.

(1) *Let $v \in V(G) \setminus V(J)$, and let N be the set of neighbours of v in $V(J)$. Then either*

- $N = A_1 \cup B_1 \cup C_1 \cup A_2 \cup B_2 \cup C_2$*, or*
- $N = B_1 \cup B_3 \cup D_1 \cup D_2 \cup C_3 \cup C_2$ *or* $C_1 \cup C_3 \cup D_1 \cup D_2 \cup B_3 \cup B_2$*, or*
- A_1 *is complete to* A_2 *and* $N = A_1 \cup A_2 \cup B_1 \cup B_2 \cup B_3$ *or* $A_1 \cup A_2 \cup C_1 \cup C_2 \cup C_3$.

For first assume that N meets both B_3 and C_3. By 4.1, N is disjoint from $A_1 \cup A_2 \cup A_3$. By 4.2 (with B_1, B_3, B_2), N includes one of B_1, B_2, and we may assume that it includes B_1 from the symmetry. By 4.1, $N \cap B_2 = \emptyset$. By 4.2 (with B_2, B_3, D_1), $D_1 \subseteq N$. By 4.2 (with A_1, B_1, B_3), $B_3 \subseteq N$. Suppose that $N \cap C_1$ is nonempty. By 4.1, $N \cap C_2 = \emptyset$; by 4.2 (with C_1, C_3, C_2), $C_1 \subseteq N$, and by 4.2 (with C_3, C_1, A_1), $C_3 \subseteq N$; but then v can be added to D_1, contrary to the maximality of $V(J)$. Thus $N \cap C_1 = \emptyset$. By 4.2 (with D_2, C_3, C_1), $D_2 \subseteq N$; by 4.2 (with C_1, C_3, C_2), $C_2 \subseteq N$; by 4.2 (with B_2, D_2, C_3), $C_3 \subseteq N$; and the second assertion of the claim holds.

So we may assume that N is disjoint from one of B_3 and C_3, say C_3. Next assume that N meets both D_1 and D_2. By 4.2 (with B_3, D_1, C_3), $B_3 \subseteq N$. By 4.2 (with B_1, D_1, C_3), $B_1 \subseteq N$, and similarly $B_2 \subseteq N$. By 4.1, N is disjoint from $A_1 \cup A_2 \cup A_3$. By 4.2 (with A_1, B_1, D_1), $D_1 \subseteq N$, and similarly $D_2 \subseteq N$. By 4.2 (with A_1, C_1, C_3), $N \cap C_1 = \emptyset$, and similarly $N \cap C_2 = \emptyset$. But then v can be added to B_3, contrary to the maximality of $V(J)$.

So we may assume that N is disjoint from both C_3 and D_2 say. We recall that $d_1 \in D_1$, and d_1 is adjacent to $d_2 \in D_2$. Suppose that $d_1 \in N$. By 4.2 (with B_1, d_1, C_3), $B_1 \subseteq N$. By 4.2 (with B_3, d_1, C_3), $B_3 \subseteq N$. By 4.2 (with C_1, d_1, d_2), $C_1 \in N$. By 4.2 (with A_1, C_1, C_3), $A_1 \subseteq N$. By 4.1, $N \cap (B_2 \cup C_2) = \emptyset$, and $N \cap (A_2 \cup A_3) = \emptyset$. By 4.2 (with B_2, B_3, D_1), $D_1 \subseteq N$. But then v can be added to B_1, contrary to the maximality of $V(J)$.

We may therefore assume that $d_1 \notin N$. Suppose next that $N \cap B_3$ is nonempty. By 4.2 (with d_1, B_3, B_2), $B_2 \subseteq N$, and similarly $B_1 \subseteq N$. By 4.2 (with d_1, B_1, A_1), $A_1 \subseteq N$, and similarly $A_2 \subseteq N$. By 4.1, A_1 is complete to A_2, and for the same reason, N is disjoint from $A_3 \cup C_1 \cup C_2 \cup D_1$. But then the final statement of (1) holds.

So we may assume that $N \cap B_3 = \emptyset$. By 4.2 (with B_3, D_1, C_3), $N \cap D_1 = \emptyset$, and so N is disjoint from all four of B_3, C_3, D_1, D_2. If N intersects none of B_1, B_2, C_1, C_2, then v can be added to A_3, contrary to the maximality of $V(J)$. So we may assume from the symmetry that N meets B_1. By 4.2 (with C_1, B_1, B_3), $C_1 \subseteq N$, and similarly $B_1 \subseteq N$; and by 4.2 (with B_3, B_1, A_1), $A_1 \subseteq N$. If N intersects either B_2 or C_2, then similarly it includes $A_2 \cup B_2 \cup C_2$, and therefore is disjoint from A_3 (by 4.1), and the first statement of the claim holds. So we may assume that N is disjoint from $B_2 \cup C_2$. But then v can be added to A_1, contrary to the maximality of $V(J)$. This proves (1).

By (1), any two vertices of B_1 are twins in G, and the same holds B_2, B_3, C_1, C_2, C_3, and so we may assume that these sets all have cardinality 1. Moreover, by (1) (D_1, D_2) is a homogeneous pair of cliques, nondominating since $A_1 \neq \emptyset$, and so by 3.3, we may assume that D_1, D_2 both have cardinality 1. If there is a vertex v satisfying the final statement of (1), then there is an induced subgraph isomorphic to $icosa(-1)$, and the claim follows from 4.4. Thus we may assume that no vertex satisfies the final statement of (1). Let U_0 be the set of all $v \in V(G) \setminus V(J)$ whose set of neighbours in $V(J)$ is $A_1 \cup A_2 \cup \{b_1, c_1, b_2, c_2\}$; and let U_1, U_2 be those with neighbours sets $\{b_1, b_3, d_1, d_2, c_3, c_2\}$ and $\{c_1, c_3, d_1, d_2, b_3, b_2\}$ respectively. Thus the sets U_0, U_1, U_2 are disjoint and have union $V(G) \setminus V(J)$. By 4.3, U_0, U_1, U_2 are cliques. If some $u_0 \in U_0$ is adjacent to some $u_1 \in U_1$, then $\{u_0, u_1, c_1, b_2\}$ is a claw, while if some $u_1 \in U_1$ is adjacent to some $u_2 \in U_2$, then $\{u_1, u_2, b_1, c_2\}$ is a claw, in either case a contradiction. Thus U_0, U_1, U_2 are anticomplete to each other. Hence for $i = 0, 1, 2$, every two vertices in U_i are twins, and so we may assume that $|U_i| \leq 1$. Now every vertex not in $A_1 \cup A_2 \cup A_3$ is either A_1-complete or A_1-anticomplete, and either A_2-complete or A_2-anticomplete, and A_3-anticomplete. Also, if $x, y \in V(G) \setminus A_1 \cup A_2 \cup A_3$ and x is $A_1 \cup A_2$-complete and y is $A_1 \cup A_2$-anticomplete, then $x \in U_0$, and either $y \in U_1 \cup U_2$ or y is one of b_3, c_3, d_1, d_2, and in either case x, y are not adjacent. If $A_3 \neq \emptyset$ then (A_1, A_3, A_2) is a breaker, and the theorem holds by 3.4, so may assume that $A_3 = \emptyset$. Consequently (A_1, A_2) is a homogeneous pair, nondominating since $D_1 \neq \emptyset$, and therefore by 3.3 we may assume that $A_i = \{a_i\}$ for $i = 1, 2$. But then $G \in \mathcal{S}_2$, and the theorem holds. This proves 4.6. ∎

5 The second line graph anomaly

Now we handle the second peculiarity that will turn up when we come to treat line graphs. Let H be an induced subgraph of G, with 11 vertices $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2$, and the following edges: for $i = 1, 2$, $\{a_i, b_i, c_i, d_i\}$ are cliques, and so are $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3\}$; every pair of a_3, b_3, c_3, d_1, d_2 are adjacent except the pair d_1, d_2; and possibly a_1, a_2 are adjacent. We call such a subgraph H a *YY-configuration.* We need to show the following.

5.1 *Let G be claw-free, and contain an YY-configuration. Then G is decomposable.*

Proof. Since there is a YY-configuration in G, we may choose nine cliques A^i_j $(1 \leq i, j \leq 3)$, with the following properties (for $1 \leq i \leq 3$, A^i denotes $A^i_1 \cup A^i_2 \cup A^i_3$, and A_i denotes $A^1_i \cup A^2_i \cup A^3_i$):

- these nine sets are nonempty and pairwise disjoint
- for $1 \le i, j, i', j' \le 3$, if $i \neq i'$ and $j \neq j'$ then A^i_j is anticomplete to $A^{i'}_{j'}$
- for $1 \le j \le 3$, A_j is a clique
- for $i = 1, 2$, A^i is a clique
- A^3_1 and A^3_2 are not complete to A^3_3
- for $1 \le j \le 3$, let S_j be the set of all vertices that are anticomplete to A_j and complete to the other two of A_1, A_2, A_3; then S_1 is not complete to S_2
- subject to these conditions, the union W of the sets A^i_j $(1 \le i, j \le 3)$ is maximal.

(To see this, take a YY-configuration, with vertices $a_1, a_2, \ldots$ as before, and let $A^1_j = \{b_j\}, A^2_j = \{c_j\}, A^3_j = \{a_j\}$ for $j = 1, 2, 3$; then d_1, d_2 belongs to S_2, S_1 respectively.) Let $Z = V(G) \setminus (W \cup S_1 \cup S_2 \cup S_3)$, and for $i = 1, 2$, let H_i be the set of vertices in A^3_i with no neighbour in A^3_3. Choose $s_1 \in S_1$ and $s_2 \in S_2$, nonadjacent.

(1) *Every vertex in $W \cup S_1 \cup S_2 \cup S_3$ with a neighbour in Z belongs to $H_1 \cup H_2$.*

For suppose that $z \in Z$, and let N be the set of neighbours of z. We will show that

$$N \cap (W \cup S_1 \cup S_2 \cup S_3) \subseteq H_1 \cup H_2.$$

Assume first that $s_1, s_2 \in N$. We claim that $A^1_3 \subseteq N$. For suppose not. 4.2 (with $A^1_3, S_2, A^2_1 \cup A^3_1$) implies that $A^2_1 \cup A^3_1 \subseteq N$, and similarly $A^2_2 \cup A^3_2 \subseteq N$. Since A^3_1 is not complete to A^3_3, 4.1 (with A^3_1, A^3_3, A^2_2) implies that $A^3_3 \not\subseteq N$. 4.2 (with A^1_j, A^3_j, A^3_3) implies that $A^1_j \subseteq N$ for $j = 1, 2$; and then three applications of 4.1 imply that $N \cap A_3 = \emptyset$. But then $z \in S_3$, a contradiction. This proves our claim that $A^1_3 \subseteq N$, and similarly $A^2_3 \subseteq N$. Suppose that $A^3_3 \not\subseteq N$. Then for $1 \le i, j \le 2$, 4.2 (with A^3_3, A^i_3, A^i_j) implies that $A^i_j \subseteq N$; and two applications of 4.1 imply that N is disjoint from A^3_1, A^3_3, contrary to 4.2 (with A^3_1, s_2, A^3_3). Thus $A^3_3 \subseteq N$. Since z cannot be added to A^3_3, N meets one of the sets A^i_j where $1 \le i, j \le 2$, and from the symmetry we may assume that $N \cap A^1_1 \neq \emptyset$. 4.1 implies that N is disjoint from A^2_2, A^3_2. If N meets A^1_2, then similarly N is disjoint from A^2_1, A^3_2, and 4.2 (with A^2_1, A^1_1, A^1_2) implies that $A^1_2 \subseteq N$, and similarly $A^1_1 \subseteq N$; but then z can be added to A^1_3, a contradiction. Thus $N \cap A^1_2 = \emptyset$. By 4.2 (with $A^1_2, A^1_1, A^2_1 \cup A^3_1$), $A^2_1 \cup A^3_1 \subseteq N$ and 4.2 (with A^1_1, A^2_1, A^2_2) implies that $A^1_1 \subseteq N$; but then $z \in S_2$, a contradiction. This completes the case when $s_1, s_2 \in N$.

Next assume that $s_1 \in N$ and $s_2 \notin N$. Suppose first that $A^1_2 \not\subseteq N$. 4.2 (with $A^1_2, s_1, A^2_3 \cup A^3_3$) implies that $A^2_3 \cup A^3_3 \subseteq N$; 4.2 (with s_2, A^1_3, A^1_2) implies that $N \cap A^1_3 = \emptyset$; 4.2 (with A^1_3, s_1, A^3_2) implies $A^3_2 \subseteq N$; 4.1 (with A^2_1, A^3_3, A^3_2) implies that $N \cap A^2_1 = \emptyset$; and this contradicts 4.2 (with A^2_1, A^2_3, A^1_3), This proves that $A^1_2 \subseteq N$. Similarly $A^2_2 \subseteq N$. If $A^3_3 \not\subseteq N$, then 4.2 (with A^3_2, A^i_2, A^i_j) implies that $A^i_j \subseteq N$, for $i = 1, 2$ and $j = 1, 3$; and then 4.1 implies that N is disjoint from both A^3_3, A^3_2, contrary to 4.2 (with A^3_3, s_1, A^3_2). Hence $A^3_2 \subseteq N$. Suppose that $N \cap (A^1_3 \cup A^2_3) = \emptyset$. Since z cannot be added to A^3_2, it follows that $N \cap (A^1_1 \cup A^2_1) \neq \emptyset$, and from the symmetry we may assume that $N \cap A^1_1 \neq \emptyset$. 4.2 (with $(A^2_1 \cup A^3_1), A^1_1, A^1_3$) implies that $A^2_1 \cup A^3_1 \subseteq N$, and similarly $A^1_1 \subseteq N$, and 4.1 implies that

$N \cap A_3^3 = \emptyset$; but then $z \in S_3$, a contradiction. Thus $N \cap (A_3^1 \cup A_3^2) \neq \emptyset$, and from the symmetry we may assume that $N \cap A_3^1 \neq \emptyset$. Suppose that $A_1^1 \not\subseteq N$. Then 4.2 (with $A_1^1, A_3^1, A_3^2 \cup A_3^3$) implies that $A_3^2 \cup A_3^3 \subseteq N$; three applications of 4.1 imply that $N \cap A_1 = \emptyset$; and 4.2 (with A_1^2, A_3^2, A_3^1) implies that $A_3^1 \subseteq N$. But then $z \in S_1$, a contradiction. This proves that $A_1^1 \subseteq N$. By 4.1, $N \cap A_j^i = \emptyset$ for $i = 2, 3$ and $j = 1, 3$; and 4.2 (with A_1^2, A_1^1, A_3^1) implies that $A_3^1 \subseteq N$. But then z can be added to A_2^1, a contradiction. This completes the case when $s_1 \in N$ and $s_2 \notin N$.

We deduce that $s_1 \notin N$, and similarly $s_2 \notin N$. 4.2 (with s_1, A_3, s_2) implies that $N \cap A_3 = \emptyset$. Suppose that $N \cap (A_1^1 \cup A_1^2) \neq \emptyset$. Then 4.2 (with A_3^1, A_1^1, A_1^2 and $A_3^2, A_1^2, A_1^1 \cup A_1^3$) implies that $A_1 \subseteq N$. Similarly if $N \cap (A_2^1 \cup A_2^2) \neq \emptyset$ then $A_2 \subseteq N$ and therefore $z \in S_3$, a contradiction; and so $N \cap (A_2^1 \cup A_2^2) = \emptyset$. But then z can be added to A_1^3, a contradiction. This proves that $N \cap (A_1^1 \cup A_1^2) = \emptyset$, and similarly $N \cap (A_2^1 \cup A_2^2) = \emptyset$. 4.2 (with $A_j^1, A_j^3 \setminus H_j, A_3^3$) implies that $N \cap A_j^3 \subseteq H_j$ for $j = 1, 2$. Consequently $N \cap W \subseteq H_1 \cup H_2$. But 4.2 (with A_1^1, S_2, A_3^2) implies that $N \cap S_2 = \emptyset$, and similarly $N \cap S_1 = N \cap S_3 = \emptyset$. This proves (1).

(2) *If $v \in V(G) \setminus H_1 \cup H_2 \cup Z$, then v has a neighbour in H_1 if and only if $v \in A_1 \cup S_2 \cup S_3$, and if so then v is complete to H_1. An analogous statement holds for H_2.*

For if $v \in A_1 \cup S_2 \cup S_3$ then v is complete to H_1, and if $v \in A_3 \cup S_1 \cup A_2^1 \cup A_2^2$ then v is anticomplete to H_1, so we may assume that $v \in A_2^3$. Let $a_2^2 \in A_2^2$. Since $v \notin H_2$, v has a neighbour $a_3^3 \in A_3^3$; and if v also has a neighbour $h_1 \in H_1$, then $\{v, h_1, a_2^2, a_3^3\}$ is a claw, a contradiction. Thus v is anticomplete to H_1. This proves (2).

We claim that there do not exist adjacent $x, y \in V(G) \setminus (H_1 \cup H_2 \cup Z))$ such that x is $H_1 \cup H_2$-complete and y is $H_1 \cup H_2$-anticomplete. For suppose that such x, y exist. By (2), $x \in S_3$, and $y \in A_3$; but then x, y are nonadjacent, a contradiction. If $Z \neq \emptyset$, then (H_1, Z, H_2) is a breaker, by (1) and (2), and the theorem holds by 3.4. We may therefore assume that $Z = \emptyset$. Now S_1, S_2, S_3 are cliques by 4.3, and so G is the hex-join of $G|W$ and $G|(S_1 \cup S_2 \cup S_3)$. This proves 5.1. ■

6 Line graphs

Our next goal is to prove that if G is claw-free and contains an induced subgraph which is a line graph $L(H)$ say, and H is sufficiently nondegenerate, then either G itself is a line graph or it is decomposable. We need to consider the possible neighbour sets in this line graph of the other vertices of G; and it is convenient to work in terms of H rather than in terms of $L(H)$. Thus, the neighbour set becomes a set of edges of H.

In this paper, a *separation* of G means a pair (A, B) of subsets of $V(G)$, such that $A \cup B = V(G)$ and every edge of G has both ends in one of A, B. A *k-separation* means a separation (A, B) such that $|A \cap B| \leq k$, and a separation (A, B) is *cyclic* if both $G|A, G|B$ contain cycles. We say that G is *cyclically 3-connected* if it is 2-connected and not a cycle, and there is no cyclic 2-separation. (For instance, we wish to consider the complete bipartite graph $K_{2,3}$ as cyclically 3-connected, but we wish to exclude the graph obtained from K_4 by deleting an edge. This differs slightly from the definition we used in [3].)

A *branch-vertex* of a graph H means a vertex with degree ≥ 3; and, if H is cyclically 3-connected, a *branch* of H means a maximal path B in H such that no internal vertex of B is a branch-vertex.

(The reason for insisting that G is cyclically 3-connected is because of our convention that all "paths" are induced subgraphs, and that is not our intention for branches; but no conflict arises when G is cyclically 3-connected.)

6.1 *Let H be a cyclically 3-connected graph with $|V(H)| \geq 7$, such that $|V(H) \setminus V(B)| \geq 4$ for every branch B of H. Let $X \subseteq E(H)$, satisfying the following:*

(Z1) *there do not exist three pairwise nonadjacent edges in X*

(Z2) *there do not exist distinct vertices t_1, t_2, t_3, t_4 of H, such that t_i is adjacent to t_{i+1} for $i = 1, 2, 3$, and the edge t_2t_3 belongs to X, and the other two edges t_1t_2, t_3t_4 do not belong to X.*

Then one of the following holds:

- *There is a subset $Y \subseteq V(H)$ with $|Y| \leq 2$ such that X is the set of all edges of H incident with a vertex in Y.*
- *There are vertices $s_1, s_2, s_3, t_1, t_2, t_3, u_1, u_2 \in V(H)$, all distinct except that possibly $t_1 = t_2$, such that the following pairs are adjacent in G: s_it_i, s_iu_1, s_iu_2 for $i = 1, 2, 3$, and s_1s_3. Moreover, X contains exactly six of these ten edges, the six not incident with s_1.*
- *There is a subgraph J of H isomorphic to a subdivision of K_4 (let its branch-vertices be $v_1, \ldots, v_4$, and let $B_{i,j}$ denote the branch between v_i, v_j); and $B_{2,3}, B_{3,4}, B_{2,4}$ all have length 1, $B_{1,2}, B_{1,3}$ have length 2, and $B_{1,4}$ has length ≥ 2. Moreover, the edges of J in X are precisely the five edges of $B_{1,2}, B_{1,3}$ and $B_{2,3}$.*

Proof. Since H is cyclically 3-connected, we have:

(1) *No vertex of H of degree 2 is in a triangle.*

(2) *If there is a vertex $y \in V(H)$ such that every edge in X is incident with y, then the theorem holds.*

For suppose y is such a vertex; let N be the set of neighbours v of y such that the edge $yv \in X$, and M the remaining neighbours of y. If $M = \emptyset$ or $N = \emptyset$ then the first statement of the theorem holds, so we assume that there exist $m \in M$ and $n \in N$. The only edge in X incident with n is ny, and by **(Z2)**, there is no edge in $E(H) \setminus X$ incident with n except possibly nm. Since n has degree ≥ 2, it follows that n has degree 2 and is in a triangle, contrary to (1). This proves (2).

(3) *If there exist two vertices y_1, y_2 of H such that every edge in X is incident with one of y_1, y_2, then the theorem holds.*

For let us choose y_1, y_2 with the given property, adjacent if possible. For $i = 1, 2$, let N_i be the set of all neighbours $v \in V(H) \setminus \{y_1, y_2\}$ of y_i such that the edge $y_iv \in X$, and let M_i be the other neighbours of y_i in $V(H) \setminus \{y_1, y_2\}$. If M_1, M_2 are both empty, then the first statement of the theorem holds, so we may assume that there exists $m_1 \in M_1$. By (2) we may assume that there exists $n_1 \in N_1$. Let a be any neighbour of n_1 different from y_1. If $an_1 \in X$ then $a = y_2$, since every edge in X is incident with one of y_1, y_2; and if $an_1 \notin X$ then $a = m_1$, by **(Z2)** applied to m_1-y_1-n_1-a.

In particular, if $n_1 \notin N_2$ then n_1 has degree 2 and belongs to a triangle, contrary to (1). It follows that $N_1 \subseteq N_2$. Suppose that $|M_1| > 1$. Then no vertex in N_1 has a neighbour in M_1, and therefore every vertex in N_1 has degree 2. Since H is cyclically 3-connected, it follows that $N_1 = \{n_1\}$; and so every edge in X is incident with one of n_1, y_2. From the choice of y_1, y_2 it follows that y_1, y_2 are adjacent, and so n_1 belongs to a triangle, contrary to (1). This proves that $M_1 = \{m_1\}$. Since H is cyclically 3-connected, every vertex in N_1 is adjacent to m_1 except possibly one. Moreover, $(N_1 \cup \{y_1, y_2, m_1\}, V(H) \setminus (N_1 \cup \{y_1\}))$ is a 2-separation of H, and so either $N_1 \cup \{y_1, y_2, m_1\} = V(H)$, or $H \setminus (N_1 \cup \{y_1\}))$ is a path of length > 1 between m_1, y_2. In the first case, it follows that $|N_1| \geq 4$ since $|V(H)| \geq 7$, and the second statement of the theorem holds. Thus we assume the second case applies. Let P be the path $H \setminus (N_1 \cup \{y_1\}))$. By hypothesis, at least 4 vertices of H do not belong to $V(P)$, and so $|N| \geq 3$. Let x be the neighbour of y_2 in P; then $x \neq m_1$. Choose $n_1' \in N_1$ adjacent to m_1; then from **(Z2)** applied to x-y_2-n_1'-m_1 we deduce that the edge xy_2 belongs to X. But then again the second statement of the theorem holds. This proves (3).

(4) *If there are three edges in X forming a cycle of length 3, then there is a fourth edge in X incident with a vertex of this cycle.*

For suppose that y_1, y_2, y_3 are vertices such that $y_1y_2, y_2y_3, y_3y_1 \in X$, and for $i = 1, 2, 3$ no other edge in X is incident with y_i. Since H is cyclically 3-connected and we may assume that $|V(H)| \geq 5$, it follows that there are two edges between $\{y_1, y_2, y_3\}$ and $V(H) \setminus \{y_1, y_2, y_3\}$, with no common end. But then both these edges belong to $E(H) \setminus X$, and **(Z2)** is violated. This proves (4).

(5) *There do not exist $Y \subseteq V(H)$ with $|Y| = 3$ and $y_4 \in V(H \setminus Y$, such that every two members of Y are joined by an edge in X, and every other edge in X is incident with y_4.*

For let $Y = \{y_1, y_2, y_3\}$, and suppose first that there is a matching of size 2 consisting of edges of $H \setminus \{y_4\}$, each with one end in Y and the other not in this set. These two edges therefore do not belong to X, and so **(Z2)** is violated. Thus there is no such matching. Consequently, there is a vertex y_5 such that every edge of H with one end in Y and the other not in this set is incident with one of y_4, y_5. It follows that $(Y \cup \{y_4, y_5\}, V(H) \setminus Y)$ is a 2-separation of H, and therefore $H \setminus Y$ is a path between y_4, y_5, contrary to the hypothesis. This proves (5).

In view of (3),(4),(5), **(Z1)** and (for instance) Tutte's theorem [4], it follows that there is a set $Y \subseteq V(H)$ with $|Y| = 5$ such that every edge in X has both ends in Y, and $H|(Y \setminus \{y\})$ has a 2-edge matching with both edges in X, for every vertex $y \in Y$. (We call this "criticality".) Criticality implies that among every three vertices in Y, some two are joined by an edge in X. Suppose that there is a 3-edge matching between $V(H) \setminus Y$ and Y. None of these three edges belongs to X, and so from **(Z2)** it follows that no two of y_1, y_2, y_3 are joined by an edge in X, contrary to criticality. We deduce that no such matching of size 3 exists. Consequently there is a set $Z \subseteq V(H)$ with $|Z| \leq 2$, such that every edge between Y and $V(H) \setminus Y$ is incident with a member of Z. By choosing Z with $Z \cup Y$ minimal, we deduce that every vertex in $Z \setminus Y$ has at least two neighbours in Y. Now $(Y \cup Z, (V(H) \setminus Y) \cup Z)$ is a 2-separation. Since $H|Y$ has a cycle, it follows that $H \setminus (Y \setminus Z)$ has no cycle; and consequently, either $Y \cup Z = V(H)$ (which implies that $|Z| = 2$, since $|V(H)| \geq 7$), or $|Z| = 2$ and $H \setminus (Y \setminus Z)$ is a path joining the two members of Z. Thus in either case, $|Z| = 2$.

Suppose first that $Y \cap Z = \emptyset$. From the choice of Z minimizing $Y \cup Z$, it follows that we can write $Z = \{z_1, z_2\}$ and $Y = \{y_1, \ldots, y_5\}$ such that z_1y_1, z_2y_2, z_2y_3 are edges. By criticality, some two of y_1, y_2, y_3 are joined by an edge in X. From **(Z2)**, this edge is not y_1y_2 or y_1y_3, so it must be y_2y_3; that is, y_2, y_3 are adjacent and X contains the edge joining them. Consequently, by **(Z2)**, z_1, y_1 are both nonadjacent to both of y_2, y_3. Since z_1 has at least two neighbours in Y, we may assume that z_1 is adjacent to y_4; and so, by the symmetry between y_1, y_4 we deduce that y_4 is nonadjacent to y_2, y_3, and exchanging z_1, z_2 implies that $y_1y_4 \in X$, and z_2 is nonadjacent to y_1, y_4. Then $(\{z_1, y_1, y_4, y_5\}, V(H) \setminus \{z_1, y_5\})$ is a cyclic 2-separation of H, a contradiction.

So $Y \cap Z$ is nonempty, and in particular $Y \cup Z \neq V(H)$, since $|V(H)| \geq 7$. Consequently $H \setminus (Y \setminus Z)$ is a path P say, joining the two vertices in Z. Let $Z = \{z_1, z_2\}$ and $Y = \{y_1, \ldots, y_5\}$. Suppose first that $Z \not\subseteq Y$; then we may assume that $z_2 = y_4$ (since we have shown that $Y \cap Z$ is nonempty), and z_1 is adjacent to y_1, y_2, and P has length ≥ 2. By criticality, some two of y_1, y_2, y_4 are joined by an edge in X, and by **(Z2)** it must be y_1y_2; and therefore, by **(Z2)** again, y_4 is nonadjacent to y_1, y_2. Consequently, by criticality, y_4 is adjacent to y_3, y_5, and the edges $y_3y_4, y_4y_5 \in X$. Thus z_1 is nonadjacent to y_3, y_5. Since H is cyclically 3-connected, we may assume that y_2y_3, y_1y_5 are edges; and **(Z2)** implies they are both in X. Thus all edges of the cycle y_1-y_2-y_3-y_4-y_5-y_1 belong to X. But then the third statement of the theorem holds.

Finally, we may assume that $Z \subseteq Y$; but then $|V(H) \setminus V(P)| = 3$, contrary to the hypothesis. This proves 6.1. ■

We need a small lemma for the next proof.

6.2 *Let H be cyclically 3-connected, and let B be a branch of H. Let $Y \subseteq V(B)$ with $|Y| \leq 2$, such that if $|Y| = 1$ then the member of Y is an internal vertex of B. Let e be an edge of H not in $E(B)$ and not incident with any vertex in Y. There is no $Z \subseteq V(H)$ with $|Z| \leq 2$ such that for every edge $f \in E(H)$, f has an end in Z if and only if either $f = e$ or f has an end in Y.*

Proof. Suppose Z is such a subset, and let N be the set of edges of H with an end in Y. Since $N \cup \{f\}$ is the set of edges with an end in Z, it follows that $N \neq \emptyset$, and therefore $Y \neq \emptyset$. Since $Y \subseteq V(B)$, it follows that $N \cap E(B) \neq \emptyset$, and therefore $Z \cap V(B) \neq \emptyset$. Let $z \in Z$ be incident with e. Since $e \notin E(B)$, z does not belong to the interior of B, and therefore is incident with an edge $e' \neq e$ and not in B. Hence $e' \in N$, and therefore is incident with a member of Y, say y; and consequently y is an end of B. There is an edge $e'' \neq e'$ incident with y and not in B, and since $e'' \in N$, it follows that $y \in Z$. But $y \neq z$ since e is not incident with any member of Y; and so $Z = \{y, z\}$, and $z \notin V(B)$ since H is cyclically 3-connected. Since y is an end of B, by hypothesis there is a second member $y' \in Y$. There is an edge incident with y' and not incident with y or z, a contradiction. This proves 6.2. ■

Let us say H is a *theta* if it is cyclically 3-connected and has exactly two branch-vertices and three branches. A subset $X \subseteq V(G)$ is *connected* if $X \neq \emptyset$ and $G|X$ is connected; and a *component* of G is a maximal connected subset of $V(G)$. The previous results of this section are combined with 4.6 and 5.1 to prove the following.

6.3 *Let H be a cyclically 3-connected graph with $|V(H)| \geq 7$, such that there is no branch B of H with $|V(H) \setminus V(B)| \leq 3$. Let G be a claw-free graph, with an induced subgraph isomorphic to $L(H)$. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

Proof. We may choose H with $|V(H)|$ maximum satisfying the hypotheses of the theorem (we call this the "maximality" of H), and to simplify notation we assume that the line graph of H is an induced subgraph of G (rather than just isomorphic to one). In particular, $E(H) \subseteq V(G)$. For each $h \in V(H)$, let $D(h)$ denote the set of edges of H incident with h in H. For each $v \in V(G) \setminus E(H)$, let $N(v)$ be the set of members of $E(H)$ adjacent to v in G. We begin with:

(1) *For each $v \in V(G) \setminus E(H)$, we may assume that there exists $Y \subseteq V(H)$ with $|Y| \leq 2$ such that $N(v) = \bigcup(D(y) : y \in Y)$, and there is a branch of H including Y.*

For $N(v) \subseteq E(H)$, and satisfies the hypotheses of 6.1, by 4.1 and 4.2. Thus one of the three conclusions of 6.1 holds. If the second holds, then G contains a YY-configuration, and so by 5.1, we deduce that G is decomposable, and the theorem holds. If the third holds, then G contains an XX-configuration (take the edges of the subgraph described in 6.1, except for those in the interior of branches, together with the vertex v), and by 4.6, either G is decomposable, or it belongs to $\mathcal{S}_1 \cup \mathcal{S}_2$. Thus we may assume that the first outcome holds. Choose $Y \subseteq V(H)$ with $|Y| \leq 2$ such that $N(v) = \bigcup(D(y) : y \in Y)$. If $|Y| \leq 1$, or $|Y| = 2$ and some branch of H contains both members of Y, then (1) holds, so we assume that $Y = \{h_1, h_2\}$ say, and no branch of H contains both h_1, h_2. Let H' be the graph obtained from H by adding the edge v incident with both h_1, h_2. Then H' is cyclically 3-connected (since h_1, h_2 do not belong to the same branch of H), and no branch of H' contains all its vertices except at most three, and yet $L(H')$ is an induced subgraph of G, a contradiction to the maximality of H. This proves (1).

For each $v \in V(G) \setminus E(H)$, let $Y(v) \subseteq V(H)$ be the set Y described in (1). For each $v \in E(H)$, let $Y(v)$ be the set consisting of the two vertices of H incident with v in H. Make the following definitions:

- For each branch-vertex t of H let $M(t) = \{v \in V(G) : Y(v) = \{t\}\}$.
- For each branch B with ends t_1, t_2 say, let $M(B) = \{v \in V(G) : Y(v) = \{t_1, t_2\}\}$.
- For each branch B and each end t of B, let

$$M(t, B) = \{v \in V(G) : Y(v) = \{t, h\} \text{ for some } h \text{ in the interior of } B\}.$$

- For each branch B with ends t_1, t_2 say, let

$$S(B) = \{v \in V(G) : \emptyset \neq Y(v) \subseteq V(B) \setminus \{t_1, t_2\}\}.$$

- Let $Z = \{v \in V(G) : Y(v) = \emptyset\}$.

From (1), we see that all these sets are pairwise disjoint (unless H is a theta, in which case all the sets $M(B)$ are equal), and have union $V(G)$.

(2) *Let B be a branch of H with ends t_1, t_2, let $v \in M(B)$, and let $u \in V(G)$ be adjacent to v. Then either:*

- *$u \in M(t_1) \cup M(t_2)$, or*

- *$u \in M(t_i, B') \cup M(B')$ for some $i \in \{1,2\}$ and some branch B' incident with t_i.*

For $Y(v) = \{t_1, t_2\}$. If $Y(u)$ contains one of t_1, t_2 then the theorem holds, so we assume not. For $i = 1, 2$, let e_i be an edge of H incident with t_i, not in B, such that e_1, e_2 have no common end. In G, v is adjacent to both e_1, e_2, and since $\{v, e_1, e_2, u\}$ is not a claw in G, it follows that u is adjacent in G to one of e_1, e_2. Consequently $Y(u)$ is nonempty, and contains a vertex not in B but adjacent to one of t_1, t_2.

Suppose that $|Y(u)| = 1$, say $Y(u) = \{t_3\}$. Since $t_1, t_2 \notin Y(u)$, it follows that $t_3 \neq t_1, t_2$. Let $B_1 \neq B$ be a branch incident with t_1 and with $t_3 \notin V(B_1)$, with ends t_1, t_4 say. Let e_1 be the edge of B_1 incident with t_1, and let e_2 be any edge incident with t_2. Since $\{v, e_1, e_2, u\}$ is not a claw of G, we deduce that for every choice of e_2, either e_2 is incident with t_3 or e_2 shares an end with e_1. In particular, choosing e_2 from B tells us that t_1, t_2 are adjacent, and so H is not a theta, and therefore $t_4 \neq t_2$. Also, the pairs t_1t_2, t_2t_3, t_1t_4, t_2t_4 are adjacent; and t_2 has degree 3 in H. By exchanging t_1, t_2 we deduce also that t_1 has degree 3 and t_1, t_3 are adjacent. Consequently H is a subdivision of K_4, and there is a branch of H with ends t_3, t_4. There are only two vertices of H not in this branch, contrary to hypothesis.

This proves that $|Y(u)| = 2$, say $Y(u) = \{s_1, s_2\}$. Let B' be a branch with $Y(u) \subseteq V(B')$. Since we have already seen that one of s_1, s_2 does not belong to B, it follows that $B' \neq B$. Suppose that B, B' share an end, say t_1, and let t_3 be the other end of B'. There is an edge e_1 of H incident with t_1, that belongs to neither of B, B'. Let e_2 be any edge incident with t_2; for each such choice, $\{v, u, e_1, e_2\}$ is not a claw in G. By choosing e_2 from B we deduce that t_1, t_2 are adjacent and therefore H is not a theta. It follows that for all choices of e_2, either e_2 has an end in $Y(u)$ (which, since H is not a theta, implies that e_2 is incident with t_3 and $t_3 \in Y(u)$), or e_2 shares an end with e_1. There is at most one choice for which the first occurs, and two for which the second occurs; and since t_2 has degree ≥ 3, we have equality throughout. More precisely, t_2 has degree 3, $t_3 \in Y(u)$, and the pairs t_1t_2, t_2t_3, t_2t_4 are adjacent, where e_1 has ends t_1, t_4. Moreover, no other choice of e_1 is possible, and so t_1 also has degree 3. Consequently H is a subdivision of K_4, and there is a branch P between t_3, t_4. By hypothesis, at least four vertices of H do not belong to P, and so B' has length ≥ 3. Let f_1 be an edge of B' incident with a vertex in $Y(u)$ but not incident with either of t_1, t_3 (this exists since B' has length ≥ 3 and one of its internal vertices is in $Y(u)$). Let f_2 be the edge of P incident with t_3. Then $\{u, v, f_1, f_2\}$ is a claw in G, a contradiction.

This proves that B, B' do not share an end, and so H is not a theta. We have already seen that one of s_1, s_2 is adjacent to one of t_1, t_2, say s_1, t_1 are adjacent. Consequently s_1 is an end of B'. Suppose that s_2 belongs to the interior of B'. Let e_1 be an edge incident with t_1, not in B and not incident with s_1; and let e_2 be any edge incident with t_2. Since $\{v, u, e_1, e_2\}$ is not a claw in G, it follows that for all choices of e_2, either e_2 is adjacent to s_1 or to an end of e_1. Consequently t_2 has degree 3, and t_2 is adjacent to s_1 and to both ends of e_1. Since this also holds for all choices of e_1, we deduce that t_1 also has degree 3. Let e_1 have ends t_1, t_3 say. Since H is cyclically 3-connected, it follows H is a subdivision of K_4 and t_3 is an end of B'. But then only two vertices of H do not belong to the branch B', contrary to hypothesis.

This proves that s_1, s_2 are both ends of B', and so $u \in M(B')$. Thus there is symmetry between u, v. Suppose that B has length 1, and let q be the edge of H incident with t_1, t_2. Let H' be the graph obtained from H by deleting q and adding a new edge v with the same ends t_1, t_2 as q. Then H' is isomorphic to H, and $L(H')$ is an induced subgraph of G, and so by (1) we may assume that there is a set $Y \subseteq V(H')$ with $|Y| \leq 2$ such that an edge of H' is adjacent to u in G if and only if it is incident in H' with a member of Y. But the edges of H' adjacent to u in G are precisely those with an end in $\{s_1, s_2\}$, together with the new edge v, and this contradicts 6.2. We may therefore assume that B has length > 1, and by symmetry we may assume the same for B'.

Let e_1 be the edge of B incident with t_1, and let e_2 be any edge of H incident with t_2. Since $\{v, u, e_1, e_2\}$ is not a claw in G, it follows that for all choices of e_2, either e_2 is incident in H with one of s_1, s_2, or it shares an end with e_1. Consequently t_2 has degree 3, and t_2 is adjacent to both s_1, s_2, and B has length 2. Similarly t_1, s_1, s_2 have degree 3, and B' has length 2, and s_1, s_2 are adjacent to both of t_1, t_2. But then $|V(H)| = 6$, a contradiction. This proves (2).

(3) *Let p_1-$\cdots$-p_k be a path of G such that $k \geq 2$, $p_1, p_k \notin Z$, and $p_2, \ldots, p_{k-1} \in Z$. Then either*

- *There is a branch B of H with ends t_1, t_2 say, such that p_1, p_k both belong to*

$$M(t_1) \cup M(t_2) \cup M(t_1, B) \cup M(t_2, B) \cup S(B),$$

or

- *$k = 2$, and there are two branches B_1, B_2 with a common end t (possibly equal), such that $p_1 \in M(t) \cup M(t, B_1) \cup M(B_1)$ and $p_2 \in M(t) \cup M(t, B_2) \cup M(B_2)$.*

For suppose first that $p_1 \in M(B)$ for some branch B. By (2), $k = 2$ and the second statement of the claim holds. So we may assume that p_1 does not belong to any $M(B)$, and the same for p_k. Since $p_1 \notin Z$, it follows that either $Y(p_1) = \{t_1\}$ for some branch-vertex t_1 of H, or there is a branch B_1 of H such that $Y(p_1) \subseteq V(B_1)$ and some internal vertex of B_1 belongs to $Y(p_1)$. Analogous statements hold for p_k. Suppose that $|Y(p_1)| = 1$ and $|Y(p_k)| = 1$, say $Y(p_1) = \{y_1\}$ and $Y(p_k) = \{y_2\}$. The graph $G|(E(H) \cup \{p_1, \ldots, p_k\})$ is the line graph of the graph H', obtained from H by adding a new branch between y_1, y_2 with edges $p_1, \ldots, p_k$. If y_1, y_2 do not belong to the same branch of H, it follows that H' is cyclically 3-connected, contrary to the maximality of H. If y_1, y_2 belong to the same branch of H then the first statement of the claim holds.

Thus we may assume that at least one of $|Y(p_1)|, |Y(p_k)| = 2$, say $|Y(p_1)| = 2$. Then $N(p_1)$ is not a clique, and since p_2 is adjacent to p_1 and G contains no claw, it follows that p_2 has a neighbour in $N(p_1)$, and in particular $p_2 \notin Z$. Thus $k = 2$.

Since $|Y(p_1)| = 2$, it follows that for some branch B_1 of H, $Y(p_1) \subseteq V(B_1)$ and some internal vertex of B_1 belongs to $Y(p_1)$. Let $Y(p_1) = \{y, y'\}$ say, where y' belongs to the interior of B_1. Next suppose that $|Y(p_2)| = 1$, say $Y(p_2) = \{z\}$. We may assume that $z \notin V(B_1)$, for otherwise the first statement of the claim holds. Let e' be an edge of B_1 incident with y' and not with y. Let e be an edge of H incident with y, not incident with z, and with no common end with e'. (This exists, since if y is an end of B_1 there are at least two edges incident with y and disjoint from e', and at most one of them is incident with z.) But then $\{p_1, p_2, e, e'\}$ is a claw in G, a contradiction. This proves that $|Y(p_2)| = 2$. Let $Y(p_2) = \{z, z'\}$ say, and let B_2 be a branch of H with $z, z' \in V(B_2)$ and with z' in the interior of B_2. We may assume that $B_2 \neq B_1$, for otherwise the first statement of the claim holds.

Suppose that $Y(p_1) \cap Y(p_2) \neq \emptyset$. It follows that $y = z$ is a common end of B_1, B_2. But then $p_1 \in M(y, B_1)$ and $p_2 \in M(y, B_2)$, and the second statement of the claim holds. We assume therefore that $Y(p_1) \cap Y(p_2) = \emptyset$.

If $p_2 \in E(H)$, then its ends in H are z, z', and therefore it has no end in $Y(p_1)$, a contradiction since p_1, p_2 are adjacent in G. Thus $p_2 \notin E(H)$, and similarly $p_1 \notin E(H)$. Next suppose that z, z'

are adjacent in H. Let q be the edge of B_2 joining them. Since $Y(p_1) \cap Y(p_2) = \emptyset$, it follows that q is not adjacent to p_1 in G. Let H' be the graph obtained from H by deleting q and replacing it by an edge p_2, joining the same two vertices z, z'. Hence $L(H')$ is also an induced subgraph of G, namely the subgraph induced on $(V(H) \setminus \{q\}) \cup \{p_2\}$. Since H' is isomorphic to H, it follows from (1) applied to H' that we may assume that there is a subset $Y \subseteq V(H')$ such that the set of members of $E(H')$ adjacent in G to p_1 equals the set of edges of H' with an end in Y. Now the set of members of $E(H')$ adjacent in G to p_1 equals $N(p_1) \cup \{p_2\}$, since q is not adjacent to p_1 in G. Moreover, $N(p_1)$ is the set of edges of H with an end in $Y(p_1)$, and since q has no end in $Y(p_1)$, this is equal to the set of edges of H' with an end in $Y(p_1)$. Consequently, the set of edges of H' with an end in Y equals the union of $\{p_2\}$ and the set of edges of H' with an end in $Y(p_1)$. But this is impossible, by 6.2. This proves that z, z' are nonadjacent, and similarly y, y' are nonadjacent.

Since y, y' are nonadjacent vertices of B_1, there are edges e, e' of B_1 incident with y, y' respectively, such that e, e' have no end in common. Since $\{p_1, p_2, e, e'\}$ is not a claw in G, it follows that p_2 is adjacent in G to one of e, e', and so some vertex of $Y(p_2)$ belongs to $V(B_1)$. Since z' is an internal vertex of B_2, we deduce that B_1, B_2 have a common end z. Similarly their common end is y, and so $y = z$, contradicting that $Y(p_1) \cap Y(p_2) = \emptyset$. This proves (3).

(4) *Let $t \in V(H)$ be a branch-vertex. If $v_1, v_2 \in V(G)$ are distinct and nonadjacent, and $t \in Y(v_1) \cap Y(v_2)$, then there are distinct branches B_1, B_2, both of length ≥ 2, with $v_i \in M(B_i)$ $(i = 1, 2)$; and every vertex of $V(H)$ adjacent to t in H either belongs to one of B_1, B_2, or has degree 3 in H and is adjacent to all the ends of B_1, B_2.*

For suppose v_1, v_2 are not adjacent. It follows that $v_1, v_2 \notin E(H)$. By (1), there are branches B_1, B_2 of H, incident with t, such that $Y(v_i) \subseteq V(B_i)$ $(i = 1, 2)$. (If H is a theta, and some $Y(v_i)$ consists of the two branch-vertices, then we can choose any branch to be B_i; in this case, choose a shortest branch.) Let B_i have ends t, t_i $(i = 1, 2)$ say. Let x be a neighbour of t, not in $V(B_1) \cup V(B_2)$. Let $y \neq t$ be a second neighbour of x. Let e, f be the edges tx, xy. Since $\{e, f, v_1, v_2\}$ is not a claw in G, it follows that f is adjacent in G to at least one of v_1, v_2; that is, $y \in Y(v_1) \cup Y(v_2)$. Since $Y(v_i) \subseteq V(B_i)$ $(i = 1, 2)$, we deduce that for some $i \in \{1, 2\}$, $y = t_i \in Y(v_i)$. If H is a theta, then x is the internal vertex of some branch of length 2; and since $v_i \in M(B_i)$, from the choice of B_i it follows that B_i has length ≤ 2. But then a branch of H contains all its vertices except two, contrary to the hypothesis. Thus, H is not a theta. Since no two branches have the same pair of ends, it follows that x is a branch-vertex; and since this holds for all choices of y, we deduce that x has degree 3 and is adjacent to both t_1, t_2, and $t_i \in Y(v_i)$ $(i = 1, 2)$. Moreover, B_1, B_2 are distinct. Suppose that say B_2 has length 1, and let q be the edge tt_2. Let H' be obtained from H by deleting q and adding a new edge v_2 incident with the same two vertices t, t_2. Then H' is isomorphic to H, and $L(H') = G|((E(H) \setminus \{q\}) \cup \{v_2\})$, and so by (1) we may assume that there exists $Y \subseteq V(H') = V(H)$ with $|Y| \leq 2$, such that the set of edges of H' with an end in Y equals the set of edges of H' that are adjacent to v_1 in G. But in the triangle $\{x, t, t_2\}$ of H', exactly one of its edges is adjacent to v_1 in G, a contradiction. This proves that B_2, and similarly B_1, has length ≥ 2, and so proves (4).

(5) *If B is a branch of H of length 1, with ends t_1, t_2, then $M(t_1)$ is anticomplete to $M(t_2)$.*

If there exists $v_1 \in M(t_1)$ adjacent to some $v_2 \in M(t_2)$, let H' be the graph obtained from H

by deleting the edge between t_1, t_2, and adding a two-edge path between these vertices, with edges v_1, v_2 (with v_i incident with t_i for $i = 1, 2$, and the middle vertex of this path being a new vertex). Then H' satisfies the hypotheses of the theorem, and $L(H')$ is an induced subgraph of G, contrary to the maximality of H. This proves (5).

For each branch B of H with ends t_1, t_2, we define $C(B), A(t_1, B), A(t_2, B)$ as follows. Let $C(B)$ be the union of $S(B)$ and the set of all $v \in Z$ such that there is a path with interior in Z from v to some vertex in $S(B)$. (Thus if B has length 1 then $C(B)$ is empty.) Let $A(t_1, B)$ be the set of all $v \in M(t_1) \cup M(t_1, B)$ with a neighbour in $C(B)$. Define $A(t_2, B)$ similarly.

(6) *For every branch B with ends t_1, t_2, every vertex in $V(G) \setminus C(B)$ with a neighbour in $C(B)$ belongs to $A(t_1, B) \cup A(t_2, B)$.*

For let $v \in V(G) \setminus C(B)$, with a neighbour in $C(B)$. From the definition of $C(B)$, $v \notin S(B) \cup Z$. Let P be a minimal path of G between $S(B)$ and v with interior in Z. By (3),

$$v \in M(t_1) \cup M(t_1, B) \cup M(t_2) \cup M(t_2, B).$$

Hence $v \in A(t_1, B) \cup A(t_2, B)$. This proves (6).

(7) *Let B be a branch with ends t_1, t_2. If $v \in V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ has a neighbour in $A(t_1, B)$, then there is a branch B' of H incident with t_1 such that $v \in M(t_1) \cup M(B') \cup M(t_1, B')$. In particular, v is either complete or anticomplete to $A(t_1, B)$.*

The second claim follows from the first and (4). To prove the first, let $v \in V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$, and assume it has a neighbour in $A(t_1, B)$. Since $A(t_1, B)$ is nonempty, it follows that t_1, t_2 are nonadjacent in H. If $t_1 \in Y(v)$, then the claim holds, so we may assume that $t_1 \notin Y(v)$. Suppose first that v is adjacent in G to every $e \in D(t_1)$ that is not in B. Since $t_1 \notin Y(v)$, it follows that $Y(v)$ contains all vertices of H that are adjacent to t_1 and not in $V(B)$. There are at least two such vertices, and $|Y(v)| \leq 2$, and so t_1 has degree 3, and its two neighbours not in B are both in $Y(v)$. By (1), there is a branch B' joining these two vertices, and $v \in M(B')$, contrary to (2). Thus there is an edge e of H not in B, such that no end of e belongs to $Y(v)$. Now v has a neighbour $a \in A(t_1, B)$. By definition of $A(t_1, B)$, a has a neighbour $c \in C(B)$. Now a is adjacent in G to v, e, c, and v, e are nonadjacent. Moreover, $v, e \notin A(t_1, B) \cup A(t_2, B) \cup C(B)$, and since $c \in C(B)$, it follows from (6) that c is nonadjacent to v, e. But then $\{a, v, e, c\}$ is a claw in G, a contradiction. This proves (7).

(8) *If there is a branch B of H with $S(B)$ nonempty, then G is decomposable, so we may assume there is no such branch (and consequently every branch has length at most 2). In particular, H is not a theta.*

For suppose that B is a branch with $S(B)$ nonempty. Let its ends be t_1, t_2. Since $S(B)$ is nonempty, it follows that B has length ≥ 2. We claim that $(A(t_1, B), C(B), A(t_2, B))$ is a breaker. To show this, in view of (6) and (7) it remains to check that:

- $A(t_1, B), A(t_2, B)$ are nonempty cliques

- there is a vertex in $V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ with a neighbour in $A(t_1, B)$ and a nonneighbour in $A(t_2, B)$; there is a vertex in $V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ with a neighbour in $A(t_2, B)$ and a nonneighbour in $A(t_1, B)$; and there is a vertex in $V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ with a nonneighbour in $A(t_2, B)$ and a nonneighbour in $A(t_1, B)$
- if $A(t_1, B)$ is complete to $A(t_2, B)$, then there do not exist adjacent $x, y \in V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$ such that x is $A(t_1, B) \cup A(t_2, B)$-complete and y is $A(t_1, B) \cup A(t_2, B)$-anticomplete.

Since B has length > 1, and $S(B) \neq \emptyset$, it follows that $M(t_1, B)$ is nonempty and is a subset of $A(t_1, B)$, and in particular, $A(t_1, B) \neq \emptyset$, and similarly $A(t_2, B) \neq \emptyset$. By (4), $A(t_1, B), A(t_2, B)$ are cliques, and so the first statement holds. For the second, let $e \in E(H) \setminus E(B)$ be incident with t_1; then e has a neighbour in $A(t_1, B)$ and a nonneighbour in $A(t_2, B)$, namely the first and last edges of B. Moreover, since H is cyclically 3-connected and at least four vertices of H do not belong to B, it follows that some edge f of H has no end in $V(B)$, and therefore is nonadjacent in G to both the first and last edges of B. The second claim follows. Thus, it remains to check the third.

Suppose then that $x, y \in V(G) \setminus (A(t_1, B) \cup A(t_2, B) \cup C(B))$; x is $A(t_1, B) \cup A(t_2, B)$-complete and y is $A(t_1, B) \cup A(t_2, B)$-anticomplete, and x, y are adjacent. By (7), $x \in M(B)$. Since x, y are adjacent, (2) implies that we may assume that $y \in M(t_1) \cup M(B') \cup M(t_1, B')$ for some branch B' incident with t_1. But then y is complete to $A(t_1, B)$, by (4). Since $A(t_1, B)$ is nonempty, it is not also anticomplete to $A(t_1, B)$, a contradiction. Consequently $(A(t_1, B), C(B), A(t_2, B))$ is a breaker. By 3.4, G is decomposable. This proves (8).

(9) *We may assume that $Z = \emptyset$.*

For suppose not, and let W be a component of $G|Z$. Since we may assume that G is connected, there are vertices not in W with neighbours in W; let X be the set of all such vertices. Thus, for each $x \in X$, $x \notin E(H)$ (since it has a neighbour in Z) and $Y(x)$ is nonempty (since W is a component of $G|Z$). Moreover, the set of neighbours of x in $E(H)$ is a clique, since G contains no claw; and consequently $|Y(x)| = 1$, say $Y(x) = \{t\}$. If t belongs to the interior of a branch B then $x \in S(B)$, contrary to (8); and so t is a branch-vertex. Suppose that there exists $x_1, x_2 \in X$ with $Y(x_i) = \{t_i\}$ $(i = 1, 2)$, where $t_1 \neq t_2$. There is a path P between x_1, x_2 with interior in W; and by (3) applied to this path, there is a branch B with ends t_1, t_2. By (8), B has length ≤ 2. Let H' be obtained from H by deleting the edges and interior vertices of B, and adding the members of $V(P)$ to H as the edges of a new branch B' between t_1, t_2, in the appropriate order. Then $L(H')$ is an induced subgraph of G, and satisfies the hypotheses of the theorem, and so by the maximality of H, we deduce that B' has length at most that of B. In particular, B' has length at most 2, and so $|V(P)| \leq 2$. But $x_1, x_2 \in V(P)$, and so x_1, x_2 are adjacent; and moreover, B has length 2. Now we recall that x_1 has a neighbour w say in W. Since $\{x_1, w, x_2, e\}$ is not a claw in G (where e is some edge of H incident with t_1 and not with t_2), it follows that x_2 is adjacent to w. Thus x_1, x_2 are the only edges of H' that are adjacent to w in G. We deduce that when H is replaced by H', and Y' denotes the function analogous to Y for H', then $Y'(w)$ contains the middle vertex of B'. But then by (8), G is decomposable. Consequently we may assume that there is no such x_2; and so there is a branch-vertex t of H such that $Y(x) = \{t\}$ for all $x \in X$. By 4.3, X is a clique. By (3) and (4), every vertex of G not in $W \cup X$ is either complete or anticomplete to X. But then the result follows from 3.2. This proves (9).

(10) *We may assume that for every branch B with ends t_1, t_2, if $v_i \in M(t_i) \cup M(t_i, B)$ for $i = 1, 2$, and v_1, v_2 are adjacent, then B has length 2 and v_1, v_2 are its two edges.*

For let F_1 be the set of vertices in $M(t_1) \cup M(t_1, B)$ with a neighbour in $M(t_2) \cup M(t_2, B)$, and define F_2 similarly. By (4), F_1, F_2 are cliques. We claim that every vertex $v \notin F_1 \cup F_2$ is either complete or anticomplete to F_i, for $i = 1, 2$. For let v have a neighbour $f_1 \in F_1$ say. We may assume that $t_1 \notin Y(v)$, for otherwise v is complete to F_1, by (4). By (3) and (9), there is a branch B' with ends t_1, t_3 say, such that $v \in M(t_3) \cup M(t_3, B')$, and in particular, $t_3 \in Y(v) \subseteq V(B') \setminus \{t_1\}$. Since $v \notin F_2$, it follows that $B' \neq B$, and therefore $t_3 \neq t_2$, since H is not a theta. Since v, f_1 are adjacent, (3) implies that $f_1 \notin M(t_1, B)$, and so $f_1 \in M(t_1)$. Let e be an edge of H incident with t_1 and not in B, B', let $f_1 \in F_1$ be adjacent in G to v, and let $f_2 \in F_2$ be adjacent in G to f_1. Then f_1 is adjacent in G to all of v, f_2, e. Since $Y(v) \subseteq V(B') \setminus \{t_1\}$, it follows that v, e are nonadjacent in G. Similarly, since $f_2 \in M(t_2) \cup M(t_2, B)$, f_2, e are nonadjacent in G. Since $\{f_1, v, f_2, e\}$ is not a claw, it follows that v, f_2 are adjacent in G. By (3), $v \notin M(t_3, B')$, and so $v \in M(t_3)$; and similarly $f_2 \in M(t_2)$; and also by (3), there is a branch B'' of H with ends t_2, t_3. Let H' be the graph obtained from H by adding a new vertex x and three new edges f_1, v, f_2, joining x to t_1, t_2, t_3 respectively. Then H' satisfies the hypotheses of the theorem, and $L(H') = G|(E(H) \cup \{v, f_1, f_2\})$, contrary to the maximality of H. This proves our claim that every vertex not in $F_1 \cup F_2$ is either complete or anticomplete to F_i, for $i = 1, 2$. Thus (F_1, F_2) is a homogeneous pair, nondominating since H is not a theta and therefore some edge of H is incident with no vertex in B; and so by 3.3 we may assume that F_1, F_2 both contain at most one element. To deduce the claim, let v_1, v_2 be as in the statement of (10); if B has length 2, then the edges of B belong to $F_1 \cup F_2$ and the claim follows. If B has length 1, then $v_i \in M(t_i)$ for $i = 1, 2$, contrary to (5). This proves (10).

From (10), we may assume that every vertex of G not in $E(H)$ belongs to $M(B)$ for some branch B, or to $M(t)$ for some branch-vertex t. If for all pairs v_1, v_2 of vertices in $V(G) \setminus E(H)$, v_1 is adjacent to v_2 if and only if $Y(v_1) \cap Y(v_2) \neq \emptyset$, then G is a line graph and the theorem holds. And we have already shown that this statement for all v_1, v_2 such that one of $|Y(v_1)|, |Y(v_2)| = 1$, by (4) and (10), and the "only if" implication holds for all v_1, v_2, by (2). From (4), we may therefore assume that there are nonadjacent $v_1, v_2 \in V(G)$, and distinct branch-vertices t_1, t_2, t_3 of H, and branches B_1, B_2 between t_1, t_3 and t_2, t_3 respectively, such that:

- $v_i \in M(B_i)$ $(i = 1, 2)$
- B_1, B_2 both have length 2, and
- every vertex of $V(H)$ adjacent to t_3 in H either belongs to one of B_1, B_2, or has degree 3 in H and is adjacent to all the ends of B_1, B_2.

Now H is not a theta. Let B_3 be the branch of H with ends t_1, t_2, if it exists. Let N be the set of all neighbours of t_3 that do not belong to B_1, B_2, let $V_1 = N \cup \{t_1, t_2, t_3\} \cup V(B_1) \cup V(B_2)$ and let $V_2 = (V(H) \setminus V_1) \cup \{t_1, t_2\}$. Since (V_1, V_2) is a 2-separation of H, we deduce that either $V(H) = V_1$, or the branch B_3 exists and $V(H) = V_1 \cup V(B_3)$. In either case, no branches of H have length > 1 except possibly B_1, B_2 and B_3 if it exists.

(11) *For $u_1, u_2 \in V(G) \setminus E(H)$, either u_1, u_2 belong to distinct sets $M(B_i)$ $(i = 1, 2, 3)$, or u_1, u_2 are adjacent if and only if $Y(u_1) \cap Y(u_2) \neq \emptyset$.*

For we have seen that if u_1, u_2 are adjacent, then $Y(u_1) \cap Y(u_2) \neq \emptyset$; and the converse holds by (4) unless $u_1 \in M(B)$ and $u_2 \in M(B')$ for distinct branches B, B', both of length ≥ 2. But B_1, B_2, B_3 are the only such branches. This proves (11).

(12) *$M(t) = \emptyset$ for all branch-vertices $t \neq t_1, t_2, t_3$ of H.*

For suppose that $x \in M(t)$ where $t \neq t_1, t_2, t_3$. We have seen that t is adjacent in H to all of t_1, t_2, t_3. Let e be the edge of H between t, t_3. Then e is adjacent in G to all of x, v_1, v_2. But v_1, v_2 are nonadjacent, and x is nonadjacent to v_1, v_2 by (2). Hence $\{e, x, v_1, v_2\}$ is a claw, a contradiction. This proves (12).

For $i = 1, 2, 3$, let $E_i = E(B_i) \cup M(B_i)$, setting $E_3 = \emptyset$ if B_3 does not exist. Thus E_1, E_2, E_3 are three cliques. For $i = 1, 2, 3$, let

$$F_i = M(t_i) \cup \bigcup (M(B) : B \neq B_1, B_2, B_3 \text{ is a branch of } H \text{ incident with } t_i).$$

From (8), (9), (10), (12) it follows that the six sets $E_1, E_2, E_3, F_1, F_2, F_3$ are pairwise disjoint and have union $V(G)$. From (4) and (11), F_1, F_2, F_3 are cliques. By (4) and (11) E_i is complete to F_i and to F_3 for $i = 1, 2$, and E_3 is complete to $F_1 \cup F_2$. By (2), E_1 is anticomplete to F_2, and E_2 is anticomplete to F_1, and E_3 is anticomplete to F_3. Thus G is expressible as a hex-join. This proves 6.3. ■

7 Prisms

A *prism* means a graph consisting of two vertex-disjoint triangles $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, and three paths P_1, P_2, P_3, where each P_i has ends a_i, b_i, and for $1 \leq i < j \leq 3$ the only edges between $V(P_i)$ and $V(P_j)$ are $a_i a_j$ and $b_i b_j$; and we say that the three paths P_1, P_2, P_3 *form* the prism. Thus a prism is just the line graph of a theta. A prism formed by paths of length $n_1, n_2, n_3 \geq 1$ is called an *(n_1, n_2, n_3)-prism.*

Our objective in this section is to handle the claw-free graphs that contain certain prisms. For big enough prisms, this is accomplished by 6.3. More precisely, we have (immediately from 6.3, taking H to be the prism):

7.1 *Let G be claw-free, with an (n_1, n_2, n_3)-prism as an induced subgraph, where either $n_1, n_2, n_3 \geq 2$, or $n_1, n_2 \geq 3$. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

In this section we prove the same thing for some slightly smaller prisms, namely the $(3, 2, 1)$-prism, the $(2, 2, 1)$-prism and the $(3, 1, 1)$-prism. We need first some lemmas about strips. A *strip* in G means a triple (A, C, B) of disjoint subsets of $V(G)$, such that

- A, B are nonempty cliques

- every vertex of $A \cup B$ belongs to a rung of the strip (a *rung* means a path between A and B with interior in C)
- for every vertex $v \in C$, there is a path from A to v with interior in C, and a path from v to B with interior in C.

Let (A_i, B_i, C_i) be a strip for $i = 1, 2$. We say they are *parallel* if

- A_1, B_1, C_1 are disjoint from A_2, B_2, C_2
- A_1 is complete to A_2 and B_1 is complete to B_2, and
- every edge between $A_1 \cup B_1 \cup C_1$ and $A_2 \cup B_2 \cup C_2$ is either between A_1 and A_2 or between B_1 and B_2.

Then $(A_1 \cup A_2, C_1 \cup C_2, B_1 \cup B_2)$ is a strip that we call the *disjoint union* of the first two strips. If a strip is not expressible as the disjoint union of two strips, we say it is *nonseparable*. We need the following lemma.

7.2 *Let G be claw-free, and let $(A_1, B_1, C_1), (A_2, B_2, C_2)$ be parallel strips. Suppose that (A_1, C_1, B_1) is nonseparable and C_1 is nonempty. Then C_1 is connected and every vertex of $A_1 \cup B_1$ has a neighbour in C_1.*

Proof. Let C_3 be a component of C_1 and $C_4 = C_1 \setminus C_3$. Let A_3 be the set of members of A_1 with a neighbour in C_3, and $A_4 = A_1 \setminus A_3$, and define B_3, B_4 similarly.

(1) *If $a \in A_3$, then no neighbour of a belongs to $B_4 \cup C_4$.*

For suppose that $x \in B_4 \cup C_4$ is a neighbour of a. By definition of A_3, a also has a neighbour $c \in C_3$; and let $a_2 \in A_2$. Since $\{a, a_2, x, c\}$ is not a claw, it follows that x is adjacent to c. Since $x \notin C_3$ and C_3 is a component of C_1, we deduce that $x \notin C_4$; and since x has a neighbour in C_3, we deduce that $x \notin B_4$, a contradiction. This proves (1).

(2) *Let R be a rung of (A_1, C_1, B_1). Then either $V(R) \subseteq A_3 \cup C_3 \cup B_3$, or $V(R) \subseteq A_4 \cup C_4 \cup B_4$.*

For suppose first that some vertex of the interior of R belongs to C_3. Then C_3 contains all the interior of R, since C_3 is a component of C_1, and so the ends of R belong to $A_3 \cup B_3$ and the claim holds. We may therefore assume that C_3 is disjoint from the interior of R. Let a be the end of R in A_1. Let r be the neighbour of a in R. If $a \in A_3$, then by (1), $r \in B_3 \cup C_3$, and since C_3 is disjoint from the interior of R, we deduce that R has length 1 and $r \in B_3$ and the claim holds. Thus we may assume that $a \notin A_3$, and similarly the other end of R is not in B_3; but then $V(R) \subseteq A_4 \cup C_4 \cup B_4$ and the claim holds. This proves (2).

(3) (A_3, C_3, B_3) *is a strip.*

For since C_3 is nonempty, and (A_1, B_1, C_1) is a strip, it follows that there is a path between C_3 and A_1 with interior in C_1 and hence in C_3; and consequently A_3 is nonempty, and similarly B_3 is nonempty. Consequently (A_3, C_3, B_3) is a strip, by (2). This proves (3).

Suppose that $A_4 \cup B_4 \neq \emptyset$. Then by (2), (A_4, C_4, B_4) is a strip, and by (1) the two strips $(A_3, C_3, B_3), (A_4, C_4, B_4)$ are parallel, contrary to hypothesis that (A_1, B_1, C_1) is nonseparable. Thus $A_4 = B_4 = \emptyset$. If there exists $v \in C_4$, then there is a path from v to A_1 with interior in C_1, which is therefore disjoint from C_3; and consequently this path has interior in C_4. Let its end in A_1 be a. By (1), $a \in A_4$, a contradiction since $A_4 = \emptyset$. This proves 7.2. ■

In several applications later in the paper, we shall have two parallel strips, and a path between them. Here is a lemma for use in that situation.

7.3 *Let G be claw-free, and for $i = 1, 2$ let R_i be a path of length ≥ 1, with ends a_i, b_i. Suppose that a_1a_2 and b_1b_2 are edges, and there are no other edges between R_1 and R_2. Let $X \subseteq V(G) \setminus \{a_1, b_1, a_2, b_2\}$ be connected, and for $i = 1, 2$ let there be a vertex in R_i with a neighbour in X. Then there is a path p_1-$\cdots$-p_k with $p_1, \ldots, p_k \in X \setminus (V(R_1) \cup V(R_2))$ such that:*

- *none of $p_1, \ldots, p_k$ belong to $R_1 \cup R_2$, and*
- *for $1 \leq i \leq k$, p_i has a neighbour in $V(R_1)$ if and only if $i = k$, and p_i has a neighbour in R_2 if and only if $i = 1$.*

Moreover, either:

1. *p_1 has exactly two neighbours in R_2 and they are adjacent, and the same for p_k in R_1, or*
2. *$k = 1$, and one of R_1, R_2 has length 1, and the other has length 2, and p_1 is complete to $V(R_1) \cup V(R_2)$, or*
3. *$k = 1$ and for $i = 1, 2$ the neighbours of p_1 in R_i are $\{a_i, b_i\}$, or*
4. *$k = 1$, and p_1 has a unique neighbour in one of R_1, R_2, and p_1 is adjacent to both $\{a_1, a_2\}$ or to both $\{b_1, b_2\}$.*

Proof. We may assume that X is minimal with the given property, and therefore X is disjoint from $V(R_1) \cup V(R_2)$, and $X = \{p_1, \ldots, p_k\}$ for some path p_1-$\cdots$-p_k such that for $1 \leq i \leq k$, p_i has a neighbour in $V(R_1)$ if and only if $i = k$, and p_i has a neighbour in R_2 if and only if $i = 1$. Let M be the set of neighbours of p_1 in $V(R_2)$, and let N be the set of neighbours of p_k in $V(R_1)$. Suppose first that $|N| = 1$. By 4.2, the vertex of N is not an internal vertex of R_1, and so we may assume that $N = \{a_1\}$. By 4.2, p_k is adjacent to a_2, and therefore $k = 1$ and $a_2 \in M$. But then the final statement of the theorem holds.

We may therefore assume that $|M|, |N| \geq 2$. If M consists of two adjacent vertices, and so does N, then the first statement of the theorem holds. So we may assume that there exist $x, y \in N$, nonadjacent. Since $\{p_k, x, y, p_{k-1}\}$ is not a claw, $k = 1$. Since $\{p_1, x, y, z\}$ is not a claw for z in the interior of R_2, it follows that $M = \{a_2, b_2\}$. Since $\{p_1, x, y, a_2\}$ is not a claw, it follows that $a_1 \in \{x, y\}$ and the same for b_1. If $|N| = 2$ then the third statement of the theorem holds, and so we may assume that N contains some vertex c from the interior of R_1. Since $\{p_1, c, a_2, b_2\}$ is not a claw, R_2 has length 1. Since $\{p_1, c, a_1, b_2\}$ is not a claw, c is adjacent to a_1 and similarly to b_1. But then R_1 has length 2 and the second statement of the theorem holds. This proves 7.3. ■

Next we show, for several different prisms, that if one is present as an induced subgraph then G is decomposable, or belongs to one of our basic classes. These proofs are quite similar, so we have extracted the main argument in the following lemma.

7.4 *Let G be claw-free, and let the three paths R_1, R_2, R_3 form a prism in G. Let R_i have ends a_i, b_i for $1 \le i \le 3$, where $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles. Suppose that R_1 has length > 1. Then one of the following holds (possibly after exchanging R_2, R_3):*

- *R_1 has length 2, R_2 has length 1, and there is a vertex v complete to $V(R_1) \cup V(R_2)$ and anticomplete to $V(R_3)$, or*
- *R_2 has length 1, and either R_3 has length 1 or R_1 has length 2, and there is a vertex v that is complete to $V(R_2)$ and anticomplete to $V(R_3)$, with exactly two neighbours in R_1, namely either the first two or last two vertices of R_1, or*
- *R_2 and R_3 both have length 1, and there is no vertex w that is complete to one of $V(R_2), V(R_3)$ and anticomplete to the other and to $V(R_1)$, or*
- *$G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

Proof. For $i = 2, 3$, let $A_i = \{a_i\}$, $B_i = \{b_i\}$ and C_i be the interior of R_i. Then (A_i, C_i, B_i) is a strip with a unique rung R_i. It follows that there is a strip (A_1, C_1, B_1) such that:

- (A_i, C_i, B_i) $(i = 1, 2, 3)$ are three parallel strips,
- R_1 is a rung of (A_1, B_1, C_1), and
- (A_1, B_1, C_1) is nonseparable.

Choose (A_1, B_1, C_1) such that W is maximal, where W denotes the union of the vertex sets of the three strips.

(1) *We may assume that every vertex $v \in V(G) \setminus (A_1 \cup B_1 \cup C_1)$ is anticomplete to C_1.*

For let $v \in V(G) \setminus (A_1 \cup B_1 \cup C_1)$, and suppose it has a neighbour in C_1. Consequently $v \notin W$. Let N be the set of neighbours of v in W. From the maximality of W, it follows that N meets one of $V(R_2), V(R_3)$. Suppose first that $a_2, a_3 \in N$. Since N meets C_1, it follows from 4.1 that $N \cap V(R_i) = \{a_i\}$ for $i = 2, 3$; but then $A_1 \subseteq N$, by 4.2 (with A_1, a_2, c_2, where c_2 is the neighbour of a_2 in R_2), and so v can be added to A_1, contrary to the maximality of W. Thus N contains at most one of a_2, a_3, and at most one of b_2, b_3 by symmetry. By 4.1, it follows that N meets exactly one of R_2, R_3, say R_2.

Now $C_1 \cup \{v\}$ is connected, and so by 7.3 there is a path p_1-$\cdots$-p_k of G with $v = p_1$ and with $p_2, \dots, p_k \in C_1$, satisfying one of the four statements of 7.3. Certainly none of $p_1, \dots, p_k$ have neighbours in R_3, and so 4.2 implies that that the fourth statement of 7.3 is impossible. Also 4.2 implies the third is impossible, since R_1 has length > 1. If the second statement of 7.3 holds, then the first statement of the theorem holds. Consequently we may assume that the first statement of 7.3 holds. Then the subgraph of G induced on $V(R_1) \cup V(R_2) \cup V(R_3) \cup \{p_1, \dots, p_k\}$ is a line graph of a graph H. If H satisfies the hypotheses of 6.3, then the theorem holds by 6.3, so we assume not. But H is a subdivision of K_4, and $|V(H)| \geq 6$. If $|V(H)| = 6$ then $k = 1$ and the second statement of the theorem holds. If $|V(H)| \geq 7$ then some branch of H contains all its vertices except at most three, and so $k = 1$ and again the second statement holds. This proves (1).

(2) *We may assume that every vertex* $v \in V(G) \setminus (A_1 \cup B_1 \cup C_1)$ *is either complete or anticomplete to* A_1.

For let $v \in V(G) \setminus (A_1 \cup B_1 \cup C_1)$, and suppose it has a neighbour and a nonneighbour in A_1. Then $v \notin W$. Let N be the set of neighbours of v in W. By (1), we may assume that $N \cap C_1 = \emptyset$. By 7.2, every vertex in A_1 has a neighbour in C_1. Since N meets A_1, 4.2 (with a_2, A_1, C_1 and a_3, A_1, C_1) implies that $a_2, a_3 \in N$. Choose $a'_1 \in A_1$ such that $a'_1 \notin N$. For $i = 2, 3$, if C_i is nonempty then 4.2 (with a'_1, a_i, C_i) implies that N meets C_i, and if $C_i = \emptyset$ then 4.2 (with a'_1-a_i-b_i) implies that $b_i \in N$. By 4.1, $N \cap (B_2 \cup C_2)$ is complete to $N \cap (B_3 \cup C_3)$; and so C_2, C_3 are empty, and $b_2, b_3 \in N$. Suppose there is a vertex w that is complete to one of $V(R_2), V(R_3)$ and anticomplete to the other and to $V(R_1)$. Thus $w \notin W$. Let w be complete to $V(R_2)$ say. By 4.2 (with a'_1-a_2-w) it follows that $w \in N$; but that contradicts 4.1, since $N \cap (A_1 \cup \{w, b_3\})$ includes a triad. Thus there is no such w; but then the third statement of the theorem holds. This proves (2).

If every vertex in $V(G) \setminus (A_1 \cup B_1 \cup C_1)$ is complete to one of A_1, B_1, then the third statement of the theorem holds. If not, then from (1) and (2), (A_1, C_1, B_1) is a breaker, and so by 3.4 G is decomposable. This proves 7.4. ■

Now we can process the little prisms.

7.5 *Let G be claw-free, with an (n_1, n_2, n_3)-prism as an induced subgraph, where $n_1 \geq 3$ and $n_2 \geq 2$. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ or G is decomposable.*

Proof. By 7.1 we may assume that $n_2 = 2$ and $n_3 = 1$. Then the result is immediate from 7.4. ■

7.6 *Let G be claw-free, with an (n_1, n_2, n_3)-prism as an induced subgraph, where $n_1, n_2 \geq 2$. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ or G is decomposable.*

Proof. By 7.5 and 7.1, we may assume that $n_1 = n_2 = 2$ and $n_3 = 1$. Let R_1, R_2, R_3 be three paths of G, forming a prism, with lengths $2, 2, 1$. Let W be the union of their vertex sets. Let R_i be a_i-c_i-b_i for $i = 1, 2$, and let R_3 have vertices a_3-b_3, where $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles. By 7.4, we may assume there is a vertex $v_1 \in V(G) \setminus W$, complete to $V(R_3)$, anticomplete to $V(R_2)$, and adjacent to c_1 and to at least one of a_1, b_1. By exchanging R_1, R_2, we may also assume there exists $v_2 \in V(G) \setminus W$ complete to $V(R_3)$, anticomplete to $V(R_1)$, and adjacent to c_2 and to at least one of a_2, b_2. Suppose first that v_1 is adjacent to both a_1, b_1. Since $\{v_1, v_2, a_1, b_1\}$ is not a claw, v_1 is not adjacent to v_2. Since $\{a_3, v_1, v_2, a_2\}$ is not a claw, v_2 is adjacent to a_2, and by symmetry v_2 is adjacent to b_2. But then the subgraph induced on these ten vertices is isomorphic to $icosa(-2)$, and the theorem follows from 4.5. We may therefore assume that v_1 is adjacent to exactly one of a_1, b_1, and v_2 to exactly one of a_2, b_2. Since $\{a_3, a_1, v_1, v_2\}$ and $\{b_3, b_1, v_1, v_2\}$ are not claws, v_1, v_2 are adjacent. Then the subgraph induced on these ten vertices is the line graph of a graph satisfying the hypotheses of 6.3, and the result follows. This proves 7.6. ■

Let $(A, \emptyset, B)$ be a strip. A *step* (in this strip) means a hole a_1-a_2-b_2-b_1-a_1 where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. We say the strip is *step-connected* if for every partition (X, Y) of A or of B with $X, Y \neq \emptyset$, there is a step meeting both X, Y. We say an (n_1, n_2, n_3)-prism is *long* if $n_1 + n_2 + n_3 \geq 5$.

7.7 *Let G be claw-free, with a long prism as an induced subgraph. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$, or G is decomposable.*

Proof. Let the paths R_1, R_2, R_3 form a long (n_1, n_2, n_3)-prism in G. By 7.6 we may assume that $n_1 \geq 3$, and $n_2 = n_3 = 1$. Let the paths R_i have ends a_i, b_i as usual, where R_1 has length ≥ 3, and R_2, R_3 have length 1.

(1) *We may assume that, for every such choice of R_1, R_2, R_3, there is no vertex w that is complete to one of $V(R_2), V(R_3)$ and anticomplete to the other and to $V(R_1)$.*

For if not then by 7.4, we may assume that there is a vertex v that is complete to $V(R_2)$ and anticomplete to $V(R_3)$, with exactly two neighbours in R_1, namely either the first two or last two vertices of R_1. From the symmetry we may assume that v is adjacent to a_1 and its neighbour in R_1. But then $G|((V(R_1) \cup V(R_2) \cup V(R_3) \cup \{v\}) \setminus \{a_2\}$ is a $(2, 2, 1)$-prism (or longer), and the result follows from 7.6. So we may assume that the statement of (1) holds.

Let $A_1 = \{a_1\}, B_1 = \{b_1\}$, and let C_1 be the interior of R_1. Now $(\{a_2, a_3\}, \emptyset, \{b_2, b_3\})$ is a step-connected strip, parallel to (A_1, C_1, B_1); and therefore we may choose a strip $(A_2, \emptyset, B_2)$ such that

- $(A_2, \emptyset, B_2)$ is step-connected, and $a_2, a_3 \in A_2$ and $b_2, b_3 \in B_2$
- the strips (A_1, C_1, B_1), $(A_2, \emptyset, B_2)$ are parallel, and
- $A_2 \cup B_2$ is maximal.

Let $W = V(R_1) \cup A_2 \cup B_2$.

(2) *Every vertex $v \in V(G) \setminus (A_2 \cup B_2)$ is either complete or anticomplete to A_2.*

For let $v \in V(G) \setminus (A_2 \cup B_2)$, and suppose it has a neighbour and a nonneighbour in A_2. Thus $v \notin W$. Let N be the set of neighbours of v in W. Since $(A_2, \emptyset, B_2)$ is step-connected, there is a step a'_2-a'_3-b'_3-b'_2-a'_2 such that $a'_2 \in N$ and $a'_3 \notin N$. 4.2 (with a'_3-a'_2-b'_2) implies that $b'_2 \in N$. Suppose that $b'_3 \in N$. Then 4.2 (with a'_3-b'_3-b_1) implies that $b_1 \in N$; 4.1 implies that $C_1 \cap N = \emptyset$; 4.2 (with a'_3, a_1, C_1) implies that $a_1 \notin N$; and 4.2 (with B_2, b_1, C_1) implies that $B_2 \subseteq N$. If we add v to B_2 then a'_2-a'_3-b'_3-v-a'_2 is a step of the enlarged strip, showing that this new strip is step-connected; but this contradicts the maximality of W. Thus $b'_3 \notin N$. Let R'_2, R'_3 be the rungs a'_2-b'_2 and a'_3-b'_3; then v is complete to $V(R'_2)$, and anticomplete to $V(R'_3)$. By (1) applied to the paths R_1, R'_2, R'_3, v has a neighbour in $V(R_1)$. Let us apply 7.3 to R_1, R'_2. Since $a'_3, b'_3 \notin N$, the third and fourth outcomes of 7.3 contradict 4.2, and so one of the first two outcomes applies. The second is impossible since R_1, R'_2 both do not have length 2, and so v has two adjacent neighbours in both R_1 and R'_2. If v is adjacent to both internal vertices of R_1, then $G|((V(R_1) \cup V(R'_2) \cup V(R'_3) \cup \{v\})$ is a line graph satisfying the hypothesis of 6.3. So we may assume that v is adjacent to a_1 and to its neighbour in

R_1. Hence $G|((V(R_1) \cup V(R'_2) \cup V(R'_3) \cup \{v\}) \setminus \{a'_2\}$ is a $(2,2,1)$-prism, and the result follows from 7.6. This proves (2).

From (1) and (2), we deduce that (A_2, B_2) is a homogeneous pair of cliques, nondominating since $C_1 \neq \emptyset$, and the result follows from 3.3. This proves 7.7. ■

8 Wheels and holes

Our goal in the next few sections is to handle claw-free graphs that contain holes of length ≥ 7. First, some definitions: An *n-hole* in G means a hole in G of length n. Let C be a n-hole, with vertices c_1-$\cdots$-c_n-c_1 in order; we call this an *n-numbering.* (We shall read these and similar subscripts modulo n, usually without saying so.) Let $v \in V(G)$, and let N be the set of neighbours of v in C, together with v if $v \in V(C)$. For $i = 1, \ldots, n$, we say that:

- v is *in position i* (relative to C, and to the given n-numbering) if either $v = c_i$ or $N = \{c_{i-1}, c_i, c_{i+1}\}$, and if $v \neq c_i$ we say v is a *clone*
- v is a *hat in position $i + \frac{1}{2}$* if $N = \{c_i, c_{i+1}\}$
- v is a *star in position $i + \frac{1}{2}$* if $n \geq 5$ and $N = \{c_{i-1}, c_i, c_{i+1}, c_{i+2}\}$
- v is a *centre* if $N = V(C)$ (and therefore $n \leq 5$)
- c is a *hub* if $n \geq 6$, and $|N| = 4$, and $N = \{a, b, c, d\}$ where ab and cd are edges, and there are no edges between $\{a, b\}$ and $\{c, d\}$.

We observe the following:

8.1 *Let C be a hole in a claw-free graph G, and let $v \in V(G) \setminus V(C)$, with at least one neighbour in $V(C)$. Then v is either a hat, clone, star, centre or hub relative to C.*

The proof is clear.

For convenient reference, here is a lemma that will have many uses later.

8.2 *Let G be claw-free, and let C be a hole in G. Let $v_1, v_2 \in V(G) \setminus V(C)$, and suppose that for $i = 1, 2$, the set of neighbours of v_i in C is the vertex set of a path P_i in C. Suppose also that no induced subgraph of G is a long prism. Then:*

1. *If P_1 is a subpath of P_2, and they have a common end, then v_1, v_2 are adjacent*
2. *If P_1 is a subpath of P_2 and they have no common end, then v_1, v_2 are nonadjacent*
3. *If neither of P_1, P_2 is a subpath of the other, and at least three vertices of P_1 do not belong to P_2, then v_1, v_2 are nonadjacent*
4. *If C has length ≥ 6 and P_1, P_2 both have length 1 and have no common end, then v_1, v_2 are nonadjacent*

5. *If P_1, P_2 both have length 1 and are different but with a common end, and C is a hole of maximum length in G, then v_1, v_2 are nonadjacent.*

Proof. Since every path has at least one vertex by definition, it follows that v_1, v_2 both have neighbours in C. The first assertion follows from 4.3. For the second, let x, y be the ends of P_2; then $\{v_2, v_1, x, y\}$ is not a claw, and so v_1, v_2 are nonadjacent. For the third, suppose three vertices of P_1 do not belong to P_2; then some two of them, say x, y, are nonadjacent, and since $\{v_1, v_2, x, y\}$ is not a claw, it follows that v_1, v_2 are nonadjacent. The fourth holds since if v_1, v_2 are adjacent, the subgraph induced on $V(C) \cup \{v_1, v_2\}$ is a long prism (since C has length ≥ 6). Finally, if P_1, P_2 share one end c, then the subgraph induced on $(V(C) \setminus \{c\}) \cup \{v_1, v_2\}$ is not a longer hole than C, and so v_1, v_2 are nonadjacent. This proves 8.2. ■

8.3 *Let G be claw-free, and let C be a hole in G of length ≥ 7, with a hub. Then either $G \in \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ or G is decomposable.*

Proof. Let w be a hub for C. Let w have neighbours a_1, a_2, b_1, b_2 in C, where a_1a_2 and b_1b_2 are edges, and a_1, b_1, b_2, a_2 lie in this order in C. Let R_1, R_2 be the two disjoint paths of C between $\{a_1, a_2\}$ and $\{b_1, b_2\}$, where R_i is between a_i, b_i for $i = 1, 2$. Thus R_1, R_2 both have length ≥ 2, and since C has length ≥ 7 we may assume that R_1 has length ≥ 3. Let $A_1 = \{a_1\}, B_1 = \{b_1\}$, and let C_1 be the interior of R_1. Choose a strip (A_2, C_2, B_2) with the following properties:

- (A_i, C_i, B_i) $(i = 1, 2)$ are parallel strips
- $a_2 \in A_2, b_2 \in B_2$ and R_2 is a rung of (A_2, C_2, B_2)
- (A_2, C_2, B_2) is nonseparable
- w is complete to $A_2 \cup B_2$ and anticomplete to C_2, and
- $W = V(R_1) \cup A_2 \cup C_2 \cup B_2$ is maximal with these properties.

(1) *We may assume that every $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ is anticomplete to C_2.*

For suppose that $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ has a neighbour in C_2. Then $v \notin W$; let N be the set of neighbours of v in W. From the maximality of W, N meets $\{w\} \cup V(R_1)$. Suppose first that $w \in N$. From 4.2 (with a_1-w-b_1), we may assume that $a_1 \in N$. From 4.1 (with C_2, w, C_1 and C_2, a_1, b_1) it follows that $N \cap C_1 = \emptyset$, and $b_1 \notin N$. From 4.2 (with A_2, a_1, C_1), it follows that $A_2 \subseteq N$. But then v can be added to A_2, contrary to the maximality of W. Thus $w \notin N$. Consequently N meets $V(R_1)$. Choose p_1-$\cdots$-p_k as in 7.3 (with R_1, R_2 exchanged), where $p_1 = v$, and $p_2, \ldots, p_k \in C_2$. Then none of $p_1, \ldots, p_k$ are adjacent to w. By 4.2, p_1 is adjacent to more than one vertex of R_1, and p_k to more than one of R_2, so the fourth outcome of 7.3 is impossible; and since R_1 has length > 1, 4.2 implies the third is impossible. The second is false since R_1 has length ≥ 3, and so the first holds. Then $G|(V(C) \cup \{w, p_1, \ldots, p_k\})$ is a line graph of a cyclically 3-connected graph H. If either $k > 1$ or the four vertices of N in the hole C are not consecutive or R_2 has length > 2, then H satisfies the hypotheses of 6.3 and the theorem holds. If $k = 1$ and the four vertices of N in C are consecutive and R_2 has length 2, we may assume that v is adjacent to a_1, a_2 and their neighbours in C. But then $G|(V(C) \cup \{v, w\} \setminus \{a_2\})$ is a $(2,2,1)$-prism or longer, and the result follows from 7.6. This proves (1).

(2) *Every $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ is either complete or anticomplete to A_2.*

For suppose that $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ has a neighbour and a nonneighbour in A_2. Then $v \notin W$; let N be the set of neighbours of v in W. By the assumption of (1), $N \cap C_2 = \emptyset$. By 7.2, every vertex in A_2 has a neighbour in C_2, and so 4.2 (with C_2, A_2, w; C_2, A_2, a_1; A_2, w, b_1; and A_2, a_1, C_1) implies that $w, a_1, b_1 \in N$, and N contains the neighbour of a_1 in R_1. But this contradicts 4.1. This proves (2).

From (1) and (2) we deduce that (A_2, C_2, B_2) is a breaker, and the result follows from 3.4. This proves 8.3. ■

8.4 *Let G be claw-free, and let C be a hole in G of length ≥ 7. Let a_1-a_2-b_2-b_1 be a path in C, and let $h, w \in V(G) \setminus V(C)$, such that the neighbours of w in C are a_1, a_2, b_2, b_1, and the neighbours of h in C are a_2, b_2. Then G is decomposable.*

Proof. Let R_1 be the path $C \setminus \{a_2, b_2\}$, and let $C_1 = V(C) \setminus \{a_1, a_2, b_1, b_2\}$. Let R_2 be the path a_2-b_2. Thus $(\{a_1\}, C_1, \{b_1\})$ is a strip, and $(\{a_2\}, \{h\}, \{b_2\})$ is another strip parallel to the first. By 8.2, w, h are nonadjacent, so we may choose a strip (A_2, C_2, B_2) with the following properties:

- (A_2, C_2, B_2) is parallel to $(\{a_1\}, C_1, \{b_1\})$
- $a_2 \in A_2, h \in C_2, b_2 \in B_2$
- (A_2, C_2, B_2) is nonseparable
- w is complete to $A_2 \cup B_2$ and anticomplete to C_2
- $W = V(R_1) \cup A_2 \cup B_2 \cup C_2$ is maximal subject to these condition.

(1) *We may assume that every $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ is anticomplete to C_2.*

For suppose that $v \in V(G) \setminus (A_2 \cup B_2 \cup C_2)$ has a neighbour in C_2. Then $v \notin W$; let N be the set of neighbours of v in W. From the maximality of W, N meets $\{w\} \cup V(R_1)$. Suppose first that $w \in N$. From 4.2 (with a_1-w-b_1), we may assume that $a_1 \in N$. From 4.1 (with C_2, w, C_1 and C_2, a_1, b_1) it follows that $N \cap C_1 = \emptyset$, and $b_1 \notin N$. From 4.2 (with A_2, a_1, C_1), it follows that $A_2 \subseteq N$. But then v can be added to A_2, contrary to the maximality of W. Thus $w \notin N$. Consequently N meets $V(R_1)$. Choose p_1-$\cdots$-p_k as in 7.3 (with R_1, R_2 exchanged), where $p_1 = v$, and $p_2, \ldots, p_k \in C_2$. Then none of $p_1, \ldots, p_k$ are adjacent to w. By 4.2, p_1 is adjacent to more than one vertex of R_1. Suppose that the fourth outcome of 7.3 holds; then p_k has a unique neighbour in R_2, say a_2. By 4.2 applied to w-a_2-h and to a_1-a_2-b_2, it follows that p_k is adjacent to h and to a_1, and hence $k = 1$ and $p_k = v$. Let c_1 be the neighbour of a_1 in R_1; then by 4.2 applied to w-a_1-c_1, v is adjacent to c_1. By 4.1, v has no neighbours in C except c_1, a_1, a_2; but then the subgraph of G induced on $(V(C) \setminus \{a_2\}) \cup \{h, v\}$ is a long prism, and the result follows from 7.7. Thus we may assume that the fourth outcome of 7.3 does not hold. Since R_1 has length > 1, 4.2 implies the third outcome is impossible. The second is false since R_1 has length ≥ 3, and so the first holds. Then $G|(V(R_1) \cup V(R_2) \cup \{w, p_1, \ldots, p_k\})$ is a line graph of a cyclically 3-connected graph H. If $k > 1$ then $G|(V(C) \cup \{p_1, \ldots, p_k\})$ is a long prism, and the result follows from 7.7; so we assume that $k = 1$. If the four vertices of N in the hole C are not consecutive, then v is a hub for C and the result follows from 8.3. We may therefore assume that v is adjacent to a_1, a_2 and their neighbours in C. But then $G|(V(C) \cup \{v, w\} \setminus \{a_2\})$ is a long prism, and the result follows from 7.7. This proves (1).

The remainder of the proof of 8.4 is identical with the latter part of the proof of 8.3, and we omit it. This proves 8.4. ■

9 Circular interval graphs

So far, our method has been to show that claw-free graphs containing subgraphs of certain types either are line graphs, or are decomposable (with a few sporadic exceptions). That is not adequate to handle all claw-free graphs containing holes of length ≥ 7, because there is another major basic class of them, the circular interval graphs. In this section we prove the following (we recall that $\mathcal{S}_3$ is the class of all circular interval graphs):

9.1 *Let G be claw-free, containing a hole of length ≥ 7. Then either $G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_3$, or G is decomposable.*

To prove this we need two lemmas. A subset $X \subseteq V(G)$ is said to be *dominating* if every vertex of G either belongs to X or has a neighbour in X; and a subgraph H of G is said to be dominating if $V(H)$ is dominating.

9.2 *Let C be a hole of maximum length (n say) in a claw-free graph G. Then either some induced subgraph of G is an (n_1, n_2, n_3)-prism, for some $n_1, n_2, n_3 \geq 1$ with $n_1 + n_2 = n - 2$, or G is decomposable, or C is dominating.*

Proof. Let Z be the set of all vertices of G that are not in $V(C)$ and have no neighbour in $V(C)$. We may assume that Z is nonempty; let W be a component of $G|Z$. Let X be the set of all vertices not in W but with a neighbour in W. Let $x \in X$; we claim that it has exactly two neighbours in $V(C)$ and they are adjacent. For if it has two nonadjacent neighbours $u, v \in V(C)$, let $w \in W$ be adjacent to x; then $\{x, u, v, w\}$ is a claw, a contradiction. From 8.1, this proves that there is an edge e of C such that the neighbours of x in $V(C)$ are precisely the ends of e. We write $e(x) = e$. Suppose there exist $x_1, x_2 \in X$ with $e(x_1) \neq e(x_2)$. Let P be a path between x_1, x_2 with interior in W. If $e(x_1), e(x_2)$ share no end, then the subgraph of G induced on $V(C) \cup V(P)$ is an $(n_1, n_2, |E(P)|)$-prism for some $n_1, n_2 \geq 1$ with $n_1 + n_2 = n - 2$, and the theorem holds. If $e(x_1), e(x_2)$ share an end c, then the subgraph induced on $V(C) \cup V(P) \setminus \{c\}$ is a hole of length $> n$, a contradiction. We may therefore assume that there are no such x_1, x_2. Thus there is an edge e of C such that every vertex in X is adjacent to both ends of e and to no other vertex of C. Let C have vertices $c_1\text{-}\cdots\text{-}c_n\text{-}c_1$ say, where e has ends c_1, c_2. By 4.3, X is a clique. Let $v \in V(G) \setminus (X \cup W)$; we claim that v is either complete or anticomplete to X. If $v \in V(C)$ this is true, so we assume $v \notin V(C)$. Suppose that v is adjacent to $x_1 \in X$ and nonadjacent to $x_2 \in X$. Let $w \in W$ be adjacent to x_1. Since $v \notin W \cup X$ it follows that v, w are nonadjacent. Since $\{x_1, w, v, c_1\}$ is not a claw, v is adjacent to c_1 and similarly to c_2. Since $\{c_2, c_3, v, x_2\}$ is not a claw, v is adjacent to c_3 and similarly to c_n; but then $\{v, x_1, c_3, c_n\}$ is a claw, a contradiction. This proves that v is either complete or anticomplete to X. By 3.2, $(X \cup W, V(G) \setminus (X \cup W))$ is decomposable. This proves 9.2. ■

Before the second lemma, we need a few definitions. Let C be a n-hole, with vertices c_1-$\cdots$-c_n-c_1 in order. We say that:

- If h_1, h_2 are adjacent hats, with no common neighbour in C, $\{h_1, h_2\}$ is a *hat-diagonal* for C
- If $n \geq 5$ and h, s are a hat and a star relative to C, and h is adjacent to c_i, c_{i+1} and s is adjacent to $c_{i-1}, c_i, c_{i+1}, c_{i+2}$ for some $i \in \{1, \dots, n\}$, we call $\{h, s\}$ a *coronet* for C.
- If $n \geq 5$ and s_1, s_2 are nonadjacent stars, adjacent respectively to $c_i, c_{i+1}, c_{i+2}, c_{i+3}$ and to $c_{i+1}, c_{i+2}, c_{i+3}, c_{i+4}$ for some i, we call $\{s_1, s_2\}$ a *crown* for C.
- If $n = 5$ or 6 and s_1, s_2 are adjacent stars, adjacent respectively to $c_i, c_{i+1}, c_{i+2}, c_{i+3}$ and to $c_{i+3}, c_{i+4}, c_{i+5}, c_{i+6}$ for some i, we call $\{s_1, s_2\}$ a *star-diagonal* for C.
- If $n = 6$ and s_1, s_2, s_3 are three pairwise adjacent stars, adjacent respectively to $c_i, c_{i+1}, c_{i+2}, c_{i+3}$, to $c_{i+2}, c_{i+3}, c_{i+4}, c_{i+5}$ and to $c_{i-2}, c_{i-1}, c_i, c_{i+1}$ for some i, we call $\{s_1, s_2, s_3\}$ a *star-triangle* for C.

The second lemma we need is the following, the main result of [2].

9.3 *Let G be claw-free, with a hole, and let n be the maximum length of holes in G. Suppose that every hole of length n is dominating, and has no hub, coronet, crown, hat-diagonal, star-diagonal, star-triangle or centre. Then either G admits a coherent W-join, or G is a circular interval graph.*

Now we are ready to prove the main result of this section.

Proof of 9.1. Let G be claw-free, and let the longest hole in G have length $n \geq 7$. By 7.7, we may assume that no induced subgraph of G is a long prism, and that G is not decomposable. By 9.2, every n-hole is dominating. By 8.3, we may assume that no n-hole has a hub, and by 8.4, we may assume that no n-hole has a coronet. If $\{s_1, s_2\}$ is a crown for an n-hole C, then G contains a long prism (obtained from $G|V(C) \cup \{s_1, s_2\}$ by deleting the middle common neighbour of s_1, s_2 in C), which is impossible. Also no n-hole has a hat-diagonal, since G has no long prism. By 9.3, we deduce that $G \in \mathcal{S}_3$. This proves 9.1. ∎

10 The icosahedron minus a triangle

Now we begin the next part of the paper. The objective of the next several sections is to prove 16.2, that every claw-free graph with a hole of length ≥ 6 either belongs to one of our basic classes or is antiprismatic, or decomposable. We begin by outlining the plan of the proof, as follows.

- We can assume G is claw-free, with a 6-hole, but with no long prism or hole of length > 6. Consequently we may assume that every 6-hole is dominating, by 9.2.
- (In 11.5) If some 6-hole has both a hub and a clone, then G is decomposable.
- (In 11.6) If some 6-hole has both a star-diagonal and a clone then G is decomposable.
- (In 13.2) Every 5-hole is dominating (or else either G is decomposable, or it belongs to one of our basic classes). Consequently, no 6-hole has a coronet.

- (In 14.1) If some 6-hole has both a hub and a hat, then either G is a line graph or it is decomposable.
- (In 14.2) If some 6-hole has both a star-diagonal and a hat, then G is decomposable.
- (In 15.3) If no 6-hole has a hub, but some 6-hole has both a star-triangle and either a hat or clone, then G is decomposable.
- (In 15.4) If no 6-hole has a hub or star-diagonal, but some 6-hole has a crown, then G is decomposable.
- (In 16.1) If no 6-hole has a hub, star-diagonal, star-triangle or crown, then either G is a circular interval graph or G is decomposable.
- To complete the proof of 16.2, we may therefore assume that some 6-hole has either a hub, a star-diagonal, or a star-triangle, and has no hat or clone. We deduce that G is antiprismatic.

The first step is to handle $icosa(-3)$, and that is the goal of this section. We recall that $icosa(-3)$ is the graph obtained from $icosa(0)$ by deleting three pairwise adjacent vertices. Thus it has nine vertices $c_1, \ldots, c_6, b_1, b_3, b_5$, where $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$ is a 6-numbering and b_1, b_3, b_5 are clones in positions $1, 3, 5$ respectively, pairwise adjacent.

10.1 *Let G be claw-free, and with no long prism or hole of length > 6, containing $icosa(-3)$ as an induced subgraph. Then G is decomposable.*

Proof. In view of the given subgraph, we can choose nine pairwise disjoint subsets $C_1, \ldots, C_6, B_1, B_3, B_5$ of $V(G)$ such that

- all nine subsets are nonempty cliques
- for $1 \leq i \leq 6$, C_i is complete to C_{i+1}, C_{i-1}, and anticomplete to $C_{i+2}, C_{i+3}, C_{i+4}$
- for $i \in \{1, 3, 5\}$, B_i is complete to C_{i-1}, C_i, C_{i+1} and anticomplete to $C_{i+2}, C_{i+3}, C_{i+4}$
- B_1, B_3, B_5 are pairwise complete
- the union $W = C_1 \cup \text{-} \cdots \text{-} \cup C_6 \cup B_1 \cup B_2 \cup B_3$ is maximal subject to these conditions.

(1) *For $v \in V(G) \setminus W$, let N be the set of neighbours of v in W; then, up to symmetry, either:*

- *N is the union of C_4, C_5, C_6, B_5 and a nonempty subset of C_1, or*
- *N is the union of C_3, C_4, C_5, C_6, B_5 and a subset of B_3, or*
- *N is the union of C_3, C_4, C_5, B_3, B_5 and a nonempty subset of C_6.*

Suppose first that none of C_1, C_3, C_5 is a subset of N. Then by 4.2, N is disjoint from C_2, C_4, C_6; by 4.2 again, it is disjoint from C_1, C_3, C_5; and by 4.2 again, it is disjoint from B_1, B_3, B_5. Thus $N = \emptyset$, and therefore G is decomposable by 9.2. We may therefore assume that $C_5 \subseteq N$.

Next assume that $C_1, C_3 \not\subseteq N$. By 4.2, $N \cap C_2 = \emptyset$; by 4.2 (with C_4, C_5, C_6), N includes one of C_4, C_6; and by 4.2 (with C_3, C_4, B_5 and C_1, C_6, B_5), $B_5 \subseteq N$. Suppose that $N \cap (B_1 \cup B_3)$ is nonempty, say $N \cap B_3 \neq \emptyset$; then 4.2 (with B_1, B_3, C_3) implies that $B_1 \subseteq N$, and similarly $B_3 \subseteq N$; 4.1 implies that N is disjoint from C_1, C_3; 4.2 (with C_2, B_3, C_4 and C_2, B_1, C_6) implies that $C_4, C_6 \subseteq N$; but then v can be added to B_5, a contradiction. Thus $N \cap (B_1 \cup B_3)$ is empty. By 4.2 (with B_1, B_5, C_4 and B_3, B_5, C_6), $C_4, C_6 \subseteq N$; by 4.1, N does not meet both C_1, C_3; if it meets neither then v can be added to C_5, a contradiction; and if it meets exactly one of C_1, C_3 then the claim holds.

Thus we may assume that N includes C_3 (as well as C_5). By 4.1, $N \cap (B_1 \cup C_1) = \emptyset$. Suppose that $C_4 \not\subseteq N$. Then by 4.2 (with C_4, C_5, C_6), N includes C_6 and similarly C_2; by 4.2 (with C_4, B_3, B_1), N is disjoint from B_3; and then 4.2 (with C_1, C_2, B_3) is violated. This proves that $C_4 \subseteq N$. By 4.1, N is disjoint from one of C_2, C_6, say C_2. Suppose that $B_5 \not\subseteq N$. Then 4.2 implies that $N \cap (B_3 \cup C_6) = \emptyset$, and then the subgraph of G induced on $W \cup \{v\} \setminus (B_1 \cup C_4)$ contains a long prism, a contradiction. Hence $B_5 \subseteq N$. By 4.2, N includes one of B_3, C_6; and if it includes B_3 then it meets C_6, for otherwise v could be added to C_4. This proves (1).

For $i = 1, 3, 5$, let X_i be the union of B_i, C_i, and the set of all $v \in V(G) \setminus W$ such that v is complete to $C_{i-1}, C_i, C_{i+1}, B_i$ and has nonneighbours in both B_{i+2}, B_{i-2}. Note that, from (1), every vertex in X_i is anticomplete to one of C_{i-2}, C_{i+2}. For $i = 2, 4, 6$, let X_i be the union of C_i and the set of all vertices in $V(G) \setminus W$ that are complete to $C_{i-1}, C_i, C_{i+1}, B_{i-1}$ and B_{i+1}. By (1), every vertex of G belongs to exactly one of the sets $X_1, \ldots, X_6$.

(2) *For $1 \leq i \leq 6$, X_i is a clique.*

For first suppose that i is odd. Let $u, v \in X_i$. From the definition of X_i, it is clear that u, v are adjacent if either of them belongs to $B_i \cup C_i$, so we may assume that $u, v \notin W$. From (1), u is anticomplete to one of B_{i+2}, B_{i-2} and has a nonneighbour in the other, and the same holds for v, and so we may assume that they have a common nonneighbour $z \in B_{i+2}$ say. But they also have a common neighbour $w \in B_i$, and since $\{w, u, v, z\}$ is not a claw, it follows that u, v are adjacent. Thus X_i is a clique if i is odd. Now let i be even, and again let $u, v \in X_i$. Choose $z \in B_{i+3}$ and $w \in B_{i+1}$; then u, v are both adjacent to w and nonadjacent to z. Since $\{w, u, v, z\}$ is not a claw, u, v are adjacent, and so again X_i is a clique. This proves (2).

(3) *For $1 \leq i \leq 6$, X_i is complete to X_{i+1}.*

For we may assume that i is odd, from the symmetry. Let $u \in X_i, v \in X_{i+1}$. If either $u \in B_i \cup C_i$ or $v \in B_{i+1}$, then u, v are adjacent from the definitions of X_i, X_{i+1}, so we may assume that $u, v \notin W$. Suppose that u, v are nonadjacent. Let z be a nonneighbour of u in B_{i-2}, and let $w \in B_i$. Since v has no neighbour in B_{i-2}, and u, v are both B_i-complete, and $\{w, u, v, z\}$ is not a claw, we deduce that u, v are adjacent. This proves (3).

(4) *For $1 \leq i \leq 6$, X_i is anticomplete to X_{i+3}.*

For we may assume that i is odd, from the symmetry. Let $u \in X_i, v \in X_{i+3}$. Now u is anticomplete to one of C_{i-2}, C_{i+2}, say C_{i-2}; and u has a nonneighbour w in B_{i+2}. Choose $z \in C_{i-2}$; then since $\{v, u, w, z\}$ is not a claw, it follows that u, v are nonadjacent. This proves (4).

From (2)–(4), we see that G is the hex-join of $G|(X_1 \cup X_3 \cup X_5)$ and $G|(X_2 \cup X_4 \cup X_6)$, and therefore is decomposable. This proves 10.1. ■

11 6-holes with clones

Let c_1-$\cdots$-c_6-c_1 be a 6-numbering of a 6-hole C. If v is a hub relative to C, we say that v is *in hub-position* i if v is adjacent to $c_{i-2}, c_{i-1}, c_{i+1}, c_{i+2}$. (Thus a hub in hub-position i is the same as a hub in hub-position $i+3$.)

11.1 *Let G be claw-free. Let C be a 6-hole in G with vertices c_1-$\cdots$-c_6-c_1 in order, and let w be a hub in hub-position i. Let $v \in V(G) \setminus (V(C) \cup \{w\})$. Then w, v are adjacent if and only if either:*

- *v is a hub in hub-position i, or*
- *v is a hat in position $i + 1\frac{1}{2}$ or in position $i - 1\frac{1}{2}$, or*
- *v is a clone in position $i+1, i+2, i-2i$ or $i-1$, or*
- *v is a star in position $i + \frac{1}{2}, i + 2\frac{1}{2}, i - \frac{1}{2}$ or $i - 2\frac{1}{2}$*

Proof. In each case listed, if v, w are nonadjacent there is a claw; and in the cases not listed, if v, w are adjacent there is a claw. We leave the details to the reader. ■

This has the following consequence.

11.2 *Let G be claw-free. Let C be a 6-hole in G with vertices c_1-$\cdots$-c_6-c_1 in order. If there are two hubs in the same hub-position, then G admits twins.*

Proof. By 11.1, any two hubs in the same hub-position are adjacent, and every other vertex is adjacent to both or neither of them. Thus they are twins. This proves 11.2. ■

Two n-numberings are *proximate* if they differ in exactly one place (and therefore they number n-holes with $n-1$ vertices in common; the exceptional vertex of each is a clone with respect to the other). Note that we regard c_1-$\cdots$-c_n-c_1 and c_2-c_3-$\cdots$-c_n-c_1-c_2 as different numberings; the choice of initial vertex is important. A nonempty set $\mathcal{C}$ of n-numberings is *connected by proximity* if the graph with vertex set $\mathcal{C}$, in which two n-numberings are adjacent if they are proximate, is connected. The *proximity distance* between two n-numberings is the length of the shortest path between them in this graph, if such a path exists, and is undefined otherwise. A *proximity component of order n* means a set $\mathcal{C}$ of n-numberings that is connected by proximity and maximal with this property.

11.3 *Let G be claw-free, and let $\mathcal{C}$ be a proximity component of order* 6. *Let $v \in V(G)$ be a hub in hub-position i for some member of $\mathcal{C}$. Then v is a hub in hub-position i for every member of $\mathcal{C}$.*

Proof. It suffices to show that if c_1-$\cdots$-c_6-c_1 and c'_1-$\cdots$-c'_6-c'_1 are proximate, and v is a hub in hub-position i for the first 6-numbering, then v is a hub in hub-position i for the second. We may assume that $i = 1$. From the symmetry we may assume that $c_j = c'_j$ for $j = 3, 4, 5$; and since v is adjacent to c_3, c_5 and not to c_4, it follows from 8.1 that v is a hub for c'_1-$\cdots$-c'_6-c'_1 in hub-position 1. This proves 11.3. ■

If $\mathcal{C}$ is a proximity component of order n, we denote the union of the vertex sets of its members by $V(\mathcal{C})$; and for $1 \le i \le n$, the set of vertices that are the ith term of some member of $\mathcal{C}$ is denoted by $A_i(\mathcal{C})$, or just A_i when there is no ambiguity. If these n sets are pairwise disjoint, we say that $\mathcal{C}$ is *pure.*

11.4 *Let G be claw-free, in which every 6-hole is dominating, and containing no 7-hole or long prism. Let $\mathcal{C}$ be a pure proximity component of order 6. Then*

- *For $1 \le i \le 6$, A_i is a clique and there are no edges between A_i and A_{i+3}*
- *If $v \in V(G)$ and $v \notin A_1 \cup \cdots \cup A_6$, then for $1 \le i \le 6$, v is either complete or anticomplete to A_i; and v is complete to either two or four of the sets $A_1, \ldots, A_6$.*
- *For $1 \le i \le 6$, every $v \in A_i$ is either complete to A_{i+1} or anticomplete to A_{i+2}*
- *For $1 \le i \le 6$, either A_i is complete to A_{i-1} or A_i is anticomplete to A_{i+2}*
- *For $1 \le i \le 6$, A_i is complete to one of A_{i-1}, A_{i+1}.*

Proof. For each vertex $v \in V(G)$, let $P(v)$ be the set of all k such that v is in position k relative to some member of $\mathcal{C}$.

(1) *For every vertex $v \in V(G)$, if k is an integer, then $k \in P(v)$ if and only if $v \in A_k$. Moreover, $|P(v)| \le 3$, and the members of $P(v)$ are consecutive multiples of $\frac{1}{2}$ modulo* 6.

For suppose first that $v \in A_k$. Then since the sets $A_1, \ldots, A_6$ are pairwise disjoint, v is the kth term of every member of $\mathcal{C}$ that contains it, and there is such a member since $v \in A_k$; and so $k \in P(v)$. For the converse, suppose that k is an integer and $k \in P(v)$. We may assume that $k = 1$. Choose a 6-numbering $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$ such that v is in position 1 relative to this 6-numbering. The 6-numbering $v\text{-}c_2\text{-}\cdots\text{-}c_6\text{-}v$ also belongs to $\mathcal{C}$, because of the maximality of $\mathcal{C}$, and since none of $c_2, \ldots, c_6$ belong to A_1, it follows that $v \in A_1$. This proves the first claim. Now note that the positions of v relative to two proximate 6-numberings differ by at most $\frac{1}{2}$; and so the members of $P(v)$ are consecutive multiples of $\frac{1}{2}$ (modulo 6). Since $P(v)$ contains at most one integer, as we have seen, it follows that $|P(v)| \le 3$. This proves (1).

To prove the first statement of the theorem, we may assume that $i = 1$. Let $u, v \in A_1$, and let $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$ containing u. Since $v \in A_1$, it follows that $1 \in P(v)$; and so $P(v) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$ by (1). In particular, v is in position $\frac{1}{2}, 1$ or $1\frac{1}{2}$ relative to $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$; and in each case, it is adjacent to u. Hence A_1 is a clique. Now let $u \in A_1$ and $v \in A_4$. As before, $P(u) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$. Choose $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$ with $c_4 = v$; then u is in position $\frac{1}{2}, 1$ or $1\frac{1}{2}$ relative to $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$, and in each case u, v are nonadjacent. This proves the first statement.

For the second statement, let $v \in V(G)$ with $v \notin A_1 \cup \cdots \cup A_6$. By 11.3 we may assume that v is not a hub relative to any member of $\mathcal{C}$. By (1), $P(v)$ contains no integer, and so $P(v) = \{i + \frac{1}{2}\}$ for some integer i. Thus v is in position $i + \frac{1}{2}$ relative to every member of $\mathcal{C}$, and it is either a hat or a star. If it is sometimes a hat and sometimes a star, then there are two proximate members of $\mathcal{C}$ such that v is a hat relative to one and a star relative to the other, which is impossible. Hence either it is a hat in position $i + \frac{1}{2}$ relative to all members of $\mathcal{C}$, or it is a star in the same position for them all, and in either case the claim follows. This proves the second statement.

For the third statement, we may assume that $i = 1$; let $v \in A_1$, and suppose it has a neighbour $a_3 \in A_3$ and a nonneighbour $a_2 \in A_2$. Choose c_1-c_2-$\cdots$-c_6-$c_1 \in \mathcal{C}$ so that $a_2 = c_2$. Since $v \in A_1$ it follows that $P(v) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$, and since v is nonadjacent to c_2, we deduce that v is a hat in position $\frac{1}{2}$ relative to c_1-c_2-$\cdots$-c_6-c_1. Since $a_3 \in A_3$, it follows that $P(a_3) \subseteq \{2\frac{1}{2}, 3, 3\frac{1}{2}\}$. If a_3 is adjacent to both c_2, c_4 then $\{a_3, c_2, c_4, v\}$ is a claw; and so a_3 is a hat in position $2\frac{1}{2}$ or $3\frac{1}{2}$ relative to c_1-c_2-$\cdots$-c_6-c_1. But then $G|\{c_1, \ldots, c_6, v, a_3\}$ is a long prism, a contradiction. This proves the third statement.

For the fourth statement, let us first prove the following.

(2) *If $1 \le i \le 6$, then every vertex in A_i is either complete to A_{i-1} or anticomplete to A_{i+2}.*

For we may assume that $i = 2$. Let $v \in A_2$, and suppose that v has a neighbour $a_4 \in A_4$ and a nonneighbour in $a_1 \in A_1$. Choose c_1-c_2-$\cdots$-c_6-c_1 and c_1'-c_2'-$\cdots$-c_6'-c_1' in $\mathcal{C}$, with $c_1 = a_1$ and $c_4' = a_4$, and choose these two 6-numberings so that their proximity distance (k say) is as small as possible. Since $v \in A_2$, it follows that $P(v) \subseteq \{1\frac{1}{2}, 2, 2\frac{1}{2}\}$. Since v is nonadjacent to c_1 we deduce that v is in position $2\frac{1}{2}$ relative to c_1-c_2-$\cdots$-c_6-c_1, and is a hat; and similarly it is in position $2\frac{1}{2}$ relative to c_1'-c_2'-$\cdots$-c_6'-c_1', and is a star. In particular, v belongs to neither of the 6-numberings; and $c_1 \ne c_1'$, and $c_4 \ne c_4'$. It follows that the two 6-numberings are not proximate, and so $k > 1$. Consequently there is a third 6-numbering c_1''-c_2''-$\cdots$-c_6''-c_1'' in $\mathcal{C}$, proximate to c_1'-c_2'-$\cdots$-c_6'-c_1', and with proximity distance to c_1-c_2-$\cdots$-c_6-c_1 less than k. From the minimality of k, it follows that c_4'' is nonadjacent to v, and therefore $c_4'' \ne a_4$; and so $c_i'' = c_i'$ for all $i \in \{1, \ldots, 6\}$ with $i \ne 4$. Consequently c_1'-v-c_3'-c_4''-c_5'-c_6'-c_1' is a 6-numbering, and therefore belongs to $\mathcal{C}$. Since c_1 is nonadjacent to v, and $P(c_1) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$, it follows that relative to this last 6-numbering, c_1 is in position $\frac{1}{2}$ and is a hat. Consequently c_1 is nonadjacent to c_3', c_5', and is adjacent to c_6'.

Suppose that c_1 is in position 1 relative to c_1'-c_2'-$\cdots$-c_6'-c_1'. Then c_1-c_2'-c_3--$\cdots$-c_6'-c_1 belongs to $\mathcal{C}$, and yet v is in position 3 relative to it, contradicting that $P(v) \subseteq \{1\frac{1}{2}, 2, 2\frac{1}{2}\}$. So c_1 is not in position 1 relative to c_1'-c_2'-$\cdots$-c_6'-c_1'. Since $P(c_1) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$ and c_1 is nonadjacent to c_3' and adjacent to c_6', we deduce that c_1 is in position $\frac{1}{2}$ relative to c_1'-c_2'-$\cdots$-c_6'-c_1' Since c_1 is nonadjacent to c_5', it follows that c_1 is nonadjacent to c_2'.

Since $\{c_2, c_2', c_4', c_1\}$ is not a claw, it follows that c_2, c_4' are nonadjacent. Since A_4 is a clique, c_4' is adjacent to c_4. Since $\{c_4', v, c_4, c_6\}$ is not a claw, c_4' is not adjacent to c_6. Thus if c_4' is in position $4\frac{1}{2}$ relative to c_1-$\cdots$-c_6-c_1, then it is a hat; but then $G|\{c_1, \ldots, c_6, v, c_4'\}$ is a long prism, a contradiction. If c_4' is in position 4 relative to c_1-c_2-$\cdots$-c_6-c_1, then v is in position 3 relative to c_1-c_2-c_3-c_4'-c_5-c_6-c_1, contradicting that $P(v) = \{1\frac{1}{2}, 2, 2\frac{1}{2}\}$. Thus, c_4' is in position $3\frac{1}{2}$ relative to c_1-c_2-$\cdots$-c_6-c_1, and therefore is a hat, since c_2, c_4' are nonadjacent. Then c_1-c_2-v-c_4'-c_4-c_5-c_6-c_1 is a 7-hole, a contradiction. Thus there is no such vertex v. This proves (2).

To complete the proof of the fourth statement of the theorem, again we may assume that $i = 2$. Suppose that $v, v' \in A_2$, and v has a neighbour $a_4 \in A_4$, and v' has a nonneighbour $a_1 \in A_1$. By (2), v', a_4 are nonadjacent, and v, a_1 are adjacent. But then $\{v, v', a_1, a_4\}$ is a claw, a contradiction. This proves the fourth statement of the theorem.

For the fifth statement, let us first prove the following:

(3) *For $1 \le i \le 6$, every vertex in A_i is either A_{i+1}-complete or A_{i-1}-complete.*

For we may assume that $i = 2$. Let $a_2 \in A_2$, and assume it has nonneighbours $a_1 \in A_1$ and $a_3 \in A_3$. Since a_1 is not complete to A_2, it is therefore anticomplete to A_3 by the third statement of the theorem; and in particular, a_1, a_3 are nonadjacent. Choose $x, y \in A_2$ adjacent to a_1, a_3 respectively. Since $\{x, a_1, a_2, a_3\}$ is not a claw, it follows that x is not adjacent to a_3, and similarly y is not adjacent to a_1. Thus a_1-x-y-a_3 is a path. Choose c_1-c_2-$\cdots$-c_6-$c_1 \in \mathcal{C}$ with $a_2 = c_2$. Now $P(a_1) \subseteq \{\frac{1}{2}, 1, 1\frac{1}{2}\}$, and a_1 is not adjacent to a_2; and so relative to c_1-c_2-$\cdots$-c_6-c_1, a_1 is a hat in position $\frac{1}{2}$. Similarly, a_3 is a hat in position $3\frac{1}{2}$. Now x has no neighbours in A_4, A_5, A_6, by respectively the third, first and fourth statements of the theorem, since x is not complete to A_3. Similarly y is anticomplete to $A_4 \cup A_5 \cup A_6$. It follows that a_1-x-y-a_3-c_4-c_5-c_6-a_1 is a 7-hole in G, a contradiction. This proves (3).

Now to prove the fifth statement of the theorem, we may assume that $i = 2$. Suppose that $a_1 \in A_1$ and $a_3 \in A_3$ both have nonneighbours in A_2. By (3) they have no common nonneighbour, and so there is a path a_1-x-y-a_3 where $x, y \in A_2$. Choose $c_i \in A_i$ for $i = 4, 5, 6$, such that c_4-c_5-c_6 is a path. By (3) and the first, third and fourth statements of the theorem, a_1-x-y-a_3-c_4-c_5-c_6-a_1 is a 7-hole in G, a contradiction. This proves the fifth statement, and therefore proves 11.4. ■

We have two applications for the previous theorem. The first is the following.

11.5 *Let G be claw-free, containing no long prism, and such that every 6-hole in G is dominating. Let C_0 be a 6-hole in G; and suppose that there is a hub for C_0, and some vertex of $V(G) \setminus V(C_0)$ is a clone with respect to C_0. Then G is decomposable.*

Proof. Let C_0 have vertices a_1-$\cdots$-a_6-a_1, and let w be a hub, adjacent to a_1, a_2, a_4, a_5 say. Let $\mathcal{C}$ be the proximity component containing C_0, and let $A_i = A_i(\mathcal{C})$ for $1 \le i \le 6$. By 11.3, w is a hub in hub-position 3 relative to every member of $\mathcal{C}$. Consequently, w is complete to $A_1 \cup A_2 \cup A_4 \cup A_5$, and anticomplete to $A_3 \cup A_6$. We observe first that:

(1) *Let $1 \le i \le 6$, let $v \in A_i$, and let N be the union of $\{v\}$ and the set of neighbours of v in G. Let c_1-$\cdots$-c_6-$c_1 \in \mathcal{C}$. If $i = 3, 6$, N contains c_i and at least one of c_{i-1}, c_{i+1}, and none of $c_{i+2}, c_{i+3}, c_{i+4}$. (Consequently, A_3 is anticomplete to $A_5 \cup A_6 \cup A_1$.) If $i = 1, 2$ N contains both of c_1, c_2, and at most one of c_4, c_5 (and symmetrically if $i = 4, 5$).*

For v belongs to some member of $\mathcal{C}$, and the claim holds for that member. Consequently it suffices to show that if c_1-$\cdots$-c_6-c_1 and c'_1-$\cdots$-c'_6-c'_1 are proximate members of $\mathcal{C}$, and the claim holds for c_1-$\cdots$-c_6-c_1 then it holds for c'_1-$\cdots$-c'_6-c'_1. Let these two 6-numberings differ in their jth entry. Assume first that $i \in \{3, 6\}$, say $i = 3$. Thus N contains at least two of c_2, c_3, c_4 and none of w, c_5, c_6, c_1. Hence if $j \in \{5, 6, 1\}$ then N contains least two of c'_2, c'_3, c'_4, and if $j \in \{2, 3, 4\}$ then N contains none of c'_5, c'_6, c'_1; and in either case, since $w \notin N$, it follows from 11.1 that N contains c'_3 and at least one of c'_2, c'_4, and contains none of c'_5, c'_6, c'_1 as required. Now assume that $i \in \{1, 2\}$, and consequently $c_1, c_2, w \in N$, and not both $c_4, c_5 \in N$. Thus if $j \in \{3, 4, 5, 6\}$ then $c'_1, c'_2 \in N$, and if $j \in \{6, 1, 2, 3\}$ then not both $c'_4, c'_5 \in N$. Since $w \in N$ and v is not a hub relative to c'_1-$\cdots$-c'_6-c'_1 (by 11.3), it follows in either case from 11.1 applied to c'_1-$\cdots$-c'_6-c'_1 that a_i is $\{c'_1, c'_2\}$-complete and not $\{c'_4, c'_5\}$-complete, as required. This proves (1).

(2) $\mathcal{C}$ *is pure.*

We must show that $A_1, \ldots, A_6$ are pairwise disjoint. The members of A_1, A_2, A_4, A_5 are adjacent to w, and those of A_3, A_6 are not. Also, by (1), members of $A_1 \cup A_2$ are $\{a_1, a_2\}$-complete and not $\{a_4, a_5\}$-complete; and members $A_4 \cup A_5$ are $\{a_4, a_5\}$-complete and not $\{a_1, a_2\}$-complete. Thus the three sets $A_3 \cup A_6, A_1 \cup A_2, A_4 \cup A_5$ are pairwise disjoint. To prove the claim, it remains to show that the intersections $A_3 \cap A_6, A_1 \cap A_2, A_4 \cap A_5$ are all empty. Now members of A_3 are adjacent to a_3 and not to a_6 by (1), and vice versa for A_6, and so $A_3 \cap A_6 = \emptyset$. Suppose that $v \in A_1 \cap A_2$ say. Since $v \in A_1$, there exists c_1-$\cdots$-c_6-$c_1 \in \mathcal{C}$ with $c_1 = v$; and since $c_6 \in A_6$, it follows that v has a neighbour x say in A_6. Similarly v has a neighbour y in A_3; and since A_3, A_6 are anticomplete by (1), it follows that $\{v, w, x, y\}$ is a claw, a contradiction. Thus $A_1 \cap A_2 = \emptyset$ and similarly $A_4 \cap A_5 = \emptyset$. This proves (2).

We deduce that the five statements of 11.4 hold. In particular, each A_i is a clique, and there are no edges between A_i and A_{i+3}, and every vertex not in $A_1 \cup \cdots \cup A_6$ is complete or anticomplete to each A_i.

(3) A_1, A_2 *are complete to each other, and so are* A_4, A_5*.*

For suppose that $x \in A_1$ is nonadjacent to $y \in A_2$. Since x belongs to some member of $\mathcal{C}$, it has a nonneighbour $z \in A_5$; and by 11.4, y is also nonadjacent to z. But then $\{w, x, y, z\}$ is a claw, a contradiction. This proves (3).

By hypothesis, there is a vertex with exactly three neighbours in C_0, and so at least one of $A_1, \ldots, A_6$ has cardinality > 1. From the symmetry we may assume that this is one of A_2, A_3, A_4. By the final statement of 11.4, A_3 is complete to one of A_2, A_4, say A_4. Let A_2' be the set of vertices in A_2 with a nonneighbour in A_3. By the third statement of 11.4, A_2' is anticomplete to A_4. Then (A_2', A_3) and $(A_2 \setminus A_2', A_4)$ are both homogeneous pairs of cliques, and they are both nondominating since $A_6 \neq \emptyset$. We may therefore assume that all four of these sets have cardinality at most one, for otherwise G is decomposable by 3.3. Hence $A_4 = \{a_4\}$ and $A_3 = \{a_3\}$. Now every vertex of A_2 has at least one neighbour in A_3, and since $|A_3| = 1$, it follows that they are all complete to A_3, that is, $A_2' = \emptyset$. Thus $A_2 = \{a_2\}$, and so all three of A_2, A_3, A_4 have cardinality 1, a contradiction. This proves 11.5. ■

Let c_1-$\cdots$-c_6-c_1 be a 6-hole. We recall that if b_1, b_2 are adjacent stars in positions $i + \frac{1}{2}, i + 3\frac{1}{2}$ for some $i \in \{1, \ldots, 6\}$, we call $\{b_1, b_2\}$ a *star-diagonal.* The subgraph formed by these eight vertices is also an induced subgraph of the icosahedron, obtained by deleting two vertices at distance two and both their common neighbours. The next result is our second application of 11.4.

11.6 *Let G be claw-free, such that every 6-hole in G is dominating, and G contains no long prism. Let C_0 be a 6-hole in G with a star-diagonal. If some vertex of $V(G) \setminus V(C_0)$ is a clone with respect to C_0, then G is decomposable.*

Proof. Let C_0 have vertices a_1-$\cdots$-a_6-a_1, and let b_1, b_2 be adjacent stars in positions $1\frac{1}{2}, -1\frac{1}{2}$ respectively, say. By 10.1, we may assume that G does not contain *icosa*(-3). By 11.5, we may assume that no vertex is a hub for C_0. Let $\mathcal{C}$ be the proximity component containing $a_1\text{-}\cdots\text{-}a_6\text{-}a_1$, and let $A_i = A_i(\mathcal{C})$ for $1 \le i \le 6$.

(1) *For every $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$, b_1, b_2 are stars in positions $1\frac{1}{2}, -1\frac{1}{2}$ respectively.*

For let $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$ and $c_1'\text{-}\cdots\text{-}c_6'\text{-}c_1'$ be proximate members of $\mathcal{C}$, differing only in their jth term say; it suffices to show that if the claim holds for $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$ then it holds for $c_1'\text{-}\cdots\text{-}c_6'\text{-}c_1'$. Let N be the union of $\{c_j'\}$ and the set of neighbours of c_j' in G. Thus $c_{j-1}, c_j, c_{j+1} \in N$, and $c_{j+2}, c_{j+3}, c_{j+4} \notin N$. From the symmetry we may assume that $j \in \{2, 3\}$. Suppose first that $j = 2$. Then we must prove that $b_1 \in N$ and $b_2 \notin N$. Now 4.2 (with $b_1\text{-}c_3\text{-}c_4$) implies that $b_1 \in N$; and 4.2 (with $c_4\text{-}b_2\text{-}c_6$) implies that $b_2 \notin N$. Next, suppose that $j = 3$; we must prove that $b_1, b_2 \in N$. If $b_1, b_2 \notin N$, then $G|\{c_1, \ldots, c_6, b_1, b_2, c_3'\}$ is isomorphic to $icosa(-3)$, a contradiction. Thus N contains at least one of b_1, b_2, and from the symmetry we may assume it contains b_1. By 4.2 (with $c_1\text{-}b_1\text{-}b_2$), it follows that $b_2 \in N$. This proves (1).

(2) *Let $1 \le i \le 6$, let $v \in A_i$, and let N be the union of $\{v\}$ and the set of neighbours of v in G. Let $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$. If $i = 3, 6$, $c_{i+3} \notin N$, and $c_{i-1}, c_i, c_{i+1} \in N$. (Consequently A_3 is anticomplete to A_6.) If $i = 1, 2$, N contains both of c_1, c_2, and at most one of c_4, c_5 (and symmetrically if $i = 4, 5$).*

For v belongs to some member of $\mathcal{C}$, and the claim is true for that member. Consequently, it suffices to show that if $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$ and $c_1'\text{-}\cdots\text{-}c_6'\text{-}c_1'$ are proximate members of $\mathcal{C}$, and the claim holds for $c_1\text{-}\cdots\text{-}c_6\text{-}c_1$, then it holds for $c_1'\text{-}\cdots\text{-}c_6'\text{-}c_1'$. Let these two 6-numberings differ in their jth entry. Assume first that $i \in \{3, 6\}$, say $i = 3$. Thus N contains b_1, b_2, c_2, c_3, c_4 and $c_6 \notin N$. If $j \neq 6$, then $c_6' = c_6 \notin N$, and by 4.2 (with $c_2'\text{-}b_1\text{-}c_6'$, $c_3'\text{-}b_1\text{-}c_6'$, and $c_4'\text{-}b_2\text{-}c_6'$), it follows that $c_2', c_3', c_4' \in N$. If $j = 6$, then $c_2', c_4' \in N$, and so $c_6' \notin N$ by 4.1. Thus in either case the claim holds. Now assume that $i = 1$. Thus $b_1 \in N$ and $b_2 \notin N$. By 4.2 (with $b_2\text{-}b_1\text{-}c_1'$), $c_1' \in N$ and similarly $c_2' \in N$. Since v is not a hub relative to $c_1'\text{-}\cdots\text{-}c_6'\text{-}c_1'$ by 11.3, it follows that N contains at most one of c_4', c_5'. This proves (2).

(3) *$\mathcal{C}$ is pure.*

We must show that $A_1, \ldots, A_6$ are pairwise disjoint. By (1), the members of $A_3 \cup A_6$ are adjacent to both b_1, b_2; the members of $A_1 \cup A_2$ are adjacent to b_1 and not to b_2; and the members of $A_4 \cup A_5$ are adjacent to b_2 and not to b_1. Consequently the three sets $A_3 \cup A_6$, $A_1 \cup A_2$, $A_4 \cup A_5$ are pairwise disjoint. By (2), the members of $A_3 \setminus \{a_3\}$ are adjacent to a_3, and the members of A_6 are not adjacent to a_3, and so $A_3 \cap A_6 = \emptyset$. Suppose that $v \in A_1 \cap A_2$ say. Since $v \in A_1$, there exists $c_1\text{-}\cdots\text{-}c_6\text{-}c_1 \in \mathcal{C}$ with $c_1 = v$; and since $c_3 \in A_3$, it follows that v has a nonneighbour x say in A_3. Similarly v has a nonneighbour y in A_6; and since A_3, A_6 are anticomplete by (1), it follows that $\{b_1, v, x, y\}$ is a claw, a contradiction. Thus $A_1 \cap A_2 = \emptyset$ and similarly $A_4 \cap A_5 = \emptyset$. This proves (3).

We deduce that the five statements of 11.4 hold. In particular, each A_i is a clique, and there are no edges between A_i and A_{i+3}, and every vertex not in $A_1 \cup \cdots \cup A_6$ is complete or anticomplete to each A_i.

(4) *We may assume (possibly after renumbering $A_1, \ldots, A_6$) that there is a vertex $h \in V(G) \setminus (A_1 \cup$*

$\cdots \cup A_6 \cup \{b_1, b_2\})$, *such that* h *is* $A_1 \cup A_2$*-complete and anticomplete to* A_3, A_4, A_5, A_6.

For since some vertex is a clone relative to C_0, at least one of the sets $A_1, \ldots, A_6$ has at least two members, and therefore from the symmetry we may assume that not all of A_1, A_3, A_5 have cardinality 1. Now A_1, A_3, A_5 are cliques, all nonempty, and their union is not equal to $V(G)$; so by 3.5, we may assume that some $h \in V(G) \setminus (A_1 \cup A_3 \cup A_5)$ does not have the property that it is complete to two of A_1, A_3, A_5 and anticomplete to the third. Consequently $h \notin A_1 \cup \cdots \cup A_6 \cup \{b_1, b_2\}$, and therefore h is complete or anticomplete to each A_i, and is complete to exactly two of $A_1, \ldots, A_6$, necessarily consecutive. If say h is complete to A_2, A_3, then by 8.1 h is adjacent to b_1 and not to b_2; and then $\{b_1, b_2, h, a_1\}$ is a claw, a contradiction. Thus h is complete to either A_1, A_2 or to A_4, A_5, and anticomplete to the other four sets. This proves (4).

(5) *For* $1 \leq i \leq 6$, A_i *is complete to* A_{i+1}; *and* A_2 *is anticomplete to* $A_4 \cup A_6$, *and* A_1 *is anticomplete to* $A_3 \cup A_5$.

For if $x \in A_1$ and $y \in A_2$, they are both adjacent to b_1 and nonadjacent to b_2, and since $\{b_1, b_2, x, y\}$ is not a claw, it follows that x, y are adjacent. Thus A_1 is complete to A_2, and similarly A_4 is complete to A_5. Now let $x \in A_2, y \in A_3$. Choose $z \in A_6$ nonadjacent to x; then since $\{b_1, x, y, z\}$ is not a claw, x, y are adjacent. Hence A_2, A_3 are complete, and similarly A_i is complete to A_{i+1} for $1 \leq i \leq 6$. Now let $x \in A_2$ and $y \in A_4$, and let h be as in (4). Then h is adjacent to x and not to y; and h is nonadjacent to b_1 by 8.2. Since $\{x, b_1, y, h\}$ is not a claw, it follows that x, y are nonadjacent. Thus A_2, A_4 are anticomplete, and similarly so are A_1, A_5. Now let $x \in A_2, y \in A_6$. Let z be a neighbour of x in A_3 (this exists, since x belongs to a member of $\mathcal{C}$). Since $\{x, y, z, h\}$ is not a claw, x, y are nonadjacent, and so A_2, A_6 are anticomplete. Similarly A_1, A_3 are anticomplete. This proves (5).

To complete the proof, suppose that A_3 is not anticomplete to A_5; then (A_3, A_5) is a homogeneous pair, nondominating since $A_1 \neq \emptyset$; and since some vertex of A_3 has a neighbour in A_5, and every vertex in A_3 has a nonneighbour in A_5, it follows that $|A_5| > 1$, and therefore G is decomposable, by 3.3. We may therefore assume that A_3 is anticomplete to A_5, and similarly A_4 is anticomplete to A_6. Hence for $1 \leq i \leq 6$, all members of A_i are twins, and since there is a clone relative to a_1-$\cdots$-a_6-a_1 and therefore one of $A_1, \ldots, A_6$ has cardinality > 1, we deduce that G is decomposable. This proves 11.6. ■

12 Generalized breakers

Let us say that a triple (A, C, B) is a *generalized breaker* in G if it satisfies:

- A, B, C are disjoint nonempty subsets of $V(G)$, and A, B are cliques
- every vertex in $V(G) \setminus (A \cup B \cup C)$ is either A-complete or A-anticomplete, and either B-complete or B-anticomplete, and C-anticomplete,
- there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a neighbour in A and a nonneighbour in B; there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a neighbour in B and a nonneighbour in A; and there is a vertex in $V(G) \setminus (A \cup B \cup C)$ with a nonneighbour in A and a nonneighbour in B.

Thus, this is the same as the definition of a breaker, except that the final condition has been removed. There is an analogue of 3.4 for generalized breakers, the following.

12.1 *Let G be claw-free, such that G contains no long prism, and every 6-hole in G is dominating. If there is a generalized breaker in G, then either G is decomposable, or $G \in \mathcal{S}_2 \cup \mathcal{S}_4 \cup \mathcal{S}_5$.*

Proof. We assume G is not decomposable. Let (A, C, B) be a generalized breaker; let $V_1 = A \cup B \cup C$, let V_0 be the set of vertices in $V(G) \setminus V_0$ that are $A \cup B$-complete, and let $V_2 = V(G) \setminus (V_0 \cup V_1)$. Let A_2 be the set of vertices in V_2 that are A-complete, and B_2 the set that are B-complete. Let X be the set of all vertices in $V_2 \setminus (A_2 \cup B_2)$ with a neighbour in V_0. By hypothesis, A, B, A_2, B_2 are nonempty, and as in the proof of 3.4, it follows that $A_2 \cup V_0$ and $B_2 \cup V_0$ are cliques. By 3.4, $X \neq \emptyset$ and A is complete to B. Since $A \cup B$ is not an internal clique cutset, it follows that $|C| = 1$, $C = \{c\}$ say. We may assume that c has a neighbour $a \in A$. Let $b \in B$ and $a_2 \in A_2$; then since $\{a, a_2, c, b\}$ is not a claw, it follows that c, b are adjacent, and therefore that c is complete to B. Similarly c is complete to A. Hence all vertices in A are twins, so $|A| = 1$ and similarly $|B| = 1$, and c is adjacent to both members of $A \cup B$. Every vertex in X has a neighbour in V_0, and since $V_0 \cup A_2 \cup X \cup B$ includes no claw, $X \cup A_2$ is a clique and similarly $X \cup B_2$ is a clique. Let $V_3 = V_2 \setminus (A_2 \cup B_2 \cup X)$. Let S be the set of vertices in V_3 with a neighbour in A_2, and T those with a neighbour in B_2. Define $Y = S \cap T, M = S \setminus Y, N = T \setminus Y$, and $Z = V_3 \setminus (S \cup T)$; thus, M, N, Y, Z are pairwise disjoint and have union V_3. Since $A_2 \cup A \cup X \cup Y$ includes no claw, it follows that $X \cup Y$ is a clique, and similarly $X \cup M$ and $X \cup N$ are cliques. Since $X \cup V_0 \cup M \cup N \cup Y$ includes no claw, $M \cup N \cup Y$ is a clique. Since $X \cup M \cup N \cup Y$ is not an internal clique cutset, $|Z| \leq 1$.

(1) *A_2 is not complete to B_2.*

For assume it is. Since $A_2 \cup B_2 \cup V_0$ is not an internal clique cutset, it follows that $|X| = 1$ and Y, M, N, Z are all empty. But then $(A \cup B, A_2 \cup B_2)$ is a homogeneous pair, coherent since V_0 is a clique, a contradiction to 3.3. This proves (1).

From (1), and since $X \cup Z \cup A_2 \cup B_2$ includes no claw, we deduce that X is anticomplete to Z. Thus (V_0, X) is a homogeneous pair, nondominating since $C \neq \emptyset$, and so 3.3 implies that V_0, X both have cardinality 1. Let $V_0 = \{v_0\}$ and $X = \{x\}$.

(2) *Every vertex in A_2 with a neighbour in B_2 is anticomplete to M, and every vertex in A_2 with a nonneighbour in B_2 is complete to M.*

For let $v \in A_2$. If v has a neighbour in M and a neighbour in B_2, then $\{v\} \cup A \cup B_2 \cup M$ includes a claw, while if v has a nonneighbour in M and a nonneighbour in B_2, then $\{x, v\} \cup B_2 \cup M$ includes a claw. This proves (2).

Let A' be the set of vertices in A_2 with a neighbour in M; then by (2), every vertex in A' is anticomplete to B_2 and is complete to M. Let B' be the set of vertices in B_2 with a neighbour in N. Let $a_1 \in A$, and $b_1 \in B$.

(3) *If M, N are both nonempty then $G \in \mathcal{S}_2 \cup \mathcal{S}_4$.*

For assume that M, N are nonempty, and choose $m \in M$ and $n \in N$. Since m has a neighbour in $a' \in A_2$, it follows that $a' \in A'$. Choose $b' \in B'$ similarly. By (2), a' is anticomplete to B_2 and b' to A_2. Suppose there exist $a'' \in A_2$ and $b'' \in B_2$, such that a'', b'' are adjacent. By (2), a'' is anticomplete to M and b'' to N, and it follows that a'-m-n-b'-b''-a''-a' is a 6-hole, and it does not dominate the vertex in C, contrary to the hypothesis. Thus A_2 is anticomplete to B_2, and so (2) implies that $A' = A_2$ and $B' = B_2$. Now a_1-a'-m-n-b'-b_1-a_1 is a 6-hole, and therefore it dominates every vertex in Z; and so m, n are Z-complete. Since this holds for all choices of m, n, we deduce that M, N are Z-complete. Hence all vertices in M are twins, and so $|M| = 1$, and similarly $|N| = 1$. Since every vertex in Y has neighbours in A_2 and in B_2, and these are therefore nonadjacent, it follows that Y is anticomplete to Z (for otherwise $Y \cup Z \cup A_2 \cup B_2$ would include a claw). Any vertex in A_2 different from a' is a clone with respect to the 6-numbering a_1-a'-m-n-b'-b_1-a_1, and yet $\{v_0, x\}$ is a star-diagonal relative to this 6-hole, and so 11.6 implies that $A_2 = \{a'\}$, and similarly $B_2 = \{b'\}$. Then $(V_0, X \cup Y)$ is a homogeneous pair, nondominating since $C \neq \emptyset$, and so $Y = \emptyset$. If Z is empty, then $G \in \mathcal{S}_4$, so we may assume that $|Z| = 1$. But then G has 10 vertices and belongs to $\mathcal{S}_2$. This proves (3).

By (3), we may assume henceforth that $N = \emptyset$.

(4) *If M is nonempty then $G \in \mathcal{S}_2 \cup \mathcal{S}_4$.*

For by (2), $A_2 \setminus A'$ is complete to B_2; and since every vertex in Y has a neighbour in B_2, it follows that Y is complete to $A_2 \setminus A'$, for otherwise $B_2 \cup B \cup Y \cup A_2$ would include a claw. Let Y' be the set of all vertices in Y with a nonneighbour in A_2 (necessarily in A'), and assume first that Y' is nonempty. Let $y' \in Y'$, and choose $a' \in A'$ such that y' is nonadjacent to a'. Choose $m \in M$ adjacent to a', and choose $b_2 \in B_2$. Since $\{x, y', b_2, a'\}$ is not a claw, y' is adjacent to b_2, and therefore y' is B_2-complete. Moreover, b_2-b_1-a_1-a'-m-y'-b_2 is a 6-hole with a star-diagonal $\{v_0, x\}$; and consequently there are no clones relative to this 6-hole, by 11.6. Since every vertex in $B_2 \setminus \{b_2\}$ is adjacent to y' and is therefore a clone with respect to the 6-hole, it follows that $B_2 = \{b_2\}$, and consequently Y is B_2-complete. Moreover, every vertex in $M \setminus \{m\}$ is also a clone, and so $M = \{m\}$. Since the same 6-hole dominates every vertex of Z it follows that Z is $M \cup Y'$-complete. For $y \in Y \setminus Y'$ and $z \in Z$, since $\{y, z, a', b_2\}$ is not a claw, it follows that y, z are nonadjacent; and so $Y \setminus Y'$ is anticomplete to Z. Since (Y', A') is a homogeneous pair, nondominating since $C \neq \emptyset$, it follows that $Y' = \{y'\}$ and $A' = \{a'\}$; and also, $(V_0, X \cup (Y \setminus Y'))$ is a homogeneous pair, nondominating since $C \neq \emptyset$, and so $Y' = Y$, that is, $Y = \{y'\}$; and all members of $A_2 \setminus A'$ are twins, so $|A_2 \setminus A'| \leq 1$. But then $G \in \mathcal{S}_2$ if $X \neq \emptyset$, and $G \in \mathcal{S}_4$ otherwise. This completes the argument when $Y' \neq \emptyset$.

Now assume that $Y' = \emptyset$, that is, Y is complete to A_2. Since $X \cup Y \cup A'$ is not an internal clique cutset, it follows that $Z = \emptyset$ and $|M| = 1$. Then (B_2, Y) is a nondominating homogeneous pair, so $|B_2| = 1$ and $|Y| \leq 1$, and Y is B_2-complete. Hence $(V_0, X \cup Y)$ is a nondominating homogeneous pair, and so $Y = \emptyset$. Then $((A_2 \setminus A') \cup V_0, B)$ is a homogeneous pair, nondominating since $M \neq \emptyset$, so $A_2 = A'$; and all vertices in A' are twins, so $|A_2| = 1$. But then G has only seven vertices, and belongs to $\mathcal{S}_4$. This proves (4).

In view of (4), we assume henceforth that M, N are both empty. Let H be the bipartite subgraph of G with vertex set $A_2 \cup B_2$ and edge set the edges of G with an end in A_2 and an end in B_2.

(5) *Every component H_0 of H has at most two vertices, and every vertex in Y is complete or anticomplete to H_0. Moreover, for every $y \in Y$, the set of members of $A_2 \cup B_2$ that are nonadjacent to y is a clique.*

For if $a_2 \in A_2$ and $b_2 \in B_2$ are adjacent, and $y \in Y$ is adjacent to a_2, then since $\{a_2, y, b_2, a\}$ is not a claw, it follows that y is adjacent to b_2. Hence y is complete or anticomplete to every component of H. If H_0 is any such component, then $(A_2 \cap H_0, B_2 \cap H_0)$ is a homogeneous pair, nondominating since $C \neq \emptyset$, and so $A_2 \cap H_0, B_2 \cap H_0$ both have cardinality at most 1. This proves the first two assertions of (5). For the final assertion, suppose that $y \in Y$ has two nonneighbours in $A_2 \cup B_2$ that are nonadjacent; since x is adjacent to all three vertices, this forms a claw, a contradiction. This proves (5).

(6) *If $Z \neq \emptyset$ then $G \in \mathcal{S}_2$.*

For we have already seen that $|Z| \leq 1$; assume that $Z = \{z\}$ say. Let Y_0 be the set of all neighbours of z; then $Y_0 \subseteq Y$. If $y \in Y_0$ then its set of neighbours in $A_2 \cup B_2$ is a clique, for y, z together with two nonadjacent neighbours would form a claw. Since y has at least one neighbour in each of A_2, B_2, it follows from (5) that y has exactly one neighbour in each, and they are adjacent. If every member of Y_0 has the same two neighbours (say a_2, b_2) in $A_2 \cup B_2$, then $\{x, a_2, b_2\}$ is an internal clique cutset; so we may assume that there exists $y' \in Y_0$, adjacent to a different pair of vertices $a_2' \in A_2, b_2' \in B_2$. Since every component of H has at most two vertices it follows that a_2, a_2', b_2, b_2' are all distinct. Since the nonneighbours of y in $A_2 \cup B_2$ form a clique by (5), it follows that $A_2 = \{a_2, a_2'\}$ and $B_2 = \{b_2, b_2'\}$. Let W, W', W'' be the sets of vertices in Y with neighbours $\{a_2, b_2\}, \{a_2', b_2'\}$ and $\{a_2, b_2, a_2', b_2'\}$ respectively. By (5), $W \cup W' \cup W'' = Y$ and $Y_0 \subseteq W \cup W'$. If $w \in W$, then since $\{y', w, z, a_2'\}$ is not a claw, it follows that w, z are adjacent. Thus any two vertices in W are twins, and therefore $W = \{y\}$. Similarly $W' = \{y'\}$. Now $(V_0, X \cup W'')$ is a nondominating homogeneous pair, and therefore $W'' = \emptyset$. Consequently $|V(G)| = 12$, and $G \in \mathcal{S}_2$. This proves (6).

We therefore assume that $Z = \emptyset$. Let $a_1, \ldots, a_n$ be the vertices in A_2 with a neighbour (necessarily unique) in B_2, and let $b_1, \ldots, b_n$ be their respective neighbours. Let $A_0 = A_2 \setminus \{a_1, \ldots, a_n\}$, and $B_0 = B_2 \setminus \{b_1, \ldots, b_n\}$. For each $y \in Y$, let M_y be the set of vertices in $A_2 \cup B_2$ that are not adjacent to y. By (5), M_y is a clique, and a union of components of H; and so either it is a subset of one of A_0, B_0, or it is one of the sets $\{a_i, b_i\}$. For $1 \leq i \leq n$ let Y_i be the set of vertices $y \in Y$ with $M_y = \{a_i, b_i\}$. Any two vertices in Y_i are twins, so each Y_i has cardinality ≤ 1. Let P be the set of vertices $y \in Y$ with $M_y \subseteq A_0$, and Q the set with $M_y \subseteq B_0$. Any vertex $y \in P \cap Q$ therefore satisfies $M_y = \emptyset$. Now (P, A_0) is a nondominating homogeneous pair, and so $|P|, |A_0| \leq 1$, and similarly $|Q|, |B_0| \leq 1$. But then $G \in \mathcal{S}_5$. This proves 12.1. ∎

13 Nondominating 5-holes

Before the main result of this section, we prove a lemma.

13.1 *Let H be a graph with seven vertices $v_1, \dots, v_7$, where v_1-$\cdots$-v_5-v_1 is a cycle of length 5, v_6 has three neighbours in this hole, and v_7 has two. Then some subgraph of H is a theta with seven vertices.*

Proof. By deleting one (appropriately chosen) edge incident with v_6, we obtain a subgraph consisting of the cycle v_1-$\cdots$-v_5-v_1, a vertex with two consecutive neighbours (say v_1, v_2) in this cycle, and a second vertex with two nonconsecutive neighbours in the cycle. Delete the edge v_1v_2 from this subgraph; the result is a 7-vertex theta. This proves 13.1. ∎

The main result of this section is the following, which will have a number of consequences.

13.2 *Let G be claw-free, containing no hole of length > 6 or long prism. If some 5-hole in G is not dominating, then either G is decomposable or $G \in \mathcal{S}_0 \cup \mathcal{S}_2 \cup \mathcal{S}_4 \cup \mathcal{S}_5$.*

Proof. We assume that G is not decomposable. Let C_0 be a 5-hole, and let c_1-$\cdots$-c_5-c_1 be a 5-numbering of it. Let Z be the set of all vertices that are $V(C_0)$-anticomplete, and assume that Z is nonempty. Let $z \in Z$, and let Y be the set of vertices in $V(G) \setminus Z$ that have a neighbour in the component of Z containing z.

(1) *Z is a stable set, and Y is a clique, and Y is the set of neighbours of z. Moreover, every member of Y is a hat relative to c_1-$\cdots$-c_5-c_1.*

For let Z_0 be the component of Z containing z, and let $y \in Y$. Then y has a neighbour in Z_0, say z_0, and has a neighbour in $\{c_1, \dots, c_5\}$ from the maximality of Z_0. For any two of its neighbours $x_1, x_2 \in \{c_1, \dots, c_5\}$, $\{y, z_0, x_1, x_2\}$ is not a claw, and so x_1, x_2 are adjacent. Hence y is a hat. To see that Y is a clique, let $y_1, y_2 \in Y$, and suppose that they are nonadjacent. y_1, y_2 are both hats, and are necessarily not in the same position, since they are nonadjacent and G is claw-free; let P be a path between y_1, y_2 with interior in Z_0. If y_1, y_2 share a neighbour in $\{c_1, \dots, c_5\}$, say c_5, then $G|(\{c_1, \dots, c_4\} \cup V(P))$ is a hole of length > 6, a contradiction. If y_1, y_2 share no neighbour in $\{c_1, \dots, c_5\}$, then $G|(\{c_1, \dots, c_5\} \cup V(P))$ is a long prism, a contradiction. Consequently Y is a clique. Since Y is not an internal clique cutset, it follows that $|Z_0| = 1$, and therefore $Z_0 = \{z\}$. In particular, Y is the set of neighbours of z, and z has no neighbours in Z. Since the latter holds for all choices of Z, it follows that Z is a stable set. This proves (1).

For $1 \le i \le 5$, let Y_i be the set of all members of Y that are hats in position $i + 2\frac{1}{2}$ relative to c_1-$\cdots$-c_5-c_1.

(2) *Let $v \in V(G) \setminus (Y \cup \{z\})$. Then for $1 \le i \le 5$, v is complete or anticomplete to Y_i. Moreover, if v is a hat relative to c_1-$\cdots$-c_5-c_1, then v is complete to Y_i if and only if v is in position $i + 2\frac{1}{2}$.*

For suppose that v has a neighbour y_1 and a nonneighbour y_2, both in Y_i. Since $v \notin Y \cup \{z\}$, it follows that v is nonadjacent to z. Now y_1, y_2 are hats in position $i + 2\frac{1}{2}$. By 4.2 applied to c_{i+2}-y_1-z, it follows that v is adjacent to c_{i+2} and similarly to c_{i+3}. By 4.2 applied to y_2-c_{i+2}-c_{i+1}, we deduce that v is adjacent to c_{i+1} and similarly to c_{i-1}. But then $\{v, y_1, c_{i+1}, c_{i-1}\}$ is a claw, a contradiction. This proves the first claim of (2). For the second claim, suppose that v is a hat, in position $j+2\frac{1}{2}$ say. Since $v \notin Y$, it follows that v, z are nonadjacent. If $j=i$ then v is Y_i-complete by 4.3. If $j \neq i$, choose $a \in \{c_{i+2}, c_{i-2}\}$ nonadjacent to v; then for $y \in Y_i$, $\{y, z, a, v\}$ is not a claw, and so y is nonadjacent to v. This proves (2).

(3) *We may assume that $Y_i \neq \emptyset$ for at least three values of $i \in \{1, \dots, 5\}$. Also, we may assume that every hat nonadjacent to z is nonadjacent to every other hat except those in the same position relative to c_1-$\cdots$-c_5-c_1.*

For if all the sets Y_i are empty except possibly for say Y_1, then G is decomposable, by (2) and 3.2 applied to $Y_1, \{z\}$. If exactly two of the sets are nonempty, say Y_i, Y_j, then $(Y_i, \{z\}, Y_j)$ is a generalized breaker by (2), and the result follows from 12.1. This proves the first assertion of (3). For the second, let h be a hat nonadjacent to z, and let h' be some other hat in a different position. Suppose that h, h' are adjacent. By (2), h' is nonadjacent to z. Choose three hats adjacent to z, all in different positions, say y_1, y_2, y_3. Then $G|\{c_1, \dots, c_5, y_1, y_2, y_3, h, h'\}$ is the line graph of a graph satisfying the hypotheses of 13.1; and so by 13.1, G contains a long prism, a contradiction. This proves (3).

(4) $|Z| = 1$.

For choose $y_1, y_2, y_3 \in Y$, all hats in different positions relative to c_1-$\cdots$-c_5-c_1. Suppose that $z' \in Z$ is different from z; then similarly there are vertices y_1', y_2', y_3', all hats in different positions, and all adjacent to z'. If say y_1' is adjacent to z, then $\{y_1', z, z', a\}$ is a claw, where $a \in \{c_1, \dots, c_5\}$ is adjacent to y_1'. Thus y_1', y_2', y_3' are nonadjacent to z, and yet they are adjacent to each other by (1), contrary to (3). This proves (4).

Let $\mathcal{C}$ be the proximity component containing c_1-$\cdots$-c_5-c_1, and for $1 \leq i \leq 5$ let $A_i = A_i(\mathcal{C})$.

(5) *z has no neighbours in $A_1 \cup \cdots \cup A_5$. Moreover, for $1 \leq i \leq 5$ and each $y \in Y_i$, if a_1-$\cdots$-a_5-a_1 belongs to $\mathcal{C}$ then y is a hat in position $i+2\frac{1}{2}$ relative to a_1-$\cdots$-a_5-a_1.*

For let a_1-$\cdots$-a_5-a_1 and a_1'-$\cdots$-a_5'-a_1' be proximate, with $a_j' \neq a_j$ say. Suppose first that z is nonadjacent to $a_1, \dots, a_5$; then since $\{a_j', a_{j-1}, a_{j+1}, z\}$ is not a claw, it follows that z is nonadjacent to a_j'. Consequently z has no neighbours in $A_1 \cup \cdots \cup A_5$. Now, with a_1-$\cdots$-a_5-a_1 and a_1'-$\cdots$-a_5'-a_1' as before, suppose that $y \in Y$ is a hat in position $i+2\frac{1}{2}$ relative to a_1-$\cdots$-a_5-a_1. If $j = i+2$, then by 8.2, a_j' is adjacent to y and therefore y is a hat in position $i+2\frac{1}{2}$ relative to a_1'-$\cdots$-a_5'-a_1'. If $j=i$, then by 8.2, a_j' is nonadjacent to y, and again y is a hat in position $i+2\frac{1}{2}$ relative to a_1'-$\cdots$-a_5'-a_1'. Thus from the symmetry we may assume that $j = i-1$. Since $\{y, a_j', z, a_{i+2}\}$ is not a claw, it follows that y, a_j' are not adjacent, and again the claim holds. This proves (5).

From (3) we may assume that there exist $y_3 \in Y_3$, and $y_5 \in Y_5$.

(6) *$A_1, \dots, A_5$ are pairwise disjoint; A_4 is anticomplete to A_1, A_2; A_1 is anticomplete to A_3; A_2 is anticomplete to A_5; and $A_1 \cup A_5, A_2 \cup A_3, A_4$ are cliques.*

By (5), y_3 is complete to $A_5 \cup A_1$ and anticomplete to $A_2 \cup A_3 \cup A_4$, and y_5 is complete to $A_2 \cup A_3$

and anticomplete to $A_1 \cup A_4 \cup A_5$. Consequently $A_5 \cup A_1, A_2 \cup A_3, A_4$ are pairwise disjoint. Let H be the bipartite subgraph of G with vertex set $A_1 \cup A_2$ and edges the edges of G between A_1 and A_2. Since $\mathcal{C}$ is a proximity component, it follows that H is connected. Let $a_4 \in A_4$, and assume that a_4 has a neighbour in $A_1 \cup A_2$. Since it also has a nonneighbour in $A_1 \cup A_2$ (because a_4 belongs to some member of $\mathcal{C}$), it follows that a_4 is adjacent to exactly one end of some edge of H; say $a_1 \in A_1$ and $a_2 \in A_2$ are adjacent, and a_4 is adjacent to a_1 and not to a_2. But then $\{a_1, a_2, a_4, y_3\}$ is a claw, a contradiction. This proves that a_4 is $A_1 \cup A_2$-anticomplete, and so A_4 is $A_1 \cup A_2$-anticomplete. Since no vertex of A_3 is A_4-anticomplete, it follows that $A_2 \cap A_3 = \emptyset$, and similarly $A_1 \cap A_5 = \emptyset$. Thus $A_1, \ldots, A_5$ are pairwise disjoint. Let $a_1 \in A_1$ and $a_3 \in A_3$, and let $a_4 \in A_4$ be adjacent to a_3. Since $\{a_3, a_1, y_5, a_4\}$ is not a claw, it follows that a_1, a_3 are nonadjacent. So A_1 is anticomplete to A_3, and similarly A_2 is anticomplete to A_5. Next, let $u, v \in A_1 \cup A_5$; since $\{y_3, z, u, v\}$ is not a claw it follows that u, v are adjacent. Consequently $A_1 \cup A_5$ and similarly $A_2 \cup A_3$ are cliques. Finally, suppose that $u, v \in A_4$ are nonadjacent. Choose $a_1\text{-}\cdots\text{-}a_5\text{-}a_1 \in \mathcal{C}$ with $a_4 = u$. Since A_4 is anticomplete to $A_1 \cup A_2$, it follows that v is nonadjacent to a_1, a_2, a_4, and therefore also to a_3, a_5, since there is no claw. But then by (4), with v, z exchanged, it follows that v has no neighbour in any member of $\mathcal{C}$, a contradiction. Thus A_4 is a clique. This proves (6).

Let $W = A_1 \cup \cdots \cup A_5$.

(7) *For every vertex $v \in V(G) \setminus W$, let N be the set of neighbours of v in W. Then either*

- *$N = \emptyset$ and $v = z$, or*
- *for some $i \in \{1, \ldots, 5\}$, $N = A_{i+2} \cup A_{i-2}$ (let H_i be the set of all such v), or*
- *for some $i \in \{1, \ldots, 5\}$, $N = W \setminus A_i$ (let S_i be the set of all such v), or*
- *N contains at least four of $a_1, \ldots, a_5$ for every $a_1\text{-}\cdots\text{-}a_5\text{-}a_1 \in \mathcal{C}$, and contains all five vertices for some choice of $a_1\text{-}\cdots\text{-}a_5\text{-}a_1$ (let T be the set of all such v).*

For we may assume that $v \neq z$. From the maximality of $\mathcal{C}$, N contains exactly two or at least four of $a_1, \ldots, a_5$ for every $a_1\text{-}\cdots\text{-}a_5\text{-}a_1 \in \mathcal{C}$; and since $\mathcal{C}$ is connected by proximity, the claim follows. This proves (7).

(8) *The sets H_i and S_i are cliques, for $1 \leq i \leq 5$, and so is T. For $1 \leq i, j \leq 5$, H_i is complete to S_j if $j = i+1$ or $i-1$, and otherwise H_i is anticomplete to S_j. Also, T is anticomplete to H_i for $1 \leq i \leq 5$.*

For H_i and S_i are cliques by 4.3, and the adjacency between the sets H_i and the sets S_j is forced by 8.2. Let $t \in T$; if t is adjacent to some $h \in H_i$, then $\{t, h, a_{i+1}, a_{i-1}\}$ is a claw (where $a_1\text{-}\cdots\text{-}a_5\text{-}a_1 \in \mathcal{C}$ is chosen so that t is adjacent to all of $a_1, \ldots, a_5$), a contradiction. Thus T is anticomplete to all the sets H_i. Let $t_1, t_2 \in T$. Since they are both adjacent to at least four of $c_1, \ldots, c_5$, they have at least three common neighbours in $\{c_1, \ldots, c_5\}$; and consequently one of these common neighbours, say a, is adjacent to one of y_3, y_5, say to y_3. Since $\{a, y_3, t_1, t_2\}$ is not a claw, it follows that t_1, t_2 are adjacent, and so T is a clique. This proves (8).

(9) *For $1 \leq i \leq 5$, if $H_i \neq \emptyset$, then T is complete to A_{i-1} and to A_{i+1}.*

For let $t \in T$ and $h \in H_i$. By (8), t, h are nonadjacent. Let $a_1\text{-}\cdots\text{-}a_5\text{-}a_1 \in \mathcal{C}$. Since t, h are nonadjacent and t has at least four neighbours in the hole $a_1\text{-}\cdots\text{-}a_5\text{-}a_1$, 8.2 implies that t, a_{i-1} are adjacent. This proves (9).

(10) *T is complete to W.*

For by (9), T is complete to $A_1 \cup A_2 \cup A_4$. Suppose that H_4 is nonempty. Then by (9), T is also complete to A_3, A_5 and the claim holds. So we may assume that $H_4 = \emptyset$. By (3), we may assume that there exists $y_1 \in Y_1$, and so T is complete to A_5 by (9). By (6) with y_5, y_1 exchanged, A_3 is complete to A_4 and anticomplete to A_5. If H_2 is nonempty, then again the claim follows from (9), so we assume that $H_2 = \emptyset$. Let T' be the set of vertices in T that are not W-complete, and assume that $T' \neq \emptyset$. Since T is complete to $A_1 \cup A_2 \cup A_4 \cup A_5$, and every vertex in T has a neighbour in A_3, we deduce that every vertex in T' has a neighbour and a nonneighbour in A_3, and in particular $|A_3| > 1$.

Let $v \in V(G) \setminus (T' \cup A_3)$; we claim that v is T'-complete or T'-anticomplete. If $v \in W$, v is T-complete, and if $v \in H_i$ for some i then v is T-anticomplete by (8). If $v \in T \setminus T'$ then v is T'-complete by (8); so we may assume that $v \in S_i$ for some $i \in \{1, 2, 3, 4, 5\}$. If $v \in S_1$ then v is T'-complete, since for $t \in T'$, $\{c_5, v, t, y_3\}$ is not a claw. Similarly v is T'-complete if $v \in S_5$. If $v \in S_2$ then v is T'-complete, since for $t \in T'$, $\{a_3, v, t, y_5\}$ is not a claw, where $a_3 \in A_3$ is adjacent to t; and similarly v is T'-complete if $v \in S_4$. If $v \in S_3$ then v is T'-complete by 4.3. This proves the claim. But every such v is also complete or anticomplete to A_3, and so (A_3, T') is a homogeneous pair, nondominating since $Z \neq \emptyset$; and therefore G is decomposable, by 3.3, since $|A_3| > 1$. This proves (10).

(11) *$A_1, \ldots, A_5$ all have cardinality 1.*

For by (10), T is complete to $A_1 \cup A_2$, and so (A_1, A_2) is a homogeneous pair, nondominating since $Z \neq \emptyset$; and hence $|A_1| = |A_2| = 1$. Suppose that there exists $y_1 \in Y_1$. Then similarly $|A_4| = |A_5| = 1$. But then all members of A_3 are twins, and so $|A_3| = 1$ and the claim holds. Thus we may assume that Y_1 is empty, and similarly $Y_2 = \emptyset$. Hence there exists $y_4 \in Y_4$, by (3). Suppose that A_4 is not complete to A_5, and choose $a_4 \in A_4$ and $a_5 \in A_5$, nonadjacent. If there exists $t \in S_1 \cup S_3 \cup T$ then $\{t, a_4, a_5, c_2\}$ is a claw, and so $T, S_1, S_3 = \emptyset$. Suppose that there exists $h \in H_2$, necessarily not adjacent to z; then by (2) it is nonadjacent to y_3, y_4. Let $a_3 \in A_3$ be adjacent to a_4. Since $\{a_3, a_4, a_5, y_5\}$ is not a claw, a_3 is nonadjacent to a_5; but then $c_2\text{-}a_3\text{-}a_4\text{-}h\text{-}a_5\text{-}y_3\text{-}y_4\text{-}c_2$ is a 7-hole, a contradiction. Thus H_2 is empty. Suppose that also A_4 is not complete to A_3; then similarly $S_2, S_5, H_1 = \emptyset$. But then (A_3, A_4, A_5) is a breaker, and the theorem holds. We may therefore assume that A_4 is complete to A_3. Let A_5' be the set of vertices in A_5 with a nonneighbour in A_4. Since A_5' is nonempty and every member of A_5' has a neighbour and a nonneighbour in A_4, it follows that $|A_4| > 1$. No member $a_5' \in A_5'$ has a neighbour $a_3' \in A_3$, because $\{a_3', a_4', a_5', y_5\}$ would be a claw, where $a_4' \in A_4$ is a nonneighbour of a_5'. Thus (A_5', A_4) is a nondominating homogeneous pair, contrary to 3.3. This proves that A_4 is complete to A_5, and similarly to A_3. Hence (A_3, A_5) is a nondominating homogeneous pair, and so A_3, A_5 both have cardinality 1; and all members of A_4 are twins, so $|A_4| = 1$. This proves (11).

(12) *Let $1 \leq i, j \leq 5$, such that $H_i \neq \emptyset$. Then if $j \in \{i, i+2, i-2\}$, S_i is complete to S_j, and*

otherwise S_i is anticomplete to S_j. Also, T is complete to $S_1, \ldots, S_5$.

By 4.3, if $i = j$ then S_i is complete to S_j. Let $h \in H_i$. Suppose that $j = i + 2$, and that $s_i \in S_i$ and $s_j \in S_j$ are nonadjacent. Then $\{c_{i-2}, s_i, h, s_j\}$ is a claw, a contradiction. Hence in this case S_i is complete to S_j, and similarly if $j = i - 2$. Now assume that $j = i + 1$, and $s_i \in S_i$ and $s_j \in S_j$ are adjacent. Then $\{s_j, h, s_i, c_i\}$ is a claw, a contradiction. Thus S_i is anticomplete to S_{i+1}, and similarly to S_{i-1}. Finally, suppose that $t \in T$ and $s_j \in S_j$ are nonadjacent, for some j with $1 \leq j \leq 5$. Now one of H_j, H_{j+2}, H_{j-2} is nonempty, and both t, s_j are anticomplete to these three sets; so there is a hat h nonadjacent to both t, s_j. But one of $c_1, \ldots, c_5$ is adjacent to all of t, s_j, h, and hence these four vertices form a claw, a contradiction. This proves (12).

(13) *Let $1 \leq i, j \leq 5$. Then if $j \in \{i, i+2, i-2\}$, S_i is complete to S_j, and otherwise S_i is anticomplete to S_j.*

Suppose first that $j = i + 1$ and S_i,S_j are not anticomplete. By (12), H_i, H_{i+1} are both empty, and since H_3, H_5 are nonempty, it follows that $i = 1$, and Y_4 is nonempty. Choose $s_1 \in S_1$ and $s_2 \in S_2$, adjacent. If there exists $s_3 \in S_3$, then by (12) s_3 is adjacent to s_1 and not to s_2 (since $H_3 \neq \emptyset$), and so $\{s_1, s_3, s_2, y_5\}$ is a claw, a contradiction. Thus S_3 is empty, and similarly S_5 is empty. But then $(S_2 \cup A_5, S_1 \cup A_3)$ is a nondominating homogeneous pair, and $|S_2 \cup A_5| \geq 2$, contrary to 3.3. This proves that S_i is anticomplete to S_{i+1} for $i \leq i \leq 5$. Now assume that $j = i + 2$, and S_i, S_j are not complete. By (12), H_i, H_j are empty, and since H_3, H_5 are nonempty, it follows that one of $i, j = 4$; and from the symmetry, we may assume that $i = 2, j = 4$. Let $s_2 \in S_2$ and $s_4 \in S_4$ be nonadjacent. But then $\{y_3, z, s_2, s_4\}$ is a claw, a contradiction. This proves (13).

To finish the proof, (13) implies that for each of the sets $H_1, \ldots, H_5, S_1, \ldots, S_5, T$, any two of its members are twins; and therefore all these sets have cardinality at most 1. From (3), (8) and (13), if $T = \emptyset$ then G is a line graph, so we assume $T = \{t\}$ say. If $H_1 \cup \cdots \cup H_5 \subseteq Y$ then $G \in \mathcal{S}_4$, so we assume that there exists $h \in H_j \setminus Y$ for some j. If there exists $y \in Y_{j-1}$ then $\{c_{j+2}, y, h, t\}$ is a claw, a contradiction. So Y_{j-1} and similarly Y_{j+1} are empty. Since Y_3, Y_5 are nonempty, it follows that $j \in \{3, 5\}$ and from the symmetry we may assume that $j = 3$. Thus Y_2, Y_4 are empty, and therefore there exists $y_1 \in Y_1$. It follows that j is unique, and so $H_i \subseteq Y$ for $i = 1, 2, 4, 5$. If there exists $s \in S_2$, then $\{s, h, t, y_1\}$ is a claw, a contradiction; so $S_2 = \emptyset$, and similarly $S_4 = \emptyset$. If $S_3 \neq \emptyset$, then $(S_3 \cup T, A_3)$ is a nondominating homogeneous pair, contrary to 3.3; and so $S_3 = \emptyset$. It follows that $G \in \mathcal{S}_0$. This proves 13.2. ∎

14 6-holes with hubs and hats

In this section we handle 6-holes that have both a hub and a hat.

14.1 *Let G be claw-free, containing no long prism and no hole of length > 6, and such that every hole of length 5 or 6 is dominating. If there is a 6-hole in G relative to which some vertex is a hub and some vertex is either a hat or a clone, then either G is a line graph, or G is decomposable.*

Proof. For a contradiction, we assume that G is not decomposable. Let C_0 be the 6-hole, and let its vertices be $a_2^1, a_3^1, a_3^2, a_1^2, a_1^3, a_2^3$ in order. Define $A_j^i = \{a_j^i\}$ for $1 \leq i, j \leq 3$ with $i \neq j$. For $1 \leq i \leq 3$

let A^i_i be the set of all hubs that are nonadjacent to a^j_k, a^k_j, where $\{i,j,k\} = \{1,2,3\}$. By hypothesis, at least one of the sets A^i_i is nonempty. By 11.2, $|A^i_i| \le 1$ for $1 \le i \le 3$, since G is not decomposable; if A^i_i is nonempty, let a^i_i be its unique member. Let W be the union of the nine sets A^i_j.

For $1 \le i \le 3$, define $A^i = A^i_1 \cup A^i_2 \cup A^i_3$, and for $1 \le j \le 3$ define $A_j = A^1_j \cup A^2_j \cup A^3_j$. For $1 \le i \le 3$, let H^i, H_i, S^i, S_i be four subsets of $V(G) \setminus W$, defined as follows. For $v \in V(G) \setminus W$, let N denote the set of neighbours of v in W; then

- $v \in H^i$ if $N = A^i$
- $v \in H_i$ if $N = A_i$
- $v \in S^i$ if $N = W \setminus A^i$
- $v \in S_i$ if $N = W \setminus A_i$.

Since there is a hat relative to C_0, it follows that one of $H^1, H^2, H^3, H_1, H_2, H_3$ is nonempty.

(1) *The twelve sets H^i, H_i, S^i, S_i $(1 \le i \le 3)$ are pairwise disjoint cliques, and they have union $V(G) \setminus W$.*

For clearly they are pairwise disjoint, and they are all cliques by 4.3. Let $v \in V(G) \setminus W$. If it is a hub relative to C_0, then it belongs to one of the sets A^i_i, and therefore belongs to W, a contradiction. Since C_0 is dominating, it follows from 8.1 that v either has two, three or four neighbours in C, and they are consecutive. If it has three, then it is a clone relative to C_0, which is impossible by 11.5 since G is not decomposable. Thus it has two or four, and by 11.1 it belongs to one of the twelve sets. This proves (1).

(2) *The six sets $H^1, H^2, H^3, H_1, H_2, H_3$ are pairwise anticomplete.*

For these are hats in different position relative to C_0; if some two are adjacent, then either G contains a hole of length > 6 or a long prism, in either case a contradiction. This proves (2).

(3) *For $1 \le i, j \le 3$, H^i is anticomplete to S_j; and H^i is complete to S^j if $j \ne i$, and anticomplete to S^i. Analogous statements hold for H_i.*

This follows from 8.2.

(4) *For $1 \le i \le 3$ one of H^i, S_i is empty, and one of H_i, S^i is empty.*

For suppose that $h^1 \in H^1$ and $s_1 \in S_1$ say. Then s_1-a^2_3-a^2_1-a^3_1-a^3_2-s_1 is a 5-hole that does not dominate h^1, a contradiction.

(5) *For $1 \le i \le 3$, S^i is anticomplete to S_i.*

For suppose that $s^1 \in S^1$ and $s_1 \in S_1$ are adjacent, say. By (4), $H^1, H_1 = \emptyset$, and so from the symmetry we may assume that there exists $h^2 \in H^2$. Then $\{s^1, s_1, h^2, a^3_1\}$ is a claw, a contradiction.

This proves (5).

(6) *For* $1 \le i \le 3$*, if* $S^i \neq \emptyset$ *and* $H_1 \cup H_2 \cup H_3 \neq \emptyset$ *then* $A^i_i = \emptyset$.

For suppose that, say, $s^1 \in S^1$ and $h \in H_1 \cup H_2 \cup H_3$, and $A^1_1 = \{a^1_1\}$. By (4), $h \notin H_1$, and so we may assume that $h \in H_2$. But then s^1-a^3_1-a^1_1-a^1_3-a^2_3-s^1 is a 5-hole not dominating h, a contradiction. This proves (6).

(7) *If* $H_1 \cup H_2 \cup H_3 \neq \emptyset$ *then* S^1, S^2, S^3 *are pairwise complete.*

For suppose that $s^1 \in S^1$ is nonadjacent to $s^2 \in S^2$ say, and let $h \in H_1 \cup H_2 \cup H_3$. By (4), $h \in H_3$. By (6), $A^1_1 = A^2_2 = \emptyset$, and so $A^3_3 = \{a^3_3\}$. But then $\{a^3_3, s^1, s^2, h\}$ is a claw, a contradiction. This proves (7).

(8) *We may assume that* $S^1 \cup S^2 \cup S^3$ *is not anticomplete to* $S_1 \cup S_2 \cup S_3$.

For suppose it is. If also S^1, S^2, S^3 are pairwise complete and S_1, S_2, S_3 are pairwise complete then G is a line graph by (1)–(3), so we may assume that, say, S^1, S^2 are not complete. By (7), $H_1, H_2, H_3 = \emptyset$. Suppose that there exists $s_j \in S_j$ for some j with $1 \le j \le 3$. Choose $s^1 \in S^1$ and $s^2 \in S^2$, nonadjacent. One of a^3_1, a^3_2 is adjacent to s_j, say x; and then $\{x, s_j, s^1, s^2\}$ is a claw, a contradiction. Thus $S_1, S_2, S_3 = \emptyset$. Now each of the three cliques S^1, S^2, S^3 is complete to two of the three cliques $A^1 \cup H^1, A^2 \cup H^2, A^3 \cup H^3$ and anticomplete to the third, and so G is the hex-join of $G|(W \cup H^1 \cup H^2 \cup H^3)$ and $G|(S^1 \cup S^2 \cup S^3)$, a contradiction. This proves (8).

(9) *For* $1 \le i \le 3$*, not both* H^i, H_i *are nonempty.*

For suppose that $h^1 \in H^1$ and $h_1 \in H_1$ say. By (4), $S_1 = S^1 = \emptyset$. By (7), S^2 is complete to S^3, and S_2 is complete to S_3. By (5), S^i is anticomplete to S_i for $i = 2, 3$. By (8) we may assume from the symmetry that there exist $s^3 \in S^3$ and $s_2 \in S_2$, adjacent. From (6), $A^2_2 = A^3_3 = \emptyset$, and so $A^1_1 = \{a^1_1\}$. By (4), $H_3 = H^2 = \emptyset$. Then $(S_2 \cup A^2_1, S^3 \cup A^1_3)$ is a homogeneous pair, nondominating since $A^3_2 \neq \emptyset$, a contradiction. This proves (9).

(10) *Not both* $H^1 \cup H^2 \cup H^3$*,* $H_1 \cup H_2 \cup H_3$ *are nonempty.*

For suppose they are; then by (9), we may assume from the symmetry that there exist $h_1 \in H_1$ and $h^2 \in H^2$. By (4), $S^1, S_2 = \emptyset$, and by (9), $H^1, H_2 = \emptyset$. By (7), S_1 is complete to S_3 and S^2 is complete to S^3. By (5), S^3 is anticomplete to S_3. Suppose first that $S^2 = \emptyset$. From (8), there exist $s^3 \in S^3$ and $s_1 \in S_1$, adjacent. From (6), $A^1_1, A^3_3 = \emptyset$. Then $(S_1 \cup A^1_2, S^3 \cup A^2_3)$ is a homogeneous pair, nondominating since $A^3_1 \neq \emptyset$, a contradiction. Hence $S^2 \neq \emptyset$, and similarly $S_1 \neq \emptyset$. From (6), $A^1_1 = A^2_2 = \emptyset$, and therefore $A^3_3 = \{a^3_3\}$. By (6) again, $S^3 = S_3 = \emptyset$. But now $(A^3_1, H_1 \cup H^2 \cup A^2_1, A^2_3)$ is a breaker, contrary to 3.4. This proves (10).

(11) *Exactly one of* $H^1, H^2, H^3, H_1, H_2, H_3$ *is nonempty.*

For by hypothesis, at least one is nonempty, say H_1. By (10), $H^1, H^2, H^3 = \emptyset$. Suppose that $H_2 \neq \emptyset$. By (4), $S^1, S^2 = \emptyset$, and by (8), S^3 is nonempty. From (4), $H_3 = \emptyset$, and from (6), $A_3^3 = \emptyset$. Then $(H_1 \cup A_1^3, H_2 \cup A_2^3)$ is a homogeneous pair, nondominating since $A_3^1 \neq \emptyset$, a contradiction. This proves (11).

In view of (11) we assume henceforth that H_3 is nonempty, and therefore H^1, H^2, H^3, H_1, H_2 are empty. By (4), $S^3 = \emptyset$.

(12) *S^1, S^2 are both nonempty, and consequently $A_1^1 = A_2^2 = \emptyset$, and $A_3^3 = \{a_3^3\}$.*

For suppose that $S^2 = \emptyset$, say. From (8), $S_1 \neq \emptyset$. From (6), $A_1^1 = \emptyset$. But then $(H_3 \cup A_3^1, A_2^1)$ is a homogeneous pair, nondominating since $A_1^2 \neq \emptyset$, a contradiction. Thus S^1, S^2 are both nonempty. By (6), $A_1^1 = A_2^2 = \emptyset$, and so $A_3^3 = \{a_3^3\}$. This proves (12).

(13) *S_3 is complete to $S^1 \cup S^2$.*

For suppose not; then from the symmetry we may assume that there exist $s_3 \in S_3$ and $s^2 \in S^2$, nonadjacent. By (12) we may choose $s^1 \in S^1$. By (7), s^1, s^2 are adjacent. If s_3, s^1 are nonadjacent, then s_3-a_2^1-s^2-s^1-a_1^2-s_3 is a 5-hole, not dominating H_3, a contradiction. If s_3, s^1 are adjacent, then $\{s^1, s_3, s^2, a_3^2\}$ is a claw, a contradiction. This proves (13).

Let S_1' be the set of vertices in S_1 with a nonneighbour in S^2, and let S_2' be the set of vertices in S_2 with a nonneighbour in S^1.

(14) *$S_1' \cup S_2'$ is anticomplete to S_3, S_1' is complete to S_2, and S_2' is complete to S_1.*

For suppose that some vertex $s_1 \in S_1'$ say has a neighbour $s_3 \in S_3$. Let $s^2 \in S^2$ be a nonneighbour of s_1. Then $\{s_3, s_1, s^2, a_1^2\}$ is a claw, a contradiction. Thus S_3 is anticomplete to S_1', and similarly to S_2'. Now suppose that some $s_1 \in S_1'$ has a nonneighbour $s_2 \in S_2$. Let $s^2 \in S^2$ be a nonneighbour of s_1; then by (5), $\{a_3^3, s_2, s_1, s^2\}$ is a claw, a contradiction. Hence S_1' is complete to S_2. Similarly S_2' is complete to S_1. This proves (14).

But now the following six sets are cliques: $S_1 \setminus S_1'$; $S_2 \setminus S_2'$; S_3; $S^1 \cup A_1$; $S^2 \cup A_2$; $H_3 \cup A_3 \cup S_1' \cup S_2'$. Every vertex belongs to exactly one of these cliques; and each of the first three cliques is complete to two of the final three, and anticomplete to the other, in the manner required for a hex-join. Consequently G is expressible as a hex-join, a contradiction. This proves 14.1. ■

There is an (easy) analogue of 14.1 for 6-holes with a star-diagonal and a hat, the following.

14.2 *Let G be claw-free, containing no long prism and no hole of length > 6, and such that every hole of length 5 or 6 is dominating. If there is a 6-hole in G with a star-diagonal, relative to which some vertex is either a hat or a clone, then G is decomposable.*

Proof. Let C_0 be the 6-hole, with vertices $c_1, \dots, c_6$-c_1. Let b_1, b_2 be adjacent stars, in positions $1\frac{1}{2}, -1\frac{1}{2}$ respectively. Let h be either a hat or clone relative to C_0. If it is a clone, the result follows from 11.6. We assume then that h is a hat. From the symmetry we may assume that it is in position

$\frac{1}{2}$ or $1\frac{1}{2}$. If it is in position $\frac{1}{2}$, then by 8.2 h is adjacent to b_1 and not to b_2, and then $\{b_1, h, b_2, c_3\}$ is a claw, a contradiction. If it is in position $1\frac{1}{2}$, then it is nonadjacent to b_1 by 8.2, and then b_1-c_4-c_5-c_6-c_1-b_1 is a nondominating 5-hole, a contradiction. This proves 14.2. ■

15 Star-triangles.

We recall that, if c_1-$\cdots$-c_6-c_1 is a 6-hole, and there are three pairwise adjacent stars in positions $1\frac{1}{2}, 3\frac{1}{2}, 5\frac{1}{2}$ respectively, we call the set of these three stars a *star-triangle* for the 6-hole. Our next goal is prove an analogue of 11.6 for star-triangles. We need the following lemma.

15.1 *Let G be claw-free, and let B_1, B_2, B_3 be disjoint cliques in G. Let $B = B_1 \cup B_2 \cup B_3$. Suppose that:*

- *$B \neq V(G)$,*
- *there are two triads $T_1, T_2 \subseteq B$ with $|T_1 \cap T_2| = 2$, and*
- *there is no triad T in G with $|T \cap B| = 2$.*

Then G is decomposable.

Proof. Since there are two triads in B sharing two vertices, it follows that there is a sequence $u_1, \ldots, u_t \in B$ with $t \geq 4$, satisfying the following:

- $u_1, \ldots, u_t$ are distinct
- u_1, u_2, u_3 are pairwise nonadjacent
- for $3 \leq s \leq t$, if $u_s \in B_i$ say, then for all $j \in \{1, 2, 3\}$ with $j \neq i$, there exists r with $1 \leq r < s$ such that $u_r \in B_j$ and u_r is nonadjacent to u_s.

Choose such a sequence with t maximum. For $i = 1, 2, 3$, let $U_i = \{u_1, \ldots, u_t\} \cap B_i$.

(1) *$\{u_1, \ldots, u_t\}$ is a union of triads.*

We prove by induction on t' that for $3 \leq t' \leq t$, there is a triad in $\{u_1, \ldots, u_{t'}\}$ containing $u_{t'}$. The result is clear if $t' = 3$, so we assume that $t' > 3$. From the symmetry, we may assume that $u_{t'} \in B_3$. Choose s with $1 \leq s < t'$ minimum such that $u_{t'}$ has a nonneighbour in $B_1 \cap \{u_1, \ldots, u_s\}$ and a nonneighbour in $B_2 \cap \{u_1, \ldots, u_s\}$. If $s \leq 3$ then $u_{t'}$ is in a triad (since u_1, u_2, u_3 are pairwise nonadjacent), so we may assume that $s > 3$. From the minimality of s, u_s is the unique nonneighbour of $u_{t'}$ in one of $B_1 \cap \{u_1, \ldots, u_s\}$, $B_2 \cap \{u_1, \ldots, u_s\}$; and so we may assume that $u_s \in B_1$ and $u_{t'}$ is complete to $B_1 \cap \{u_1, \ldots, u_{s-1}\}$. By hypothesis, there exists r with $1 \leq r < s$ such that $u_r \in B_2$ and u_r is nonadjacent to u_s. Since $s > 3$, there exist p, q with $1 \leq p, q \leq s - 1$, such that $\{u_p, u_q, u_r\}$ is a triad. (This is clear if $r \leq 3$, taking $\{p, q, r\} = \{1, 2, 3\}$, and follows by the inductive hypothesis otherwise.) Hence $u_p, u_q \in B_1 \cup B_3$, and therefore are adjacent to $u_{t'}$, from the minimality of s. Since $\{u_{t'}, u_p, u_q, u_r\}$ is not a claw, $u_{t'}$ is not adjacent to u_r. But then $\{u_r, u_s, u_{t'}\}$ is a triad. This proves (1).

(2) *Every vertex not in $U_1 \cup U_2 \cup U_3$ is complete to two of U_1, U_2, U_3 and anticomplete to the third.*

For let $v \in V(G) \setminus U_1 \cup U_2 \cup U_3$. Suppose first that $v \in B$, say $v \in B_1$. Thus from the maximality of the sequence, v is complete to at least one of U_2, U_3, since otherwise we could set $u_{t+1} = v$. We assume v is complete to U_2 say. Let $x \in U_3$. By (1), there is a triad $T \subseteq U_1 \cup U_2 \cup U_3$ containing x, and therefore $T \setminus \{x\} \subseteq U_1 \cup U_2$. Since G is claw-free, v is not complete to T, and so v, x are not adjacent. Hence v is anticomplete to U_3, as required. Now assume that $v \notin B$. Let N be the set of neighbours of v in B. By hypothesis, $B \setminus N$ is a clique (for otherwise there would be a triad with exactly two vertices in B). In particular, N contains at least two of u_1, u_2, u_3, since they are pairwise nonadjacent; and N does not contain all three, since G is claw-free. Consequently N contains exactly two of u_1, u_2, u_3. From the symmetry we may assume that when $s = 3$, N includes $\{u_1, \ldots, u_s\} \cap B_1$ and $\{u_1, \ldots, u_s\} \cap B_2$ and is disjoint from $\{u_1, \ldots, u_s\} \cap B_3$; and therefore we may choose s with $3 \le s \le t$, maximum such that the same statement holds. Suppose that $s < t$. If $u_{s+1} \in U_3$, then since N includes no triad by 4.1, it follows from (1) that $u_{s+1} \notin N$, contrary to the maximality of s. Thus $u_{s+1} \in B_1 \cup B_2$, and from the symmetry we may assume that $u_{s+1} \in B_1$. But u_{s+1} has a nonneighbour in $\{u_1, \ldots, u_s\} \cap B_3$, from the definition of the sequence, and since $B \setminus N$ is a clique it follows that $u_{s+1} \in N$, again contrary to the maximality of N. This proves that $s = t$, and therefore proves (2).

Now since $t \ge 4$, it follows that one of U_1, U_2, U_3 has cardinality > 1, and so from 3.5, G is decomposable. This proves 15.1. ■

15.2 *Let G be claw-free, and let $A = \{a_1, a_2, a_3\}$ be a dominating triangle. Suppose that there are distinct vertices $u_1, u_2, u_3, u_4 \in V(G) \setminus A$ such that:*

- *$u_1, \ldots, u_4$ each have exactly two neighbours in A, and*
- *$G|\{u_1, \ldots, u_4\}$ has at most one edge.*

Then G is decomposable.

Proof. For $i = 1, 2, 3$ let B_i be the set of all vertices in $V(G) \setminus A$ that are nonadjacent to a_i and adjacent to the other two members of A. From 4.3 it follows that B_1, B_2, B_3 are cliques. Let $B = B_1 \cup B_2 \cup B_3$. From the hypothesis, there are two triads included in B that have two vertices in common, and so the first two hypotheses of 15.1 hold. For the third, let $v \in V(G) \setminus B$, and suppose that there is a triad $\{v, b_1, b_2\}$, where $b_1 \in B_1$ and $b_2 \in B_2$. By 4.2 (with b_1-a_3-b_2) it follows that v is not adjacent to a_3. Since $v \notin B_3$, it is not adjacent to both a_1, a_2, and from the symmetry we may assume that v is not adjacent to a_2. From 4.2 (with a_2-a_1-b_2) it follows that v is not adjacent to a_1, contrary to the hypothesis that A is dominating. Thus all the hypotheses of 15.1 hold, and the result follows. This proves 15.2. ■

15.3 *Let G be claw-free, such that every 5- and 6-hole in G is dominating, and no 6-hole in G has a hub. Let C_0 be a 6-hole in G, with a star-triangle. If some vertex of $V(G) \setminus V(C_0)$ is a hat or a clone with respect to C_0, then G is decomposable.*

Proof. Let C_0 have vertices c_1-$\cdots$-c_6-c_1, and let $A = \{a_1, a_3, a_5\}$ be a star-triangle, where a_1, a_3, a_5 are in positions $1\frac{1}{2}, 3\frac{1}{2}, 5\frac{1}{2}$ respectively.

(1) *There is no hat in position $1\frac{1}{2}, 3\frac{1}{2}$, or $5\frac{1}{2}$ relative to c_1-$\cdots$-c_6-c_1.*

For suppose that h is a hat in position $1\frac{1}{2}$ say. Then the 5-hole a_1-c_3-c_4-c_5-c_6-a_1 is not dominating, a contradiction. This proves (1).

(2) *A is dominating.*

For suppose that $v \in V(G) \setminus A$, with no neighbour in A. Then $v \notin V(C_0)$, and so, since there is no hub for C_0, it follows that v is a hat, clone or star relative to C_0. By (1) and 8.2, v is not a hat; and by 8.2 it is not a clone, and not a star in position $1\frac{1}{2}, 3\frac{1}{2}$ or $5\frac{1}{2}$. Thus we may assume v is a star in position $2\frac{1}{2}$ say; but then v-c_3-a_2-c_5-c_6-c_1-v is a 6-hole, and a_1 is a hub for it, a contradiction. This proves (2).

For $i = 1, 3, 5$, let B_i be the set of all vertices in $V(G)\setminus A$ that are anticomplete to a_i and complete to the other two members of A. Thus $c_4, c_5 \in B_1, c_6, c_1 \in B_3$, and $c_2, c_3 \in B_5$. By hypothesis, some vertex $v \in V(G)\setminus V(C_0)$ is either a hat or a clone with respect to C_0, say either a hat in position $\frac{1}{2}$ or a clone in position 1 without loss of generality. By 8.2, v is adjacent to a_1 and nonadjacent to a_3. Since $\{a_1, v, a_5, c_3\}$ is not a claw, v is adjacent to a_5 and so $v \in B_3$. But then $c_1, c_3, c_5, v \in B_1 \cup B_3 \cup B_5$, and $G|\{c_1, c_3, c_5, v\}$ has only one edge, namely vc_1, and so the result follows from (1) and 15.2. This proves 15.3. ■

15.4 *Let G be claw-free, such that every 5-hole in G is dominating, and there is no 6-hole with a hub or with a star-diagonal. Suppose that some 6-hole has a crown. Then G is decomposable.*

Proof. Let C be a 6-hole with vertices c_1-$\cdots$-c_6-c_1 in order, and let s_1, s_2 be nonadjacent stars in positions $2\frac{1}{2}, 3\frac{1}{2}$ respectively. Thus the strip $(\{s_1, c_2\}, \emptyset, \{s_2, c_4\})$ is step-connected and parallel to the strip $(\{c_1\}, \{c_6\}, \{c_5\})$. Choose a step-connected strip $(A, \emptyset, B)$ with $s_1, c_2 \in A$ and $s_2, c_4 \in B$, with $A \cup B$ maximal such that c_3 is $A \cup B$-complete and the strips $(A, \emptyset, B)$, $(\{c_1\}, \{c_6\}, \{c_5\})$ are parallel. Suppose that $v \in V(G) \setminus (A \cup B)$, and v has both a neighbour and a nonneighbour in A. Then $v \notin \{c_1, c_3, c_5, c_6\}$. Let N be the set of neighbours of v. Choose a step a_1-a_2-b_2-b_1-a_1 in the strip $(A, \emptyset, B)$ such that $a_1 \in N$ and $a_2 \notin N$. By 4.2, $b_1 \in N$. Suppose that $b_2 \in N$. Then 4.2 implies that $c_5 \in N$; 4.1 implies that $c_6 \notin N$; 4.2 implies that $c_1 \notin N$; 4.2 implies that $B \subseteq N$ and $c_3 \in N$; and then v can be added to B, contrary to the maximality of $A \cup B$. Thus $b_2 \notin N$. Since c_1-c_6-c_5-b_2-a_2-c_1 is dominating, we may assume from the symmetry that $c_1, c_6 \in N$. If $c_5 \notin N$, then v-c_6-c_5-b_2-a_2-a_1-v is a 6-hole, and b_1 is a hub for it, a contradiction. Thus $c_5 \in N$; but then 4.1 implies that $c_3 \notin N$, and so c_1-c_6-c_5-b_1-c_3-a_2-c_1 is a 6-hole, with a star-diagonal $\{a_1, v\}$, again a contradiction. So there is no such vertex v. We deduce from the symmetry that (A, B) is a homogeneous pair, nondominating because of c_6, and so by 3.3, G is decomposable. This proves 15.4. ■

16 6-holes in non-antiprismatic graphs

The next lemma, a consequence of 9.3, is complementary to the last few results.

16.1 *Let G be claw-free, containing no hole of length > 6 or long prism, and such that every hole of length 5 or 6 is dominating. Suppose that G contains a 6-hole, but there is no 6-hole in G with a hub, a star-diagonal, or a star-triangle. Then either $G \in \mathcal{S}_3$, or G is decomposable.*

Proof. Since every 5-hole is dominating, no 6-hole has a coronet; by hypothesis, no 6-hole has a hub, star-diagonal or star-triangle; by 15.4, we may assume that none has a crown; and none has a hat-diagonal since G contains no long prism. By 9.3, this proves 16.1. ■

We recall that G is *prismatic* if for every triangle A, every vertex $v \in V(G) \setminus A$ has a unique neighbour in A; and G is *antiprismatic* if its complement is prismatic. We combine 16.1 with the previous results, to prove the next theorem, which has been the goal of the last several sections.

16.2 *Let G be claw-free, with a hole of length ≥ 6. Then either G is antiprismatic, or $G \in \mathcal{S}_0 \cup \cdots \cup \mathcal{S}_5$, or G is decomposable.*

Proof. By 7.7, 9.1, 9.2 and 13.2, we may assume that G has no hole of length > 6 or long prism, and every hole of length 5 or 6 is dominating.

(1) *We may assume that there is a 6-hole C in G such that no vertex of G is a hat or clone relative to C.*

For by hypothesis there is a hole of length ≥ 6, and therefore of length 6. If there is no 6-hole in G with a hub, a star-diagonal, or a star-triangle, then either $G \in \mathcal{S}_3$, or G is decomposable, by 16.1. Thus we may assume that there is a 6-hole C with either a hub, a star-diagonal, or a star-triangle, choosing C with a hub if possible. By 14.1 and 14.2, if C has a hub or a star-diagonal, then we may assume that no vertex is a hat or clone with respect to C. If C has a star-triangle and has no hub, then no 6-hole has a hub, and so by 15.3, again we may assume that no vertex is a hat or clone with respect to C. This proves (1).

(2) *There do not exist four pairwise nonadjacent vertices in G.*

For suppose that $a_1, \ldots, a_4$ are pairwise nonadjacent. Not all of $a_1, \ldots, a_4$ belong to C; and each a_i that does not belong to C has exactly four neighbours in C, since C is dominating and no vertex is a clone or hat relative to C. We may assume that $a_1 \notin V(C)$. Since it has four neighbours in C and is nonadjacent to a_2, a_3, a_4, at most two of a_2, a_3, a_4 belong to C, and we may assume that $a_2 \notin V(C)$. By 4.3, a_1, a_2 do not have exactly the same four neighbours in C, and so at most one vertex of C is nonadjacent to both a_1, a_2; and so not both $a_3, a_4 \in V(C)$, and we may assume that $a_3 \notin V(C)$. Then a_1, a_2, a_3 each have four neighbours in C. But they have no common neighbour, and therefore every vertex of C is adjacent to exactly two of them. Consequently $a_4 \notin V(C)$, and therefore a_4 also has four neighbours in C; and so some three of $a_1, \ldots, a_4$ have a common neighbour in $V(C)$, a contradiction. This proves (2).

Let C have vertices c_1-$\cdots$-c_6-c_1 in order.

(3) *If there exist stars s_1, s_2, s_3, each in position $1\frac{1}{2}$ or $2\frac{1}{2}$, such that s_3 is nonadjacent to both s_1, s_2, then G is decomposable.*

For suppose that such s_1, s_2, s_3 exist. s_1, s_3 are in different positions, by 8.2, and so are s_2, s_3, and therefore s_1, s_2 are in the same positions. Choose A, B with $A \cup B$ maximal such that:

- A is a set of stars in position $1\frac{1}{2}$
- B is a set of stars in position $2\frac{1}{2}$
- $s_1, s_2, s_3 \in A \cup B$
- let H be the graph with $V(H) = A \cup B$, in which x, y are adjacent if and only if x, y are nonadjacent in G and exactly one of x, y belongs to A; then H is connected.

We claim that (A, B) is a homogeneous pair. For suppose that $v \notin A \cup B$ has a neighbour and a nonneighbour in A say. Since H is connected, we may choose $a_1, a_2 \in A$ and $b \in B$ such that v is adjacent to a_1 and not to a_2, and b is nonadjacent in G to both a_1, a_2. Since v has a neighbour and a nonneighbour in A, it follows that $v \notin V(C)$, and therefore v has exactly four neighbours in C. Since v has a nonneighbour in A, it is not a star in position $1\frac{1}{2}$ or a hub in hub-position 2; and from the maximality of $A \cup B$, it is not a star in position $2\frac{1}{2}$. Consequently v is adjacent to c_5. Since $\{v, a_1, b, c_5\}$ is not a claw, v is not adjacent to b. But v is adjacent to one of c_1, c_2, c_3, say c_i, and then $\{c_i, a_2, b, v\}$ is a claw, a contradiction. This proves that (A, B) is a homogeneous pair, nondominating because of c_5, and so G is decomposable, by 3.3. This proves (3).

(4) *If there exist a hub t in hub-position 1, and stars s_2, s_3, s_4, each in positions $2\frac{1}{2}$ or $5\frac{1}{2}$, such that s_4 is nonadjacent to s_2, s_3, then G is decomposable.*

For choose A, B with $A \cup B$ maximal such that:

- A is a set of stars in position $2\frac{1}{2}$
- B is a set of stars in position $5\frac{1}{2}$
- $s_2, s_3, s_4 \in A \cup B$
- let H be the graph with $V(H) = A \cup B$, in which x, y are adjacent if and only if x, y are nonadjacent in G and exactly one of x, y belongs to A; then H is connected.

We claim that (A, B) is a homogeneous pair. For let $v \in V(G) \setminus A \cup B$, and suppose it has a neighbour and a nonneighbour in A say. Thus $v \notin V(C)$. Since H is connected, we may choose $a_1, a_2 \in A$ and $b \in B$ such that v is adjacent to a_1 and not to a_2, and b is nonadjacent to both a_1, a_2. By 11.1, v is not a hub, and not a star in position $2\frac{1}{2}$; and by the maximality of $A \cup B$, v is not a star in position $5\frac{1}{2}$. Hence v is a star in some other position. Consequently v is adjacent to t by 11.1, and v is adjacent to one of c_1, c_4, say c_1. By 11.1, t is nonadjacent to all of a_1, a_2, b. If v is nonadjacent to b, then $\{c_1, v, a_2, b\}$ is a claw, while if v is adjacent to b, then $\{v, a_1, b, t\}$ is a claw, in either case a contradiction. Thus (A, B) is a homogeneous pair. By 11.1, t has no neighbours in $A \cup B$, and so (A, B) is nondominating. By 3.3, G is decomposable. This proves (4).

(5) *If there exist stars $s_1, \ldots, s_4$, each in position $1\frac{1}{2}$, $3\frac{1}{2}$ or $5\frac{1}{2}$, and all pairwise nonadjacent except*

for s_3s_4, then G is decomposable.

For let B_1, B_2, B_3 be the set of all stars in positions $1\frac{1}{2}$, $3\frac{1}{2}$ and $5\frac{1}{2}$ respectively. By 4.3, B_1, B_2, B_3 are all cliques. Let $B = B_1 \cup B_2 \cup B_3$. Because of $s_1, \ldots, s_4$, there are two triads in B with two vertices in common. Suppose that T is a triad with $|T \cap B| = 2$; say $T = \{v, b_1, b_2\}$, where $v \notin B$ and $b_1 \in B_1, b_2 \in B_2$. Since every vertex of C is adjacent to one of b_1, b_2 it follows that $v \notin V(C)$, and therefore v has four neighbours in C. Since $\{c_2, v, b_1, b_2\}$ is not a claw, v is not adjacent to c_2 and similarly not to c_3; and so it is a star in position $5\frac{1}{2}$, contradicting that $v \notin B$. Thus there is no such triad. By 15.1, it follows that G is decomposable. This proves (5).

We may assume that G is not antiprismatic. Therefore there are four vertices $a_1, \ldots, a_4$, pairwise nonadjacent except possibly for a_3a_4. By (2), a_3, a_4 are adjacent. Suppose first that $a_1, a_2 \in V(C)$. Then at least one of a_3, a_4 is not in $V(C)$, say a_3, and therefore a_3 is adjacent to every vertex of C except a_1, a_2. Since a_1, a_2 are nonadjacent, it follows that a_3 is a hub, and so we may assume that $a_1 = c_1, a_2 = c_4$. Then every other vertex of C is adjacent to one of a_1, a_2, and so $a_4 \notin V(C)$; and therefore a_4 is also a hub, in the same hub-position as c_3. Then G is decomposable, by 11.2.

We may therefore assume that not both $a_1, a_2 \in V(C)$, say $a_1 \notin V(C)$. Consequently a_1 has four neighbours in $V(C)$. Assume that $a_2, a_3 \in V(C)$. Then since a_1, a_2, a_3 are pairwise nonadjacent, it follows that a_1 is a hub, and we may assume that $a_2 = c_1, a_3 = c_4$. Since a_4 is adjacent to a_3 and $a_4 \notin V(C)$, it follows that a_4 is a star in position $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, or $5\frac{1}{2}$, or a hub in hub-position 2 or 3. Since a_4 is nonadjacent to $a_2 = c_1$, we may assume from the symmetry that a_4 is a star in position $2\frac{1}{2}$; but then it is adjacent to a_1 by 11.1, a contradiction.

This proves that not both $a_2, a_3 \in V(C)$. Assume that $a_2 \in V(C)$, say $a_2 = c_1$. Then $a_3 \notin V(C)$, and similarly $a_4 \notin V(C)$. Each of a_1, a_3, a_4 is adjacent to four of $c_2, \ldots, c_6$, and is therefore either a star in position $3\frac{1}{2}$ or $4\frac{1}{2}$, or a hub in hub-position 1. If any of them is a hub in hub-position 1, then it is adjacent to both the others by 11.1, a contradiction; and so all three are stars. But then the result follows by (3). So we may assume that $a_2 \notin V(C)$.

Since a_1, a_2 do not have exactly the same neighbours in C by 4.3, it follows that at least one of $a_3, a_4 \notin V(C)$, say a_3. Hence a_1, a_2, a_3 each has four neighbours in $V(C)$, and yet they have no common neighbour. Consequently each vertex of C is adjacent to exactly two of a_1, a_2, a_3, and therefore $a_4 \notin V(C)$. Thus a_4 also has exactly four neighbours in C, and no vertex is adjacent to all of a_1, a_2, a_4, and therefore a_3, a_4 have the same neighbours in C. By 11.2 we may assume that a_3, a_4 are not hubs. If one of a_1, a_2 is a hub, then G is decomposable by (4); and if a_1, a_2 are not hubs, then G is decomposable by (5). This proves 16.2. ■

17 Stable sets of size 4

In this section we finish the case that $\alpha(G) \geq 4$. We have already handled such graphs that have a hole of length at least 6, so it suffices to prove the following.

17.1 *Let G be claw-free, such that G has no hole of length > 5, every 5-hole in G is dominating, $\alpha(G) \geq 4$, and G is not decomposable. Then G is either a line graph or a circular interval graph.*

The proof of 17.1 falls into several parts, as follows. Let G satisfy the hypotheses of 17.1. We shall prove the following.

- (In 17.7) If some 5-hole has a coronet, then G is a line graph.
- (In 17.8) If G contains a $(1,1,1)$-prism, then G is a line graph.
- (In 17.9) If G has a 5-hole, but no 5-hole has a coronet, and G contains no $(1,1,1)$-prism, then G is a circular interval graph.
- (In 17.10) If G has a 4-hole but no 5-hole, then G is a line graph.
- (In 17.11) It is impossible that G has no holes at all.

We begin with a few lemmas.

17.2 *Let B be a clique in a claw-free graph G, and let $a_1, a_2 \in V(G)\setminus B$ be nonadjacent. If a_1, a_2 are not B-complete and not B-anticomplete, then there is a path of length 3 between a_1, a_2 with interior in B.*

Proof. For $i = 1, 2$, let N_i be the set of neighbours of a_i in B. By hypothesis, $N_i \neq \emptyset, B$. Suppose that $N_1 \subseteq N_2$. Since $N_1 \neq \emptyset$, there exists $x \in N_1$; and since $N_2 \neq B$, there exists $y \in B \setminus N_2$. But then $\{x, y, a_1, a_2\}$ is a claw, a contradiction. Thus $N_1 \not\subseteq N_2$, and similarly $N_2 \not\subseteq N_1$. Choose $n_1 \in N_1 \setminus N_2$, and $n_2 \in N_2 \setminus N_1$. Then a_1-n_1-n_2-a_2 is a path. This proves 17.2. ■

17.3 *Let G be claw-free, with no hole of length > 5, not decomposable, and such that every 5-hole is dominating. Let the paths a_1-b_1, a_2-b_2 and a_3-c_3-b_3 form a prism in G, where $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles. Then there is a 5-hole in G with a centre, and every neighbour of c_3 that is nonadjacent to a_1, b_1, a_2, b_2 is adjacent to both of a_3, b_3.*

Proof. Choose a step-connected strip $(A, \emptyset, B)$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$, parallel to the strip $(\{a_3\}, \{c_3\}, \{b_3\})$, and maximal with this property. Since c_3 is anticomplete to $A \cup B$ and G is not decomposable, 3.3 implies that (A, B) is not a nondominating homogeneous pair. Thus we may assume that there exists $v \in V(G) \setminus (A \cup B)$ with a neighbour and a nonneighbour in A. Then $v \notin \{a_3, b_3, c_3\}$. Choose a step a_1'-a_2'-b_2'-b_1'-a_1' such that v is adjacent to a_1' and not to a_2'. By 4.2, v is adjacent to b_1'. If v is adjacent to b_2', then by 4.2 v is adjacent to b_3; by 4.1 v is nonadjacent to c_3; and by 4.2 v is nonadjacent to a_3. But then v can be added to B, contrary to the maximality of $A \cup B$. Thus v is nonadjacent to b_2'. Since the 5-hole a_3-c_3-b_3-b_2'-a_2'-a_3 is dominating, v has a neighbour in the path a_3-c_3-b_3, and therefore is adjacent to at least two adjacent vertices of this path. In particular, v is adjacent to c_3. Since v-c_3-b_3-b_2'-a_2'-a_1'-v is not a 6-hole, v is adjacent to b_3 and similarly to a_3. Hence v is a centre for the 5-hole a_3-c_3-b_3-b_1'-a_1'-a_3. Now suppose that d is a neighbour of c_3, nonadjacent to a_1, b_1, a_2, b_2. Hence d has a nonneighbour in A. If d also has a neighbour in A, then by exchanging v, d we deduce that d is adjacent to both a_3, b_3 as required. Thus we may assume that d has no neighbour in A, and similarly none in B. From the symmetry, we may assume that d is adjacent to a_3. By 4.2 (with d-a_3-a_2'), v is adjacent to d; and by 4.1 (with $\{d, a_1', b_3\}$) it follows that d is adjacent to b_3 as required. This proves 17.3. ■

17.4 *Let G be claw-free, with no hole of length > 5, and such that every 5-hole is dominating. Let C be a 4-hole. If there exist adjacent vertices of $G \setminus V(C)$, both with no neighbour in $V(C)$, then G is decomposable.*

Proof. Let C have vertices c_1-$\cdots$-c_4-c_1 in order. Let $Z \subseteq V(G) \setminus V(C)$ be maximal such that Z is connected and no vertex in Z has a neighbour in $V(C)$, with $|Z| > 1$. Let Y be the set of vertices of $V(G) \setminus Z$ with a neighbour in Z. Then from the maximality of Z, every vertex of Y has a neighbour in $V(C)$; and since G is claw-free, it follows that every vertex in Y is a hat relative to C. Let $Y = Y_1 \cup \cdots \cup Y_4$, where for $i = 1, \ldots, 4$, Y_i is the set of vertices in Y that are adjacent to c_i, c_{i+1} (reading subscripts modulo 4).

(1) $Y_1, \ldots, Y_4$ *are cliques; and for* $1 \le i \le 4$, Y_i *is complete to* Y_{i+1}.

The first assertion follows from 4.3. For the second, suppose that $y_1 \in Y_1$ and $y_2 \in Y_2$ say are nonadjacent, and let P be a path between y_1, y_2 with interior in Z. Then y_1-c_1-c_4-c_3-y_2-P-y_1 is a hole of length ≥ 6, a contradiction. This proves (1).

(2) *We may assume that if* $y, y' \in Y$ *are nonadjacent then every vertex in* Z *is adjacent to both* y, y'.

For let $y \in Y_1, y' \in Y_3$ say (without loss of generality, by (1)). Let P be a path between y, y' with interior in Z. Since the hole y-c_2-c_3-y'-P-y has length ≤ 5, it follows that P has length 2, and the hole has length 5. Let z be the middle vertex of P. Since every 5-hole is dominating, every vertex in $Z \setminus \{z\}$ has a neighbour in P, and therefore is adjacent to z and to at least one of y, y'. If some $z' \in Z \setminus \{z\}$ is nonadjacent to one of y, y', then the three paths c_1-c_4, c_2-c_3 and P form a prism satisfying the hypotheses of 17.3, and the result follows. Thus we may assume that they are all adjacent to both y, y'. This proves (2).

(3) *For* $1 \le i < j \le 4$, *if* $y_i \in Y_i$ *and* $y_j \in Y_j$ *then* y_i, y_j *have the same neighbours in* Z.

For if y_i, y_j are nonadjacent this follows from (2). If they are adjacent, suppose that $z \in Z$ is adjacent to y_i and not to y_j, and choose $c \in V(C)$ adjacent to y_i and not to y_j; then $\{y_i, z, y_j, c\}$ is a claw, a contradiction. This proves (3).

If Y is a clique then Y is an internal clique cutset and the theorem holds. Thus by (1), we may assume that Y_1 is not complete to Y_3 (and therefore Y_1, Y_3 are nonempty). By (2) and (3) it follows that Y is complete to Z, and therefore Z is a clique by 4.3; but then all members of Z are twins. This proves 17.4. ■

17.5 *Let* G *be claw-free, let* C *be a dominating* 5*-hole in* G, *and let* $X \subseteq V(G)$ *be stable with* $|X| = 4$. *Then there is a* 5*-numbering of* C *such that either*

- *there are three hats in* X, *in positions* $1\frac{1}{2}, 2\frac{1}{2}$ *and* $3\frac{1}{2}$, *or*
- X *consists of two hats in positions* $1\frac{1}{2}$ *and* $2\frac{1}{2}$ *and two clones in positions* $4, 5$, *or*
- X *consists of three hats in positions* $1\frac{1}{2}, 2\frac{1}{2}$ *and* $4\frac{1}{2}$ *and a star in position* $4\frac{1}{2}$.

Proof. Let C have vertices c_1-$\cdots$-c_5-c_1 and let $X = \{v_1, \ldots, v_4\}$. Each member of $X \setminus V(C)$ has at least two neighbours in $V(C)$, since C is dominating; and on the other hand, every vertex of C

is adjacent to at most two members of X, since G is claw-free. At most two members of X belong to C, so we may assume that $v_1, v_2 \in X \setminus V(C)$. But v_1, v_2 both have at least two neighbours in C, and since they are not hats in the same position by 4.3and v_3, v_4 are nonadjacent, it follows that not both $v_3, v_4 \in V(C)$. Thus we assume that $v_3 \notin V(C)$. Suppose that $v_4 \in V(C)$, say $v_4 = c_5$. Then each of c_1, c_4 is adjacent to at most one of v_1, v_2, v_3, and each of c_2, c_3 is adjacent to at most two of v_1, v_2, v_3. On the other hand, v_1, v_2, v_3 each have at least two neighbours in C. Hence equality holds, and therefore v_1, v_2, v_3 are hats in positions $1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$, as required. We may therefore assume that $v_4 \notin V(C)$. Now $c_1, \dots, c_5$ are each adjacent to at most two of members of X, and every member of X is adjacent to at least two of $c_1, \dots, c_5$. Consequently at least two members of X are hats, say v_1, v_2. Suppose that no two members of X are hats in consecutive positions. Then we may assume that v_1, v_2 are in positions $1\frac{1}{2}, 3\frac{1}{2}$, and v_3, v_4 are not hats; and from counting the edges between $V(C)$ and X, it follows that v_3, v_4 are clones, in positions $1, 4$. But since they are nonadjacent to v_1, v_2, this contradicts 8.2. Thus at least two members of X are hats in consecutive positions; and so we may assume that v_1, v_2 are hats in positions $1\frac{1}{2}, 2\frac{1}{2}$ respectively. If v_3, v_4 are not hats, then they are clones in positions $4, 5$ and the theorem holds. Thus we may assume that v_3 is a hat. If it is in position $3\frac{1}{2}$ or $\frac{1}{2}$ then the theorem holds, so we may assume it is in position $4\frac{1}{2}$. If v_4 is a hat, then it is in position $3\frac{1}{2}$ or $\frac{1}{2}$ and the theorem holds; and by 8.2 is it not a clone. So we may assume it is a star, and hence in position $4\frac{1}{2}$; but then the theorem holds. This proves 17.5. ■

17.6 *Let G be claw-free, such that G has no hole of length > 5, every 5-hole in G is dominating, and $\alpha(G) \geq 4$. Then no 5-hole in G has a centre; and G does not contain a $(2,1,1)$-prism.*

Proof. For suppose first that $c_1\text{-}\cdots\text{-}c_5\text{-}c_1$ is a 5-hole C, with a centre z. Since $\alpha(G) \geq 4$, we may assume by 17.5 that there are nonadjacent hats h_1, h_2 in positions $1\frac{1}{2}, 2\frac{1}{2}$ say. Since $\{z, h_1, c_3, c_5\}$ is not a claw, z is not adjacent to h_1, and similarly it is not adjacent to h_2. But then $\{c_2, z, h_1, h_2\}$ is a claw, a contradiction. This proves that no 5-hole has a centre. The second assertion of the theorem follows from 17.3. This proves 17.6. ■

The following completes the first step of the proof of 17.1.

17.7 *Let G be claw-free, such that G has no hole of length > 5, every 5-hole in G is dominating, $\alpha(G) \geq 4$, and G is not decomposable. If some 5-hole has a coronet then G is a line graph.*

Proof. Let $c_1\text{-}\cdots\text{-}c_5\text{-}c_1$ be a 5-numbering of a 5-hole C, such that there is a hat h and a star s both in position $1\frac{1}{2}$. By 8.2, h and s are nonadjacent. Let $\mathcal{C}$ be the proximity component of order 5 containing C.

(1) *For every $a_1\text{-}\cdots\text{-}a_5\text{-}a_1$ in $\mathcal{C}$, h is a hat and s is a star, both in position $1\frac{1}{2}$.*

For it suffices to show that if two 5-numberings are proximate, and the claim is true for one of them, then it is true for the other. Thus, suppose that $a_1\text{-}\cdots\text{-}a_5\text{-}a_1$ is a 5-numbering and h is a hat and s is a star, both in position $1\frac{1}{2}$, relative to $a_1\text{-}\cdots\text{-}a_5\text{-}a_1$. Let $1 \leq i \leq 5$, and let a_i' be a clone in position i relative to $a_1\text{-}\cdots\text{-}a_5\text{-}a_1$. We must show that a_i and a_i' have the same neighbours in $\{h, s\}$. If $i = 1$, then a_1' is adjacent to s, h by 8.1. If $i = 4$, then a_4' is nonadjacent to h by 8.1, and nonadjacent to s by 17.6, since otherwise s would be a centre for $a_1\text{-}a_2\text{-}a_3\text{-}a_4'\text{-}a_5\text{-}a_1$. Thus from the symmetry we may assume that $i = 5$. Since $\{a_5', h, s, a_4\}$ is not a claw, it follows that a_5' is nonadjacent to at least one of h, s. Since $\{a_1, a_5', h, s\}$ is not a claw, a_5' is adjacent to at least one of h, s. If a_5' is adjacent to h and not to s, then the 5-hole h-a_2-s-a_5-a_5'-h has a centre a_1, contrary to 17.6. Thus a_5' is adjacent to s and not to h. This proves (1).

For $1 \le i \le 5$, let $A_i = A_i(\mathcal{C})$. From (1), $A_1 \cup A_2$ is complete to both h, s; $A_3 \cup A_5$ is complete to s and anticomplete to h; and A_4 is anticomplete to both h, s. Let $W = A_1 \cup \cdots \cup A_5$. For each $v \in V(G) \setminus \{h, s\}$, let $P(v)$ be the set of all k such that v is in position k relative to some member of $\mathcal{C}$. (Note that since every 5-hole is dominating, and none has a centre, it follows that v has a position relative to each member of $\mathcal{C}$.) If two 5-numberings are proximate, then the positions of v relative to them differ by at most $\frac{1}{2}$, and it follows that $P(v)$ is a set of consecutive $\frac{1}{2}$-integers modulo 5, that is, $P(v)$ is an "interval".

(2) *The sets $A_1, \dots, A_5$ are pairwise disjoint; and every vertex in $V(G) \setminus W$ is either complete to four of $A_1, \dots, A_5$ and anticomplete to the fifth, or complete to two consecutive of $A_1, \dots, A_5$ and anticomplete to the other three.*

For certainly the sets $A_1 \cup A_2$, $A_3 \cup A_5$ and A_4 are pairwise disjoint. Suppose that there exists $v \in A_1 \cap A_2$. Then $1, 2 \in P(v)$, and v is adjacent to h, s. Hence $3, 4, 5 \notin P(v)$, by (1), and since $P(v)$ is an interval, it follows that $1\frac{1}{2} \in P(v)$. So relative to some member of $\mathcal{C}$, v is a hat or star in position $1\frac{1}{2}$. But by 8.2, a hat in position $1\frac{1}{2}$ is nonadjacent to s, and a star in position $1\frac{1}{2}$ is nonadjacent to h, in either case a contradiction. This proves that $A_1 \cap A_2 = \emptyset$. Now assume that there exists $v \in A_3 \cap A_5$. Thus $3, 5 \in P(v)$, and by (1) v is adjacent to s and not to h. By (1) $1, 2, 4 \notin P(v)$, contradicting that $P(v)$ is an interval. This proves that $A_1, \dots, A_5$ are pairwise disjoint. Now if $v \in V(G) \setminus W$, it follows that $P(v)$ contains no integer, and so $P(v)$ has only one member, since it is an interval; and the final assertion of (2) follows. This proves (2).

(3) *$A_i = \{c_i\}$ for $i = 1, 2$.*

For if $a_1 \in A_1$ and $a_4 \in A_4$, then since $\{a_1, a_4, h, s\}$ is not a claw it follows that a_1, a_4 are nonadjacent. Thus $A_1 \cup A_2$ is anticomplete to A_4. Let a_1-$\cdots$-a_5-a_1 be in $\mathcal{C}$, and suppose that some $v \in A_1$ is adjacent to a_3. Since v is anticomplete to A_4 as we saw, it follows that v is adjacent to a_2; by 8.2 v is not a hat, since it is adjacent to h, and so v is adjacent to a_1; and by 8.2, v is nonadjacent to a_5 since it is adjacent to h. Hence v is in position 2 relative to a_1-$\cdots$-a_5-a_1, and hence $v \in A_1 \cap A_2$, contrary to (2). This proves that A_1 is anticomplete to A_3, and similarly A_2 is anticomplete to A_5. Now let a_1-$\cdots$-a_5-a_1 be in $\mathcal{C}$, and suppose that some $a_1' \in A_1$ is nonadjacent to a_5. Then $\{s, a_1', a_5, a_3\}$ is a claw, a contradiction. Consequently A_1 is complete to A_5, and similarly A_2 to A_3. Since every vertex in $V(G) \setminus W$ is either complete or anticomplete to A_i for $i = 1, 2$, it follows that (A_1, A_2) is a homogeneous pair, nondominating since $A_4 \neq \emptyset$; and so by 3.3, A_1, A_2 both have cardinality 1, since G is not decomposable. This proves (3).

(4) *A_3, A_4, A_5 are cliques.*

For if $a_3, a_3' \in A_3$ then they are adjacent since $\{s, a_3, a_3', c_1\}$ is not a claw, and so A_3 is a clique, and similarly so is A_5. Now let a_1-$\cdots$-a_5-a_1 be in $\mathcal{C}$, and let $a_4' \in A_4$ be different from a_4. Since A_4

is disjoint from A_3, A_5, it follows that $3, 5 \notin P(a'_4)$; and since $4 \in P(a'_4)$ and $P(a'_4)$ is an interval, it follows that $P(a'_4) \subseteq \{3\frac{1}{2}, 4, 4\frac{1}{2}\}$. In particular, relative to $a_1\text{-}\cdots\text{-}a_5\text{-}a_1$, a'_4 has position one of $3\frac{1}{2}, 4, 4\frac{1}{2}$, and therefore is adjacent to a_4. This proves that A_4 is a clique, and therefore proves (4).

For $i = 3, 5$, let A'_i be the set of members of A_i with a nonneighbour in A_4.

(5) *A'_3 is complete to A'_5; A'_3 is anticomplete to $A_5 \setminus A'_5$; and $A_3 \setminus A'_3$ is anticomplete to A'_5.*

For suppose that $a_3 \in A'_3$ and $a_5 \in A'_5$ are nonadjacent. Each of them is not A_4-complete and not A_4-anticomplete, and therefore by 17.2, there is a path between them of length 3 with interior in A_4. But also $a_5\text{-}c_1\text{-}c_2\text{-}a_3$ is a path, and the union of these two paths is a 6-hole, contrary to hypothesis. This proves the first assertion of (5). Now suppose that $a_3 \in A'_3$ and $a_5 \in A_5 \setminus A'_5$ are adjacent. Choose $a_4 \in A_4$ nonadjacent to a_3. Since $a_5 \notin A'_5$, it follows that a_4, a_5 are adjacent; but then $\{a_5, a_3, a_4, c_1\}$ is a claw, a contradiction. Thus A'_3 is anticomplete to $A_5 \setminus A'_5$, and the third assertion of (5) follows by symmetry. This proves (5).

(6) *One of A'_3, A'_5 is empty.*

For suppose they are both nonempty. Choose $a'_3 \in A'_3$ and $a'_5 \in A'_5$. Choose $a_4, a'_4 \in A_4$ with a_4 adjacent to a'_3 and a'_4 nonadjacent to a'_3. Since $\{a'_5, a'_3, a'_4, c_1\}$ is not a claw, a'_4 is nonadjacent to a'_5, and since $\{a'_3, a'_5, a_4, c_2\}$ is not a claw, a_4 is adjacent to a'_5. Let $\overline{G}$ be the complement of G. Since $\mathcal{C}$ is connected by proximity, it follows that $\overline{G}|(A_3 \cup A_5)$ is connected, and so $A'_3 \cup (A_5 \setminus A'_5)$ is not complete to $A'_5 \cup (A_3 \setminus A'_3)$. Hence by (4) and (5), there exist $a_3 \in A_3 \setminus A'_3$ and $a_5 \in A_5 \setminus A'_5$, nonadjacent. But then $a_3\text{-}a'_4\text{-}a_5\text{-}a'_5\text{-}a'_3\text{-}a_3$ is a 5-hole with a centre a_4, contrary to 17.6. This proves (6).

(7) *$A_i = \{c_i\}$ for $1 \leq i \leq 5$.*

For from (6) we may assume that $A'_5 = \emptyset$. Then (A'_3, A_4) and $(A_3 \setminus A'_3, A_5)$ are both homogeneous pairs, by (5), and they are both nondominating because of h, and so by 3.3, $A'_3, A_4, A_3 \setminus A'_3, A_5$ all have cardinality at most 1. In particular $A_4 = \{c_4\}$ and $A_5 = \{c_5\}$. Since every vertex in A_3 has a neighbour in A_4, it follows that $A'_3 = \emptyset$, and therefore $A_3 = \{c_3\}$. From (3), this proves (7).

For $1 \leq i \leq 5$ let H_i be the set of all hats in position $i + 2\frac{1}{2}$, and let S_i be the set of all stars in this position. Thus $h \in H_4$ and $s \in S_4$. From 4.3, each H_i and each S_i is a clique. From (7), we see that $V(G)$ is the union of $H_1, \ldots, H_5, S_1, \ldots, S_5$ and $V(C)$.

(8) *The following hold:*

- *For $1 \leq i, j \leq 5$, H_i is complete to S_j if $j = i+1$ or $j = i-1$, and otherwise H_i is anticomplete to S_j*
- *For $1 \leq i < j \leq 5$, H_i is anticomplete to H_j*
- *For $1 \leq i \leq 5$, if $H_i \neq \emptyset$ then S_i is anticomplete to S_{i-1}, S_{i+1}*
- *For $1 \leq i \leq 5$, if $H_i \neq \emptyset$ then S_i is complete to S_{i-2}, S_{i+2}*

- *For $1 \leq i \leq 5$, if $H_i, S_i \neq \emptyset$ then S_{i-1} is complete to S_{i+1}.*

For the first claim follows from 8.2. No two hats in consecutive positions are adjacent, by 8.2, and no two hats in distinct nonconsecutive positions are adjacent, by 17.6, since the union of two such adjacent hats with C would be a $(2,1,1)$-prism. Hence the second claim holds. The third and fourth claims are trivial if $S_i = \emptyset$, so we may assume that S_i, H_i are both nonempty; and therefore, since S_4, H_4 are nonempty by hypothesis, we may assume that $i = 4$. Since $S_3 \cup S_4 \cup \{h, c_4\}$ includes no claw, S_3 is anticomplete to S_4, and similarly S_4 to S_5. This proves the third claim. Since $\{c_1, s\} \cup S_3 \cup S_5$ includes no claw, S_3 is complete to S_5. Since $\{c_1, h\} \cup S_2 \cup S_4$ includes no claw, S_2 is complete to S_4 and similarly S_1 is complete to S_4. This proves the fourth claim. For the final claim, suppose that $s_i \in S_i$. By the third claim, S_i is anticomplete to S_{i-1}, S_{i+1}, and since $\{c_{i+2}, s_i\} \cup S_{i-1} \cup S_{i+1}$ includes no claw, S_{i-1} is complete to S_{i+1}. This proves (8).

If S_i is anticomplete to S_{i+1} and complete to S_{i+2} for $1 \leq i \leq 5$, then by (8), G is a line graph and the theorem holds. Therefore, in view of (8), we may assume the following (for a contradiction):

(9) *Either S_i is not anticomplete to S_{i+1} for some $i \in \{1,2,5\}$ or S_i is not complete to S_{i+2} for some $i \in \{1,5\}$.*

(10) *$H_1, H_2 = \emptyset$.*

For suppose that there exists $h_1 \in H_1$ say. Since $S_2 \cup S_3 \cup \{s, h_1\}$ includes no claw, S_2 is anticomplete to S_3. By (8), S_1 is anticomplete to S_5, S_2 and complete to S_3. By (9), there exist $s_2 \in S_2$ and $s_5 \in S_5$, nonadjacent. Then s-c_2-s_5-c_4-c_5-s is a 5-hole; and relative to this 5-numbering, c_3, h are a star and a hat both in position $2\frac{1}{2}$, and s_2 is a clone in position 5, contrary to (7) applied to this 5-hole. Thus there is no such h_1, and similarly $H_2 = \emptyset$. This proves (10).

By (7), there are no clones relative to c_1-$\cdots$-c_5-c_1, and so by 17.5 there are hats in at least three positions. By (10) it follows that H_3, H_5 are both nonempty. From (8), S_3 is complete to S_1 and anticomplete to S_2; and S_5 is complete to S_2 and anticomplete to S_1. By (9), S_1 is not anticomplete to S_2. Since $S_1 \cup S_2 \cup S_3 \cup H_5$ includes no claw, $S_3 = \emptyset$, and similarly $S_5 = \emptyset$. But then $(S_1 \cup \{c_3\}, S_2 \cup \{c_5\})$ is a homogeneous pair, nondominating because of h, contrary to 3.3. Hence our assumption in (9) was false. This proves 17.7. ∎

Let the paths a_i-b_i $(i = 1,2,3)$ form a $(1,1,1)$-prism. For $1 \leq i \leq 3$, a *hat on a_i-b_i* means a vertex adjacent to a_i, b_i and nonadjacent to the other four vertices in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. The following completes the second step of the proof of 17.1.

17.8 *Let G be claw-free, such that G has no hole of length > 5, every 5-hole in G is dominating, $\alpha(G) \geq 4$, and G is not decomposable. If G contains a $(1,1,1)$-prism then G is a line graph.*

Proof. By 17.7, we may assume that no 5-hole has a coronet.

(1) *G contains a $(1,1,1)$-prism with a hat.*

For let the paths a_i-b_i $(i = 1,2,3)$ form a $(1,1,1)$-prism, where $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ are triangles. Suppose first that $A \cup B$ is dominating. By hypothesis, $\alpha(G) \geq 4$, and so there exist pairwise disjoint vertices $v_1, \ldots, v_4$. For $1 \leq i \leq 4$, let N_i be the set of neighbours of v_i in $A \cup B$, together with v_i itself if $v_i \in A \cup B$. Thus each $|N_i| \geq 2$, and if $|N_i| = 2$ then v_i is a hat, so we may assume that $|N_i| \geq 3$ for each i. If $|N_i| = 3$, then $v_i \notin A \cup B$ and $N_i = A$ or B; and so by 4.3, $|N_i| = 3$ for at most two values of i. Consequently $|N_1| + |N_2| + |N_3| + |N_4| \geq 14$, and therefore we may assume that a_1 belongs to N_i for at least three values of i. But then G contains a claw, a contradiction. So if $A \cup B$ is dominating then (1) holds.

Now assume that $A \cup B$ is not dominating. Let $z \in V(G)$ have no neighbours in $A \cup B$, and let N be the set of neighbours of z. For $n \in N$, let $Y(n)$ be the set of neighbours of n in $A \cup B$. By 17.4, $Y(n)$ is nonempty; and since G is claw-free, $Y(n)$ is a clique. We claim we may assume that either $Y(n) = A$ or $Y(n) = B$. For we may assume that $a_1 \in Y(n)$. If $b_1 \in Y(n)$ then since $Y(n)$ is a clique, it follows that n is a hat as required. We assume then that $b_1 \notin Y(n)$. By 4.2, $a_2, a_3 \in Y(n)$, and since $Y(n)$ is a clique, we deduce that $Y(n) = A$. Thus for every $n \in N$, $Y(n) = A$ or $Y(n) = B$. Suppose there exist $m, n \in N$ with $Y(m) = A$ and $Y(n) = B$. If m, n are not adjacent, then the paths m-z-n, a_1-b_1 and a_2-b_2 form a $(2,1,1)$-prism, contrary to 17.6. If m, n are adjacent, then the paths m-n, a_1-b_1, a_2-b_2 form a $(1,1,1)$-prism with a hat z on m-n, as required. Thus we may assume that $Y(n) = A$ for all $n \in N$. By 4.3, N is a clique. Let X be the set of all vertices in $V(G) \setminus (N \cup \{z\})$ with a neighbour in N. We claim that X is a clique. Let $x_1, x_2 \in X$, and assume they are nonadjacent. Thus not both $x_1, x_2 \in A$, say $x_1 \notin A$. Since some vertex in N is adjacent to x_1 and to z, 4.2 implies that x_1 is complete to A, and therefore $x_2 \notin A$. If x_1, x_2 have a common neighbour $n \in N$, then $\{n, z, x_1, x_2\}$ is a claw, a contradiction. Thus x_1, x_2 have no common neighbour in N. Let $n_1, n_2 \in N$ be adjacent to x_1, x_2 respectively. Since $\{a_i, n_2, x_1, b_i\}$ is not a claw, it follows that x_1 is adjacent to b_i for $1 \leq i \leq n$ and similarly x_2 is complete to B. Hence b_1-x_1-n_1-n_2-x_2-b_1 is a 5-hole with a centre a_1, contrary to 17.6. Thus X is a clique, and therefore X is an internal clique cutset (unless $N = \emptyset$, when G is expressible as a 0-join). Hence G is decomposable, a contradiction. This proves (1).

(2) *G contains a $(1,1,1)$-prism with hats on two different paths.*

For by (1) we may choose paths a_i-b_i $(i = 1,2,3)$ forming a $(1,1,1)$-prism, where $A = \{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ are triangles, such that there is a hat h on a_3-b_3. Choose a step-connected strip $(A, \emptyset, B)$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$, parallel to $(\{a_3\}, \{h\}, \{b_3\})$, and with $A \cup B$ maximal with this property. Since (A, B) is not a nondominating homogeneous pair, by 3.3, we may assume there is a vertex $v \notin A \cup B$ with a neighbour and a nonneighbour in A. Let N be the set of neighbours of v, and let a_1'-a_2'-b_2'-b_1'-a_1' be a step with $a_1' \in N$ and $a_2' \notin N$. By 4.2, $b_1' \in N$. If $b_2' \in N$, then by 4.2, $b_3 \in N$; by 4.1, $h \notin N$; by 4.2, $B \subseteq N$; and by 4.2, $a_3 \notin N$; and then v can be added to B, contrary to the maximality of $A \cup B$. Thus $b_2' \notin N$. Suppose that $h \in N$. Since v-h-a_3-a_2'-b_2'-b_1'-v is not a 6-hole, it follows that $a_3 \in N$, and similarly $b_3 \in N$. But then v-a_3-a_2'-b_2'-b_1'-v is a 5-hole, and $\{a_1', h\}$ is a coronet for it, a contradiction. Thus $h \notin N$. From 4.2, $a_3, b_3 \notin N$; and so h, v are both hats for the prism formed by a_1'-b_1', a_2'-b_2' and a_3-b_3, on different paths. This proves (2).

From (2), we may choose $k \geq 3$, and disjoint cliques $A_1, \ldots, A_k, B_1, \ldots, B_k$ and $C_1, \ldots, C_k$ with the following properties (let $A = A_1 \cup \cdots \cup A_k$, $B = B_1 \cup \cdots \cup B_k$ and $C = C_1 \cup \cdots \cup C_k$):

- $A_1, \ldots, A_{k-1}, B_1, \ldots, B_{k-1}$ and $C_1, \ldots, C_{k-1}$ are all nonempty; and if $k = 3$ then A_3, B_3 are both nonempty

- A and B are cliques
- for $1 \le i, j \le k$ with $i \neq j$, A_i is anticomplete to B_j
- for $1 \le i \le k-1$, A_i is complete to B_i
- every vertex in A_k has a neighbour in B_k, and every vertex in B_k has a neighbour in A_k; and if C_k is nonempty then A_k, B_k are both nonempty and are complete to each other
- for $1 \le i \le k$, C_i is complete to $A_i \cup B_i$, and anticomplete to $A \cup B \setminus (A_i \cup B_i)$
- $A \cup B \cup C$ is maximal with these properties.

Note that if C_k is nonempty then there is symmetry between C_k and $C_1, \ldots, C_{k-1}$ (this will be used in the case analysis below).

(3) *$C_1, \ldots, C_k$ are pairwise anticomplete.*

For suppose not; then from the symmetry we may assume that $c_1 \in C_1$ is adjacent to $c_2 \in C_2$. Choose $a_i \in A_i$ and $b_i \in B_i$ for $i = 1, 2, 3$, such that a_3, b_3 are adjacent (this is possible even if $k = 3$). Then c_1-c_2-b_2-b_3-a_3-a_1-c_1 is a 6-hole, a contradiction. This proves (3).

(4) *For every $v \in V(G) \setminus (A \cup B \cup C)$, let N be the set of neighbours of v in $A \cup B \cup C$; then $N = \emptyset, A, B$ or $A \cup B$.*

For suppose first that $N \cap C \neq \emptyset$; there exists $c_1 \in N \cap C_1$, say. Suppose that N meets both $A \setminus A_1$ and $B \setminus B_1$. By 4.1, $N \cap (A \setminus A_1)$ is complete to $N \cap (B \setminus B_1)$, and so there exists i with $2 \le i \le k$ such that $N \cap A \subseteq A_1 \cup A_i$ and $N \cap B \subseteq B_1 \cup B_i$. Choose $a_i \in N \cap A_i$ and $b_i \in N \cap B_i$, necessarily adjacent. Choose $j \neq i$ with $2 \le j \le k$, and choose $a_j \in A_j$ and $b_j \in B_j$, adjacent. For $a_1 \in A_1$, v-c_1-a_1-a_j-b_j-b_i-v is not a 6-hole, and so $a_1 \in N$. But then v-a_1-a_j-b_j-b_i-v is a 5-hole, and $\{a_i, c_1\}$ is a coronet for it, a contradiction. Hence N does not have nonempty intersection with both $A \setminus A_1$ and $B \setminus B_1$. Suppose next that N meets $A \setminus A_1$ (and therefore does not meet $B \setminus B_1$). If $A \setminus A_1 \not\subseteq N$, we may choose distinct i, j with $2 \le i, j \le k$, such that $a_i \in N$ and $a_j \notin N$; but then $\{a_i, a_j, v\} \cup B_i$ includes a claw, a contradiction. Thus $A \setminus A_1 \subseteq N$. 4.2 (with A_1, A_2, B_2) implies that $A_1 \subseteq N$. 4.1 (with C_1, C_2, A_3) implies that $N \cap C_2 = \emptyset$, and similarly $N \cap C \subseteq C_1$. If $b_1 \in B_1$ is nonadjacent to v, then v-c_1-b_1-b_3-a_3-v is a 5-hole (where $a_3 \in A_3$ and $b_3 \in B_3$ are adjacent), and it does not dominate the vertices in C_2, a contradiction. Thus $B_1 \subseteq N$. By 4.2 (with C_1, B_1, B_2), $C_1 \subseteq N$; but then v can be added to A_1, a contradiction. Finally, if N meets neither of $A \setminus A_1$ and $B \setminus B_1$, then $A_2 \cup A_3 \cup B_2 \cup B_3$ includes a 4-hole that does not dominate either of v, c_1, contrary to 17.4. This proves that $N \cap C = \emptyset$.

Next assume that $N \cap A_1 \neq \emptyset$. 4.2 (with C_1, A_1, A_i) implies that $A \setminus A_1 \subseteq N$. In particular, $N \cap A_2 \neq \emptyset$, and so 4.2 (with C_2, A_2, A_1) implies that $A \subseteq N$. If N intersects $B \setminus B_k$, then the same argument implies that $B \subseteq N$ and the theorem holds. We assume then that $N \cap B \subseteq B_k$. If $N \cap B_k = \emptyset$ then again the theorem holds; and otherwise v can be added to A_k, a contradiction.

Thus we may assume that $N \cap A \subseteq A_k$ and $N \cap B \subseteq B_k$; and since we may assume that $N \neq \emptyset$, it follows that $C_k = \emptyset$. By 4.2 (with $A_1, N \cap A_k, B_k \setminus N$), it follows that $N \cap A_k$ is anticomplete to $B_k \setminus N$, and similarly $N \cap B_k$ is anticomplete to $A_k \setminus N$. Also, $N \cap A_k$ is complete to $N \cap B_k$, for otherwise G contains a $(2,1,1)$-prism, contrary to 17.6. Let $C'_k = \{v\}$, $A'_k = A_k \cap N$, $B'_k = B_k \cap N$, $A'_{k+1} = A_k \setminus N$, and $B'_{k+1} = B_k \setminus N$ (and set $A'_i = A_i$ and so on, for $1 \leq i < k$); then this contradicts the maximality of $A \cup B \cup C$. This proves (4).

Let A_0, B_0, M, Z be the sets of vertices $v \in V(G) \setminus (A \cup B \cup C)$ whose set of neighbours in $A \cup B \cup C$ is $A, B, A \cup B$ and $\emptyset$ respectively. By 4.3, A_0, B_0, M are cliques. Suppose that there exist adjacent $a \in A_0$ and $b \in B_0$. If $C_k = \emptyset$, we can add a to A_k and b to B_k, and if $C_k \neq \emptyset$, we can define $A_{k+1} = \{a\}$ and $B_{k+1} = \{b\}$, in either case contradicting the maximality of $A \cup B \cup C$. Thus A_0 is anticomplete to B_0. Since $A_1 \cup C_1 \cup A_0 \cup M$ includes no claw, M is complete to A_0 and similarly to B_0. Suppose that there exists $z \in Z$, and let N be the set of neighbours of z. Then by 17.4, $N \subseteq A_0 \cup B_0 \cup M$, and $N \cap M = \emptyset$ since $M \cap A_1 \cup B_2 \cup \{z\}$ includes no claw. If N meets both A_0 and B_0, then G contains a $(2,1,1)$-prism, contrary to 17.6, so we may assume that $N \subseteq A_0$. Since G is claw-free and Z is stable by 17.4, no other member of Z has a neighbour in N. Hence every vertex in $V(G) \setminus (N \cup \{z\})$ is $\{z\}$-anticomplete, and either complete or anticomplete to N. By 3.2, applied to $N, \{z\}$, it follows that G is decomposable, a contradiction. This proves that $Z = \emptyset$. Moreover, (A_k, B_k) is a homogeneous pair, nondominating since $C_1 \neq \emptyset$, and so A_k, B_k both have cardinality ≤ 1, and therefore A_k is complete to B_k. But then G is a line graph. This proves 17.8. ■

The following completes the third step of the proof of 17.1.

17.9 *Let G be claw-free, such that G has a 5-hole, G has no hole of length > 5, every 5-hole in G is dominating, $\alpha(G) \geq 4$, and G is not decomposable. If no 5-hole has a coronet, and G contains no $(1,1,1)$-prism, then G is a circular interval graph.*

Proof. By 9.3 it suffices to show that no 5-hole has a coronet, crown, hat-diagonal, star-diagonal or centre. Let C be a 5-hole. By hypothesis, C has no coronet. Also, if $\{s_1, s_2\}$ is a crown for C, then $G|(V(C) \cup \{s_1, s_2\})$ contains a $(1,1,1)$-prism (delete the middle of the three common neighbours of s_1, s_2 in C), a contradiction. C has no hat-diagonal since by 17.6, G contains no $(2,1,1)$-prism. By 17.6, C has no centre; so it remains to prove that C has no star-diagonal.

Suppose that it does; let C have vertices $c_1\text{-}\cdots\text{-}c_5\text{-}c_1$ in order, and let s_1, s_2 be adjacent stars, adjacent respectively to $c_1, \ldots, c_4$ and to c_3, c_4, c_5, c_1. Since C has no coronet, there are no hats in positions $2\frac{1}{2}, 4\frac{1}{2}$; and there is not both a hat and a star in position $3\frac{1}{2}$. Consequently, the first and third outcomes of 17.5 are impossible, and so 17.5 implies that there is a stable set X with $|X| = 4$, consisting of two hats x_1, x_2 in positions $\frac{1}{2}$ and $1\frac{1}{2}$ respectively, and two clones x_3, x_4 in positions $3, 4$ respectively. By 8.2, s_1 is adjacent to x_2, x_3 and not to x_1, and s_2 is adjacent to x_1, x_4 and not x_2. If x_3 is adjacent to s_2 then $\{s_2, x_1, x_3, x_4\}$ is a claw, while if x_3 is not adjacent to s_2 then $\{s_1, s_2, x_2, x_3\}$ is a claw, in either case a contradiction. Hence C has no star-diagonal, and 9.3 implies that G is a circular interval graph. This proves 17.9. ■

For the fourth step of the proof of 17.1, we use the following.

17.10 *Let G be claw-free, such that G has a hole of length 4, G has no hole of length > 4, $\alpha(G) \geq 4$, and G is not decomposable. Then G is a line graph.*

Proof. By 17.8, we may assume that G contains no $(1,1,1)$-prism. Let $c_1\text{-}\cdots\text{-}c_4\text{-}c_1$ be a 4-hole. It is dominating, by 9.2, since G contains no $(1,1,1)$-prism. By hypothesis, there is a stable set X with $|X| = 4$. Thus each member of X either belongs to $\{c_1, \ldots, c_4\}$ or has at least two neighbours in this set. If say $c_1 \in X$, then $c_2, c_4 \notin X$, and each is adjacent to at most one member of $X \setminus \{c_1\}$, which is impossible. Thus $c_1, \dots, c_4 \notin X$. Also, $c_1, \dots, c_4$ each are adjacent to at most two members of X, and so equality holds, and therefore each member of X is a hat relative to c_1-$\cdots$-c_4-c_1, all in different positions. Let $X = \{x_1, \dots, x_4\}$, where x_i is a hat adjacent to c_i, c_{i+1}.

Consequently there are four nonempty cliques $A_1, \dots, A_4$, pairwise disjoint, such that:

- A_i is complete to A_{i+1} and anticomplete to A_{i+2} for $1 \le i \le 4$ (reading subscripts modulo 4)
- x_i is complete to A_i, A_{i+1} and anticomplete to A_{i+2}, A_{i+3}, for $1 \le i \le 4$.

Choose $A_1, \dots, A_4$ with maximal union W. Let B be the set of all vertices $v \in V(G) \setminus W$ that are W-complete. For $i = 1, 2, 3, 4$, let H_i be the set of all $v \in V(G) \setminus W$ such that v is complete to $A_i \cup A_{i+1}$ and anticomplete to $A_{i+2} \cup A_{i+3}$. Thus $x_i \in H_i$ $(1 \le i \le 4)$.

(1) $V(G) = W \cup B \cup H_1 \cup H_2 \cup H_3 \cup H_4$.

For suppose that $v \in V(G) \setminus W$. We claim that $v \in B \cup H_1 \cup H_2 \cup H_3 \cup H_4$. For let N be the set of neighbours of v. Since every 4-hole is dominating, we may assume that $A_1 \subseteq N$. 4.2 (with A_4, A_1, A_2) implies that N includes one of A_4, A_2, and from the symmetry we may assume that $A_2 \subseteq N$. Suppose that N intersects but does not include A_3. Choose $a_3, a_3' \in A_3$ such that $a_3 \in N$ and $a_3' \notin N$. Then 4.2 (with x_1, A_2, a_3') implies that $x_1 \in N$; 4.1 implies that $x_4 \notin N$; 4.2 (with a_3', A_4, x_4) implies that $N \cap A_4 = \emptyset$; 4.2 (with x_2, a_3, A_4) implies that $x_2 \in N$; and then v-x_2-a_3'-a_4-a_1-v is a 5-hole (where $a_1 \in A_1$ and $a_4 \in A_4$), a contradiction. Thus N either includes A_3 or is disjoint from A_3, and the same holds for A_4. If N is disjoint from both A_3, A_4 then $v \in H_1$ as claimed, and if N includes both A_3, A_4 then $v \in B$ as claimed. We assume therefore that N includes just one of them, say A_3, and is disjoint from A_4. By 4.2, $x_1, x_2 \in N$, and by 4.1, $x_3, x_4 \notin N$, and so v can be added to A_2, contrary to the maximality of W. This proves (1).

It follows from (1) that for $1 \le i \le 4$, all members of A_i are twins, and therefore $|A_i| = 1$, and so $A_i = \{c_i\}$. For $1 \le i \le 4$, there are no edges between H_i and H_{i+1}, since G has no 5-hole, and there is no edge between H_i and H_{i+2} since G contains no $(1,1,1)$-prism. Thus $H_1, \dots, H_4$ are pairwise anticomplete. By 4.3, each H_i is a clique. Let B_1 be the set of all $v \in B$ that are complete to $H_1 \cup H_3$ and anticomplete to $H_2 \cup H_4$, and let B_2 be those that are complete to $H_2 \cup H_4$ and anticomplete to $H_1 \cup H_3$. We claim that $B = B_1 \cup B_2$. For let $b \in B$, and let N be the set of its neighbours. 4.2 (with H_1, c_2, H_2) implies that N includes one of H_1, H_2, say H_1. By 4.1, N is disjoint from at least two of H_2, H_3, H_4. By 4.2 (with H_2, c_3, H_3 and H_3, c_3, H_4), $H_3 \subseteq N$, and so $N \cap (H_2 \cup H_4) = \emptyset$. Thus $v \in B_1$. This proves that $B = B_1 \cup B_2$. Consequently all members of H_i are twins, and so $H_i = \{x_i\}$ for $1 \le i \le 4$. Now if $b_1 \in B_1$ and $b_2 \in B_2$ then $\{b_1, b_2, x_1, x_3\}$ is not a claw, and so b_1, b_2 are nonadjacent. Thus B_1 is anticomplete to B_2. By 4.3, B_1, B_2 are cliques, and so for $i = 1, 2$, all members of B_i are twins. Hence $|B_1|, |B_2| \le 1$. But then G is a line graph. This proves 17.10. ■

Finally, we handle graphs without any holes at all, in the following.

17.11 *Let G be claw-free, such that G has no holes and $\alpha(G) \ge 4$. Then G is decomposable.*

Proof. For a contradiction, suppose that G is not decomposable.

(1) *There do not exist distinct* $x_1, \dots, x_4 \in V(G)$ *such that* x_1x_2, x_3x_4 *are edges and* $\{x_1, x_2\}$ *is anticomplete to* $\{x_3, x_4\}$.

For suppose that such $x_1, \dots, x_4$ exist. Choose connected sets A_1, A_2 with $A_1 \cup A_2$ maximal such that $x_1, x_2 \in A_1$, $x_3, x_4 \in A_4$, $A_1 \cap A_2 = \emptyset$, and A_1 is anticomplete to A_2. Let X be the set of vertices in $V(G) \setminus (A_1 \cup A_2)$ with a neighbour in $A_1 \cup A_2$. We claim that X is a clique; for let $u, v \in X$. By the maximality of $A_1 \cup A_2$, both u, v have neighbours in both A_1 and A_2; and so for $i = 1, 2$, there is a path P_i between u, v with interior in A_i. If u, v are nonadjacent, $P_1 \cup P_2$ is a hole, a contradiction. This proves that X is a clique, and therefore it is an internal clique cutset, since $|A_1|, |A_2| > 1$, a contradiction. This proves (1).

Say a subset $Y \subseteq V(G)$ is *split* if $|Y| \geq 4$ and every connected subset $C \subseteq Y$ satisfies $|C| \leq |Y| - 2$. Since $\alpha(G) \geq 4$, there is a split subset $Y \subseteq V(G)$. Choose Y maximal, and let the components of $G|Y$ be $C_1, \dots, C_k$. Let $V(G) \setminus Y = X$. For each $x \in X$, we observe that x has neighbours in at most two of $C_1, \dots, C_k$, since G is claw-free; and if it has neighbours in at most one of $C_1, \dots, C_k$, then $Y \cup \{x\}$ is split, a contradiction. Thus each $x \in X$ has neighbours in exactly two of $C_1, \dots, C_k$. We claim that X is a clique. For let $u, v \in X$, and suppose they are nonadjacent. Choose $1 \leq i < i' \leq k$ such that u has neighbours in C_i and in $C_{i'}$, and define j, j' similarly for v. If $\{i, i'\} \neq \{j, j'\}$, then we may assume that $i \neq j, j'$ and $j \neq i, i'$; choose $a \in C_i$ adjacent to u and $b \in C_j$ adjacent to v, and then the existence of x, u, y, b is contrary to (1). Hence $\{i, i'\} = \{j, j'\}$; but then there are paths joining u, v with interior in C_i and in $C_{i'}$, and their union is a hole, a contradiction. This proves that X is a clique. Since Y is split and $|Y| \geq 4$, there is a partition Y_1, Y_2 of Y such that $|Y_1|, |Y_2| \geq 2$ and Y_1 is anticomplete to Y_2; and so X is an internal clique cutset, a contradiction. Thus G is decomposable. This proves 17.11, and therefore completes the proof of 17.1. ■

18 Non-antiprismatic graphs

In view of 17.1, to complete the proof of 2.1 it remains to study graphs G with $\alpha(G) \leq 3$ that are not antiprismatic, and that is the topic of this section. We need a number of lemmas before the main theorem. A vertex $v \in V(G)$ is *simplicial* if all neighbours of G are pairwise adjacent. An edge of G is a *leaf-edge* if one of its ends has degree 1.

18.1 *Let G be claw-free, with $\alpha(G) \leq 3$, and let a_0, b_0 be simplicial vertices of G, nonadjacent and with no common neighbour. Then either*

- *G admits a nondominating or coherent W-join or twins, or*
- *G is a linear interval graph, and a_0, b_0 are the first and last vertices of the corresponding linear order of $V(G)$, or*
- *G is the line graph of some graph H, and a_0, b_0 are both leaf-edges of H, or*
- *there is a graph H with $E(H) = V(G)$, such that a_0, b_0 are leaf-edges of H, and there is a path of H of length 4 with edges a_0, a, b, b_0, such that G is obtained from $L(H)$ by deleting the edge ab, (and consequently G is expressible as a hex-join), or*

- *G is 2-simplicial of antihat type.*

In particular, either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ or G is decomposable.

Proof. We assume that G does not admit a nondominating or coherent W-join or twins. Let A, B and C be the sets of all vertices different from a_0, b_0 that are adjacent to a_0, to b_0 and to neither of a_0, b_0 respectively. Thus $V(G) = A \cup B \cup C \cup \{a_0, b_0\}$. Moreover, $A \cup \{a_0\}$ and $B \cup \{b_0\}$ are cliques since a_0, b_0 are simplicial; and C is a clique since if $c_1, c_2 \in C$ are nonadjacent then $\{a_0, b_0, c_1, c_2\}$ is a stable set, contradicting that $\alpha(G) \leq 3$.

(1) *$A, B \neq \emptyset$. Moreover, if $a \in A$ and $b \in B$ are adjacent, they have the same neighbours in C.*

For suppose that $A = \emptyset$, say. Thus a_0 has degree 0. Then $(B \cup \{b_0\}, C)$ is a homogeneous pair of cliques, nondominating, and so 3.3 implies that $B = \emptyset$. But then a_0, b_0 are twins, a contradiction. This proves the first claim. For the second, note that if $c \in C$ is adjacent to $a \in A$ and not to b say, then $\{a, a_0, b, c\}$ is a claw, a contradiction. This proves (1).

(2) *Every vertex in A has at most one neighbour in B, and vice versa.*

For let H be the graph with vertex set $A \cup B$ and edge set the edges of G with one end in A and one in B. Let X be any component of H; then by (1), $(X \cap A, X \cap B)$ is a homogeneous pair of cliques, coherent since all X-complete vertices belong to C, and so $|X \cap A|, |X \cap B| \leq 1$. This proves (2).

(3) *Every vertex in $A \cup B$ has a neighbour in C; and in particular, $C \neq \emptyset$.*

For let A_0 be the set of vertices in A with no neighbour in C, and define B_0 similarly. By (1), there are no edges between A_0 and $B \setminus B_0$, and no edges between B_0 and $A \setminus A_0$. Consequently, $(A_0 \cup \{a_0\}, B_0 \cup \{b_0\})$ is a homogeneous pair of cliques, coherent since a_0, b_0 have no common neighbours. By 3.3 it follows that A_0, B_0 are empty. This proves (3).

(4) *G is connected and we may assume that it admits no 1-join.*

For A, B are nonempty, and by (3) C is nonempty, and since C is a clique, (3) implies that G is connected. Suppose that G admits a 1-join; let $V(G) = P_1 \cup Q_1 \cup P_2 \cup Q_2$, where P_1, Q_1, P_2, Q_2 are all nonempty and pairwise disjoint, and $Q_1 \cup Q_2$ is a clique, and P_1 is anticomplete to $P_2 \cup Q_2$, and P_2 is anticomplete to $P_1 \cup Q_1$. Suppose first that both of P_1, P_2 are cliques. Then (P_1, Q_2) is a homogeneous pair of cliques, nondominating since P_2 is nonempty, and so 3.3 implies that $|P_1| = |Q_1| = 1$, and similarly $|P_2| = |Q_2| = 1$, and therefore $|V(G)| = 4$, contrary to (1) and (3). We may therefore assume that one of P_1, P_2 is not a clique, say P_1. Since $\alpha(G) \leq 3$, it follows that $P_2 \cup Q_2$ is a clique; and since G does not admit twins, it follows that $|P_2| = |Q_2| = 1$. Let $P_2 = \{p_2\}, Q_2 = \{q_2\}$ say. If $x, y \in P_1$ are nonadjacent, then x, y have no common neighbour in Q_1 (since if q_1 were a common neighbour then $\{q_1, q_2, x, y\}$ would be a claw) and they have no common nonneighbour in Q_1 (for if q_1 were a common nonneighbour then $\{x, y, q_1, p_2\}$ would be stable, contradicting $\alpha(G) \leq 3$). Hence the set of neighbours of x in Q_1 is precisely the complement in Q_1 of the set of neighbours of y in Q_1. Since this holds for all nonadjacent pairs $x, y \in P_1$, it follows that there is no odd length cycle in the complement graph of $G|P_1$, and so this graph is bipartite; and consequently P_1 is the union of two cliques of G, say X, Y. Now not both a_0, b_0 belong to $Q_1 \cup P_2 \cup Q_2$, since a_0, b_0 are nonadjacent and have no common neighbour; so we may assume that $a_0 \in X$. The set of nonneighbours of a_0 in P_1 is a clique (because it is a subset of Y); and so we may assume that $X = (A \cup \{a_0\}) \cap P_1$ and $Y = P_1 \setminus X$. For $y \in Y$, since y, a_0 are nonadjacent, it follows that the set of neighbours of y in Q_1 is $Q_1 \setminus A$; and so $Y \cup (Q_1 \setminus A)$ is a clique, and Y is anticomplete to $Q_1 \cap A$. Let X_1 be the set of vertices in X with a neighbour in $Q_1 \setminus A$, and $X_2 = X \setminus X_1$. If $x_1 \in X_1$ and $y \in Y$, let $q_1 \in Q_1 \setminus A$ be adjacent to x_1; then since $\{q_1, x_1, y, q_2\}$ is not a claw, it follows that x_1, y are adjacent. Hence X_1 is complete to Y. Consequently (X_2, Y) is a homogeneous pair of cliques, nondominating since $P_2 \neq \emptyset$, and so by 3.3, $|X_2|, |Y| \leq 1$ and hence $X_2 = \{a_0\}$. Moreover, $(X_1, Q_1 \setminus A)$ is also a nondominating homogeneous pair of cliques, and so 3.3 implies that $|X_1|, |Q_1 \setminus A| \leq 1$. Also, every two members of $Q_1 \cap A$ are twins, and so $|Q_1 \cap A| \leq 1$. Hence $|V(G)| \leq 7$, and G is the line graph of some graph H, and since b_0 is simplicial it follows that a_0, b_0 are both leaf-edges of H, and the theorem holds. This proves (4).

(5) *We may assume that G admits no internal clique cutset.*

For suppose it does. By (4) and 3.1, we may assume that G is a linear interval graph. Let $v_1, \ldots, v_n$ be the corresponding ordering of the vertex set. If $\{a_0, b_0\} = \{v_1, v_n\}$ then the theorem holds, so we may assume that $a_0 = v_h$ and $b_0 = v_j$ say for some h, j with $1 \leq h < j < n$. Since G is connected and $j > 1$, it follows that v_{j-1}, v_{j+1} are adjacent to v_j. Choose i, k with $1 \leq i < j < k \leq n$ such that v_i, v_k are adjacent to v_j, with i minimum and k maximum. Since v_h, v_j are nonadjacent, it follows that $h < i$. Since v_j is simplicial, all of $v_i, \ldots, v_{j-1}$ are adjacent to all of $v_{j+1}, \ldots, v_k$. Since v_j, v_k are not twins, it follows that $k < n$. Since $\alpha(G) \leq 3$, $\{v_1, \ldots, v_{i-1}\}$ is a clique. For all m with $j < m \leq n$, if v_m is adjacent to any of $v_i, \ldots, v_j$, then it is adjacent to v_j, and therefore $m \leq k$; and it follows that v_m is adjacent to all of $v_i, \ldots, v_j$. We deduce that $(\{v_1, \ldots, v_{i-1}\}, \{v_i, \ldots, v_j\})$ is a homogeneous pair, nondominating since $k < n$. By 3.3, this is impossible. This proves (5).

Let there be k edges between A and B. By (5), C is not an internal clique cutset, and so $k > 0$. Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, where for $1 \leq i \leq k$ a_i is adjacent to b_i, and there are no other edges between A and B. Define $A' = \{a_{k+1}, \ldots, a_m\}$, and $B' = \{b_{k+1}, \ldots, b_n\}$. For each $c \in C$, let

$$I_c = \{i : 1 \leq i \leq k \text{ and } c \text{ is adjacent to } a_i, b_i.\}$$

(6) *If $c, c' \in C$, and $i \in I_c \setminus I_{c'}$, then a_i, b_i are the only vertices in $A \cup B$ that are adjacent to c and not to c'. In particular, $|I_c \setminus I_{c'}| \leq 1$.*

For suppose that a_j is adjacent to c and not to c', say, where $j \neq i$. Then $\{c, c', a_j, b_i\}$ is a claw by (2), a contradiction. This proves (6).

Let j be the maximum cardinality of the sets I_c ($c \in C$). By (6), $|I_c| = j$ or $j-1$ for all $c \in C$. By (3) $j \geq 1$. Let

$$P = \{c \in C : |I_c| = j - 1\}$$

and $Q = C \setminus P$. Let Z be the set of vertices in $A' \cup B'$ with a neighbour in Q. By (6), if $p \in P$ and $q \in Q$, then $I_p \subseteq I_q$, and every vertex in $A' \cup B'$ that is adjacent to q is also adjacent to p. In particular, Z is complete to P. By definition, Q is nonempty. Now there are four cases:

- P is empty and $I_{q_1} = I_{q_2}$ for all $q_1, q_2 \in Q$
- There exist $q_1, q_2 \in Q$ with $I_{q_1} \neq I_{q_2}$
- There exist $p_1, p_2 \in P$ with $I_{p_1} \neq I_{p_2}$, and
- P is nonempty, $I_{q_1} = I_{q_2}$ for all $q_1, q_2 \in Q$, and $I_{p_1} = I_{p_2}$ for all $p_1, p_2 \in P$.

We treat these cases separately.

(7) *If P is empty and $I_{q_1} = I_{q_2}$ for all $q_1, q_2 \in Q$ then the theorem holds.*

For then by (3), $j = k$ and $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ is complete to C, and so $(\{a_1, \dots, a_k\}, \{b_1, \dots, b_k\})$ is a coherent homogeneous pair of cliques. By 3.3, $k = 1$. By (5), we may assume that $C \cup \{a_1\}$ is not an internal clique cutset, and so $m = 1$, and similarly $n = 1$. Hence all members of C are twins, and so $|C| = 1$, and the third outcome of the theorem holds. This proves (7).

(8) *If there exist $q_1, q_2 \in Q$ with $I_{q_1} \neq I_{q_2}$ then the theorem holds.*

For then let X be the set of neighbours of q_1 in $A' \cup B'$. By (5), q_1, q_2 have the same neighbours in $A' \cup B'$. Thus X is the set of neighbours of q_2 in $A' \cup B'$. For any third member $q \in Q$, I_q is different from one of I_{q_1}, I_{q_2}, and so by the same argument, X is the set of neighbours of q in $A' \cup B'$. Consequently Q is complete to X and anticomplete to $(A' \cup B') \setminus X$. Hence $X = Z$, and therefore X is complete to P and hence to C.

Choose $q_1, q_2 \in Q$ with $I_{q_1} \neq I_{q_2}$, and let $Y = I_{q_1} \cap I_{q_2}$. Now $I_p = Y$ for every $p \in P$, by (6). Suppose that there exists $q_3 \in Q$ with $Y \not\subseteq I_{q_3}$. (Hence $P = \emptyset$.) Let $Y' = I_{q_1} \cup I_{q_2}$. Since $|I_q \cup I_{q'}| \leq j + 1$ for all $q, q' \in Q$, it follows that $|Y'| = j + 1$ and $I_{q_3} \subseteq Y'$; and since no subset $Y'' \subseteq Y'$ with $|Y''i| \leq j - 1$ has intersection of cardinality $\geq j - 1$ with each of $I_{q_1}, I_{q_2}, I_{q_3}$, it follows that $I_q \subseteq Y'$ for all $q \in Q$. By (3), $j + 1 = k$. Moreover, there do not exist $q, q' \in Q$ with $I_q = I_{q'}$, since then q, q' would be twins. Consequently, G is 2-simplicial of antihat type, and the theorem holds.

We may therefore assume that $Y \subseteq I_q$ for all $q \in Q$. If $p \in P$ has a neighbour $a \in A \setminus Z$ and $b \in B \setminus Z$ then $\{p, q_1, a, b\}$ is a claw, a contradiction; so $P = P_1 \cup P_2$ where P_1, P_2 are the sets of vertices in P anticomplete to $A' \setminus Z, B' \setminus Z$ respectively. Since $I_p = Y$ for all $p \in P$, it follows that $(P_1, A' \setminus Z)$ is a homogeneous pair, nondominating because of b_0, and so $|P_1|, |A' \setminus Z| \leq 1$; and similarly $|P_2|, |B' \setminus Z| \leq 1$. Moreover $(\{a_i : i \in Y\} \cup (A' \cap Z), \{b_i : i \in Y\} \cup (B' \cap Z))$ is a coherent homogeneous pair of cliques, and so by 3.3, $|Y| \leq 1$, that is, $j \leq 2$; and moreover, either $j = 1$ or $A' \cap Z = B' \cap Z = \emptyset$. But then in either case G is the line graph of a graph H such that a_0, b_0 are both leaf-edges of H, and the theorem holds. This proves (8).

(9) *If there exist $p_1, p_2 \in P$ with $I_{p_1} \neq I_{p_2}$, then the theorem holds.*

For let $Y = I_{p_1} \cup I_{p_2}$; then $|Y| = j$. By (6), $I_q = Y$ for all $q \in Q$. Choose $q \in Q$; then by (6), $I_p \subseteq I_q$ and therefore $I_p \subseteq Y$, for all $p \in P$. By (3), $j = k$, and so Q is complete to $(A \setminus A') \cup (B \setminus B')$. Let W be the set of neighbours of p_1 in $A' \cup B'$. By (6), W is also the set of neighbours in $A' \cup B'$ of p_2. Moreover, if $p \in P$ then I_p is different from one of I_{p_1}, I_{p_2}, and so W is the set of neighbours of p in $A' \cup B'$. We deduce that P is complete to W and anticomplete to $(A' \cup B') \setminus W$. But by (3), every vertex in $A' \cup B'$ has a neighbour in C, and Z is complete to P; so every vertex in $A' \cup B'$ has a neighbour in P, and therefore belongs to W. We deduce that P is complete to $A' \cup B'$. If $q \in Q$ has nonneighbours $a' \in A'$ and $b' \in B'$, then $\{p_1, q, a', b'\}$ is a claw, a contradiction; so every member of Q is either complete to A' or complete to B'. Let Q_1 be those complete to B', and Q_2 those complete to A'. Since (Q_1, A') is a homogeneous pair of cliques, nondominating because of b_0, 3.3 implies that $|Q_1|, |A'| \leq 1$, and similarly $|Q_2|, |B'| \leq 1$. Now there do not exist $p, p' \in P$ with $I_p = I_{p'}$, because they would be twins. Consequently, if $|Q| \leq 1$ then G is 2-simplicial of antihat type; so we may assume that $Q_1 = \{q_1\}$ and $Q_2 = \{q_2\}$, and $Q_1 \cap Q_2 = \emptyset$. In particular, q_1 is not complete to A', and so A' is nonempty; let $A' = \{a'\}$ say, where q_1, a' are nonadjacent. Similarly, $B' = \{b'\}$ where b', q_2 are nonadjacent. But then again, G is 2-simplicial of antihat type. This proves (9).

In view of (7)–(9), we may henceforth assume that P is nonempty, $I_{q_1} = I_{q_2}$ for all $q_1, q_2 \in Q$, and $I_{p_1} = I_{p_2}$ for all $p_1, p_2 \in P$. Let $I_p = Y$ for all $p \in P$. Then $|Y| = j - 1$, and $(\{a_i : i \in P\}, \{b_i : i \in P\})$ is a coherent homogeneous pair of cliques, and so 3.3 implies that $j \leq 2$. By (3), $k = j$. If some $q \in Q$ has nonneighbours $a' \in A' \cap Z$ and $b' \in B' \cap Z$, then $\{p, q, a', b'\}$ is a claw where $p \in P$, a contradiction. Thus $Q = Q_1 \cup Q_2$, where Q_1, Q_2 are the sets of members of Q which are complete to $B' \cap Z$ and to $A' \cap Z$ respectively. Since $(Q_1, A' \cap Z)$ is a homogeneous pair, nondominating because of b_0, 3.3 implies that $|Q_1|, |A' \cap Z| \leq 1$, and similarly $|Q_2|, |B' \cap Z| \leq 1$. If some $p \in P$ has neighbours $a' \in A' \setminus Z$ and $b' \in B' \setminus Z$ then $\{p, q, a', b'\}$ is a claw, where $q \in Q$, a contradiction. Thus $P = P_1 \cup P_2$, where P_1, P_2 are the sets of members of P that are anticomplete to $B' \setminus Z$ and to $A' \setminus Z$ respectively. Since $(P_1, A' \setminus Z)$ is a nondominating homogeneous pair of cliques, 3.3 implies that $|P_1|, |A' \setminus Z| \leq 1$, and similarly $|P_2|, |B' \setminus Z| \leq 1$.

(10) *If $|Q| \geq 2$ then the theorem holds.*

For in this case it follows that $Q_1, Q_2 \neq Q$. Since $Q_1 \cup Q_2 = Q$ and $|Q_1|, |Q_2| \leq 1$, we deduce that $Q = \{q_1, q_2\}$, where $Q_i = \{q_i\}$ for $i = 1, 2$. Since $q_1 \notin Q_2$, there exists $a' \in A' \cap Z$ nonadjacent to q_1. Suppose that there exists $p \in P \setminus P_1$. Since $p \notin P_1$, p has a neighbour $b' \in B' \setminus Z$; but then $\{p, q_1, a', b'\}$ is a claw, a contradiction. This proves that $P_1 = P$, and similarly $P_2 = P$. Hence $|P| = 1$, $P = \{p\}$ say. Since $p \in P_1$, p has no neighbours in $B' \setminus Z$; but every vertex in $B' \setminus Z$ is adjacent to p, by (3), and so $B' \subseteq Z$. Similarly $A' \subseteq Z$, and so G is 2-simplicial of antihat type, and the theorem holds. This proves (10).

In view of (10) we may assume that $|Q| = 1$. Since every vertex in Z has a neighbour in Q, it follows that Q is complete to Z, and so $Q_1 = Q_2 = Q$. If $Z = \emptyset$ then G is a line graph of some graph of which a_0, b_0 are both leaf-edges, and the theorem holds. Thus we may assume that Z is nonempty.

If $Y \neq \emptyset$, let $1 \in Y$, say; then $((Z \cap A') \cup \{a_1\}, (Z \cap B') \cup \{b_1\})$ is a coherent homogeneous pair of cliques, and therefore G is decomposable by 3.3 (since Z is nonempty) and the theorem holds. Thus we may assume that Y is empty. If not both $Z \cap A', Z \cap B'$ are nonempty, then G is a line graph and the theorem holds; so let $Z \cap A' = \{a'\}$ and $Z \cap B' = \{b'\}$ say. Then G is the hex-join of $G|Z$ and $G|(V(G) \setminus Z)$; and if we add the edge $a'b'$ to G we obtain the line graph of a graph in which a_0, b_0 are leaf-edges and there is a path with edge set $\{a_0, a', b', b_0\}$, and again the theorem holds. This proves 18.1. ■

18.2 *Let G be claw-free, such that there is no hole in G of length > 5, every hole of length 5 is dominating, and $\alpha(G) \leq 3$. Let C be a 5-hole in G with vertices c_1-$\cdots$-c_5-c_1, and let there be hats in positions $1\frac{1}{2}, 2\frac{1}{2}$ respectively. Then G is decomposable.*

Proof. For $i = 1, \ldots, 5$, let C_i be the set of all clones in position i, and let $H_{i+\frac{1}{2}}, S_{i+\frac{1}{2}}$ be the set of all hats and stars in position $i+\frac{1}{2}$ respectively. By 8.2, $H_{i-\frac{1}{2}}$ is anticomplete to $H_{i+\frac{1}{2}}$ for $i = 1, \ldots, 5$. By hypothesis, we may choose $h_1 \in H_{1\frac{1}{2}}$ and $h_2 \in H_{2\frac{1}{2}}$.

(1) *There is no centre for C.*

For suppose that z is a centre for C. Since $\{z, h_1, c_3, c_5\}$ is not a claw, z is not adjacent to h_1, and similarly z is not adjacent to h_2. But then $\{c_2, h_1, h_2, z\}$ is a claw, a contradiction. This proves (1).

(2) *$H_{\frac{1}{2}}, H_{3\frac{1}{2}}$ are empty; at least one of $H_{4\frac{1}{2}}, S_{4\frac{1}{2}}$ is empty; and C_4 is complete to C_5.*

For suppose that there exists $h_3 \in H_{3\frac{1}{2}}$ say. Since $H_{2\frac{1}{2}}$ is anticomplete to $H_{3\frac{1}{2}}$, it follows that h_2, h_3 are nonadjacent. Since every 5-hole is dominating, h_1-h_3-c_4-c_5-c_1-h_1 is not a 5-hole (because h_2 has no neighbours in it), and so h_1, h_3 are nonadjacent. But then $\{h_1, h_2, h_3, c_5\}$ is stable, contradicting that $\alpha(G) \leq 3$. This proves the first assertion of (2). Suppose that $h \in H_{4\frac{1}{2}}$ and $s \in S_{4\frac{1}{2}}$. By 8.2, s is nonadjacent to h, h_1, h_2. If h is nonadjacent to both h_1, h_2 then $\{s, h, h_1, h_2\}$ is stable, a contradiction; if h is adjacent to say h_1 and not h_2 then s-c_4-h-h_1-c_1-s is a 5-hole and h_2 has no neighbour in it, a contradiction; while if h is adjacent to both h_1, h_2 then $\{h, h_1, h_2, c_4\}$ is a claw, a contradiction. Thus not both $H_{4\frac{1}{2}}, S_{4\frac{1}{2}}$ are nonempty, and this proves the second assertion of (2). For the third assertion, suppose that $x \in C_4$ and $y \in C_5$ are nonadjacent. By 8.2, x is nonadjacent to h_1 and y is nonadjacent to h_2. Since $\{x, y, h_1, h_2\}$ is not stable, we may assume that x is adjacent to h_2; but then x-c_4-y-c_1-c_2-h_2-x is a 6-hole, a contradiction. Thus C_4 is complete to C_5. This proves (2).

Let

$$\begin{aligned}
B_1 &= H_{1\frac{1}{2}} \cup C_1 \cup \{c_1\} \cup S_{\frac{1}{2}} \cup S_{2\frac{1}{2}} \\
B_2 &= H_{2\frac{1}{2}} \cup C_3 \cup \{c_3\} \cup S_{3\frac{1}{2}} \cup S_{1\frac{1}{2}} \\
B_3 &= C_4 \cup C_5 \cup \{c_4, c_5\} \cup S_{4\frac{1}{2}} \cup H_{4\frac{1}{2}} \\
B &= B_1 \cup B_2 \cup B_3.
\end{aligned}$$

(3) *B_1, B_2, B_3 are cliques.*

For by 8.2, $H_{1\frac{1}{2}} \cup C_1 \cup \{c_1\} \cup S_{\frac{1}{2}}$ is a clique, and $S_{2\frac{1}{2}}$ is a clique. We must show that $S_{2\frac{1}{2}}$ is complete to $H_{1\frac{1}{2}} \cup C_1 \cup \{c_1\} \cup S_{\frac{1}{2}}$. Let $s \in S_{2\frac{1}{2}}$. Certainly s is adjacent to c_1, and s is nonadjacent to h_2 and complete to $H_{1\frac{1}{2}}$, by 8.2 (and in particular, s is adjacent to h_1). Suppose that $x \in S_{\frac{1}{2}}$. By 8.2, x is not adjacent to h_2, and since $\{c_2, x, h_2, s\}$ is not a claw it follows that s, x are adjacent. Thus s is complete to $S_{\frac{1}{2}}$. Now suppose that $x \in C_1$. By 8.2, x is adjacent to h_1. Since $\{x, h_1, h_2, c_5\}$ is not a claw, x is not adjacent to h_2. Since $\{c_2, x, h_2, s\}$ is not a claw, s is adjacent to x, and so s is complete to C_1. This proves that B_1 is a clique, and similarly B_2 is a clique. By 4.3, the sets $C_4 \cup \{c_4\}, C_5 \cup \{c_5\}, S_{4\frac{1}{2}}, H_{4\frac{1}{2}}$ are cliques; by (2), it follows that $C_4 \cup C_5 \cup \{c_4, c_5\}$ and $S_{4\frac{1}{2}} \cup H_{4\frac{1}{2}}$ are cliques; and by 8.2, $C_4 \cup C_5 \cup \{c_4, c_5\}$ is complete to $S_{4\frac{1}{2}} \cup H_{4\frac{1}{2}}$, and therefore B_3 is a clique. This proves (3).

(4) *There is no triad T with $|T \cap B| = 2$.*

For suppose that $\{x, y, z\}$ is a triad, where $x, y \in B$ and $z \notin B$. Since C is dominating and has no centre, and $H_{\frac{1}{2}}, H_{3\frac{1}{2}}$ are empty, it follows that $z \in C_2 \cup \{c_2\}$. Thus $x, y \neq c_1, c_3$ and by 8.2, z is complete to all of $H_{1\frac{1}{2}}$, $H_{2\frac{1}{2}}$, $S_{1\frac{1}{2}}$, $S_{2\frac{1}{2}}$, and so $x, y \notin H_{1\frac{1}{2}} \cup H_{2\frac{1}{2}} \cup S_{1\frac{1}{2}} \cup S_{2\frac{1}{2}}$. If $x \in C_1$, then x is adjacent to h_1 by 8.2, and so x-h_1-z-c_3-c_4-c_5-x is a 6-hole, a contradiction. Thus $x \notin C_1$, and similarly $x, y \notin C_1 \cup C_3$.

Since B is the union of the three cliques B_1, B_2, B_3, and there is symmetry between B_1, B_2, we may assume that $x \in B_1$, and therefore $x \in S_{\frac{1}{2}}$. Moreover, $y \in B_2 \cup B_3$, and so

$$y \in C_4 \cup C_5 \cup \{c_4, c_5\} \cup S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}} \cup H_{4\frac{1}{2}}.$$

Since $x \in S_{\frac{1}{2}}$, it follows that $z \neq c_2$, and so $z \in C_2$; and $y \neq c_4, c_5$. By 8.2, $y \notin C_5 \cup H_{4\frac{1}{2}}$. Since x, y, z have no common neighbour (since G is claw-free) it follows that y is nonadjacent to c_1, c_2, and so $y \notin S_{3\frac{1}{2}} \cup S_{4\frac{1}{2}}$. We deduce that $y \in C_4$. By 8.2, x is adjacent to h_1, and y is nonadjacent to h_1; but then x-h_1-z-c_3-y-c_5-x is a 6-hole, a contradiction. This proves (4).

Now $\{h_1, h_2, c_4\}$ and $\{h_1, h_2, c_5\}$ are triads, both contained in B and sharing two vertices. From 15.1, we deduce that G is decomposable. This proves 18.2. ■

Let G be a graph. We say a triple (A_1, A_2, A_3) is a *spread* in G if

- A_1, A_2, A_3 are nonempty cliques, pairwise disjoint and pairwise anticomplete
- $|A_1| + |A_2| + |A_3| \geq 4$
- there is no vertex in $V(G) \setminus (A_1 \cup A_2 \cup A_3)$ that is complete to one of A_1, A_2, A_3 and anticomplete to the other two.

If (A_1, A_2, A_3) is a spread, no vertex has neighbours in all three of A_1, A_2, A_3 since G is claw-free. For $1 \leq i, j \leq 3$ with $i \neq j$, let $M_{i,j}$ be the set of all vertices in $V(G) \setminus (A_i \cup A_j)$ complete to $A_i \cup A_j$, and let $N_{i,j}$ be the set of all vertices in $V(G) \setminus (A_i \cup A_j)$ that are complete to A_i and have both a neighbour and a nonneighbour in A_j. Thus $M_{i,j} = M_{j,i}$ but $N_{i,j}$ and $N_{j,i}$ are disjoint. A spread (A_1, A_2, A_3) is *poor* if $M_{1,2} = N_{1,2} = N_{2,1} = \emptyset$.

18.3 *Let G be claw-free, with $\alpha(G) \leq 3$, with no hole of length > 5, and such that every 5-hole in G is dominating; and let (A_1, A_2, A_3) be a spread. Then*

- *the sets A_1, A_2, A_3, $M_{i,j}$ $(1 \leq i < j \leq 3)$ and $N_{i,j}$ $(1 \leq i \neq j \leq 3)$ are pairwise disjoint and have union $V(G)$*
- *if $i, j, k \in \{1, 2, 3\}$ are distinct, then $N_{i,j}$ is anticomplete to $M_{j,k} \cup N_{j,k}$*
- *if $i, j \in \{1, 2, 3\}$ are distinct, then $N_{i,j}$ is a clique*
- *if $i, j, k \in \{1, 2, 3\}$ are distinct, and some vertex has neighbours in both A_j and A_k, then $N_{i,j}$ is complete to $N_{i,k}$*
- *if $i, j, k \in \{1, 2, 3\}$ are distinct, and some vertex has neighbours in both A_j and A_k, then either $N_{j,i}$ is complete to $N_{k,i}$ or G is decomposable.*

Proof. For the first claim, clearly these sets are pairwise disjoint. Let $v \in V(G) \setminus (A_1 \cup A_2 \cup A_3)$; we must show that v belongs to one of the given sets. Since no vertex has neighbours in all of A_1, A_2, A_3, we may assume that v has no neighbour in A_3. If it has both a nonneighbour $a_1 \in A_1$ and a nonneighbour $a_2 \in A_2$, then $\{v, a_1, a_2, a_3\}$ is a stable set of size 4 (for any $a_3 \in A_3$), contradicting that $\alpha(G) \leq 3$. Thus we may assume that v is A_1-complete. From the third condition in the definition of a spread, v has a neighbour in A_2. If v is A_2-complete then $v \in M_{1,2}$, and otherwise $v \in N_{1,2}$, and in either case the theorem holds. This proves the first claim of the theorem.

For the second claim, suppose that $x \in N_{i,j}$ is adjacent to $y \in M_{j,k} \cup N_{j,k}$. Choose $a_i \in A_i$, choose $a_j \in A_j$ nonadjacent to x, and choose $a_k \in A_k$ adjacent to y. Then $\{y, x, a_i, a_k\}$ is a claw, a contradiction. This proves the second statement.

For the third, let $i, j, k \in \{1, 2, 3\}$ be distinct, and suppose that $x, y \in N_{i,j}$ are nonadjacent. Let $a_i \in A_i$ and $a_k \in A_k$. By 17.2, there is a path x-p-q-y with $p, q \in A_j$. Then x-p-q-y-a_i-x is a 5-hole, not dominating a_k, a contradiction. This proves the third claim.

For the fourth claim, suppose that $x \in N_{i,j}$ is nonadjacent to $y \in N_{i,k}$, and there exists $z \in V(G)$ with neighbours in A_j, A_k. The set $A_j \cup \{z\} \cup A_k$ is connected, and x, y both have neighbours in it, and so there is a path P between x, y with interior in $A_j \cup \{z\} \cup A_k$, necessarily using z. Let $a_i \in A_i$; then P can be completed to a hole C via y-a_i-x. Since G has no hole of length > 5, C has length ≤ 5, and so P has length ≤ 3. Since z belongs to P, we may assume that no vertex of A_j is in P. Let $a_j \in A_j$ be a nonneighbour of x. Then a_j has at most one neighbour in C, and therefore it has none; and since every 5-hole is dominating, it follows that C has length 4. Consequently P is x-z-y. Now z is complete to one of A_j, A_k, say A_j; and so $z \in M_{j,k} \cup N_{j,k}$, and yet $x \in N_{i,j}$ and x, z are adjacent, contrary to the second assertion above. This proves the fourth claim.

For the fifth claim, suppose that $n_i \in N_{i,k}$ is nonadjacent to some $n_j \in N_{j,k}$. By hypothesis there exist $a_i \in A_i$ and $a_j \in A_j$, and a vertex z adjacent to a_i, a_j. Hence there is a path P between n_i, n_j with interior in $\{a_i, a_j, z\}$. Since n_i, n_j are both not complete and not anticomplete to A_k, it follows from 17.2 that there is a path Q of length 3 between n_i, n_j with interior in A_k. The union of P, Q is a hole, and since G has no hole of length > 5 it follows that P has length 2, and therefore n_i, n_j

are adjacent to z. Relative to this 5-hole, a_i, a_j are hats in consecutive positions, and therefore G is decomposable by 18.2. This proves the fifth claim, and therefore proves 18.3. ■

18.4 *Let G be claw-free, with $\alpha(G) \leq 3$, with no hole of length > 5 and such that every 5-hole in G is dominating. If G has a poor spread then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ or G is decomposable.*

Proof. We assume that G is not decomposable. Choose a poor spread (A_1, A_2, A_3) with $|A_3|$ maximum, and define $M_{i,j}$ etc as before.

(1) *$N_{3,1}, N_{3,2}$ are both empty.*

For suppose that $N_{3,1}$ is nonempty, and choose $x \in N_{3,1}$ with as few neighbours in A_1 as possible. Let Y be the set of vertices in A_1 adjacent to x. Let X be the set of all vertices in $N_{3,1}$ that are complete to Y and anticomplete to $A_1 \setminus Y$; thus, $x \in X$. Define $A'_3 = A_3 \cup X$, $A'_1 = A_1 \setminus Y$, and $A'_2 = A_2$. We claim that (A'_1, A'_2, A'_3) is a poor spread. For certainly A'_1, A'_2 are cliques, and so is A'_3 from the third statement of 18.3; and since $Y \neq A_1$, it follows that A'_1, A'_2, A'_3 are all nonempty. Moreover, A'_1, A'_2, A'_3 are pairwise anticomplete. Suppose that $v \in V(G) \setminus A'_1 \cup A'_2 \cup A'_3$, and is complete to one of A'_1, A'_2, A'_3, say A_i, and anticomplete to the other two. Consequently $v \notin A_1 \setminus Y, A_2, A_3$. Moreover, every vertex in Y is X-complete and A_3-anticomplete, and therefore has both a neighbour and a nonneighbour in A'_3; and consequently $v \notin Y$. If $i = 1$, then v is anticomplete to both A_2, A_3, contrary to the first assertion of 18.3. If $i = 2$, then by the first assertion of 18.3, v has a neighbour $y \in A_1$, which is impossible since (A_1, A_2, A_3) is a poor spread. If $i = 3$, then by the first statement of 18.3, it follows that v has a neighbour in $A_1 \cup A_2$; and since v has no neighbour in $A'_1 \cup A'_2$ it follows that every neighbour of v in $A_1 \cup A_2$ belongs to Y, and in particular $v \in N_{3,1}$. From the choice of x we deduce that v is Y-complete, and therefore $v \in X$, a contradiction. This proves that (A'_1, A'_2, A'_3) is a spread. Since (A_1, A_2, A_3) is poor, no vertex of G has neighbours in both A_1, A_2. Consequently, no vertex of G has neighbours in both A'_1, A'_2, and therefore the spread (A'_1, A'_2, A'_3) is poor. But this contradicts the maximality of $|A_3|$. Hence $N_{3,1} = \emptyset$, and similarly $N_{3,2} = \emptyset$. This proves (1).

From (1), it follows that all members of A_1 are twins, and so we may assume that $A_1 = \{a_1\}$ say, and similarly $A_2 = \{a_2\}$ say. For $i = 1, 2$, let P_i be the set of members of $M_{i,3}$ with a nonneighbour in $N_{i,3}$, and let Q_i be the set of members of $N_{i,3}$ with a nonneighbour in $M_{i,3}$. Note that, by the second assertion of 18.3, $N_{1,3}$ is anticomplete to $M_{2,3}$, and $N_{2,3}$ is anticomplete to $M_{1,3}$.

(2) *P_1 is complete to $M_{2,3}$, and P_2 is complete to $M_{1,3}$. Moreover, Q_1 is complete to $N_{2,3}$, and Q_2 is complete to $N_{1,3}$.*

For if $p_1 \in P_1$ has a nonneighbour $x \in M_{2,3}$, choose $q_1 \in Q_1$ nonadjacent to p_1, and let $a_3 \in A_3$ be adjacent to q_1. Then $\{a_3, p_1, q_1, x\}$ is a claw, a contradiction. This proves the first assertion, and the second follows by symmetry. For the third, suppose that $q_1 \in Q_1$ has a nonneighbour $x \in N_{2,3}$; let $p_1 \in P_1$ be nonadjacent to q_1, and let $a_3 \in A_3$ be adjacent to q_1. Then a_1-p_1-a_3-q_1-a_1 is a 4-hole, and since x, a_2 are adjacent and a_2 has no neighbour in this 4-hole, it follows that x has a neighbour in this 4-hole, by 17.4. But x is nonadjacent to q_1, p_1, a_1, and so it is adjacent to a_3, and therefore

$\{a_3, p_1, q_1, x\}$ is a claw, a contradiction. This proves the third claim, and the fourth follows by symmetry. This proves (2).

(3) *Either $M_{1,3}$ is complete to $N_{1,3}$ or $M_{2,3}$ is complete to $N_{2,3}$.*

For suppose not; then P_1, Q_1, P_2, Q_2 are all nonempty. For $i = 1, 2$ choose $p_i \in P_i$ and $q_i \in Q_i$, nonadjacent. By (2), p_1 is adjacent to p_2 and q_1 to q_2. But then a_1-p_1-p_2-a_2-q_2-q_1-a_1 is a 6-hole, a contradiction. This proves (3).

(4) *$N_{1,3}, N_{2,3}$ are both nonempty, and $M_{1,3}, M_{2,3}$ are both cliques.*

For suppose that, say, $N_{2,3} = \emptyset$. Since G admits no 0-join, it follows that a_2 has degree > 0, and so there exists $m \in M_{2,3}$. Let S, T be the set of all $v \in M_{1,3} \cup N_{1,3} \cup A_3$ that are $M_{2,3}$-complete and $M_{2,3}$-anticomplete respectively. Thus $A_3 \subseteq S$ and $N_{1,3} \subseteq T$. We claim that (S, T) is a homogeneous pair of cliques. First let us see that they are cliques. If $s_1, s_2 \in S$ are nonadjacent, then $\{m, s_1, s_2, a_2\}$ is a claw, a contradiction; so S is a clique. If $t_1, t_2 \in T$ are nonadjacent, then since $N_{1,3}$ is a clique, it follows that at least one of $t_1, t_2 \in M_{1,3}$, and therefore t_1, t_2 have a common neighbour in A_3, say a_3; but then $\{a_3, s, t, m\}$ is a claw, a contradiction. This proves that S, T are both cliques. Now suppose that $v \in V(G) \setminus (S \cup T)$. We claim that v is either S-complete or S-anticomplete, and either T-complete or T-anticomplete. If $v \in A_1 \cup A_2 \cup M_{2,3}$ the claim holds, so we may assume that $v \in M_{1,3} \cup N_{1,3} \cup A_3$, and therefore $v \in M_{1,3}$. Since $v \notin T$, it has a neighbour $x \in M_{2,3}$ say; and since every $s \in S$ is adjacent to x, and $\{x, s, v, a_2\}$ is not a claw, it follows that v is complete to S. Since $v \notin S$, it has a nonneighbour $y \in M_{2,3}$. If $t \in T$, choose $a_3 \in A_3$ adjacent to t; then since $\{a_3, v, t, y\}$ is not a claw, it follows that v, t are adjacent, and so v is T-complete. This proves that (S, T) is a homogeneous pair of cliques, nondominating because $A_2 \neq \emptyset$. Now $A_3 \subseteq S$, and by hypothesis $|A_3| \geq 2$, since A_1, A_2 both have only one member; and so G is decomposable, by 3.3, a contradiction. Hence $N_{1,3}, N_{2,3}$ are both nonempty. If there exist $x, y \in M_{1,3}$, nonadjacent, choose $z \in N_{2,3}$, let $a_3 \in A_3$ be a neighbour of z, and then $\{a_3, x, y, z\}$ is a claw, a contradiction. Thus $M_{1,3}$ is a clique, and similarly $M_{2,3}$ is a clique. This proves (4).

(5) *$M_{i,3} \subseteq P_i$ for $i = 1, 2$.*

For by (3) and the symmetry, we may assume that $M_{2,3}$ is complete to $N_{2,3}$. Define $V_1 = (M_{1,3} \setminus P_1) \cup M_{2,3}$, and $V_2 = V(G) \setminus V_1$. If $V_1 = \emptyset$ then the claim holds, so we may assume that $V_1 \neq \emptyset$; and clearly $V_2 \neq \emptyset$. We claim that G is the hex-join of $G|V_1$ and $G|V_2$. For V_1 is the union of the two cliques $M_{1,3} \setminus P_1$ and $M_{2,3}$, and V_2 is the union of the three cliques $N_{2,3} \cup A_2$, $N_{1,3} \cup A_1$ and $P_1 \cup A_3$. Since $M_{1,3} \setminus P_1$ is anticomplete to $N_{2,3} \cup A_2$ and complete to $N_{1,3} \cup A_1$ and $P_1 \cup A_3$, and $M_{2,3}$ is anticomplete to $N_{1,3} \cup A_1$ and complete to $N_{2,3} \cup A_2$ and $P_1 \cup A_3$, it follows that G is a hex-join and therefore decomposable, a contradiction. This proves (5).

(6) *$M_{1,3} = M_{2,3} = \emptyset$.*

For from (3) $P_2 = \emptyset$, and therefore from (5) $M_{2,3} = \emptyset$. Suppose that $M_{1,3} \neq \emptyset$. By (5), $P_1 \neq \emptyset$, and therefore $Q_1 \neq \emptyset$. If $x \in N_{1,3}$ and $y \in N_{2,3}$ are adjacent, and $a_3 \in A_3$, then since $\{x, a_1, a_3, y\}$ and

$\{y, a_3, a_2, x\}$ are not claws, it follows that a_3 is adjacent to both or neither of x, y. Consequently x, y have the same neighbours in A_3, for every such adjacent pair x, y. Choose $x \in Q_1$, and let Z be the set of neighbours of x in A_3. By (2), x is complete to $N_{2,3}$, and therefore every vertex in $N_{2,3}$ is complete to Z and anticomplete to $A_3 \setminus Z$. In particular, every vertex in A_3 is either complete or anticomplete to $N_{2,3}$. We claim that every vertex $x \in V(G) \setminus N_{2,3}$ is either complete or anticomplete to $N_{2,3}$. For suppose not; then $x \in N_{1,3} \setminus Q_1$. Since x has a neighbour in $N_{2,3}$, it follows as before that x is complete to Z and anticomplete to $A_3 \setminus Z$. Let $y \in N_{2,3}$ be nonadjacent to x. Choose $z \in Z$, and $a_3 \in A_3$ nonadjacent to x. (a_3 exists since $x \in N_{1,3}$.) Thus $a_3 \notin Z$, and so y is nonadjacent to a_3; but then $\{z, a_3, x, y\}$ is a claw, a contradiction. This proves our claim that every vertex in $V(G) \setminus N_{2,3}$ is either complete or anticomplete to $N_{2,3}$. Hence every vertex in $V(G) \setminus (N_{2,3} \cup A_2)$ is either complete or anticomplete to $N_{2,3}$, and anticomplete to A_2. By 3.2 it follows that G is decomposable, a contradiction. $M_{1,3} = \emptyset$. This proves (6).

From (6), it follows that G satisfies the hypotheses of 18.1, and the result follows. This proves 18.4. ■

Now we can prove the main result of this section.

18.5 *Let G be claw-free, with $\alpha(G) \leq 3$, with no hole of length > 5 and such that every 5-hole in G is dominating. If G is not antiprismatic then either $G \in \mathcal{S}_0 \cup \mathcal{S}_3 \cup \mathcal{S}_6$ or G is decomposable.*

Proof. We assume that G is not decomposable. Since G is not antiprismatic, and $\alpha(G) \leq 3$, it follows that there are three cliques A_1, A_2, A_3, all nonempty and pairwise disjoint and anticomplete, such that $|A_1 \cup A_2 \cup A_3| \geq 4$. Choose them with $A_1 \cup A_2 \cup A_3$ maximal; then (A_1, A_2, A_3) is a spread. Choose a spread (A_1, A_2, A_3) with $|A_3|$ maximal, and define the sets $M_{i,j}, N_{i,j}$ as before. It follows that $|A_3| \geq 2$. By 18.4, we may assume that the spread (A_1, A_2, A_3) is not poor, and nor are the spreads (A_2, A_3, A_1), (A_3, A_1, A_2).

(1) *$N_{1,2} \cup N_{2,1} \cup M_{1,2}$ is a clique.*

For suppose that there are two nonadjacent vertices in this set, say x, y. Since x, y both have neighbours in A_1, and both have neighbours in A_2, there is a hole C containing x, y with $V(C) \subseteq A_1 \cup A_2 \cup \{x, y\}$. No vertex of A_3 has a neighbour in C, and since G has no hole of length > 5 and every 5-hole is dominating, it follows that C has length 4. Since $|A_3| \geq 2$, we deduce from 17.4 that G is decomposable, a contradiction. This proves (1).

(2) *$N_{3,1} = N_{3,2} = \emptyset$; $N_{1,2}$ is complete to $M_{1,3}$, and $N_{2,1}$ is complete to $M_{2,3}$.*

For suppose that there exists $x \in N_{3,1}$. Choose $a_1 \in A_1$ nonadjacent to x. Then the cliques $\{a_1\}$, A_2, and $A_3 \cup \{x\}$ are pairwise disjoint, and there are no edges between them, and consequently they may be enlarged to form a spread, contradicting the maximality of $|A_3|$. Hence $N_{3,1} = N_{3,2} = \emptyset$. Now suppose that $x \in N_{1,2}$ has a nonneighbour $y \in M_{1,3}$. Let $a_2 \in A_2$ be a nonneighbour of x. Then the three cliques $\{x\}$, $\{a_2\}$ and $A_3 \cup \{y\}$ again may be enlarged to form a spread, contrary to the maximality of $|A_3|$. This proves (2).

(3) *$N_{1,2}, N_{2,1} = \emptyset$, and A_1, A_2 both have cardinality 1.*

For we claim that $(N_{1,2} \cup A_1, N_{2,1} \cup A_2)$ is a homogeneous pair of cliques. Certainly both sets are cliques, so let $v \in V(G)$ with $v \notin N_{1,2} \cup N_{2,1} \cup A_1 \cup A_2$. We will show that v is either complete or anticomplete to $N_{1,2} \cup A_1$. Now v belongs to one of the sets $A_3, M_{1,3}, N_{1,3}, M_{2,3}, N_{2,3}, M_{1,2}$, by (2). If $v \in A_3 \cup M_{2,3} \cup N_{2,3}$ then it is anticomplete to $N_{1,2} \cup A_1$, by 18.3, and if $v \in M_{1,3} \cup N_{1,3} \cup M_{1,2}$ then v is complete to $N_{1,2} \cup A_1$, by (1), (2) and the final assertion of 18.3, since (A_2, A_3, A_1) is not poor. This proves that v is either complete or anticomplete to $N_{1,2} \cup A_1$, and similarly it is either complete or anticomplete to $N_{2,1} \cup A_2$. Hence $(N_{1,2} \cup A_1, N_{2,1} \cup A_2)$ is a homogeneous pair of cliques, nondominating because $A_3 \neq \emptyset$, and so by 3.3, the claim follows. This proves (3).

For $i = 1, 2$, let $A_i = \{a_i\}$.

(4) *Either $M_{1,3}$ is complete to $N_{1,3}$, or $M_{2,3}$ is complete to $N_{2,3}$.*

For suppose that for $i = 1, 2$ there exist $m_i \in M_{i,3}$ and $n_i \in N_{i,3}$, nonadjacent. Hence n_1, n_2 are adjacent, by the final assertion of 18.3. If m_1, m_2 are adjacent, then m_1-a_1-n_1-n_2-a_2-m_2-m_1 is a 6-hole, a contradiction. Thus m_1, m_2 are nonadjacent, and so m_1-a_1-n_1-n_2-a_2-m_2 is a path P of length 5. Choose $a_3 \in A_3$ nonadjacent to n_1. Then since $\{n_2, n_1, a_3, a_2\}$ is not a claw, a_3 is not adjacent to n_2; and so P can be completed to a 7-hole via m_2-a_3-m_1, a contradiction. This proves (4).

For $i = 1, 2$, let X_i be the set of all vertices in $M_{1,2}$ with a nonneighbour in $N_{i,3}$. Let $X_0 = M_{1,2} \setminus (X_1 \cup X_2)$.

(5) *$X_1 \cap X_2 = \emptyset$, X_1 is complete to $M_{2,3}$ and X_2 is complete to $M_{1,3}$.*

For suppose first that $x \in X_1 \cap X_2$. For $i = 1, 2$, let $n_i \in N_{i,3}$ be nonadjacent to x. Then x-a_1-n_1-n_2-a_2-x is a 5-hole, and a_3 has at most one neighbour in it, where $a_3 \in A_3$ is a nonneighbour of n_1, a contradiction. This proves the first assertion. Now suppose that $x_1 \in X_1$ is nonadjacent to $m \in M_{2,3}$. Choose $n \in N_{1,3}$ nonadjacent to x_1; and choose $a_3 \in A_3$ adjacent to n. Then x_1-a_2-m-a_3-n-a_1-x_1 is a 6-hole, a contradiction. Thus X_1 is complete to $M_{2,3}$ and similarly X_2 is complete to $M_{1,3}$. This proves (5).

(6) *At least one of $M_{1,3}, M_{2,3}$ is nonempty.*

For suppose not. Then by 18.3, $N_{1,3} \cup N_{2,3}$ is an internal clique cutset, and therefore G is decomposable, a contradiction.

(7) *It is not the case that $M_{1,3}$ is complete to $N_{1,3}$ and $M_{2,3}$ is complete to $N_{2,3}$.*

For let $B_1 = A_1 \cup N_{1,3} \cup X_2$, $B_2 = A_2 \cup N_{2,3} \cup X_1$, and $B_3 = A_3$. Then B_1, B_2, B_3 are disjoint cliques, and their union is not $V(G)$, by (6); and since $\{a_1, a_2, a_3\}$ is a triad for each $a_3 \in A_3$, and there are at least two such vertices a_3, it follows from 15.1 that there is a triad $\{t_1, t_2, t_3\}$ with $t_1, t_2 \in B_1 \cup B_2 \cup B_3$ and $t_3 \notin B_1 \cup B_2 \cup B_3$. Not both $t_1, t_2 \in B_3$, so we may assume from the symmetry that $t_1 \in B_1$. Consequently $t_3 \notin X_0 \cup M_{1,3}$, since $X_0 \cup M_{1,3}$ is complete to B_1 by hypothesis, (1) and (5), and so $t_3 \in M_{2,3}$. Hence t_3 is complete to $B_2 \cup B_3$ by (5) and the hypothesis, and therefore $t_1, t_2 \in B_1$, a contradiction since t_1, t_2 are nonadjacent. This proves (7).

In view of (4) and (7), we may assume that $M_{1,3}$ is complete to $N_{1,3}$ and $M_{2,3}$ is not complete to $N_{2,3}$.

(8) *$X_1 = \emptyset$, that is, $N_{1,3}$ is complete to $M_{1,2}$.*

For suppose not, and choose $n \in N_{1,3}$ and $m \in M_{1,2}$, nonadjacent. Then $m \in X_1$, and so $m \notin X_2$ by (5). Choose $x \in N_{2,3}$ and $y \in M_{2,3}$, nonadjacent. Since $m \notin X_2$, it follows that m, x are adjacent. Thus x-n-a_1-m-x is a 4-hole C. Choose $a_3 \in A_3$ nonadjacent to x. Since $\{n, a_3, x, a_1\}$ is not a claw, it follows that a_3 is not adjacent to n. Since $\{m, x, y, a_1\}$ is not a claw, m, y are not adjacent. But a_3, y are adjacent, and neither of them has any neighbours in the 4-hole; and therefore G is decomposable, by 17.4, a contradiction. This proves (8).

(9) *Not both $M_{1,3}, X_2$ are nonempty.*

For suppose they are, and choose $m_1 \in M_{1,3}$ and $m_2 \in X_2$. Choose $n \in N_{2,3}$ nonadjacent to m_2, and choose $a_3 \in A_3$ adjacent to n, and $a_3' \in A_3$ nonadjacent to n. By the second assertion of (5), m_1, m_2 are adjacent; but then m_2-m_1-a_3-n-a_2-m_2 is a 5-numbering of a 5-hole, and a_1, a_3' are hats in positions $1\frac{1}{2}, 2\frac{1}{2}$ respectively, and 18.2 implies that G is decomposable, a contradiction. This proves (9).

Let Y be the set of all vertices in $M_{2,3}$ with a nonneighbour in $N_{2,3}$.

(10) *Y is complete to $M_{1,3}$; Y is a clique; and Y is anticomplete to $M_{1,2} \setminus X_2$.*

For suppose that $y \in Y$ is nonadjacent to $m \in M_{1,3}$. Choose $x \in N_{2,3}$ nonadjacent to y, and choose $a_3 \in A_3$ adjacent to x; then $\{a_3, x, y, m\}$ is a claw, a contradiction. Thus Y is complete to $M_{1,3}$. Now suppose that there exist nonadjacent $y_1, y_2 \in Y$. Since the spread (A_3, A_1, A_2) is not poor, one of $M_{1,3}, N_{1,3}$ is nonempty. If there exists $n \in N_{1,3}$, let $a_3 \in A_3$ be adjacent to n; then $\{a_3, n, x, y\}$ is a claw, a contradiction. Thus there exists $m \in M_{1,3}$, adjacent to y_1, y_2 since Y is complete to $M_{1,3}$. But then $\{m, a_1, y_1, y_2\}$ is a claw, a contradiction. Thus Y is a clique. Finally suppose that $y \in Y$ has a neighbour $m \in M_{1,2} \setminus X_2$. Let $x \in N_{2,3}$ be nonadjacent to y; then $\{m, x, y, a_1\}$ is a claw, a contradiction. This proves (10).

Let $B_1 = A_1 \cup N_{1,3} \cup X_2$, $B_2 = A_2 \cup N_{2,3}$ and $B_3 = A_3 \cup Y$. From (5), (10) and 18.3, these three sets are all cliques.

(11) *$B_1 \cup B_2 \cup B_3 = V(G)$.*

For suppose not. Since $\{a_1, a_2, a_3\}$ is a triad for all $a_3 \in A_3$, 15.1 implies that there is a triad $\{t_1, t_2, t_3\}$ with $t_1, t_2 \in B_1 \cup B_2 \cup B_3$ and $t_3 \notin B_1 \cup B_2 \cup B_3$. It follows that $t_3 \in X_0 \cup M_{1,3} \cup (M_{2,3} \setminus Y)$. Now X_0 is complete to $B_1 \cup B_2$, and $M_{1,3}$ is complete to $B_1 \cup B_3$, by (5) and (10); and therefore $t_3 \in M_{2,3} \setminus Y$. Hence t_3 is complete to B_2, and so we may assume that $t_1 \in B_1$ and $t_2 \in B_3$. Since t_3 is complete to A_3, it follows that $t_2 \in Y$. If there exists $n \in N_{1,3}$, let $a_3 \in A_3$ be adjacent to n, and then $\{a_3, n, t_2, t_3\}$ is a claw, a contradiction. Thus $N_{1,3} = \emptyset$. Since (A_3, A_1, A_2) is not poor, there exists $m_1 \in M_{1,3}$. By (9), $X_2 = \emptyset$, and so $X_0 = M_{1,2}$. Choose $m_2 \in M_{1,2}$. Let $Z = A_2 \cup A_3 \cup M_{2,3} \cup N_{2,3}$; thus, A_1 is anticomplete to Z. Let P be the set of all vertices in Z complete to $M_{1,2}$ and anticomplete to $M_{1,3}$, and let Q be the set of all vertices in Z that are complete to $M_{1,3}$ and anticomplete to $M_{1,2}$. Since m_1, m_2 exist, it follows that $P \cap Q = \emptyset$. Moreover, $A_2 \cup N_{2,3} \subseteq P$, and $A_3 \cup Y \subseteq Q$, by (10). If $p_1, p_2 \in P$ are nonadjacent, then $\{m_2, a_1, p_1, p_2\}$ is a claw, while if $q_1, q_2 \in Q$ are nonadjacent then $\{m_1, a_1, q_1, q_2\}$ is a claw, in either case a contradiction; thus, P, Q are cliques. We claim that (P, Q) is a homogeneous pair of cliques. For let $v \in V(G) \setminus (P \cup Q)$. We claim that v is either complete or anticomplete to P, and either complete or anticomplete to Q. This is true if $v \notin Z$, so we assume that $v \in Z$, and consequently $v \in Z \setminus (P \cup Q) \subseteq M_{2,3} \setminus Y$. Suppose first that v has a nonneighbour $p \in P$. Since v is complete to $A_2 \cup A_3 \cup N_{2,3}$, it follows that $p \in M_{2,3}$. If v has a neighbour $x \in M_{1,2}$, then $\{x, a_1, p, v\}$ is a claw, while if v has a nonneighbour $x \in M_{1,3}$ then $\{a_3, x, p, v\}$ is a claw, in either case a contradiction; and otherwise v is complete to $M_{1,3}$ and anticomplete to $M_{1,2}$, and therefore belongs to Q, a contradiction. Thus v is complete to P. Suppose that v has a nonneighbour $q \in Q$. Since v is complete to $A_2 \cup A_3 \cup N_{2,3}$, it follows that $q \in M_{2,3}$. If v has a neighbour $x \in M_{1,3}$ then $\{x, a_1, v, q\}$ is a claw, and if v has a nonneighbour $x \in M_{1,2}$ then $\{a_2, x, v, q\}$ is a claw, in either case a contradiction; and otherwise v is anticomplete to $M_{1,3}$ and complete to $M_{1,2}$, and therefore belongs to P, a contradiction. This proves that (P, Q) is a homogeneous pair, nondominating since $A_1 \neq \emptyset$. Since $A_3 \subseteq P$ and $|A_3| \geq 2$, 3.3 implies that G is decomposable, a contradiction. This proves (11).

From (11) it follows that $X_0 = M_{1,3} = \emptyset$ and $Y = M_{2,3}$. Since (A_3, A_1, A_2) is not poor, $N_{1,3}$ is nonempty; and since (A_2, A_3, A_1) is not poor, one of $M_{2,3}, N_{2,3}$ is nonempty. If $M_{2,3}$ is nonempty then $Y \neq \emptyset$, and therefore $N_{2,3} \neq \emptyset$ from the definition of Y. Consequently $N_{1,3}, N_{2,3}$ are both nonempty. If $x \in N_{1,3}$ and $y \in N_{2,3}$, then x, y are adjacent by 18.3; and if $a_3 \in A_3$, then since $\{x, a_1, a_3, y\}$ and $\{y, a_2, a_3, x\}$ are not claws, it follows that a_3 is adjacent to both or neither of x, y. Consequently x, y have the same neighbours in A_3. Since this holds for all choices of x, y, and since $N_{1,3}, N_{2,3}$ are both nonempty, it follows that there exists $Z \subseteq A_3$ such that every vertex in $N_{1,3} \cup N_{2,3}$ is complete to Z and anticomplete to $A_3 \setminus Z$. Since every vertex in $N_{1,3}$ has a neighbour and a nonneighbour in A_3, it follows that $Z \neq \emptyset$, and $Z \neq A_3$. Since all vertices in Z are twins, and all vertices in $A_3 \setminus Z$ are twins, it follows that $|Z| = 1$ and $|A_3| = 2$. Let $a_3 \in A_3 \setminus Z$, and $z \in Z$. If $x, y \in M_{2,3}$ are nonadjacent then $\{z, x, y, n\}$ is a claw, where $n \in N_{1,3}$. It follows that a_1, a_3 are both simplicial vertices, with no common neighbour, and the theorem follows from 18.1. This proves 18.5. ■

Finally, let us explicitly prove the main theorem.

Proof of 2.1. If some hole has length ≥ 6, the result follows from 16.2, so we assume that every hole has length at most 5. By 9.2, we may assume that every 5-hole is dominating. If $\alpha(G) \geq 4$, the result follows from 17.1, and otherwise it follows from 18.5. This proves 2.1. ■

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