Clique covers in graphs with stability number two

Sepehr Hajebi
University of Waterloo, Waterloo, Canada

Paul Seymour[1]
Princeton University, Princeton, NJ 08544

May 9, 2024; revised June 9, 2024

[1]Supported by AFOSR grant FA9550-22-1-0234, and NSF grant DDMS-2154169.

Abstract

It is an old conjecture that for every graph G with no stable set of size three, there is a set of at most $|G|$ complete subgraphs with union G. This remains open, but it would imply that, for each integer $k \geq 0$, if G has no stable set of size three and $F \subseteq E(G)$ with $|F| = k|G| + 1$, then some complete subgraph of G has at least $k + 1$ edges in F. This too remains open in general, but we prove it for $k = 0, 1, 2, 3$.

1 Introduction

A clique X of a graph G *covers* an edge uv of G if $u, v \in X$, and a *clique cover* of G is a collection of cliques of G that together cover all the edges. The *size* of a clique cover is the number of cliques in the collection. The size of the largest stable set of a graph G is denoted by $\alpha(G)$.

There is a long-standing conjecture, possibly dating back to the 1980's (but we do not know where it originated), that:

1.1 Conjecture: *If G is a graph with $\alpha(G) \le 2$ then G admits a clique cover of size $|G|$.*

(See [1, 2] for some related work.) It is easy to prove that every such graph has a clique cover of size a little less than $2|G|$ (proved below), but that is the best we know at the moment.

If the conjecture is true, then for every subset F of the edges of G, some clique covers at least $|F|/|G|$ edges in F, and it seems a nice question to try to prove this for small sets F. Let $k+1 = \lceil |F|/|G| \rceil$; then $|F| \ge k|G| + 1$, and we want a clique that covers at least $k+1$ edges in F. That brings us to the problem of this paper:

1.2 Conjecture: *Let G be a graph with $\alpha(G) = 2$, and let $k \ge 0$ be an integer. Then for every $F \subseteq E(G)$ with $|F| \ge k|G| + 1$, there is a clique of G that covers at least $k+1$ edges of F.*

This too is still open in general, but we have been able to prove it when $k = 0, 1, 2, 3$. For $k = 0, 1$ it is trivial, but even for $k = 2$ it is not so obvious, and for $k = 3$ it took us several days (though in the end we found a moderately slick proof).

Before we start the main proof, let us prove the claim above:

1.3 *For every graph G with $\alpha(G) \le 2$, there is a clique cover of size at most $2|G| - |G|^{1/2} + 1$; and there is a list of $3|G|$ cliques covering every edge of G at least twice.*

Proof. Choose $F \subseteq E(G)$ maximal such that $G \setminus F$ has no stable set of size three, and let $G' = G \setminus F$. For each $w \in V(G)$, let C_w be the set of vertices of G that are nonadjacent in G' to w (thus, C_w is a clique of G', and hence of G, because $\alpha(G') \le 2$); and let D_w be the set consisting of w and all $v \in C_w$ such that $vw \in F$ (thus, D_w is also a clique of G). For every edge uv of G', the maximality of F implies that there is a vertex $w \in V(G)$ nonadjacent in G' to both of u, v, and so the cliques C_w $(w \in V(G))$ cover all edges of G'; but the cliques D_w $(w \in V(G))$ cover all edges in F. Thus the cliques C_w $(w \in V(G))$ and D_w $(w \in V(G))$ cover all edges of G. Indeed, one could do better: there is no triangle of edges in F, and so there exists $X \subseteq V(G)$, with $|X| \ge |G|^{1/2} - 1$, such that no edge of F has both ends in X; and we could just take the cliques C_w $(w \in V(G))$ and D_w $(w \in V(G) \setminus X)$ to cover all edges of G. This proves the first claim. The cliques D_w $(w \in V(G))$ cover every edge in F twice (once from each end), and so the list containing each C_v twice and each D_v once covers every edge twice. This proves 1.3. ∎

2 The main proof

In this section we prove 1.2 for $k = 0, 1, 2, 3$.

2.1 *Let G be a graph with $\alpha(G) = 2$, and let $k \in \{0, 1, 2, 3\}$. Then for every $F \subseteq E(G)$ with $|F| \ge k|G| + 1$, there is a clique of G that covers at least $k+1$ edges of F.*

Proof. We say a clique C of G *works* if it covers at least $k+1$ edges in F. We assume that no clique works. An F*-neighbour* of a vertex x means a neighbour v joined to x by an edge in F; and we denote the number of F-neighbours of x by $d(x)$.

If $k = 0$, choose $u, v \in V(G)$ joined by an edge of F; then the clique $\{u, v\}$ works (a contradiction). Next suppose $k = 1$. Since $|F| \geq |G| + 1$, some vertex x has at least three F-neighbours v_1, v_2, v_3. Since $\alpha(G) \leq 2$, some two of v_1, v_2, v_3 are adjacent, say v_1, v_2; and then the clique $\{x, v_1, v_2\}$ works.

Now suppose that $k = 2$. We will use a "discharging" argument. Let the initial charge of each vertex v be $d(v) - 4$. Thus the sum of all charges is strictly positive. For each edge $uv \in F$, such that $d(u) \geq 5$ and $d(v) \leq 3$, we move charge $4/d(v) - 1$ from u to v (that is, reduce the charge on u by $4/d(v) - 1$ and increase it by the same amount on v). (Note that $d(v) \geq 1$ since $uv \in F$.) This is called "discharging". The sum of all charges remains positive, and so there is a vertex x that has positive charge after discharging. If $d(x) \leq 4$, then the initial charge of x is $d(x) - 4$, and it receives at most $d(x)(4/d(x) - 1)$ by discharging (zero if $d(x) = 0$), so its final charge is still non-positive, a contradiction. Thus $d(x) \geq 5$.

Let R be the set of F-neighbours of x. Since $\alpha(G) \leq 2$, no three vertices in R are pairwise non-adjacent; and if some three of them are pairwise adjacent, say v_1, v_2, v_3, then the clique $\{x, v_1, v_2, v_3\}$ works. So we assume not. But $|R| \geq 5$, and so $|R| = 5$ and $G[R]$ is a cycle of length five, with vertices v_1-v_2-$\cdots$-v_5-v_1 in order, say. If $v_1v_2 \in F$, then the clique $\{x, v_1, v_2\}$ works; so we may assume that no edges of F have both ends in R. If each of $v_1, \ldots, v_5$ is incident with at most three edges in F, then x sends a charge of at least $4/3 - 1 = 1/3$ to each of them, which is impossible; so we assume that v_1 say has at least four F-neighbours, say x, w_1, w_2, w_3. If w_1, w_2, w_3 are all nonadjacent to x, then $\{w_1, w_2, w_3, v_1\}$ is a clique and works. If w_1 say is adjacent to x, then w_1 is also adjacent to one of v_2, v_5, say v_2, and then the clique $\{x, v_1, v_2, w_1\}$ works.

Henceforth we assume that $k = 3$. Observe first that if $uv \in F$ and $d(v) \geq 7$, then u is adjacent to at least two F-neighbours of v (because otherwise R contains a clique of size four, and adding v to it works). We use a different discharging rule, as follows. Let the initial charge on each vertex v be $d(v) - 6$, so total charge is positive. For each edge $uv \in F$ of G such that $d(u) \geq 7$ and u is adjacent to at most one F-neighbour of v (and hence $d(v) \leq 6$), we send a charge of p from u to v, where

- $p = 2$ if u is adjacent to no F-neighbours of v;
- $p = 1/2$ if u is adjacent to exactly one F-neighbour of v.

There is a vertex x which has positive charge after discharging. Let R be the set of F-neighbours of x, and N the set of all neighbours of x. Let $R = \{v_1, \ldots, v_{|R|}\}$ say.

(1) $d(x) \geq 7$.

If $d(x) \leq 2$, then since x receives a charge of at most two from each F-neighbour and has initial charge at most -4, this is impossible. If $d(x) = 3$, some two of v_1, v_2, v_3 are adjacent, say v_1, v_2, and then x receives a charge of at most $1/2$ from each of v_1, v_2 and at most two from v_3, and since its initial charge is -3, this is impossible. If $d(x) = 4$, we may assume that x receives a charge of more than $1/2$ from one of its F-neighbours, say from v_1, and so v_1 is nonadjacent to v_2, v_3, v_4. Hence $\{v_2, v_3, v_4\}$ is a clique, and therefore each has two neighbours in R and so send a charge of zero to x; and since x has an initial charge of -2, this is impossible.

If $d(x) = 5$, we may assume that v_1 sends charge into x and so has at most one F-neighbour in R; so we may assume that v_1 is nonadjacent to v_3, v_4, v_5. Consequently v_3, v_4, v_5 are pairwise adjacent, and so send zero charge into x. Hence x receives charge only from v_1, v_2, and since its initial charge is -2, we may assume that v_1 sends more than one to x. But then v_1 has no neighbour in R, and so $\{v_2, \ldots, v_5\}$ is a clique and $\{x, v_2, \ldots, v_5\}$ works.

If $d(x) = 6$, we may assume that v_1 sends charge into x and so has at most one neighbour in R, and hence it has at least four non-neighbours in R, say $v_3, \ldots, v_6$, and the clique $\{x, v_3, \ldots, v_6\}$ works. This proves (1).

For each $v \in R$, let $r(v)$ be the number of neighbours of v in R.

(2) *$r(v) \geq |R| - 4$ for each $v \in R$.*

Let $v \in R$ and let X be the set of vertices in $R \setminus \{v\}$ nonadjacent to v. Then X is a clique, and since $X \cup \{x\}$ does not work, and covers at least $|X|$ edges in F, it follows that $|X| \leq 3$. This proves (2).

(3) *$r(v) + d(v) \leq 5$ for each $v \in R$, and if $r(v) + d(v) = 5$, no F-neighbour of v belongs to R.*

Let H be the graph induced by G on the union of the set of F-neighbours of v in N and the set of neighbours of v in R. Let H' be obtained from H by replacing by two adjacent twins each vertex in R that is joined to v by an edge in F. So $|H'| = r(v) + d(v)$. Suppose that H' has three pairwise adjacent vertices. Then either H has three pairwise adjacent vertices, p, q, r say, or there is an edge pq of H such that $p \in R$ and $pv \in F$. In the first case the clique $\{x, v, p, q, r\}$ works (because $xv \in F$, and either $xp \in R$ or $vp \in R$, and the same for q, r). In the second case the clique $\{x, v, p, q\}$ works (because $xv, vp, xp \in F$ and one of $xq, vq \in F$). So we may assume that H' has no triangle and so $|H'| \leq 5$, since $\alpha(H') \leq 2$. Suppose that $|H'| = 5$; then H' is a cycle of length five, and so no two of its vertices are twins; and therefore, no F-neighbour of v is in R. This proves (3).

If no vertex in R has an F-neighbour in N, then x sends a charge of 2 to each vertex in R, and since its initial charge is only $|R| - 4$, this is impossible. So we may assume that $d(v_1) > 0$, and so by (3), $r(v_1) \leq 4$. By (2), $|R| \leq 8$. Since the charge on x after discharging is still positive, not every vertex in R receives $1/2$ from x; and so we may assume that $d(v_1) \geq 2$. By (3), $r(v_1) \leq 3$, and (2) implies that $|R| = 7$. Since the initial charge on x is 1, there is no $v \in R$ such that x sends a charge of 2 to v, and at most one vertex $v \in R$ such that x sends a charge of $1/2$ to v. Hence there is a set $A \subseteq R$ with $|A| = 6$ such that x sends no charge to any member of A. Let $A = \{v_1, \ldots, v_6\}$. Consequently, for $1 \leq i \leq 6$, $r(v_i) \geq 2$, and so $d(v_i) \leq 3$ by (3), and $d(v_i) = 3$ by (2). Since $|R| = 7$, and each of the six vertices in A has degree three in $G[R]$, it follows that v_7 has even degree in R. There is a triangle X in A, since $|A| = 6$ and contains no triple of pairwise nonadjacent vertices. If v_7 is adjacent to each vertex in X then $X \cup \{x, v_7\}$ works; so we assume v_7 is nonadjacent to a vertex in X, and so its degree in $G[R]$ is at most five. Since its degree is even and at least three, v_7 has degree four in $G[R]$. Let v_7 be adjacent to v_1, v_2, v_3, v_4 say. As before, no three of v_1, v_2, v_3, v_4 are pairwise adjacent (because we could add v_7, x to the triangle and obtain a clique that works), and no three are pairwise nonadjacent; and consequently some two edges of $G[\{v_1, v_2, v_3, v_4\}]$ are disjoint, say v_1v_2 and v_3, v_4. Since v_5, v_6 each have degree three in $G[R]$ and are nonadjacent to v_7, there are at least four edges between $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6\}$. On the other hand, each of v_1, v_2, v_3, v_4 is adjacent to at most one of v_5, v_6, since it has degree three in $G[R]$ and is adjacent to v_7. Thus each of v_1, v_2, v_3, v_4 has exactly one neighbour in $\{v_1, v_2, v_3, v_4\}$, and so there are no edges between $\{v_1, v_2\}$ and $\{v_3, v_4\}$. Since each vertex in R has an F-neighbour in N, and in particular, v_7 has such an F-neighbour, say n, and $r(v_7) = 4$, (3) implies that $n \notin R$. Since $\alpha(G) = 2$, either n is adjacent to both v_1, v_2, or n is adjacent to both v_3, v_4, and from the symmetry we may assume the first. But then the clique $\{x, v_7, n, v_1, v_2\}$ works. This proves 2.1. ∎

3 Concluding remarks

We have tried to prove 1.2 when $k = 4$, but failed so far. A sensible discharging rule is: start with an initial charge of $d(v) - 8$ for each vertex v; and then for each edge $uv \in F$ where $d(u) \geq 9$, send a charge of p from u to v, where

- $p = 3$ if u is adjacent to no F-neighbours of v;
- $p = 1$ if u is adjacent to exactly one F-neighbour of v;
- $p = 1/2$ if u is adjacent to exactly two F-neighbours of v and they are nonadjacent to each other;
- $p = 1/3$ if u is adjacent to exactly two F-neighbours of v and they are adjacent;
- $p = 0$ if u is adjacent to more than two F-neighbours of v.

This does quite well; if x has positive charge after discharging, we can show that $d(x) \in \{9, 10\}$, but are not able to finish.

Going back to 1.1 itself: it is easy to prove 1.1 for graphs G with vertex set the union of two cliques (indeed, there is a clique cover of size at most $|G|/2 + 2$); but what if $V(G)$ is the union of three cliques? Then it is easy to produce a clique cover of size $|G| + 3$, and Javadi and Hajebi [2] showed that there is one of size at most $|G| + 1$; but we have not yet proved $|G|$ in this case.

References

[1] T. Nguyen, A. Scott, P. Seymour and S. Thomassé, "Clique covers of H-free graphs", *European J. Combinatorics*, **118** (2024), 103909, `arXiv:2211.12065`.

[2] R. Javadi and S. Hajebi, "Edge clique cover of claw-free graphs", *J. Graph Theory* **90** (2018), 311–405.