Thomassen's theorem on the two-linkage problem in acyclic digraphs: a shorter proof

Paul Seymour[1]
Princeton University, Princeton, NJ 08544

September 14, 2024; revised October 11, 2024

[1]Supported by AFOSR grant FA9550-22-1-0234, and NSF grant DMS-2154169.

Abstract

Let G be an acyclic digraph, and let $a, b, c, d \in V(G)$, where a, b are sources, c, d are sinks, and every other vertex has in-degree and out-degree at least two. In 1985, Thomassen showed that there do not exist disjoint directed paths from a to c and from b to d, if and only if G can be drawn in a closed disc with a, b, c, d drawn in the boundary in order. We give a shorter proof.

1 Introduction

Digraphs in this paper are finite, and without loops or parallel edges. A digraph is *acyclic* if it has no directed cycle, and a *dipath* is a directed path. The *k-linkage problem* is, given two k-tuples $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ of vertices of a digraph G, to decide whether there are k pairwise vertex-disjoint dipaths $P_1, \ldots, P_k$ in G, where P_i is from a_i to b_i for each i. The k-linkage problem is NP-complete [1] for general digraphs G, even if $k = 2$, but solvable in polynomial time [1] if G is acyclic.

The 2-linkage problem in acyclic digraphs is particularly nice, because it comes equipped with a theorem characterizing the digraphs in which the problem is infeasible. It is easy to reduce the characterization question to the case handled by the following beautiful theorem of Thomassen [2]:

1.1 *Let G be an acyclic digraph, and let $S = \{a, b\}$ be its set of sources, and $T = \{c, d\}$ its set of sinks. Suppose that $S \cap T = \emptyset$, and every vertex not in $S \cup T$ has in-degree and out-degree both at least two. Then exactly one of the following holds:*

- *there are vertex-disjoint dipaths P, Q of G, where P is from a to c and Q is from b to d;*
- *G can be drawn in a closed disc with a, b, c, d drawn in the boundary in order.*

Thomassen's proof was about five pages long. Our objective is to give a shorter proof of a slightly stronger statement.

Let G be a digraph, and let $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_\ell\}$ be disjoint subsets of $V(G)$. Let P be a dipath from $s_i \in S$ to $t_j \in T$, and let P' be a dipath from $s_{i'} \in S$ to $t_{j'} \in T$. We say the pair (P, P') is a *cross relative to* $(s_1, \ldots, s_k), (t_1, \ldots, t_\ell)$ if P, P' are vertex-disjoint and either $i < i'$ and $j > j'$, or $i > i'$ and $j < j'$. We will prove the following (1.1 is the case when $k = \ell = 2$):

1.2 *Let G be an acyclic digraph, and let $S = \{s_1, \ldots, s_k\}$ be its set of sources, and $T = \{t_1, \ldots, t_\ell\}$ its set of sinks. Suppose that $S \cap T = \emptyset$, and every vertex not in $S \cup T$ has in-degree and out-degree both at least two. Then exactly one of the following holds;*

- *there is a cross in G relative to $(s_1, \ldots, s_k), (t_1, \ldots, t_\ell)$;*
- *G can be drawn in a closed disc with $s_1, s_2, \ldots, s_k, t_\ell, t_{\ell-1}, \ldots, t_1$ drawn in the boundary in order.*

Proof. That not both statements hold is clear; so we assume that the first bullet is false, and will prove that the second is true by induction on $|V(G)|+|E(G)|$. Let $C = V(G) \setminus (S \cup T)$. The following observation will be useful:

(1) *For all $u, v \in C \cup T$ (possibly equal), there are two dipaths from $\{u, v\}$ to T, vertex-disjoint if $u \neq v$, and vertex-disjoint except for v if $u = v$. The same holds for all $u, v \in S \cup C$ and dipaths from S to $\{u, v\}$.*

Let $W = \{u\} \cap \{v\}$. If the first statement is false, then by Menger's theorem, there is a set $X \subseteq V(G) \setminus W$ with $|X| \leq 1$ such that every dipath from $\{u, v\}$ to T has a vertex in X. Not both $u, v \in X$ (since $X \cap W = \emptyset$), so we assume that $v \notin X$. Choose a maximal dipath P with first vertex v and with no vertex in X, and let p be the last vertex of P. Thus $p \notin S$ since $v \notin S$ and

S is the set of sources; and $p \notin T$ from the choice of X. So $p \in C$, and therefore has out-degree at least two; and so has an out-neighbour $q \notin X$. Since G is acyclic, $q \notin V(P)$, and so we could add q to P and make a longer dipath, contrary to the maximality of P. This proves the first statement, and the second follows similarly. This proves (1).

(2) *We may assume that there is no edge between S, T.*

Suppose that s_i is adjacent to t_j, where $1 \leq i \leq k$ and $1 \leq j \leq \ell$. Let A be the set of $v \in C$ such that there is a dipath from $\{s_1, \ldots, s_{i-1}\}$ to v, and let A' be the set such that there is a dipath from $\{s_{i+1}, \ldots, s_k\}$ to v. Similarly, let B be the set of $v \in C$ such that there is a dipath from v to $\{t_1, \ldots, t_{j-1}\}$, and let B' be the set of $v \in C$ such that there is a dipath from v to $\{t_{j+1}, \ldots, t_\ell\}$. From (1), $A \cup A' = B \cup B' = C$. But if $v \in A$, then $v \notin B'$, since otherwise the union of a dipath from $\{s_1, \ldots, s_{i-1}\}$ to v and one from v to $\{t_{j+1}, \ldots, t_\ell\}$ is a dipath that makes a cross with the dipath s_i-t_j. Hence $A \cap B' = \emptyset$, and similarly $A' \cap B = \emptyset$; so $A = B$ and $A' = B'$. Define $V = A \cup \{s_1, \ldots, s_{i-1}\} \cup \{t_1, \ldots, t_{j-1}\}$, and $V' = A' \cup \{s_{i+1}, \ldots, s_k\} \cup \{t_{j+1}, \ldots, t_\ell\}$.

Thus $V, V', \{s_i, t_j\}$ are pairwise disjoint and have union $V(G)$. There is no edge from V to V', since if uv is such an edge then $u \notin T$ and $v \notin S$, and the union of a dipath from $\{s_1, \ldots, s_{i-1}\}$ to u, the edge uv, and a dipath from v to $\{t_{j+1}, \ldots, t_\ell\}$ is a dipath that makes a cross with the dipath s_i-t_j. Similarly there is no edge from V' to V. Let H be obtained from G by deleting the edge s_it_j. The result follows from the inductive hypothesis, applied to $H \setminus V'$ with the sequences $(s_1, \ldots, s_i)$, $(t_1, \ldots, t_j)$ and applied to $H \setminus V$ with the sequences $(s_i, \ldots, s_k), (t_j, \ldots, t_\ell)$. This proves (2).

If $C = \emptyset$, then $E(G) = \emptyset$ by (2) and the result is true, so we assume that $C \neq \emptyset$. Hence we may choose $v \in C$ with no in-neighbour in C, since G is acyclic. But v has at least two in-neighbours in G, and they all belong to S; let the in-neighbours of v be $\{s_i : i \in I\}$, where $I \subseteq \{1, \ldots, k\}$. Thus $|I| \geq 2$; let h, j be the smallest and largest members of I.

(3) *For $h < i < j$, s_i has no out-neighbour except possibly v.*

Suppose that s_iu is an edge where $u \neq v$. Hence $u \in C \cup T$, and by (1) there are two vertex-disjoint dipaths P, Q from $\{u, v\}$ to T. One of P, Q, say Q, has first vertex v, and so P has first vertex u. Let Q_1 be obtained from Q by adding the edges s_hv, let Q_2 be obtained from Q by adding s_jv, and let P' be obtained from P by adding s_iu. Then one of $(P', Q_1), (P', Q_2)$ is a cross relative to $(s_1, \ldots, s_k), (t_1, \ldots, t_\ell)$ (which one depends on the order of the ends of P, Q in T), a contradiction. This proves (3).

Let G' be obtained from G by deleting $s_{h+1}, \ldots, s_{j-1}$ and the edges s_hv, s_jv, and let $S' = (S \setminus \{s_{h+1}, \ldots, s_{j-1}\}) \cup \{v\}$. Thus $S' \cap T = \emptyset$, and from (3), S' is the set of sources of G', and every vertex of G' not in $S' \cup T$ has out-degree and in-degree at least two. Suppose that in G' there is a cross (P, Q) relative to $(s_1, \ldots, s_h, v, s_j, \ldots, s_k), (t_1, \ldots, t_\ell)$. One of P, Q has first vertex v (since there is no cross in G relative to $(s_1, \ldots, s_k), (t_1, \ldots, t_\ell)$), say P. Since $h \neq j$, one of $s_h, s_j \notin V(Q)$, say s_i. Let P' be obtained by adding s_iv to P. Then (P', Q) is a cross in G relative to $(s_1, \ldots, s_k), (t_1, \ldots, t_\ell)$, a contradiction. So there is no such cross. From the inductive hypothesis, the result holds for G' and the sequences $(s_1, \ldots, s_h, v, s_j, \ldots, s_k), (t_1, \ldots, t_\ell)$; but then it holds for G, by (3). This proves 1.2. ∎

References

[1] S. Fortune, J. Hopcroft and J. Wyllie, "The directed subgraph homeomorphism problem", *Theoret. Comput. Sci.* **10** (1980), 111–121.

[2] C. Thomassen, "The 2-linkage problem for acyclic digraphs", *Discrete Math.* **55** (1985), 73–87.