

Rough structure. I. Exclusion among series-parallel graphs

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Abstract

This is the first in a series of papers about rough structure theorems: graph properties that are implied by the exclusion of some graph as a minor, and imply the exclusion of some larger graph of the same type. Here we look at the structure of the series-parallel graphs that do not contain a given series-parallel graph as a minor. We prove that for every series-parallel graph H , there exists m such that every simple two-connected series-parallel graph that does not contain H as a minor admits a tree-decomposition of height at most m , where each “torso” is either an outerplanar graph or a planar dual of one. We prove similar theorems about the series-parallel graphs that exclude an outerplanar graph, the series-parallel graphs that exclude a dual of an outerplanar graph, the graphs of tree-width three that exclude a series-parallel graph, and the general graphs that exclude a fan.

1 Introduction

All graphs in this paper are finite, and may have loops or parallel edges. A *tree-decomposition* $\mathcal{T}$ of a graph G is a pair $(T, (W_t : t \in V(T)))$, where T is a tree, and $W_t \subseteq V(G)$ for each $t \in V(T)$, such that

- $G = \bigcup_{t \in V(T)} G[W_t]$, and
- if $t, t', t'' \in V(T)$ and t' lies on the path between t, t'' , then $W_t \cap W_{t''} \subseteq W_{t'}$.

(See Figure 1.)

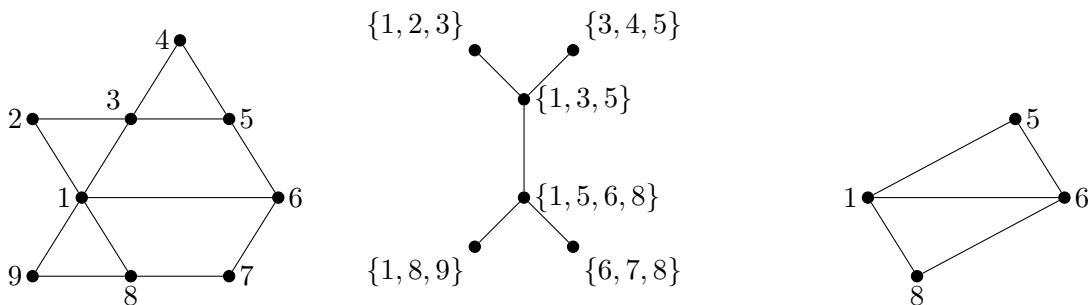


Figure 1: A graph and a tree-decomposition of it. The right-hand graph is one of the torsos.

The *width* of $\mathcal{T}$ is the maximum of $|W_t| - 1$ over all $t \in V(T)$, and the *tree-width* of G is the minimum width of its tree-decompositions. The simple graphs of tree-width at most one are the forests, and the graphs of tree-width at most two are the series-parallel graphs. There is a well-known theorem of Robertson and Seymour [8]:

1.1 *For every planar graph H , there exists ℓ such that every graph not containing H as a minor has tree-width at most ℓ .*

(A graph is a *minor* of another if the first can be obtained from a subgraph of the second by edge-contraction.) See [3] for a simpler proof, and [1] for the best dependence between H and ℓ known.

This is best possible in a sense; because for every non-planar graph H there is no such ℓ (to see this, observe that large planar grids have large tree-width and do not contain H as a minor if H is non-planar). And it has a converse in a sense; because for all ℓ , there is a planar graph H such that every graph with tree-width at most ℓ does not contain H as a minor. (To see this, choose a grid with tree-width more than ℓ , and observe that every minor of a graph with tree-width at most ℓ also has tree-width at most ℓ .)

There is a similar theorem about excluding a forest, also due to Robertson and Seymour [7] (a graph has *path-width* at most k if it admits a tree-decomposition $(T, (W_t : t \in V(T)))$ of width at most k where T is a path):

1.2 *For every forest H there exists ℓ such that every graph not containing H as a minor has path-width at most ℓ .*

(See Diestel [2] for a short proof.) Again, this is “best possible”, and has a “converse”, in the same senses as before, since uniform binary trees of large depth have large path-width. Such theorems are called “rough structure theorems”.

Let us say an *ideal* of graphs is a class of graphs closed under isomorphism and under taking minors: thus, the class of all planar graphs is an ideal. The first theorem above can be reformulated as:

1.3 *Let $\mathcal{J}$ be the ideal of all planar graphs, and let $\mathcal{I}$ be an ideal of graphs. Then $\mathcal{J} \not\subseteq \mathcal{I}$ if and only if there exists ℓ such that every graph in $\mathcal{I}$ has tree-width at most ℓ .*

This formulation makes its “if-and-only-if” nature more clear, and similar reformulations in terms of ideals exist for all rough structure theorems.

It is helpful in this paper to assign designate one vertex of the tree T as a *root*; and then a tree-decomposition $(T, (W_t : t \in V(T)))$ is called a *rooted tree-decomposition*. There are more rough structure theorems; for instance, here are two that are well known (the *height* of a rooted tree-decomposition $(T, (W_t : t \in V(T)))$ is the length of the longest path in T starting from the root):

1.4 *For every path H there exists m such that every graph not containing H as a minor admits a rooted tree-decomposition of width at most m and height at most m .*

1.5 (Erdős and Pósa [6]:) *If every component of H has one vertex and one edge (a loop), there exists ℓ such that every graph not containing H as a minor can be obtained from a forest by adding at most ℓ vertices (joined arbitrarily).*

And there are many more, some of which are substantially more complicated: for instance the “graph minors structure theorem” of Robertson and Seymour [9] is such a result, that we do not state here.

In this series of papers we will look for rough structure theorems that lie between 1.1 and 1.2; we want to exclude types of planar graph that are more complicated than forests, and see what we get. A natural question is, is there a rough structure theorem for excluding a series-parallel graph? We have solved this, and will give the answer in a later paper, but it requires a good many preliminaries; and there is a toy version of this question that we answer first, because the answer is much simpler and very pretty. The first goal of this paper is to prove a rough structure theorem that describes the series-parallel graphs that do not contain a given series-parallel graph as a minor.

Some terminology: if $\mathcal{T} = (T, (W_t : t \in V(T)))$ is a tree-decomposition of G and $t \in V(T)$, the *torso* of $\mathcal{T}$ at t is the graph obtained from $G[W_t]$ by adding edges to make $W_s \cap W_t$ a clique for all $s \in V(T)$ adjacent to t . (More precisely, add a minimal set of edges to make $W_s \cap W_t$ a clique for all such s ; do not introduce parallel edges.) If $e = st$ is an edge of T , the *adhesion* at e is the set $W_s \cap W_t$. A tree-decomposition has *adhesion* k if all its adhesions have size exactly k . (Many of the results in this paper are about tree-decompositions with adhesion two.)

2 Statement of results

Let us state our results. A planar drawing of a graph is *outerplanar* if there is a region that is incident with each vertex, and a graph is *outerplanar* if it admits an outerplanar drawing. Rik Sengupta [10] proved a result related to our question. Let us say an *apex-forest* is a graph obtained from a forest by adding one vertex, joined arbitrarily. (Thus, apex-forests are series-parallel.) Sengupta proved:

2.1 *For every apex-forest H , there exists m such that every simple two-connected series-parallel graph with no H minor admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso is outerplanar.*

This is a rough structure theorem, because it has a converse as usual: for all m , there is an apex-forest H such that every two-connected series-parallel graph that admits such a tree-decomposition contains no H minor.

If we take an outerplanar drawing of some connected graph, and construct the dual graph, we obtain an apex-forest. But a connected outerplanar graph in general admits planar drawings that are not outerplanar drawings, and, correspondingly, not all their duals are apex-forests. (See Figure 2.) A connected graph is *innerplanar* if it admits a planar drawing that is the dual of a drawing of some outerplanar graph. Let us say a disconnected graph is innerplanar if we can add edges to make a connected innerplanar graph. (One can check that the innerplanar graphs are those that do not contain K_4 or C_3^2 as a minor, where C_3^2 is obtained from a cycle of length three by replacing each edge by two parallel edges.)

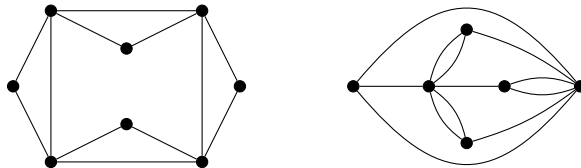


Figure 2: The left graph is outerplanar, but drawn with a non-outerplanar drawing. Its dual, on the right, is innerplanar but not an apex-forest.

Our main result is:

2.2 *For every series-parallel graph H , there exists m such that every simple two-connected series-parallel graph with no H minor admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso is either outerplanar or innerplanar.*

We will use the same machinery to show:

2.3 *For every outerplanar graph H , there exists m such that every simple two-connected series-parallel graph with no H minor admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso is innerplanar.*

2.4 *For every innerplanar graph H , there exists m such that every simple two-connected series-parallel graph with no H minor admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso is either outerplanar or an apex-forest.*

A *fan* is a simple graph obtained from a path by adding a vertex joined to each vertex of the path (see Figure 3), and its *length* is the length of the path. We will prove:

2.5 *For every fan H , there exist ℓ, m such that every simple two-connected graph G with no H minor admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso either is a cycle or is three-connected and has no path of length ℓ .*

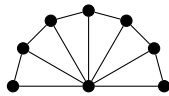


Figure 3: A fan of length six.

(Here, G is a general graph, not necessarily series-parallel; but we can apply the same machinery because the result was already known for three-connected graphs, and all we need to do is to allow for 2-separations). It follows easily from 1.4 (we omit the details) that

2.6 *For every fan H , there exist ℓ, m such that every simple two-connected graph G with no H minor admits a rooted tree-decomposition of height at most m , and such that each torso either is a cycle or has at most ℓ vertices.*

The result 2.2 seemed very pretty, and that motivated us to ask, what if the graph we are decomposing has tree-width three rather than two, does something similar happen? It does indeed, although the proof is tougher. We will show:

2.7 *For every series-parallel graph H , there exists m such that if G is a simple three-connected graph with tree-width at most three and with no H minor, then G admits a rooted tree-decomposition with adhesion three, of height at most m , and such that each torso is of one of four kinds that we explain later.*

This can be extended to allow G to be two-connected rather than three-connected, but we postpone that to a later paper since it requires some new methods.

Finally, we prove two other rough structure theorems, both easy, by proofs that do not use the method of “scores” that we used for all the earlier rough structure theorems. One is for excluding a “stopsign”, and one is for excluding a “caterpillar”.

All these theorems are indeed rough structure theorems, and therefore have “rough” converses. For instance, in the case of 2.2, for all m there is a series-parallel graph H such that for every simple two-connected graph G , if G admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso is either outerplanar or innerplanar, then G has no H minor. We sketch the proofs of the converses in Section 14. (In most cases, given such a decomposition, one can delete the vertices of the bag of the root, and win by induction on m .)

3 Innerplanar graphs

Let a, b be distinct vertices of a connected graph C , and let a', b' be distinct vertices of a connected graph C' . Let G be obtained from the disjoint union of C, C' by identifying $a = a'$ and $b = b'$, and let H be obtained from the disjoint union of C, C' by identifying $a = b'$ and $b = a'$. The operation that takes G to H is called a *Whitney twist*.

A *Whitney flip* is different; it is an operation that takes a planar drawing of a graph to another drawing of the same graph, best shown by a figure. (See Figure 4.)

We start with a planar drawing (of a loopless two-connected graph, say, although the operation can be extended to other graphs), and choose a closed disc Δ in the plane whose boundary F passes

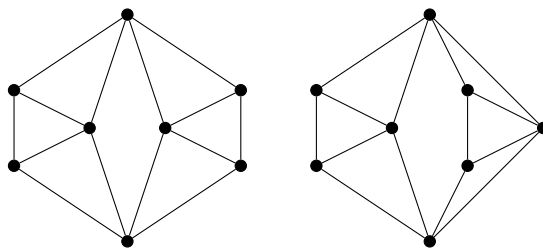


Figure 4: A Whitney flip.

through two vertices u, v and two regions of the drawing (and nothing else). Take a homeomorphism of Δ to itself that fixes u, v and exchanges the two components of $F \setminus \{u, v\}$, and replace the part of the drawing inside Δ by its image under this homeomorphism. This gives a new drawing of the same graph, and this operation is called a *Whitney flip*. (An internet search shows that the two terms are often confused, and sometimes called a Whitney “switch”.) Whitney proved that given two drawings of the same two-connected loopless graph in the sphere, we can take one drawing to the other by a sequence of Whitney flips, up to a homeomorphism of the sphere to itself. (This is not explicitly stated in Whitney’s papers as far as we know, but follows from results of [11].)

3.1 *A two-connected loopless graph is innerplanar if and only if it can be transformed by Whitney twists to an apex-forest.*

Proof. The dual of a two-connected loopless drawing is also two-connected and loopless (see [11]). By Whitney’s theorem, a drawing D of a two-connected loopless graph is a drawing of an outerplanar graph if and only if there is a sequence of drawings $D_1, \dots, D_n$ each obtained from its predecessor by a Whitney flip, with $D_1 = D$ and D_n an outerplanar drawing. Thus, a graph G is innerplanar if and only if it has a drawing D such that the dual drawing admits such a sequence; that is, (in terms of dual drawings), if and only if there is a sequence of drawings $E_1, \dots, E_n$, with $E_1 = D$ and E_n the dual of an outerplanar drawing, where each E_i is obtained from its predecessor by a Whitney twist (since a drawing is taken to a second drawing by a Whitney flip if and only if the dual of the first is obtained from the dual of the second by a Whitney twist). But the dual of a two-connected, loopless outerplanar drawing is a two-connected loopless apex-forest. This proves 3.1. ■

Let G be a graph and $X \subseteq E(G)$; adding a triangle to (G, X) results in (G', X') where

- G' is obtained from G by choosing an edge uv of G in X , and adding a new vertex w adjacent to both u, v .
- X' is obtained by choosing f to be one of the edges uw, vw , and setting $X' = X \cup \{f\}$.

(See Figure 5 for a graph that can be grown in this way: start with the longest horizontal edge.)

3.2 *If there is a sequence $(G_1, X_1), \dots, (G_n, X_n)$ where G_1 is a two-vertex graph with one edge (not a loop) and X_1 contains its edge, and (G_i, X_i) is obtained from (G_{i-1}, X_{i-1}) by adding a triangle for each $i \geq 2$, then G_n is innerplanar.*

Proof. For $i \geq 2$, let w_i be the vertex in $V(G_i) \setminus V(G_{i-1})$, and let $u_i v_i$ be the edge of G_{i-1} that forms a triangle with w_i . Let $V(G_1) = \{w_0, w_1\}$. Then $V(G_n) = \{w_0, \dots, w_n\}$, and

$$\{w_0, \dots, w_{i-1}\}, \{u_i, v_i, w_i\}$$

can be extended to a 2-separation of G_n , for each $i \geq 2$. By performing a Whitney twist at this 2-separation if necessary, for each $i \geq 2$, we can arrange that all edges in X_n have a common end w_0 ; but then we have converted G_n to an apex-forest by Whitney twists, and so G_n is innerplanar. This proves 3.2. ■

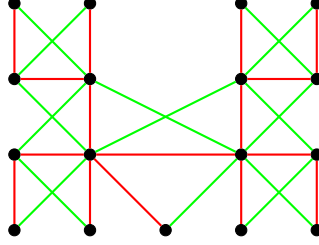


Figure 5: An innerplanar graph. The red tree is an inner tree.

Here is a third way to view innerplanar graphs. Take a tree T , and for each vertex $v \in V(T)$, take a tree T_v with vertex set the set of neighbours of v in T . (The trees T_v are necessarily edge-disjoint from T and from each other.) Let G be the union of T and all the trees T_v ($v \in V(T)$). Then G is innerplanar, simple, and edge-maximal with these properties, and conversely, every edge-maximal simple innerplanar graph can be constructed in this way. If G can be grown by adding edges to a tree T in this way, we call T an *inner tree* for G . (See Figure 5.)

4 Rooted minors

A k -rooted graph $(G, r_1, \dots, r_k)$ consists of a graph G and k distinct vertices $r_1, \dots, r_k \in V(G)$. For the moment we will only need $k = 2$. If (G_1, u_1, v_1) and (G_2, u_2, v_2) are 2-rooted graphs, their *parallel connection* is the 2-rooted graph (G, u, v) where G is obtained from the disjoint union of G_1, G_2 by identifying u_1 with u_2 (making u) and v_1 with v_2 (making v). (If u_i, v_i are adjacent in G_i for both $i = 1, 2$, then these edges become parallel edges in the parallel connection.) The parallel connection of (G, u, v) with itself is called its *parallel doubling*. Similarly, the *series connection* of (G_1, u_1, v_1) and (G_2, u_2, v_2) is the 2-rooted graph (G, u_1, v_2) , where G is obtained from the disjoint union of G_1, G_2 by identifying v_1 with u_2 . (The order matters: the series connection of (G_2, u_2, v_2) and (G_1, u_1, v_1) is different in general.) The series connection of (G, u, v) with itself is called its *series doubling*. (See Figure 6.)

For $k \geq 1$ we define a 2-rooted graph SP_k as follows. SP_1 is the 2-rooted graph (G, u, v) where G consists of u, v and an edge between them. For even $k \geq 2$, SP_k is the series doubling of SP_{k-1} , and for odd $k \geq 2$, SP_k is the parallel doubling of SP_{k-1} . We show $\text{SP}_1, \dots, \text{SP}_5$ in Figure 7.

Thus, SP_k is a series-parallel graph (more exactly, if $(G, u, v) = \text{SP}_k$ then G is a series-parallel graph; we will usually not bother with this distinction); and it is easy to see that for every series-parallel graph H there exists k such that H is a minor of SP_k .

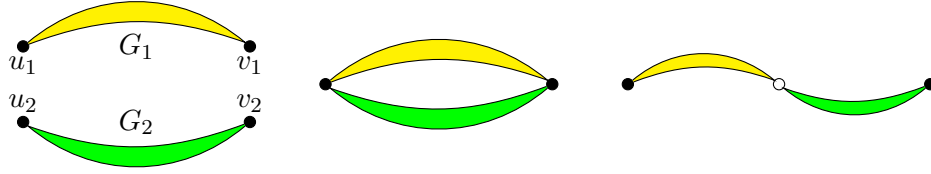


Figure 6: Parallel and series connection.

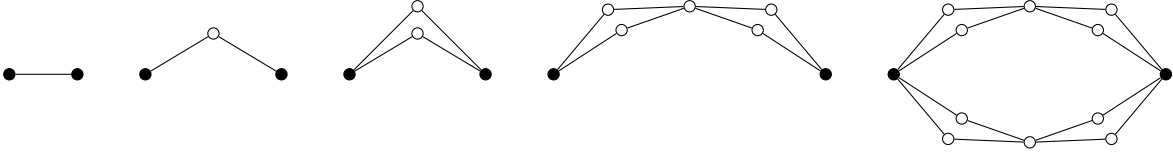


Figure 7: $SP_1, \dots, SP_5$.

If G, H are graphs and H is simple, a *model* of H in G is a function η with domain $V(H) \cup E(H)$, such that

- for each $v \in V(H)$, $\eta(v)$ is a non-null connected subgraph of G , and $\eta(u), \eta(v)$ are vertex-disjoint for all distinct $u, v \in V(H)$;
- for each edge $e \in E(H)$ with ends u, v , $\eta(e)$ is an edge of G with one end in $V(\eta(u))$ and one end in $V(\eta(v))$.

It is easy to see that H is isomorphic to a minor of G if and only if there is a model of H in G . If $(G, r_1, \dots, r_k)$ and $(H, s_1, \dots, s_k)$ are k -rooted graphs, we say the second is a *rooted minor* of the first if there is a model η of H in G such that $r_i \in V(\eta(s_i))$ for $1 \leq i \leq k$, and in this case we say that η is a model of $(H, s_1, \dots, s_k)$ in $(G, r_1, \dots, r_k)$.

We will need to grow models of graphs into models of larger graphs, and the notation can be cumbersome. Here is a slight simplification: let η be a model of H in G , and let P be a path of G with one end in $\eta(u)$ for some $u \in V(H)$, and otherwise disjoint from each subgraph $\eta(v)$ ($v \in V(H)$). Define $\eta'(u) = \eta(u) \cup P$, and $\eta'(v) = \eta(v)$ for all $v \in V(H) \setminus \{u\}$, and $\eta'(e) = \eta(e)$ for each $e \in E(H)$. Then η' is also a model of H in G , and for brevity we say such a model is obtained from η by “contracting P ”. We use similar terminology in some more complicated situations, and we hope that the meaning will be clear.

5 Refined tree-decompositions.

We need some more terminology and notation to handle a tree-decomposition $(T, (W_t : t \in V(T)))$. We direct all edges of T away from the root. Thus, if we speak of an edge $t_1 t_2$ of T , we imply that t_1 lies in the path of T between the root and t_2 . We call t_1 the *parent* of t_2 (in T), and t_2 a *child* of t_1 . More generally, if s belongs to the path of T between the root and t , we say s is an *ancestor* of t and t is a *descendant* of s .

A *triangle* means a cycle of length three. A graph is *three-connected* if it has at least four vertices and there is no $X \subseteq V(G)$ with $|X| \leq 2$ such that $G \setminus X$ is not connected. A rooted tree-decomposition of adhesion two is *refined* if

- if $s, t \in V(T)$ are adjacent then $W_s \not\subseteq W_t$;
- for every path P of T with one end the root of T , no two edges of P have the same adhesion. (To clarify, we are just asking that the two adhesions are not the same set; they might have the same size.)
- each torso is either three-connected or a triangle.

We need:

5.1 *Let G be a simple two-connected graph. Then G admits a refined rooted tree-decomposition.*

Proof. Choose a tree-decomposition $(T, (W_t : t \in V(T)))$ with adhesion two, such that the set of all pairs $\{u, v\}$ such that $\{u, v\} \subseteq W_t$ for some $t \in V(T)$ is minimal. We call this set its *set of nonseparated pairs*. Such a tree-decomposition is *acceptable*. Choose a root r for T . The *height-sum* of T is the sum of the lengths of all paths of T that have the root r as one end. Let $\mathcal{T} = (T, (W_t : t \in V(T)))$ be an acceptable rooted tree-decomposition such that the height-sum of T is minimum. We will show that $\mathcal{T}$ satisfies the three bullets in the definition of “refined”.

If there are adjacent $s, t \in V(T)$ with $W_s \subseteq W_t$, contract the edge st to a vertex u and define $W_u = W_t$. This operation does not change the set of nonseparated pairs; and so it gives another acceptable tree-decomposition with smaller height-sum, a contradiction. Thus $\mathcal{T}$ satisfies the first bullet.

If there are distinct edges e_1, e_2 of T with the same adhesion, and e_1 is between e_2 and the root r , let $e_i = s_i t_i$ for $i = 1, 2$, where there is a path of T passing through r, s_1, t_1, s_2, t_2 in order (possibly $r = s_1$ or $s_2 = t_1$). Let T' be the tree obtained from T by deleting e_2 and adding a new edge $s_1 t_2$. Then $(T', (W_t : t \in V(T)))$ is another acceptable tree-decomposition with smaller height-sum, again a contradiction. So $\mathcal{T}$ satisfies the second bullet.

Let $t \in V(T)$, and suppose that the torso of $\mathcal{T}$ at t is not three-connected and not a triangle. Consequently, there exists $X \subseteq W_t$ with $|X| \leq 2$, and a partition (Y_1, Y_2) of $W_t \setminus X$ such that $Y_1, Y_2 \neq \emptyset$ and there are no edges of the torso between Y_1, Y_2 . Hence we may partition the set of edges of T incident with t into two sets E_1, E_2 , such that for $i = 1, 2$, if $st \in E_i$ then $W_s \cap W_t \subseteq Y_i \cup X$. Let T' be obtained from T by: deleting t ; adding two new vertices t_1, t_2 , adjacent to each other; and for $i = 1, 2$ and each $st \in E_i$, adding an edge between s, t_i . Then T' is a tree; define $W_{t_i} = X \cup Y_i$ for $i = 1, 2$. Then $(T', (W_t : t \in V(T)))$ is a tree-decomposition of G , all its adhesions have size two, and its set of nonseparated pairs is strictly reduced, a contradiction. This proves that every torso is either three-connected or a complete graph, and so $\mathcal{T}$ satisfies the third bullet. This proves 5.1. $\blacksquare$

If $(T, (W_t : t \in V(T)))$ is a rooted tree-decomposition of G , and $e = t_1 t_2 \in E(T)$, then $T \setminus e$ has two components T_1, T_2 say, where the root belongs to $V(T_1)$ (and hence $t_i \in V(T_i)$ for $i = 1, 2$). We define $B_e = \bigcup_{t \in V(T_1)} W_t$ and $A_e = \bigcup_{t \in V(T_2)} W_t$ (“before” e and “after” e); thus, (A_e, B_e) is a separation of G , and $A_e \cap B_e$ is the adhesion at e .

If T is a tree with root r , and S is a subtree of T , the *root* of S is defined to be the vertex of S closest to r in T . If the root of S is a leaf, the corresponding leaf-edge is a *root-edge* for S . A vertex of

S is *internal* if it has degree at least two in S . If S has an internal vertex, S^- is the subtree induced on the set of all internal vertices of S . Let $(T, (W_t : t \in V(T)))$ be a rooted tree-decomposition with adhesion two of some graph G ; if S is a subtree of T with a root-edge and an internal vertex, we define the S -torso $H(S)$ to be the graph defined as follows. Let G_1 be the union of $G[W_s]$ over all $s \in V(S^-)$. The S -torso is obtained from G_1 by adding a new edge f_e joining the two members of $A_e \cap B_e$ (if they are currently nonadjacent), for each edge $e \in E(T)$ that has exactly one end in $V(S^-)$.

Here is a general proof method that we will use repeatedly. We have some two-connected graph G , and a refined rooted tree-decomposition $(T, (W_t : t \in V(T)))$ for G . Each edge $e \in E(T)$ has a “score” $\text{sc}(e)$. Scores are ordered by some partial order, and for each path P of T with one end the root of T , the scores of the edges of P monotonically (non-strictly) decrease with distance from the root, and take only a bounded number of different values, say at most k values. Let S be a subtree of T such that:

- S has a root-edge;
- every edge of S has the same score; and
- S is maximal with these properties.

We call such a subtree S a *plateau*. Every edge of T therefore belongs to a unique plateau (because if two plateaus share an edge, their union also satisfies the first two bullets above, contrary to their maximality). It follows that every path of T with one end the root contains edges of at most k plateaus.

Let F be the set of all edges of T that are leaf-edges of some plateau (including the root-edge of each plateau). Thus, each component of $T \setminus F$ is either a singleton vertex, or obtained from some plateau by deleting its leaves. Let T' be the rooted tree obtained from T by contracting each component C of $T \setminus F$ to a vertex t_C , and setting the root of T' to be the root of T (which is a singleton component of $T \setminus F$). Thus, $E(T') = F$. For each component C of $T \setminus F$:

- if $|V(C)| = 1$, let $V(C) = \{t\}$ say, and define $W'_{t_C} = W_t$;
- if $|V(C)| > 1$, let C be obtained from a plateau S by deleting its leaves, and define $W'_{t_C} = \bigcup_{t \in V(S^-)} W_t$.

Then $(T', (W'_t : t \in V(T')))$ is a rooted tree-decomposition of G with adhesion two, and its height is at most $2k$, and each of its torsos is either a torso of some vertex of T in $(T, (W_t : t \in V(T)))$, (and hence either three-connected or a triangle) or equal to $H(S)$ for some plateau S with an internal vertex. We have proved:

5.2 *Let G be a two-connected graph, let $(T, (W_t : t \in V(T)))$ be a refined tree-decomposition for G ; and for each edge $e \in E(T)$, let $\text{sc}(e)$ be an integer, the “score” of e . Suppose that for every path P of T with one end the root of T :*

- *if $e, e' \in E(P)$ and e is between e' and the root, then $\text{sc}(e') \leq \text{sc}(e)$; and*
- *the edges of P have at most k different scores.*

Then G admits a rooted tree-decomposition with adhesion two and of height at most $2k$, and each of its torsos is either three-connected, or a triangle, or the S -torso $H(S)$ of some plateau S with an internal vertex.

6 The proof of 2.2.

To prove 2.2, we may assume that $H = \text{SP}_k$ for some choice of k . Let G be a simple two-connected series-parallel graph with no SP_k minor. Choose a refined rooted tree-decomposition as in 5.1. Since G is series-parallel, no torso is three-connected, and so $|W_t| = 3$ for each $t \in V(T)$. For each $e \in E(T)$, we say the *score* $\text{sc}(e)$ of e is the maximum d such that SP_d is a rooted minor of $(G[A_e], u, v)$, where $A_e \cap B_e = \{u, v\}$. (We will have to change the definition of “score” to prove 2.3 and 2.4, but this is the version we need for 2.2.) From the symmetry of SP_k exchanging its roots, the order of u, v makes no difference.

If $e, f \in E(T)$ and e belongs to the path of T between the root and f , then the score of e is at least the score of f , because there are two disjoint paths between $A_e \cap B_e$ and $A_f \cap B_f$, so if SP_d is a rooted minor of $(G[A_f], u, v)$ (where $A_f \cap B_f = \{u, v\}$) then it is also a rooted minor of $(G[A_e], w, x)$ where $A_e \cap B_e = \{w, x\}$, as we see by contracting the two paths. Thus, for every path of T with one end the root, the scores of its edges decrease monotonically with distance from the root. Since every edge of T has score between 1 and $k - 1$ (because SP_k is not a minor of G), it follows that every path of T with one end the root contains edges of at most k plateaus. By 5.2 (taking Q to be the set of integers $\{1, \dots, k - 1\}$ ordered by size), G admits a rooted tree-decomposition with adhesion two and of height at most $2k$, and each of its torsos is either a triangle, or the S -torso $H(S)$ of some plateau S with an internal vertex. Hence, to complete the proof of 2.2, it remains only to check that each S -torso $H(S)$ is outerplanar or innerplanar.

6.1 *With $G, k, T, (W_t : t \in V(T))$ as above, let S be a plateau with an internal vertex and with root r_0 , and let d be the common score of the edges of S . If d is even then the S -torso is outerplanar.*

Proof. Let $s \in V(S) \setminus \{r_0\}$, and let r be its parent (thus $r \in V(S)$) and $f = rs$. Let $W_s = \{v_1, v_2, v_3\}$, where $A_f \cap B_f = \{v_1, v_2\}$. Since $(T, (W_t : t \in V(T)))$ is refined, every edge $e \neq f$ of T incident with s satisfies $A_e \cap B_e = \{v_1, v_3\}$ or $A_e \cap B_e = \{v_2, v_3\}$.

(1) *There is at most one edge $e \in E(S)$ incident with s with $A_e \cap B_e = \{v_1, v_3\}$, and the same for $\{v_2, v_3\}$.*

Since d is even, SP_{d+1} is the parallel doubling of SP_d . Suppose that $e, e' \in E(T)$ are distinct edges of S both incident with s and with $A_e \cap B_e = A_{e'} \cap B_{e'} = \{v_1, v_3\}$. Since $e \in E(S)$, and so e has score d , it follows that SP_d is a rooted minor of $(G[A_e], v_1, v_3)$, and similarly it is a rooted minor of $(G[A_{e'}], v_1, v_3)$. By combining the corresponding two models, it follows that the parallel doubling SP_{d+1} of SP_d is a rooted minor of $(G[A_e \cup A_{e'}], v_1, v_3)$. Let η be the corresponding model of SP_{d+1} , and let h be the root of SP_{d+1} with $v_3 \in \eta(h)$. Since G is two-connected, there is a path of $G \setminus \{v_1\}$ between v_2, v_3 , say P , and by contracting this path we convert η to a model of SP_{d+1} in $G[A_f]$; and so SP_{d+1} is a rooted minor of $(G[A_f], v_1, v_2)$, a contradiction. This proves (1).

Now every edge of the S -torso has both ends in W_s for some $s \in V(S) \setminus \{r_0\}$. Let J be the graph with vertex set $\bigcup_{s \in V(S) \setminus \{r_0\}} W_s$ and with edges every pair u, v of distinct vertices that both belong to some W_s for some $s \in V(S) \setminus \{r_0\}$. From (1), J is outerplanar, and hence so is the S -torso. This proves 6.1. ■

6.2 With $G, k, T, (W_t : t \in V(T))$ as above, let S be a plateau with an internal vertex and with root r_0 , and let d be the common score. If d is odd then the S -torso is innerplanar.

Proof. Let $s \in V(S) \setminus \{r_0\}$, and let r be its parent (thus $r \in V(S)$) and $f = rs$. Let $W_s = \{v_1, v_2, v_3\}$, where $A_f \cap B_f = \{v_1, v_2\}$. Since $(T, (W_t : t \in V(T)))$ is refined, every edge $e \neq f$ of T incident with s satisfies $A_e \cap B_e = \{v_1, v_3\}$ or $A_e \cap B_e = \{v_2, v_3\}$.

(1) *There do not exist $e, e' \in E(S)$ incident with s with $A_e \cap B_e = \{v_1, v_3\}$ and $A_{e'} \cap B_{e'} = \{v_2, v_3\}$.*

Since d is odd, SP_{d+1} is the series doubling of SP_d . If such e, e' exist, then by combining the two models we obtain a model of SP_{d+1} in $(G[A_f], v_1, v_2)$, a contradiction. This proves (1).

Let J be the graph with vertex set $\bigcup_{s \in V(S) \setminus \{r_0\}} W_s$ and with edges every pair u, v of distinct vertices that both belong to some W_s for some $s \in V(S) \setminus \{r_0\}$. By (1) and 3.2, it follows that J is innerplanar, and hence so is the S -torso. This proves 6.2. ■

In view of 6.1 and 6.2, this completes the proof of 2.2.

7 When the S -torso is a cycle

To prove 2.3 and 2.5, we need the following lemma:

7.1 *Let G be a two-connected graph, and let $(T, (W_t : t \in V(T)))$ be a refined rooted tree-decomposition. Let S be a subtree of T with a root-edge $e_0 = r_0 s_0$ and an internal vertex, and let $H(S)$ denote the S -torso. If $H(S)$ is not a cycle, there are edges e, e' of S , such that*

- *the head of e is the tail of e' ;*
- *there are two disjoint paths P_1, P_2 of G between $A_e \cap B_e$ and $A_{e'} \cap B_{e'}$; and*
- *there is a path of $G[A_e \cap B_{e'}]$ between $V(P_1), V(P_2)$.*

Proof. Suppose there are no such e, e' (we call this “the supposition”). We will prove that $H(S)$ is a cycle, by induction on $|V(S)|$.

(1) $|W_s| = 3$ for each $s \in V(S^-)$.

Let $s \in V(S^-)$. Thus $s \neq r_0$, and is not a leaf of S . Choose edges e, e' of S incident with s , such that e has head s and e' has tail s . Let H be the torso of $(T, (W_t : t \in V(T)))$ at s . Let $X = A_e \cap B_e$ and $X' = A_{e'} \cap B_{e'}$; then there are edges $f_e, f_{e'}$ of H joining the two vertices in X and joining the two vertices in X' respectively. Suppose that H is three-connected. Then $|V(H)| \geq 4$, so deleting two edges of H does not disconnect H ; and hence $H \setminus \{f_e, f_{e'}\}$ is connected. Since G is two-connected, there are two disjoint paths Q_1, Q_2 of H between X, X' . Since $H \setminus \{f_e, f_{e'}\}$ is connected, there is a path Q_0 of $H \setminus \{f_e, f_{e'}\}$ between $V(Q_1), V(Q_2)$ with no internal vertex in $V(Q_1) \cup V(Q_2)$. All vertices of these paths belong to W_s , but some of their edges might not be edges of G . For each edge g of Q_0, Q_1 or Q_2 that is not an edge of G , there is an edge e of T incident with s with $A_e \cap B_e$

equal to the set of ends of g , and there is a path R_g of $G[A_e]$ joining the two ends of g . By replacing each edge g of Q_0 that is not in $E(G)$ by the path R_g , we obtain a path P_0 of G joining the ends of Q_0 . Define P_1, P_2 similarly; then P_0, P_1, P_2 violate the supposition. Thus H is not three-connected; and so by the third property of “refined”, $|W_s| = 3$. This proves (1).

We recall that $r_0 s_0$ is the root-edge of S . Since S has an internal vertex, it follows that s_0 is internal, and so $|W_{s_0}| = 3$ by (1). Let $W_{s_0} = \{v_1, v_2, w\}$, where $\{v_1, v_2\}$ is the adhesion at e_0 . For each edge $e \neq e_0$ of T incident with s_0 , the adhesion at e is one of the sets $\{v_1, w\}, \{v_2, w\}$, since no such edge has adhesion $\{v_1, v_2\}$ by the second property of “refined”. Let E_i be the set of edges of T incident with s_0 with adhesion $\{v_i, w\}$, for $i = 1, 2$. For each edge $e \in E_1 \cup E_2$, since G is two-connected and $A_e \neq A_e \cap B_e$ (by the first property of the definition of “refined”), there is a path of $G[A_e]$ between the two members of $A_e \cap B_e$, say P_e .

Suppose that there is an edge e_1 of S that belongs to E_i , for some $i \in \{1, 2\}$. By the supposition, applied to the edges e_0, e_1 , there is no path in $G[A_{e_0} \cap B_{e_1}]$ between v_i, w . Consequently, v_i, w are nonadjacent in G ; and moreover, there is no path P_e for $e \in E_i \setminus \{e_1\}$ and so $E_i = \{e_1\}$. In summary, either $E_i \cap E(S) = \emptyset$, or $|E_i| = 1$ and $E_i \subseteq E(S)$. In either case, at most one edge of S belongs to E_i , so s_0 has degree at most three in S .

Since $|E(S)| \geq 2$, at least one of E_1, E_2 contains an edge of S ; so we may assume that $|E_1| = 1$ and $E_1 \subseteq E(S)$, say. For $i = 1, 2$, let F_i be the set of $e \in E_i \cap E(S)$ such that both ends of e are internal vertices of S . If $F_1 = F_2 = \emptyset$, then the S -torso $H(S)$ has vertex set W_{s_0} ; and since $|E_1| = 1$, the edge $v_1 w$ of $H(S)$ is not parallel to any other edge of $H(S)$, and therefore $H(S)$ is a cycle. Thus we may assume that $F_1 \neq \emptyset$. Now there are three cases, depending on whether no edge of S belongs to E_2 , or some edge of S is in E_2 but $F_2 = \emptyset$, or $F_2 \neq \emptyset$.

If no edge of S belongs to E_2 , then s_0 has degree two in S ; let S_1 be the tree $S \setminus \{r_0\}$. If some edge of S belongs to E_2 , let S_1, S_2 be the maximal subtrees of $S \setminus \{r_0\}$ with leaf-edges e_1, e_2 , where e_i is the unique edge of S in E_i for $i = 1, 2$. From the inductive hypothesis, applied to S_1 , the S_1 -torso $H(S_1)$ is a cycle, and it contains the edge f_{e_1} joining the two vertices of $A_{e_1} \cap B_{e_1}$. But v_1, w are nonadjacent in G , and $E_1 = \{e_1\}$, so the edge f_{e_1} does not belong to $H(S)$. Hence, if $F_2 = \emptyset$, then $H(S)$ is obtained from the cycle $H(S_1)$ by deleting f_{e_1} and adding v_2 and two new edges $v_1 v_2, v_2 w$; and so it is a cycle. Thus we may assume that $F_2 \neq \emptyset$. But then both $H(S_1), H(S_2)$ are cycles, and the edges f_{e_1}, f_{e_2} do not belong to $H(S)$, and so $H(S)$ is obtained from the union of $H(S_1), H(S_2)$ by deleting f_{e_1}, f_{e_2} and adding $v_1 v_2$. This proves 7.1. ■

8 The proof of 2.3.

Next we prove 2.3. The proof is similar to that of 2.2, except we have to change the definition of “score” and use 7.1. Let us define $OP_1 = SP_1$, and for $d \geq 2$ even, let OP_d be the series doubling of OP_{d-1} . For $d \geq 2$ odd, let OP_d be obtained from OP_{d-1} by adding an edge between its two roots. (See Figure 8.)

Every outerplanar graph is a minor of OP_k for some k , so it suffices to exclude OP_k . Let G be a two-connected simple series-parallel graph that does not contain OP_k as a minor for some k . Take a refined tree-decomposition $(T, (W_t : t \in V(T)))$ of G of width two as before. Now the score of each edge e of T is defined as the maximum d such that OP_d is a rooted minor of $(G[A_e], u, v)$, where $A_e \cap B_e = \{u, v\}$. As before, the order of u, v here does not matter. Define plateaus as before; so G

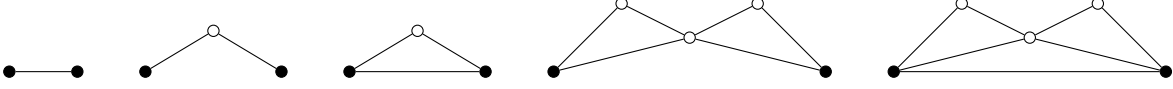


Figure 8: $OP_1, \dots, OP_5$.

admits a rooted tree-decomposition with adhesion two, with height at most $2k$; and each of its torsos is either a triangle, or the S -torso for some plateau S with an internal vertex. Hence, to complete the proof of 2.3, it remains only to check that for each plateau S with an internal vertex, the S -torso is innerplanar.

8.1 *With $G, k, T, (W_t : t \in V(T))$ as above, let S be a plateau with an internal vertex, and let d be the common score. If d is even then the S -torso is a cycle and hence innerplanar.*

Proof. If e, e' are edges of S and the head of e equals the tail of e' , then since e, e' have the same score d (which is even), the conclusion of 7.1 is false. Hence the result is immediate from applying 7.1 to S . ■

8.2 *With $G, k, T, (W_t : t \in V(T))$ as above, let S be a plateau with an internal vertex, and let d be the common score. If d is odd then the S -torso is innerplanar.*

Proof. This proof is exactly like that of 6.2. ■

9 The proof of 2.5

If (G, u, v) is a 2-rooted graph, its *left-legging* is the series connection of SP_1 and (G, u, v) , that is, the rooted graph (H, w, v) , where H is obtained from G by adding a new vertex w and the edge uw . Similarly, its *right-legging* is the series connection of (G, u, v) and SP_1 . (See Figure 9.)

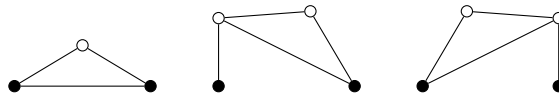


Figure 9: Leggings.

Let us define $FP_1 = SP_1$; for $d \geq 2$, even, let FP_d be the right-legging of FP_{d-1} , and for $d \geq 2$, odd, let FP_d be obtained from FP_{d-1} by adding an edge between the roots of FP_{d-1} . (See Figure 10.)

We also need the following theorem, proposition 3.8 of Ding, Dziobiak, and Wu [4]:

9.1 *For every k there exists ℓ such that every three-connected graph with a path of length at least ℓ contains a k -wheel (and hence FP_k) as a minor.*

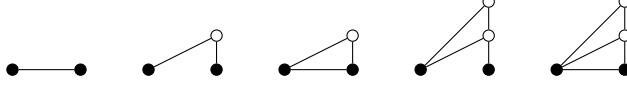


Figure 10: $FP_1, \dots, FP_5$.

To prove 2.5, let G be a simple two-connected graph not containing FP_k as a minor. Take a refined rooted tree-decomposition $(T, (W_t : t \in V(T)))$ as in 5.1. For each edge e of T , let $A_e \cap B_e = \{u, v\}$; we define the *score* of e to be $d_1 + d_2$ where d_1 is maximum such that FP_{d_1} is a rooted minor of $(G[A_e], u, v)$, and d_2 is maximum such that FP_{d_2} is a rooted minor of $(G[A_e], v, u)$. Thus, all scores are less than $4k$. By 5.2, G admits a rooted tree-decomposition with adhesion two, and of height at most $8k$, such that each of its torsos is three-connected (and therefore has no path of length ℓ , where ℓ is as in 9.1), or a triangle, or equal to $H(S)$ for some plateau with an internal vertex. Thus, it remains to show that for every plateau S with an internal vertex, $H(S)$ is a cycle. Let S be such a plateau, and let d be the common score of the edges of S . Let e_0 be the root-edge of S , let $A_{e_0} \cap B_{e_0} = \{u_0, v_0\}$, and choose d_1, d_2 as in the definition of score, with $d_1 + d_2 = d$.

(1) *For every edge e of S , let P_1, P_2 be disjoint paths between the adhesions of e_0 and of e , where $u_0 \in V(P_1)$. Let the adhesion at e be $\{u, v\}$, where $u \in V(P_1)$. Then d_1 is maximum such that FP_{d_1} is a rooted minor of $(G[A_e], u, v)$, and d_2 is maximum such that FP_{d_2} is a rooted minor of $(G[A_e], v, u)$.*

Let d'_1, d'_2 be maximum such that $FP_{d'_1}$ is a rooted minor of $(G[A_e], u, v)$, and d'_2 is maximum such that $FP_{d'_2}$ is a rooted minor of $(G[A_e], v, u)$. Thus $d'_1 + d'_2 = d$, since e, e_0 have the same score. But since $FP_{d'_1}$ is a rooted minor of $(G[A_e], u, v)$, by contracting the paths P_1, P_2 we deduce that $FP_{d'_1}$ is a rooted minor of $(G[A_{e_0}], u_0, v_0)$, and so $d'_1 \leq d_1$. Similarly $d'_2 \leq d_2$; but $d_1 + d_2 = d'_1 + d'_2$, and the claim follows. This proves (1).

If one of d_1, d_2 is even, then it follows immediately from 7.1 that $H(S)$ is a cycle. If they are both odd, then since S has an internal vertex, it has two edges e, e' where the head of e equals the tail of e' . Let s be their common end and choose two disjoint paths P_1, P_2 of the torso of s in $(T, (W_t : t \in V(T)))$ between the adhesion $\{u, v\}$ of e and the adhesion $\{u', v'\}$ of e' , where $u, u' \in V(P_1)$. If P_2 has length at least one, then the model of FP_{d_1} in $(G[A_e], u, v)$ can be extended to show that FP_{d_1+1} is a rooted minor of $(G[A_e], u, v)$, a contradiction; and similarly P_1 has length zero. But then the two adhesions are equal, contradicting that the tree-decomposition is refined. This proves 2.5. $\blacksquare$

10 The proof of 2.4.

If (G, u, v) is a 2-rooted graph, its *reverse* is (G, v, u) . We define $IP_1 = SP_1$, and for $d \geq 2$, even, IP_d denotes the right-legging of IP_{d-1} . For $d \geq 2$, odd, IP_d is the parallel connection of IP_{d-1} and its reverse. (See Figure 11.)

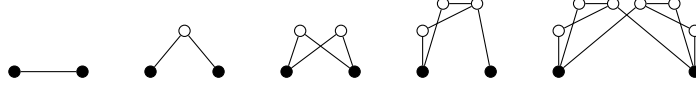


Figure 11: $IP_1, \dots, IP_5$.

Every innerplanar graph is a minor of IP_d for some d . To prove 2.4, let G be a two-connected simple series-parallel graph with no IP_k minor, for some k . Choose a refined rooted tree-decomposition $(T, (W_t : t \in V(T)))$ as in 5.1. For each edge $e \in E(T)$, let $\{u, v\}$ be the adhesion at e ; we say the score $sc(e)$ of e is $d_1 + d_2$, where d_1 is maximum such that IP_{d_1} is a rooted minor of $(G[A_e], u, v)$, and d_2 is maximum such that the reverse of IP_{d_2} is a rooted minor of $(G[A_e], u, v)$. If $d > 1$, then the reverse of IP_d contains IP_{d-1} as a rooted minor. Moreover, if d is odd then IP_d is isomorphic to its own reverse. Consequently d_1, d_2 differ by at most one, and if they are different then the larger is even. So, either $d_1 = d_2$, or they differ by one and the larger is even.

Thus, all scores are between 2 and $2k - 2$; and as usual, for every path P of T with one end the root of T , the scores of the edges in P monotonically decrease with distance from the root. By 5.2, G admits a rooted tree-decomposition with adhesion two and with height at most $4k$, such that each of its torsos is either a triangle or the S -torso of some plateau S with an internal vertex. Thus, to prove 2.4 it suffices to show that:

10.1 *For every plateau S with an internal vertex, the S -torso $H(S)$ is either an apex-forest, or outerplanar.*

Proof. Let e_0 be the root-edge of S , let d be the common score, let $A_{e_0} \cap B_{e_0} = \{u_0, v_0\}$, and choose d_1, d_2 as in the definition of score, with $d_1 + d_2 = d$. Let r_0 be the root of S . As in the proof of 2.5, we have:

(1) *For every edge e of S , let P_1, P_2 be disjoint paths between the adhesions of e_0 and of e , where $u_0 \in V(P_1)$. Let the adhesion at e be $\{u, v\}$, where $u \in V(P_1)$. Then d_1 is maximum such that IP_{d_1} is a rooted minor of $(G[A_e], u, v)$, and d_2 is maximum such that the reverse of IP_{d_2} is a rooted minor of $(G[A_e], u, v)$.*

As we saw, either $d_1 = d_2$, or they differ by one and the larger is even. As in the proof of 2.5, not both d_1, d_2 are odd, since S has an internal vertex. Suppose they are both even, and hence equal. We proceed as in the proof of 6.1, but we write out the proof because there are slight differences. Let $s \in V(S) \setminus \{r_0\}$, and let r be its parent (thus $r \in V(S)$) and $f = rs$. Let $W_s = \{v_1, v_2, v_3\}$, where $A_f \cap B_f = \{v_1, v_2\}$. Since $(T, (W_t : t \in V(T)))$ is refined, every edge $e \neq f$ of T incident with s satisfies $A_e \cap B_e = \{v_1, v_3\}$ or $A_e \cap B_e = \{v_2, v_3\}$.

(2) *There is at most one edge $e \in E(S)$ incident with s with $A_e \cap B_e = \{v_1, v_3\}$, and the same for $\{v_2, v_3\}$.*

Since d_1 is even, IP_{d_1+1} is the parallel connection of IP_{d_1} and its reverse. Suppose that $e, e' \in E(T)$ are distinct edges of S both incident with s and both with adhesion $\{v_1, v_3\}$. Since $e \in E(S)$, it follows that IP_{d_1} is a rooted minor of $(G[A_e], v_1, v_3)$. Since $d_2 = d_1$, the reverse of IP_{d_1} is a rooted

minor of $(G[A_{e'}], v_1, v_3)$. By combining the corresponding two models, it follows that the parallel connection IP_{d_1+1} of IP_{d_1} and its reverse is a rooted minor of $(G[A_e \cup A_{e'}], v_1, v_3)$. Let η be the corresponding model of IP_{d_1+1} , and let h be the root of IP_{d_1+1} with $v_3 \in \eta(h)$. Since G is two-connected, there is a path of $G \setminus \{v_1\}$ between v_2, v_3 , say P . By contracting this path, we convert η to a model of IP_{d_1+1} in $G[A_f]$, and so IP_{d_1+1} is a rooted minor of $(G[A_f], v_1, v_2)$, a contradiction. This proves (2).

Now every edge of the S -torso has both ends in W_s for some $s \in V(S) \setminus \{r_0\}$. Let J be the graph with vertex set $\bigcup_{s \in V(S) \setminus \{r_0\}} W_s$ and with edges every pair u, v of distinct vertices that both belong to some W_s for some $s \in V(S) \setminus \{r_0\}$. From (2), J is outerplanar, and hence so is the S -torso.

Thus we may assume that d_1 is even and $d_2 = d_1 - 1$ is odd. Let the adhesion at e_0 be $\{u_0, v_0\}$, where d_1 is maximum such that IP_{d_1} is a rooted minor of $(G[A_{e_0}], u_0, v_0)$.

(3) For each $e \in E(S)$, the adhesion at e contains u_0 .

Suppose that this is false for some $e \in E(S)$, with adhesion $\{u, v\}$ say. Choose two disjoint paths P_1, P_2 between the adhesions of e and e_0 , with $u, u_0 \in V(P_1)$ (exchanging the names of u, v if necessary). Thus, P_1 has positive length. But IP_{d_2} (indeed, IP_{d_1} , but all we need is IP_{d_2}) is a rooted minor of $(G[A_e], u, v)$. By adding P_1 to the corresponding model, we deduce that the left-legging of IP_{d_2} (that is, the reverse of IP_{d_1}) is a rooted minor of $(G[A_{e_0}], u_0, v_0)$, contradicting that $d_2 < d_1$. This proves (3).

From (3), we deduce that $H(S)$ is obtained from a graph of tree-width one by adding a vertex, and so is an apex-forest. This proves 10.1. ■

11 The proof of 2.7.

A graph is *near-outerplanar* if it can be drawn in the plane such that every region has a vertex in common with the infinite region. (See Figure 12.) If such a graph G is simple and three-connected, the boundary of the infinite region is a cycle C , $G \setminus V(C)$ is a tree, and all its other edges are between this tree and $V(C)$.

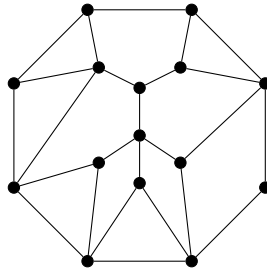


Figure 12: A near-outerplanar graph.

Near-outerplanar graphs have tree-width at most three, and form one of the building blocks of 2.7. Here is the full statement of 2.7:

11.1 For every series-parallel graph H there exists k such that every simple three-connected graph of tree-width 3 and with no H minor admits a tree-decomposition with adhesion three and height at most k , such that each torso is either:

- a near-outerplanar graph; or
- obtained from an outerplanar graph by adding a vertex; or
- obtained from an innerplanar graph by adding a vertex; or
- a graph with path-width at most three.

Proof. We may assume that H is SP_k . Let G be a simple three-connected graph with tree-width three and with no H minor; then G admits a tree-decomposition of width three and adhesion three. As in the proof of 5.1, we may choose a rooted tree-decomposition $(T, (W_t : t \in V(T)))$, of width three and adhesion three, such that

- if $s, t \in V(T)$ are adjacent then $W_s \not\subseteq W_t$; and
- for every path P of T with one end the root of T , no two edges of P have the same adhesion.

We need to define the score of edges of T . For each $e \in E(T)$ and all distinct $u, v \in A_e \cap B_e$, we define $\text{insec}(u, v, e)$ to be the largest d such that SP_d is a rooted minor of (G', u, v) , where $G' = G[(A_e \setminus B_e) \cup \{u, v\}]$; and we define $\text{outsec}(u, v, e)$ to be the largest d such that SP_d is a rooted minor of $(G[A_e], u, v)$. Thus, $\text{insec}(u, v, e) \leq \text{outsec}(u, v, e) \leq k - 1$, since G has no SP_k minor. We use this to define the score of edges of T , as follows. Let $e \in E(T)$ with adhesion $\{u, v, w\}$; then

$$\text{sc}(e) = \text{insec}(u, v, e) + \text{outsec}(u, v, e) + \text{insec}(u, w, e) + \text{outsec}(u, w, e) + \text{insec}(v, w, e) + \text{outsec}(v, w, e).$$

We observe, first, that, as in the proof of 2.2:

(1) Let e, e' be distinct edges of T where e belongs to the path between e' and the root of T ; and choose three disjoint paths P_1, P_2, P_3 between the two adhesions. For $i = 1, 2, 3$, let v_i, v'_i be the vertices of P_i that belong to the adhesion at e and the adhesion at e' , respectively. Then $\text{insec}(v_i, v_j, e) \geq \text{insec}(v'_i, v'_j, e')$ and $\text{outsec}(v_i, v_j, e) \geq \text{outsec}(v'_i, v'_j, e')$ for all distinct $i, j \in \{1, 2, 3\}$. Hence $\text{sc}(e') \leq \text{sc}(e)$, and if $\text{sc}(e') = \text{sc}(e)$, then $\text{insec}(v_i, v_j, e) = \text{insec}(v'_i, v'_j, e')$ and $\text{outsec}(v_i, v_j, e) = \text{outsec}(v'_i, v'_j, e')$ for all distinct $i, j \in \{1, 2, 3\}$.

We define plateaus as before; and for each plateau S with an internal vertex, we define the S -torso $H(S)$ as follows. Let G_1 be the union of $G[W_s]$ over all $s \in V(S^-)$. The S -torso is obtained from G_1 by adding new edges joining every two of the three members of $A_e \cap B_e$ (if they are currently nonadjacent), for each edge $e \in E(T)$ that has exactly one end in $V(S^-)$. We deduce as before that G admits a rooted tree-decomposition with adhesion three and of height at most $12k$, such that each of its torsos is either $H(S)$ for some plateau S with an internal vertex, or is a complete graph K_4 . We need to figure out the possibilities for the graphs $H(S)$, and they are handled in the following results. Let e_0 be the root-edge of S , and let the adhesion at e_0 be $\{u_1, u_2, u_3\}$. For $h = 1, 2, 3$, define

$$\begin{aligned} c_h &= \text{outsec}(u_i, u_j, e_0) \\ d_h &= \text{insec}(u_i, u_j, e_0) \end{aligned}$$

where $\{i, j\} = \{1, 2, 3\} \setminus \{h\}$.

We need to show that $H(S)$ is one of the four bullets of the theorem.

(2) *If $c_1 \neq d_1$ then $H(S)$ is as in one of the four bullets.*

Suppose that $c_1 \neq d_1$. Hence SP_{c_1} is contained as a rooted minor of $(G[A_{e_0}], u_2, u_3)$, but not of $(G[A_{e_0} \setminus \{u_1\}], u_2, u_3)$. Suppose that for some $e \in E(S)$, the adhesion at e does not contain u_1 . Let the adhesion at e be $\{v_1, v_2, v_3\}$ where there are disjoint paths P_1, P_2, P_3 such that P_i links u_i with v_i for $i = 1, 2, 3$. Since $\text{outsc}(v_2, v_3, e) = c_1$, SP_{c_1} is contained as a rooted minor of $(G[A_e], v_2, v_3)$. But since $u_1 \notin A_e$, and u_1, v_1 do not belong to P_2, P_3 , it follows that SP_{c_1} is contained as a rooted minor of $(G[A_{e_0} \setminus \{u_1\}], u_2, u_3)$, contradicting that $d_1 < c_1$. So u_1 belongs to all the adhesions at e (in $(T, (W_t : t \in V(T)))$) for every edge e of S . But then $H(S) \setminus \{u_1\}$ is either outerplanar or innerplanar, by 6.1 and 6.2, depending on the parity of d_1 , and so $H(S)$ is indeed as in one of the four bullets. This proves (2).

So we may assume that $c_i = d_i$ for $i = 1, 2, 3$. Henceforth we write sc for both insc and outsc . Let r_0 be the root of S .

(3) *Let $s \in V(S)$. If $s_1, s_2 \in V(S)$ are two of its children, then the adhesions of ss_1, ss_2 are different.*

Since s has more than one child in S , it follows that $s \neq r_0$; let r be the parent of s . Let $W_s = \{v_1, v_2, v_3, w\}$ where $\{v_1, v_2, v_3\}$ is the adhesion at rs . We may assume that $\text{sc}(v_1, v_2, rs) = d_3$, and $\text{sc}(v_1, v_3, rs) = d_2$, and $\text{sc}(v_2, v_3, rs) = d_1$. The adhesion at ss_1 is a subset of W_s of size three, and not $\{v_1, v_2, v_3\}$ from the choice of $(T, (W_t : t \in V(T)))$. Suppose that the adhesions of ss_1 and ss_2 are equal; then we may assume they both equal $\{v_1, v_2, w\}$. Since G is three-connected, there is a path P_3 between w, v_3 not containing v_1 or v_2 . From P_3 and the two one-vertex paths with vertices v_1, v_2 respectively, we deduce from (1) that for $i = 1, 2$, $\text{sc}(v_1, v_2, ss_i) = d_3$, and $\text{sc}(v_1, w, ss_i) = d_2$, and $\text{sc}(v_2, w, ss_i) = d_1$.

There is a model of SP_{d_3} in $(G[A_{ss_1} \setminus \{w\}], v_1, v_2)$, and there is a path Q of $G[A_{ss_2} \setminus \{v_2\}]$ between v_1, w . By contracting Q and P_3 , we obtain a model of SP_{d_3} in $(G[A_{rs}], v_2, v_3)$, and we deduce that $\text{sc}(v_2, v_3, rs) \geq d_3$. Since $d_1 = c_1 = \text{sc}(v_2, v_3, rs)$ it follows that $d_1 \geq d_3$. Similarly, $d_2 \geq d_3$. Now there are two cases. Suppose that d_3 is even; then SP_{d_3+1} is the parallel doubling of SP_{d_3} . By combining the models of SP_{d_3} in $(G[A_{ss_i} \setminus \{w\}], v_1, v_2)$ for $i = 1, 2$, we obtain a model of the parallel doubling of SP_{d_3} (that is, SP_{d_3+1}) in $(G[A_{rs}], v_1, v_2)$, contradicting that $\text{outsc}(v_1, v_2, rs) = d_3$. Now we suppose that d_3 is odd, and so SP_{d_3+1} is the series doubling of SP_{d_3} . Since $d_1, d_2 \geq d_3$, there are models of SP_{d_3} in $(G[A_{ss_1} \setminus \{v_1\}], v_2, w)$ and in $(G[A_{ss_2} \setminus \{v_2\}], v_1, w)$. By combining them, we obtain a model of the series doubling of SP_{d_3} (that is, SP_{d_3+1}) in $(G[A_{rs}], v_1, v_2)$, contradicting that $\text{sc}(v_1, v_2, rs) = d_3$. This proves (3).

If S is a path, then $H(S)$ has path-width three and satisfies one of the bullets of the theorem, so henceforth we assume that S is not a path.

(4) *$d_1 = d_2 = d_3 = d$ say, and d is even.*

Since S is not a path, there exists $s \in V(S)$, with children $s_1, s_2 \in V(S)$. Let r be the parent of s , and let $W_s = \{v_1, v_2, v_3, w\}$ where $\{v_1, v_2, v_3\}$ is the adhesion at rs . By (3), the adhesions of ss_1, ss_2 are different, so we may assume that the adhesion at ss_i is $W_s \setminus \{v_i\}$ for $i = 1, 2$. (See Figure 13.)

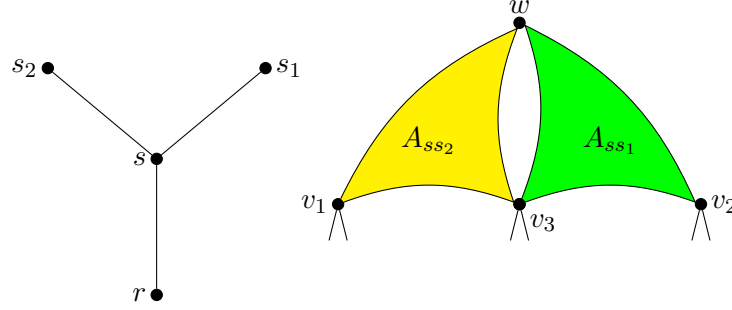


Figure 13: Two children with different adhesions.

We may assume that $sc(v_1, v_2, rs) = d_3$, and $sc(v_1, v_3, rs) = d_2$, and $sc(v_2, v_3, rs) = d_1$. As in (3), $sc(v_2, v_3, ss_1) = d_1$, and $sc(v_2, w, ss_1) = d_3$, and $sc(v_3, w, ss_1) = d_2$, and similarly $sc(v_1, v_3, ss_2) = d_2$, and $sc(v_1, w, ss_2) = d_3$, and $sc(v_3, w, ss_2) = d_1$. By contracting a path in $G[A_{ss_2} \setminus \{w\}]$ between v_1, v_3 , we convert a model of SP_{d_1} in $(G[A_{ss_1} \setminus \{w\}], v_2, v_3)$ to a model of SP_{d_1} in $(G[A_{rs}], v_1, v_2)$, and so $sc(v_1, v_2, rs) \geq d_1$; and consequently $d_3 \geq d_1$, and similarly $d_3 \geq d_2$. But by contracting a path in $G[A_{ss_2} \setminus \{v_1\}]$ between w, v_3 , we convert a model of SP_{d_3} in $(G[A_{ss_1} \setminus \{v_3\}], v_2, w)$ to a model of SP_{d_3} in $(G[A_{rs}], v_2, v_3)$, and so $sc(v_2, v_3, rs) \geq d_3$; and therefore $d_1 \geq d_3$, and similarly $d_2 \geq d_3$. So $d_1 = d_2 = d_3 = d$ say.

By combining a model of SP_d in $(G[A_{ss_2} \setminus \{v_3\}], v_1, w)$ with a model of SP_d in $(G[A_{ss_1} \setminus \{v_3\}], v_2, w)$, we obtain a model of the series doubling of SP_d in $(G[A_{rs}], v_1, v_2)$. Since there is no model of SP_{d+1} in $(G[A_{rs}], v_1, v_2)$, we deduce that d is even. This proves (4).

(5) *Every vertex in S has at most two children in S .*

Suppose that $s \in V(S)$ has three children s_1, s_2, s_3 in S ; and let r be the parent of s in S , and $W_s = \{v_1, v_2, v_3, w\}$. By (3), we may assume that the adhesion at ss_i is $W_s \setminus \{v_i\}$ for $i = 1, 2, 3$. By combining the models of SP_d in $(G[A_{ss_1} \setminus \{v_2\}], v_3, w)$ and in $(G[A_{ss_2} \setminus \{v_1\}], v_3, w)$ we obtain a model of the parallel doubling of SP_d , that is, of SP_{d+1} , in

$$(G[(A_{ss_1} \setminus \{v_2\}) \cup (A_{ss_2} \setminus \{v_1\})], v_3, w) = (G[(A_{ss_1} \cup A_{ss_2}) \setminus \{v_1, v_2\}], v_3, w).$$

By contracting a path of $G[A_{ss_3} \setminus \{v_2\}]$ between w, v_1 , we convert this model to a model of SP_{d+1} in $(G[A_{rs}], v_1, v_3)$, a contradiction. This proves (5).

(6) *Let $s, s' \in V(S)$, with parents r, r' respectively, where s is in the path of S between e_0 and s' . Let s have two children s_1, s_2 in S , and let s' have two children $s'_1, s'_2 \in S$. Let $W_s = \{v_1, v_2, v_3, w\}$, where the adhesion at rs is $\{v_1, v_2, v_3\}$, and the adhesion of ss_i is $W_s \setminus \{v_i\}$ for $i = 1, 2$; and let $W_{s'} = \{v'_1, v'_2, v'_3, w'\}$, where the adhesion at $r's'$ is $\{v'_1, v'_2, v'_3\}$, and the adhesion of $s's'_i$ is $W_{s'} \setminus \{v'_i\}$*

for $i = 1, 2$. For every set of three vertex-disjoint paths between $\{v_1, v_2, v_3\}$ and $\{v'_1, v'_2, v'_3\}$, one of the three paths joins v_3 and v'_3 .

We may assume that $s \neq s'$. Since s' is a descendant of s and not equal to s , it is a descendant of one of s_1, s_2 and we assume s' is a descendant of s_1 without loss of generality. Suppose that P_1, P_2, P_3 are vertex-disjoint paths between $\{v_1, v_2, v_3\}$ and $\{v'_1, v'_2, v'_3\}$, and none of them joins v_3, v'_3 . Let $v_i \in V(P_i)$ for $i = 1, 2, 3$. So $w \in V(P_1)$, since each of P_1, P_2, P_3 has a vertex in $\{w, v_2, v_3\}$. Let P'_1 be the subpath of P_1 between w and the end of P_1 in $\{v'_1, v'_2, v'_3\}$. (So any or all of P'_1, P_2, P_3 might have length zero.) Since P_3 does not contain v'_3 , we may assume (by exchanging v'_1, v'_2 if necessary) that $v'_1 \in V(P_3)$; and then there are two cases: $v'_3 \in V(P_1)$ and $v'_2 \in V(P_2)$; and $v'_3 \in V(P_2)$ and $v'_2 \in V(P_1)$. (See Figure 14.) In either case, let Q be a path in $G[A_{s's'_1} \setminus \{w'\}]$ between v'_2, v'_3 ;

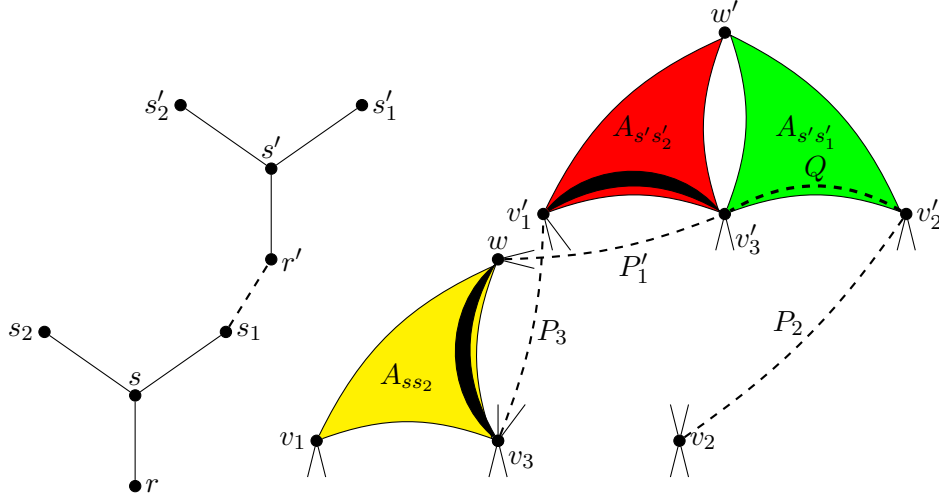


Figure 14: For the proof of (6). The black bananas represent models of SP_d . The paths P'_1 and P_2 might link w to v'_2 and v_2 to v'_3 instead of what is drawn.

so $P'_1 \cup Q \cup P_2$ is a path between w, v_2 passing through v'_3 and disjoint from P_3 . By contracting $P'_1 \cup Q \cup P_2$ and P_3 , we convert the model of SP_d in $(G[A_{ss2} \setminus \{v_1\}], v_3, w)$ and the model of SP_d in $(G[A_{s's'_2} \setminus \{w'\}], v'_1, v'_3)$ to a model of SP_{d+1} in $(G[A_{rs}], v_2, v_3)$, which is impossible. This proves (6).

Let us define (inductively on the length of the path between e_0, e) a numbering, called the “standard numbering”, of the three members of the adhesion at e , for each edge $e \in E(S)$, as follows. The adhesion at e_0 has standard numbering (u_1, u_2, u_3) . For each edge $e \in E(S)$ with $e \neq e_0$, let s be the end of e in the interior of the path between e, e_0 , let r be the parent of s , and let $e' = rs$. We assume inductively that the standard numbering of the adhesion at e' is defined, say (v_1, v_2, v_3) . Since the adhesions at e and at e' are both subsets of W_s , and $|W_s| = 4$, the adhesion of e contains two of v_1, v_2, v_3 , say the two different from v_h ; and its third member is the vertex of W_s different from v_1, v_2, v_3 , say w . Then the standard numbering of the adhesion at e is defined to be the sequence obtained from (v_1, v_2, v_3) by replacing v_h by w .

It follows immediately that if $e \in E(S)$ and (v_1, v_2, v_3) is the standard numbering of the adhesion

at e , and P_1, P_2, P_3 are pairwise vertex-disjoint paths of G (or of $H(S)$) between the adhesions of e_0 and e , and $u_i \in V(P_i)$ for $i = 1, 2, 3$, then $v_i \in V(P_i)$ for $i = 1, 2, 3$.

If $s \in V(S)$ with parent r and two children in S , we say that “ s divides according to the standard numbering” if we can number the children s_1, s_2 such that the adhesion of ss_i is $W_s \setminus \{v_i\}$ for $i = 1, 2$, where $W_s = \{v_1, v_2, v_3, w\}$, and (v_1, v_2, v_3) is the standard numbering of the adhesion at rs .

(7) *We may assume that every vertex of S with two children divides according to the standard numbering.*

Choose $s \in V(S)$ minimal (that is, as close to r_0 as possible) with two children: and then we may choose the standard numbering such that s divides according to the standard numbering. From (6), it follows that every vertex of S with two children also divides according to the standard numbering. This proves (7).

For each $e = rs \in E(S)$, we define J_e as follows. If s is a leaf of S , let J_e be a triangle with vertex set the adhesion at e . If s is not a leaf of S , let (v_1, v_2, v_3) be the standard numbering of the adhesion at e , and let J_e be the subgraph of $H(S)$ defined as follows. Let X be the set of vertices of S^- that are descendants of s . Start with the union of the graphs $G[W_{s'}]$ over all $s' \in X$, and for every edge f of T different from e with exactly one end in X , add edges (if they are not already present) to make the adhesion at f a clique. It follows that J_e is a subgraph of $H(S)$.

(8) *For every edge $e = rs$ of S , if (v_1, v_2, v_3) is the standard numbering of the adhesion at e , and there is a child s_i of s in S with $i \in \{1, 2\}$ such that the adhesion at ss_i is $W_s \setminus \{v_i\}$, then v_1, v_2 are nonadjacent in J_e .*

Suppose that v_1, v_2 are adjacent in J_e . The edge v_1v_2 of J_e might not be an edge of G , but if not then there is an edge $f = ba$ of T such that $b \in V(S^-)$ and is a descendant of s , and $a \notin V(S^-)$, such that v_1, v_2 belong to the adhesion of f . In this case there is a path Q say of $G[A_f]$ between v_1, v_2 with interior disjoint from the adhesion at f .

Since S is not a path, there is a vertex in S with two children in S ; choose such a vertex, s' say, with minimum distance from r_0 . Then s' is either an ancestor or a descendant of s . Let r' be its parent, and let (v'_1, v'_2, v'_3) be the standard numbering of the adhesion at $r's'$.

Suppose first that s' is a descendant of s . Choose three disjoint paths P_1, P_2, P_3 of G between the adhesions of rs and $r's'$, where P_i joins v_i, v'_i for $i = 1, 2, 3$. (See Figure 15.) In this case, suppose that $f = ba$ and Q exist. We claim that the interior of Q is disjoint from $A_{s's'_1}, A_{s's'_2}$ and the interior of P_3 . Now a is not a descendant of s'_1 or of s'_2 , since the adhesions of $s's'_1, s's'_2$ do not include $\{v_1, v_2\}$. Thus, the interior of Q is disjoint from $A_{s's'_1}, A_{s's'_2}$, and it remains to show that it is disjoint from the interior of P_3 . Suppose that some vertex x belongs to the interior of Q and to the interior of P_3 . Thus, $x \neq v_1, v_2, v_3, v'_1, v'_2, v'_3$. Let T_a, T_b be the two components of $T \setminus \{f\}$, where the root of T is in T_b . Since $x \in A_f$ and x does not belong to the adhesion at f (because it is in the interior of Q), it follows that every $t \in V(T)$ with $x \in W_t$ belongs to T_a . But $rs, r's'$ are both edges of T_b , since a is not an ancestor of s' (because $a \notin V(S^-)$). Since the subpath of P_3 between v_3, x has an end in the adhesion of rs and an end x , it contains a vertex in the adhesion of f ; and the same holds for the subpath of P_3 between x, v'_3 . These two vertices are not the same (because they are different from x), and not equal to v_1 or v_2 ; and so the adhesion at f has size at least four, a contradiction.

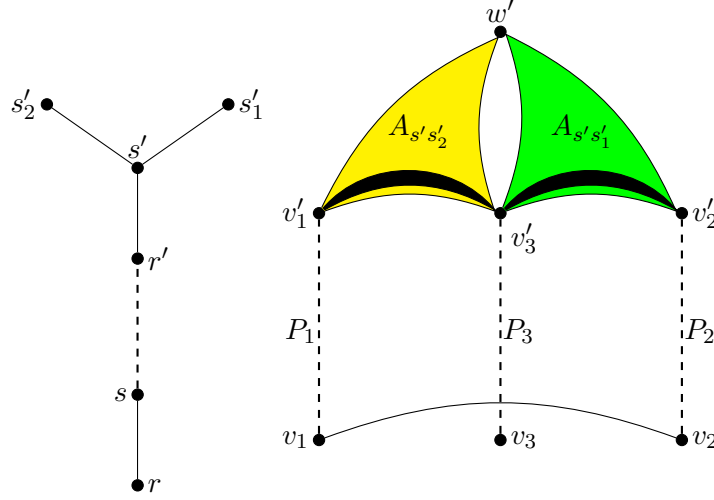


Figure 15: For the proof of (8).

This proves that the interior of Q is disjoint from $A_{s's'_1}, A_{s's'_2}$ and the interior of P_3 . Now Q might intersect P_1 or P_2 , but that will not matter.

By contracting P_1, P_2, P_3 and the edge v_1v_2 (or the path Q if it exists), we convert the model of SP_d in $(G[A_{s's'_2} \setminus \{w'\}], v'_1, v'_3)$ and the model of SP_d in $(G[A_{s's'_1} \setminus \{w'\}], v'_2, v'_3)$ to a model of SP_{d+1} in $(G[A_{rs}], v_2, v_3)$, a contradiction.

Now suppose that s is a descendant of s' , and hence of one of s'_1, s'_2 say s'_1 . For this case we need to use the child of s . Let $W_s = \{v_1, v_2, v_3, w\}$. We may assume that s has a child s_1 in S , and the adhesion of ss_1 is $W_s \setminus \{v_1\}$.

Let P_1, P_2, P_3 be three disjoint paths between the adhesions of $s's'_1$ and rs , where P_3 joins v'_3, v_3 . (See Figure 16.) Suppose that the edge $f = ba$ and Q exist; we claim that the interior of Q is disjoint from $A_{ss_1}, A_{s's'_2}$ and the interior of P_3 . Now a is not a descendant of either of s_1, s'_2 (because the adhesions of ss_1 and $s's'_2$ do not include $\{v_1, v_2\}$), and a is not an ancestor of s_1 (since $a \notin V(S^-)$), so the interior of Q is disjoint from $A_{ss_1}, A_{s's'_2}$. Suppose that x belongs to the interior of Q and to the interior of P_3 . Thus, $x \neq v_1, v_2, v_3, v'_1, v'_2, v'_3$. The same argument as before shows that two vertices of P_3 belong to the adhesion at f , a contradiction. This proves that the interior of Q is disjoint from $A_{ss_1}, A_{s's'_2}$ and the interior of P_3 .

Now there are two cases: either P_1 joins w', v_1 and P_2 joins v'_2, v_2 , or P_1 joins w', v_2 and P_2 joins v'_2, v_1 . In either case, the union of P_1, P_2 and the edge v_1v_2 (or the path Q) is a connected subgraph containing w', v'_2 .

By contracting all of P_1, P_2, P_3 and the edge v_1v_2 (or Q), we convert the model of SP_d in $(G[A_{s's'_2} \setminus \{v'_1\}], v'_3, w')$ and the model of SP_d in $(G[A_{ss_1} \setminus \{w\}], v_2, v_3)$ to a model of SP_{d+1} in $(G[A_{r's'}], v'_2, v'_3)$, a contradiction. This proves (8).

Let $e = rs \in E(S)$, and let (v_1, v_2, v_3) be the standard numbering of the adhesion at rs . We say e behaves if:

- there is a path P_e of J_e between v_1, v_2 , and a tree M_e of J_e containing v_3 , such that $V(P_e), V(M_e)$

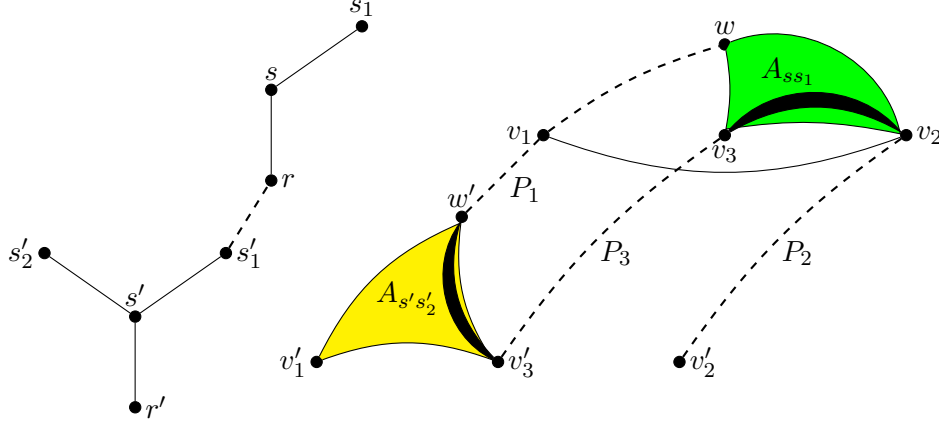


Figure 16: Also for the proof of (8). The paths P_1, P_2 might link w' to v_2 and v_2' to v_1 instead of what is drawn.

are disjoint and have union $V(J_e)$;

- every edge of J_e either belongs to one of P_e, M_e , or has one end in $V(P_e)$ and the other in $V(M_e)$; and
- there is a planar drawing of J_e with v_1, v_2, v_3 all incident with the infinite region.

We claim:

(9) *Every edge rs of S behaves.*

We prove that $e \in E(S)$ behaves, by induction on $|V(J_e)|$. Let $e = rs$, and let $W_s = \{v_1, v_2, v_3, w\}$, where (v_1, v_2, v_3) is the standard numbering of the adhesion at e . If s is a leaf of S , then from the definition of J_e , w is not in $V(J_e)$, and v_1, v_2, v_3 are pairwise adjacent in J_e , and so e behaves (taking P_e to be the path of length one between v_1, v_2 , and M_e the one-vertex tree with vertex v_3).

Next, suppose that s has exactly one child s_1 in S , and the adhesion of ss_1 is $W_s \setminus \{v_1\}$. Let $f = ss_1$. Then v_1, w are adjacent in $H(S)$ and hence in J_e since G is three-connected. From the inductive hypothesis, ss_1 behaves, so P_f and M_f exist, satisfying the bullets in the definition of “behaves”. The ends of P_f are w, v_2 ; let P_e be obtained by adding v_1 and the edge v_1w to P_f . Also, $v_3 \in V(M_f)$; let $M_e = M_f$. The only edges of J_e that are not edges of J_f are v_1w and possibly v_1v_3 or v_3w (not v_1v_2 , by (8)); and so e behaves.

Next, suppose that s has exactly one child s_3 in S , and the adhesion of ss_3 is $W_s \setminus \{v_3\}$. Let $f = ss_3$. Then v_3, w are adjacent in $H(S)$ and hence in J_e since G is three-connected. From the inductive hypothesis, ss_3 behaves, so P_f and M_f exist. The ends of P_f are v_1, v_2 ; let $P_e = P_f$. Also, $w \in V(M_f)$; let M_e be obtained from M_f by adding v_3 and the edge v_3w . The only edges of J_e that are not edges of J_f are between v_3 and $\{v_1, v_2, w\}$, and so e behaves.

Finally, suppose that s has two children s_1, s_2 , and the adhesions at ss_i are $W_s \setminus \{v_i\}$ for $i = 1, 2$. Let $f_i = ss_i$ for $i = 1, 2$. From the inductive hypothesis, f_1 and f_2 both behave, and so

$P_{f_1}, M_{f_1}, P_{f_2}, M_{f_2}$ exist and satisfy the bullets above. Since P_{f_1} has ends v_2, w and P_{f_2} has ends v_1, w , their union $P_{f_1} \cup P_{f_2}$ is a path between v_1, v_2 ; let P_e be this path. Let $M_e = M_{f_1} \cup M_{f_2}$. Every edge of J_e is an edge of one of J_{f_1} or of J_{f_2} except possibly v_1v_3 or v_2v_3 (because v_1, v_2 are not adjacent by (8)); and so e behaves. This proves (9).

From (9), we deduce that $H(S)$ is near-outerplanar. This proves 11.1. ■

12 The stopsign.

Let H be a graph obtained from a cycle of length at least three and a path by identifying one end of the path with a vertex of the cycle. We call such a graph a *stopsign*. Every stopsign is a subgraph of a fan, so we can use 2.6 to give information about the graphs that do not contain a given stopsign. We claim:

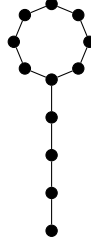


Figure 17: A stopsign.

12.1 *For every stopsign H , there exist ℓ, m such that every simple connected graph with no H minor admits a rooted tree-decomposition $\mathcal{T} = (T, (W_t : t \in V(T)))$ such that either:*

- *$\mathcal{T}$ has height at most m , and one torso is a cycle, and all other torsos have at most $\ell + 1$ vertices; or*
- *$\mathcal{T}$ has width at most ℓ , and every path of T of length m contains an edge with adhesion of size 1.*

Proof. Let F be a fan containing H as a minor. Choose ℓ_1, m_1 such that setting $\ell = \ell_1$ and $m = m_1$ satisfies 2.6. Choose ℓ_2, m_2 such that every graph with no path of length $|V(F)|^2$ admits a tree-decomposition of height at most m_2 and width at most ℓ_2 . Let $\ell = \max(\ell_1, \ell_2, |V(F)|) + 1$ and $m = \max(m_1, m_2) + 2$. We claim that ℓ, m satisfy 12.1.

To see this, let G be a simple connected graph with no H minor. Suppose first that G has no cycle of length at least $|V(F)|$. Then each maximal two-connected component has no path of length $|V(F)|^2$ (by a theorem of Dirac [5]), and so admits a tree-decomposition of height at most m_2 and width at most ℓ_2 . By combining these tree-decompositions, we deduce that G admits a tree-decomposition of width at most ℓ_2 such that every path of its tree of length $m_2 + 1$ contains an edge with adhesion of size 1, and so the theorem holds.

Thus we may assume that some maximal two-connected component G' has a cycle of length at least $|V(F)|$. For each $v \in V(G')$, let B_v be the component of $G \setminus \{v\}$ that contains $G' \setminus \{v\}$, and let $A_v = G \setminus V(B_v)$ (thus, $v \in V(A_v)$). Hence the subgraphs A_v ($v \in V(G')$) are pairwise vertex-disjoint, and their union, together with G' , equals G . Since G does not contain H , it follows that for each $v \in V(G')$, there is no path in A_v of length $|V(H)|$ with one end v , and hence there is no path in A_v of length $2|V(F)|$. Consequently, A_v admits a tree-decomposition of height at most m_2 and width at most ℓ_2 .

Now, since G' is two-connected and does not contain F as a minor, 2.6 implies that G' admits a tree-decomposition $\mathcal{T} = (T, (W_t : t \in V(T)))$ of height at most m_1 such that for every $t \in V(T)$, either $|W_t| \leq \ell_1 + 1$ or the torso of $\mathcal{T}$ at t is a cycle. If each $|W_t| \leq \ell$, then by combining this tree-decomposition with that of A_v for each $v \in V(G')$, we deduce that G admits a tree-decomposition of width at most ℓ such that every path of length more than $\max(m_1, m_2)$ in T contains an edge with adhesion of size one, and so the theorem holds. Thus, we assume that at least one torso of $\mathcal{T}$ is a cycle of length at least $\ell \geq |V(F)|$. Since G' is two-connected, there exists $X = \{x_1, \dots, x_n\} \subseteq V(G')$ with $n \geq \ell$, such that (reading subscripts modulo n):

- for each component D of $G' \setminus X$, there exists $i \in \{1, \dots, n\}$ such that x_i, x_{i+1} have neighbours in $V(D)$ and no other members of X have neighbours in $V(D)$; and
- for $1 \leq i \leq n$, either $x_i x_{i+1}$ is an edge of G' , or there is a component D of $G' \setminus X$ such that x_i, x_{i+1} have neighbours in $V(D)$.

Consequently, the same is true for G , except that there may be components D of $G \setminus X$ such that only one vertex of X has a neighbour in $V(D)$. Hence, for $1 \leq i \leq n$ there is a subgraph C_i of G with the following properties, again, reading subscripts modulo n (see Figure 18):

- $x_i, x_{i+1} \in V(C_i)$ and there is a path in C_i between x_i, x_{i+1} ;
- for $1 \leq i < j \leq n$, if $j \neq i + 1$ and $(i, j) \neq (1, n)$ then $C_i \cap C_j$ is null;
- $C_i \cap C_{i+1} = \{x_{i+1}\}$ for $1 \leq i \leq n$; and
- $C_1 \cup \dots \cup C_n = G$.

Choose n maximum such that there is such a set $X = \{x_1, \dots, x_n\}$ and $C_1, \dots, C_n$. It follows that for $1 \leq i \leq n$, either x_i, x_{i+1} are adjacent, or there are two paths of C_i between x_i, x_{i+1} with disjoint interiors. We claim that for $1 \leq i \leq n$, C_i has no path of length $|V(F)|^2$. Suppose that $i = 1$ say, and there is a path P in C_1 of length $|V(F)|^2$. There is a path in C_1 between x_1, x_2 , say Q . There is no path R of C_1 with $|V(R)| \geq |V(F)|$ with one end in Q and no other vertex in Q , since G does not contain H as a minor. (Let us call this fact the “forbidden path condition”.) If $P \cap Q$ is null, then the union of P and a path of C_1 between P, Q gives a path that violates the forbidden path condition, a contradiction. Thus $P \cap Q$ is non-null. If any component of $P \setminus V(Q)$ has at least $|V(F)| - 1$ vertices, then again the forbidden path condition is violated. So there is no such component. Let $|V(P \cap Q)| = k$; then $P \setminus V(Q)$ has at most $k + 1$ components, each with at most $|V(F)| - 2$ vertices, and so

$$|V(F)|^2 \leq |V(P)| \leq k + (k + 1)(|V(F)| - 2) < (k + 1)(|V(F)| - 1).$$

It follows that $k > |V(F)|$, and in particular $|V(Q)| > |V(F)|$. Hence every path of C_1 between x_1, x_2 has at least $|V(F)| + 1$ vertices. So x_1, x_2 are nonadjacent, and there are two paths of C_1 between

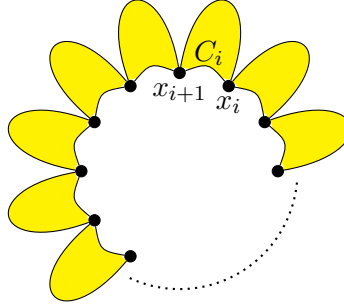


Figure 18: For 12.1.

x_1, x_2 with disjoint interiors, both of length at least $|V(F)|$, and again the forbidden path condition is violated. This proves that C_i has no path of length $|V(F)|^2$, for $1 \leq i \leq n$. Hence C_i admits a tree-decomposition of height at most m_2 and width at most ℓ_2 , for $1 \leq i \leq n$; and so it admits a tree-decomposition of height at most m_2 and width at most $\ell_2 + 1$, such that some bag contains both x_i, x_{i+1} . Consequently, G admits a rooted tree-decomposition of height at most $m_2 + 1$ such that the torso at the root is a cycle and all others have at most $\ell_2 + 2$ vertices, and the theorem holds. This proves 12.1. ■

13 Excluding caterpillars.

A *caterpillar* is a tree such that some path contains a neighbour of every vertex not in the path. We know a rough structure theorem for excluding a tree, namely 1.2, but what if the tree we are excluding is a caterpillar, is there a rough structure theorem for that? Just saying that path-width is bounded is not enough: it excludes no caterpillars, because all caterpillars have path-width at most one. But there is a rough structure theorem that we prove in this section. We first need some definitions.

For $d \geq 1$, let L_d be the caterpillar formed by taking a path with $d + 2$ vertices and adding d new vertices, one adjacent to each internal vertex of the path. (See Figure 19.) Every caterpillar is

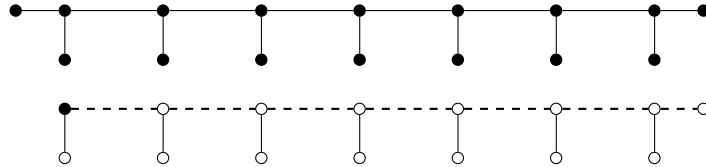


Figure 19: The caterpillar L_7 , and a one-rooted graph of type 7.

a minor of L_d for some d , so for our purposes it suffices to exclude L_d .

For $d \geq 1$, we say a one-rooted graph (H, v) has *type d* if H is a tree and can be obtained from a path with ends v, v' by adding d new vertices, all adjacent to distinct vertices of the path, and not adjacent to v' .

A *tree-partition* of a graph G is a pair $(T, (W_t : t \in V(T)))$, such that

- T is a rooted tree;
- the sets W_t ($t \in V(T)$) are pairwise disjoint subsets of $V(G)$ with union $V(G)$;
- for each edge $e = uv$ of G , let $s, t \in V(T)$ with $u \in W_s$ and $v \in W_t$; then one of s, t is an ancestor of the other.

Its *height* is the height of T .

We claim:

13.1 *For every $d \geq 1$, there exists ℓ_d such that every simple connected graph with no L_d minor admits a tree-partition $(T, (W_t : t \in V(T)))$ of height at most $d - 1$, such that for each $t \in V(T)$, if the distance in T between t and the root is less than $d - 1$ then $|W_t| \leq \ell_d$, and if this distance is $d - 1$ then $G[W_t]$ is a path or cycle.*

Proof. Choose k such that every graph with no L_d minor has tree-width less than k . We will prove, by induction on h , that for all h with $1 \leq h \leq d$, every simple connected graph with no L_h minor admits a tree-partition $(T, (W_t : t \in V(T)))$ of height at most $h - 1$, such that for each $t \in V(T)$, if the distance in T between t and the root is less than $h - 1$ then $|W_t| \leq k$, and if this distance is $h - 1$ then $G[W_t]$ is a path or cycle. Consequently, this will show that setting $\ell_d = k$ satisfies the theorem.

Every simple connected graph with no L_1 minor is a path or cycle, so the result holds when $h = 1$, taking $|V(T)| = 1$. We assume that $h \geq 2$, and the claim holds for $h - 1$. Now G contains L_h as a minor if and only if it has a subgraph that is a subdivision of L_h , since L_h has maximum degree three; and the same for L_{h-1} .

(1) *Either there are two vertex-disjoint subdivisions of L_{h-1} in G , or there exists $X \subseteq V(G)$ with $|X| \leq k$ such that there is no subdivision of L_{h-1} in $G \setminus X$.*

This is true by the Erdős-Pósa property of planar minors proved in [8], but the proof is short so we give it anyway. Take a tree-decomposition of G of width less than k . For each subdivision S of L_{h-1} in G , let T_S be the subtree induced on the set of $t \in V(T)$ with $W_t \cap V(S) \neq \emptyset$. If there exist subdivisions S, S' of L_{h-1} such that $T_S, T_{S'}$ are vertex-disjoint, then S, S' are vertex-disjoint. If not, then by the Helly property of subtrees of a tree, there exists $t \in V(T)$ such that $t \in V(T_S)$ for every subdivision S of L_{h-1} in G ; and then there is no subdivision of L_{h-1} in $G \setminus W_t$. This proves (1).

We also need:

(2) *Let S be a subdivision of L_{h-1} ; then for every $u \in V(S)$, the one-rooted graph (S, u) has a one-rooted subgraph of type $\lceil h/2 \rceil$.*

The proof is easy case-checking (using that $h \geq 2$) and we leave it to the reader.

Now for the inductive proof of the claim: let G be a connected simple graph with no L_h minor. Suppose that there are two vertex-disjoint subdivisions S, S' of L_{h-1} in G , and let P be a minimal path in G between them, with ends $u \in V(S)$ and $u' \in V(S')$. By (2), (S, u) has a one-rooted subgraph of type $\lceil h/2 \rceil$, and the same for (S', u') ; and joining these by P gives a subgraph that contains a subdivision of L_h , a contradiction. So there do not exist two vertex-disjoint subdivisions of L_{h-1} in G . By (1), there exists $X \subseteq V(G)$ with $|X| \leq k$ such that there is no subdivision of L_{h-1} in $G \setminus X$, and hence $G \setminus X$ has no L_{h-1} minor. From the inductive hypothesis, each component C of $G \setminus X$ admits a rooted tree-partition $(T_C, (W_t : t \in V(T_C)))$ of height at most $h - 2$, such that for each $t \in V(T_C)$, if the distance in T_C between t and the root is less than $h - 2$ then $|W_t| \leq k$, and if this distance is $h - 2$ then $G[W_t]$ is a path or cycle. Add a new vertex t_0 adjacent to the root of T_C for each C , making a tree T ; and define $W_{t_0} = X$. Then $(T, (W_t : t \in V(T)))$ satisfies the theorem. This proves 13.1. $\blacksquare$

14 The converses

As promised earlier, in this section we sketch the proof of the “rough converse” to 2.2, 2.3, 2.4, 2.7, 2.5, 12.1 and 13.1. Let $\text{SPP}_1 = \text{SP}_1$, and for $d \geq 2$ let SPP_d be obtained from the parallel doubling of the series doubling of SPP_{d-1} by adding an edge between its roots. (See Figure 20.) The *level* of an edge e of SPP_d is d if e joins the two roots of SPP_d , and otherwise e belongs to one of the four copies of SPP_{d-1} used to construct SPP_d , and its level is its level in that subgraph. Thus, SPP_3 has one edge of level 3, four edges of level two, and 16 with level one.

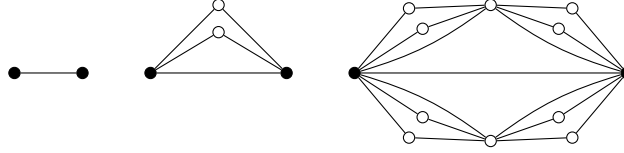


Figure 20: $\text{SPP}_1, \dots, \text{SPP}_3$.

The graph SPP_d is series-parallel and chordal, and every series-parallel graph is a minor of SPP_d for all large enough d . For 2.2, we need to show:

14.1 *For all $m \geq 0$ there is a series-parallel graph H , such that no simple two-connected graph with an H minor admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso is either outerplanar or innerplanar.*

That follows from:

14.2 *For all $m \geq 0$, no simple two-connected graph with an SPP_{2m+3} minor admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso is either outerplanar or innerplanar.*

Proof sketch. Now SPP_d is a 2-rooted graph; let H_d be the corresponding unrooted graph. We proceed by induction on m . The result is true when $m = 0$, since H_3 is neither innerplanar nor

outerplanar, so we assume that $m \geq 1$. Suppose there is a simple two-connected graph G with an H_{2m+3} minor that admits a rooted tree-decomposition with adhesion two, of height at most m , and such that each torso is either outerplanar or innerplanar. Every minor of G therefore admits a rooted tree-decomposition with all adhesions of size at most two, of height at most m , and such that each torso is either outerplanar or innerplanar. In particular, H_{2m+3} itself admits such a tree-decomposition $(T, (W_t : t \in V(T)))$ say; and since H_{2m+3} is two-connected, we may assume that all adhesions have size exactly two. Let $f_1, \dots, f_t$ be the edges of T incident with the root r_0 of T , and for $1 \leq i \leq t$, let $\{u_i, v_i\}$ be the adhesion at f_i . By choosing $(T, (W_t : t \in V(T)))$ with T minimal, it follows that $A_{f_i} \setminus B_{f_i} \neq \emptyset$ and $B_{f_i} \setminus A_{f_i} \neq \emptyset$, and so $\{u_i, v_i\}$ is a minimal cutset. Since H_{2m+3} is chordal, u_i, v_i are adjacent. Now $H_{2m+3}[A_{f_i}]$ is two-connected and admits a rooted tree-decomposition with adhesion two, of height at most $m - 1$, and such that each torso is either outerplanar or innerplanar; and so from the inductive hypothesis, it has no H_{2m+1} minor. If the edge $u_i v_i$ has level $k \geq 2$, then $H_{2m+3}[C \cup \{u_i, v_i\}]$ contains H_{k-1} as a subgraph, for each component C of $H_{2m+3} \setminus \{u_i, v_i\}$; and it follows that $u_i v_i$ has level at most $2m + 1$, for $1 \leq i \leq t$. But then $H_{2m+3}[W_{r_0}]$ contains H_3 as a subgraph, contradicting that it is either outerplanar or innerplanar. This proves 14.2. $\blacksquare$

What about the other converses? For 2.3 and 2.4, the same method works, just replacing the sequence SPP_d with an appropriate sequence of chordal outerplanar or chordal innerplanar graphs. For 2.7 we use SPP_d again, but we have to work a little harder, because there the adhesions have size at most three, and SPP_d is only two-connected. But again we can arrange that the adhesions are minimal cutsets, and so are cliques, and we can give them levels as before.

For a converse to 2.5, it suffices to show:

14.3 *For all $\ell, m \geq 1$ there is a fan H such that no simple two-connected graph G with an H minor admits a tree-decomposition $(T, (W_t : t \in V(T)))$ of height at most m such that for each $t \in V(T)$, either $|W_t| \leq \ell$ or the torso at t is a cycle.*

Proof sketch. To see this, we may assume that $\ell \geq 3$ by increasing ℓ if necessary. Choose n such that no graph with a path of length n admits a tree-decomposition of width at most ℓ and height at most m . Let H be a fan of length n . It suffices to show that H does not admit a decomposition as in the theorem. Suppose that it does, that is, it admits a tree-decomposition $\mathcal{T} = (T, (W_t : t \in V(T)))$ of height at most m such that for each $t \in V(T)$, either $|W_t| \leq \ell$ or the torso at t is a cycle. Since H is a chordal graph, any torso that is a cycle has length three, and hence has three vertices, and hence at most ℓ vertices. Thus $\mathcal{T}$ has width at most ℓ , and so H has no path of length n , a contradiction. This proves 14.3. $\blacksquare$

For 12.1, we show:

14.4 *For all $\ell, m \geq 1$ there is a stopsign H such that no simple connected graph G with an H minor admits a rooted tree-decomposition $\mathcal{T} = (T, (W_t : t \in V(T)))$ such that either*

- *$\mathcal{T}$ has height at most m , and one torso is a cycle, and all other torsos have at most $\ell + 1$ vertices; or*
- *$\mathcal{T}$ has width at most ℓ , and every path of T of length m contains an edge with adhesion of size 1.*

Proof sketch. Choose t such that no graph with a path of length t admits a tree-decomposition of width at most ℓ and height at most m . Let H be a stopsign made by identifying one end of a path of length $t + 1$ with a vertex of a cycle of length $t + 1$. As usual, it suffices to show that H does not admit a tree-decomposition as in the theorem. Suppose H admits a tree-decomposition $\mathcal{T}$ such that either:

- $\mathcal{T}$ has height at most m , and one torso is a cycle, and all other torsos have size at most $\ell + 1$; or
- $\mathcal{T}$ has width at most ℓ , and every path of T of length m contains an edge with adhesion of size 1.

In the second case, all cycles have bounded length, so if t is sufficiently large then H admits no such decomposition. In the first case, H admits a decomposition as shown in Figure 18: there are subgraphs $C_1, \dots, C_n$ for some $n \geq 3$, arranged in a circular order, each sharing one vertex with the next and otherwise disjoint, with union H , and every path in each C_i has length less than t . In particular, no C_i has a cycle of length at least $t + 1$, so the (unique) cycle of H contains all the vertices that belong to more than one of the C_i 's. But there is a path of H of length t disjoint from the cycle of H , so it is a subgraph of some C_i , which is impossible. This proves 14.4. ■

For a converse to 13.1, we need:

14.5 *For every $\ell, m \geq 0$, let $d(\ell, m) = 3(2(\ell + 1))^m$; then no simple connected graph with an $L_{d(\ell, m)}$ minor admits a tree-partition $(T, (W_t : t \in V(T)))$ of height at most m , such that for each $t \in V(T)$, if the distance in T between t and the root is less than m then $|W_t| \leq \ell$, and if this distance is m then $G[W_t]$ is a path or cycle.*

Proof sketch. It suffices to show that $L_{d(\ell, m)}$ does not admit such a tree-partition. This is true if $m = 0$, and if $m \geq 1$ it follows immediately from an inductive hypothesis on m , by deleting the bag of the root of T , using that if X is a set of vertices of size at most ℓ , then $L_{d(\ell, m)} \setminus X$ contains a copy of $L_{d(\ell, m-1)}$. This proves 14.5. ■

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