

Triangle-free graphs with no six-vertex induced path

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Abstract

The graphs with no five-vertex induced path are still not understood. But in the triangle-free case, we can do this and one better; we give an explicit construction for all triangle-free graphs with no six-vertex induced path. Here are two examples: the 16-vertex Clebsch graph, and the graph obtained from a complete bipartite graph by subdividing a perfect matching. We show that every connected triangle-free graph with no six-vertex induced path is an induced subgraph of one of these two (modulo some twinning and duplication).

1 Introduction

Graphs in this paper are without loops and multiple edges, and finite unless we say otherwise. If G is a graph and $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on X ; and we say that G *contains* H if some induced subgraph of G is isomorphic to H . If G has no induced subgraph isomorphic to H , we say G is *H -free*, and if \mathcal{C} is a set of graphs, G is *\mathcal{C} -free* if G is H -free for all $H \in \mathcal{C}$. We denote the k -vertex path graph by P_k . Our objective in this paper is to find a construction for all $\{P_6, K_3\}$ -free graphs.

Constructing all P_5 -free graphs remains open, although it has been heavily investigated, mostly because P_5 is one of the minimal graphs for which the Erdős-Hajnal conjecture [2, 3] is unsolved. It is also not clear whether we know how to construct all K_3 -free graphs (it depends on what exactly counts as a “construction”). But we can construct all $\{P_5, K_3\}$ -free graphs, and indeed all $\{P_6, K_3\}$ -free graphs, and the answer is surprisingly pretty.

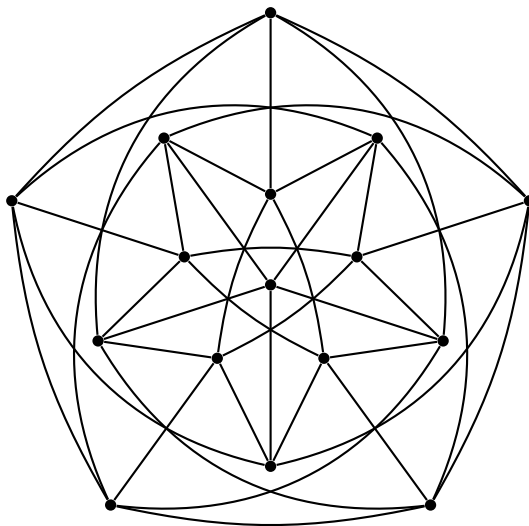


Figure 1: The Clebsch graph

The Clebsch graph, shown in figure 1, is the most interesting of the $\{P_6, K_3\}$ -free graphs. Here are three alternative definitions for it.

- Take the five-dimensional cube graph, and identify all pairs of opposite vertices.
- Take the elements of the field $GF(16)$, and say two of them are adjacent if their difference is a cube.
- Start with the Petersen graph; for each stable subset X of cardinality four (there are five such subsets) add a new vertex adjacent to the vertices in X ; and then add one more vertex adjacent to the five new vertices. (This third definition is the least symmetric but in practise we found it the most helpful).

For every edge uv of the Clebsch graph, the subgraph induced on the set of vertices nonadjacent to u, v is a three-edge matching, and from this it follows that the graph is P_6 -free. We say G is *Clebschian* if G is contained in the Clebsch graph.

There is another kind of $\{P_6, K_3\}$ -free graph that we need to discuss, the following. Take a complete bipartite graph $K_{n,n}$ with bipartition $\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\}$, and subdivide each edge $a_i b_i$; that is, for $1 \leq i \leq n$ we delete the edge $a_i b_i$, and add a new vertex c_i adjacent to a_i, b_i . This graph, H say, is $\{P_6, K_3\}$ -free, and we say a graph G is *climbable* if it is isomorphic to an induced subgraph of H for some n .

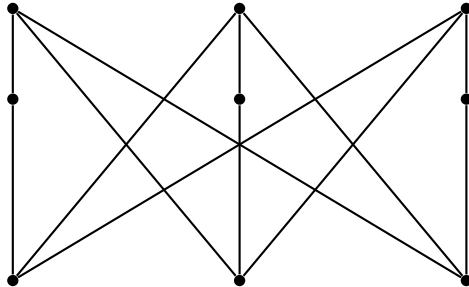


Figure 2: A climbable graph

Our aim is to prove something like “every $\{P_6, K_3\}$ -free G is either climbable or Clebschian”, but by itself this is not true. For instance, we have to assume G is connected, because otherwise the disjoint union of two Clebsch graphs would be a counterexample. But just assuming connectivity is not enough. For instance, if v is a vertex of a $\{P_6, K_3\}$ -free graph, then we could add a new vertex with the same neighbours as v , and the enlarged graph would still be $\{P_6, K_3\}$ -free. Let us say two vertices are *twins* if they are nonadjacent and have the same neighbour sets. Thus we need to assume that G has no twins. There are two other “thickening” operations of this kind that we define later.

Our main result is the following (although some definitions have not yet been given):

1.1 *Let G be a connected $\{P_6, K_3\}$ -free graph without twins. Then either*

- *G is Clebschian, climbable, or a V_8 -expansion; or*
- *G admits a nontrivial simplicial homogeneous pair.*

There has already been work on this and similar questions:

- In [7], Randerath, Schiermeyer and Tewes proved that every connected $\{P_6, K_3\}$ -free graph which is not 3-colourable, and in which no vertex dominates another, is an induced subgraph of the Clebsch graph.
- Brandstädt, Klemmt and Mahfud [1] proved that every $\{P_6, K_3\}$ -free graph can be decomposed in a certain way that implies that all such graphs have bounded clique-width.
- Lozin [6] gave a construction for all bipartite graphs not containing what he called a “skew star”, obtained from P_6 by adding one more vertex adjacent to its third vertex.

- Lokshantov, Vatshelle and Villanger [5] found a polynomial-time algorithm to find a stable set of maximum weight in a P_5 -free graph; and more recently Grzesik, Klimošová, Pilipczuk and Pilipczuk [4] claim to have the same for P_6 -free graphs.

2 An overview of the proof

The proof of 1.1 divides into a number of cases, and here we give a high-level survey of how it all works. We define some “thickening” operations, that when applied to $\{P_6, K_3\}$ -free graphs, will produce larger graphs that are still $\{P_6, K_3\}$ -free. One of the thickening operations involves substituting a bipartite P_6 -free graph for an edge, so we need to understand the bipartite P_6 -free graphs, and it is convenient to postpone this until 5.3. Except for that, it suffices to understand the $\{P_6, K_3\}$ -free graphs that cannot be built from a smaller graph by means of one of the thickening operations (let us temporarily call such graphs “unthickened”).

Let Q_5 be the graph that is the disjoint union of a P_3 and a P_2 ; and let V_8 be the graph obtained from a cycle of length eight by making opposite pairs of vertices adjacent. The proof of 1.1 is in two stages. First, we figure out the connected unthickened triangle-free graphs that contain Q_5 but not P_6 . One such graph is V_8 ; and it turns out every connected unthickened triangle-free graph that contain Q_5 but not P_6 is contained in V_8 .

Then, second, we find a construction for all unthickened connected $\{Q_5, K_3\}$ -free graphs. We prove that every such graph is either Clebschian, or climbable, or bipartite. (We figure out directly how to construct all bipartite Q_5 -free graphs in section 5.)

Our main approach is via the size of the largest induced matching in G , denoted by $\mu(G)$. If M is an induced matching, let $V(M)$ be the vertices incident with edges in M , and let N_M be the set of vertices not in $V(M)$ but adjacent to a vertex in $V(M)$. There are three easy but very useful observations (5.1 and 6.2): if G is $\{Q_5, K_3\}$ -free and M is a matching in G , then

- every vertex in N_M has exactly one neighbour in *every* edge of M ;
- if u, v in N_M are adjacent, then they are joined to opposite ends of each edge of M ;
- if u, v in N_M are nonadjacent, they are joined to the same end of all edges of M except at most two.

This tells us most when M is large and N_M is not stable; for instance, if G is unthickened, it tells us (shown in 9.1) that either $|M| \leq 4$ or N_M is stable. So two special cases need attention, when $\mu(G)$ is small, and when N_M is stable. We need to do the case when N_M is stable first, because we need some of those lemmas to handle the case $\mu(G) = 2$.

There are three essentially different ways to choose vertices of a n -cube pairwise with Hamming distance at most two (namely, choose vertices all with distance at most one from some given vertex; choose the vertices of a square; or choose the even vertices of a 3-cube). The case when N_M is stable breaks into three subcases correspondingly. The first subcase more-or-less implies that G is climbable, the second implies G is Clebschian, and the third implies that G contains the Petersen graph, which makes it easy to handle. This is all done in section 7.

Now we turn to the cases when $\mu(G)$ is small. The case $\mu(G) \leq 1$ is easy, and it turns out that the $\mu(G) = 2$ case is the most difficult, and we devote section 8 to it. To analyze the graphs G with $\mu(G) = 2$, we need to look at another parameter, $\nu(G)$, the size of the largest antimatching in G

(this is a bipartite induced subgraph whose bipartite complement is a perfect matching); and handle the cases $\mu(G) = 2, \nu(G) > 2$ and $\mu(G) = 2, \nu(G) \leq 2$ separately (in 8.1, 8.2 respectively). From now on we can assume $\mu(G) \geq 3$, and the proof gets easier.

If N_M is not stable, and G is unthickened, then $|M| \leq 4$, and since we have already handled the cases when $|M| \leq 2$, we just have to figure out what happens when $3 \leq |M| \leq 4$. In both cases G turns out to be Clebschian (shown in 9.2, 9.3). That will complete the proof of the main theorem.

One annoyance is that the thickening operations are designed to not introduce P_6 , but they might introduce Q_5 . Thus the construction that we give for all unthickened $\{Q_5, K_3\}$ -free graphs does not directly give a construction for all $\{Q_5, K_3\}$ -free graphs. This can easily be remedied, and we do so in section 10.

3 Thickening

A k -path in G means an induced subgraph isomorphic to P_k . Two induced subgraphs H, J of G are *anticomplete* to each other if $V(H) \cap V(J) = \emptyset$, and there is no edge of G between $V(H)$ and $V(J)$. They are *complete* to each other if $V(H) \cap V(J) = \emptyset$ and every vertex in H is adjacent to every vertex in J . We use the same terminology for subsets of $V(G)$ instead of induced subgraphs, and for single vertices instead of subsets. We also say H *misses* J if H is anticomplete to J .

As we mentioned, adding twins to a graph with no 6-path will not introduce a 6-path. Let us say a subset $X \subseteq V(G)$ is a *homogeneous set* if X is stable and every vertex of G not in X is either complete or anticomplete to X . (The requirement that X is stable is not the standard definition, but we add it here to make the concept useful for constructing triangle-free graphs.) If G has a homogeneous set X of cardinality at least two, then G is obtained from a smaller graph by adding twins; so let us say a homogeneous set X of G is *nontrivial* if $|X| \geq 2$.

There is another, similar concept, ‘‘homogeneous pairs’’. A pair (A, B) of disjoint stable subsets of $V(G)$ is a *homogeneous pair* if every vertex not in $A \cup B$ is either complete or anticomplete to A , and either complete or anticomplete to B . (Again, the requirement that A, B are stable is not standard.) It is *nontrivial* if $|A| + |B| \geq 3$ and there is an edge between A and B , and $A \cup B \neq V(G)$ (this again is slightly nonstandard; the requirement of an edge between A, B is not usual, but convenient for us).

Let (A, B) be a nontrivial homogeneous pair in G , let $J = G[A \cup B]$, and let us identify the vertices of A into one vertex a , and identify those in B to one vertex b (and thus a, b will be adjacent). The graph we produce, H say, is contained in G (because there is an edge of G between A and B), and so is J , and from the hypotheses they both have fewer vertices than G . We can regard G as being constructed from H by ‘‘substituting J for the edge ab ’’.

This operation, applied to two triangle-free graphs H, J , produces a triangle-free graph G , but it might well introduce 6-paths; we need to restrict it further to make it safe. A homogeneous pair (A, B) of G is *simplicial* if every vertex in $V(G) \setminus (A \cup B)$ with a neighbour in A is adjacent to every vertex in $V(G) \setminus (A \cup B)$ with a neighbour in B . Then, with H, J as above, if both H, J are $\{P_6, K_3\}$ -free, then G is also $\{P_6, K_3\}$ -free (proved below); and so nontrivial simplicial homogeneous pairs can be used safely to construct larger $\{P_6, K_3\}$ -free graphs from smaller ones. A special case is when there are no vertices in $V(G) \setminus (A \cup B)$ with a neighbour in B ; in this case we say the homogeneous pair is *pendant*.

To regard this as a construction, we need to understand how to construct the building blocks,

and in particular, how to build all bipartite P_6 -free graphs. That issue is addressed in section 5.

Let us prove the claim above.

3.1 *Let (A, B) be a homogeneous pair in a graph G , and let H be obtained from G by identifying the vertices of A into one vertex a , and identifying those in B to one vertex b , where a, b are adjacent. Let $J = G[A \cup B]$. If H, J are both $\{P_6, K_3\}$ -free then so is G .*

Proof. Since A, B are stable, and the edge ab is not in a triangle of H , it follows that G is K_3 -free. Let C be the set of vertices in $V(G) \setminus (A \cup B)$ that are complete to A and anticomplete to B , and define D similarly with A, B exchanged. Suppose that there is a 6-path P in G . Define $A' = A \cap V(P)$, and define B', C', D' similarly. Since there is no 6-path in H , it follows that $|A' \cup B'| \geq 2$; and since no two vertices of P are twins in P , it follows that $A', B' \neq \emptyset$.

Suppose that A', B' are anticomplete. Each vertex of A' has a neighbour in P and hence in C' ; and so $|D'| \leq 1$. Since each vertex of B' has a neighbour in P , it follows that $|D'| = 1$, and similarly $|C'| = 1$. But then $|A'| = |B'| = 1$, and since the vertices in $C' \cup D'$ have no more neighbours in P , it follows that $|V(P)| = 4$, a contradiction.

Consequently there exist adjacent vertices $u \in A'$ and $v \in B'$. Hence not both C', D' are nonempty, so we may assume that $D' = \emptyset$. Since J is P_6 -free, $C' \neq \emptyset$; and so v has only one neighbour in A . Hence v is an end of P , and indeed, so is every vertex in B' , since each has a neighbour in A' . In particular $|B'| \leq 2$. If $A' = \{u\}$ then $B' = \{v\}$, contradicting that H is P_6 -free; so there exists $u' \neq u \in A'$. So $|C'| = 1$, and the vertex in C' , w say, has no neighbours in P except u, u' . Consequently $V(P) \subseteq A' \cup B' \cup \{w\}$. Since $|B'| \leq 2$, it follows that $|A'| \geq 3$, contradicting that w has degree two in P . Thus there is no such P . This proves 3.1. ■

The second way we use homogeneous pairs is to construct a class of $\{P_6, K_3\}$ -free graphs, as follows. A bipartite graph J with bipartition (A, B) is an *antisubmatching* (relative to the given bipartition) if every vertex in A has at most one nonneighbour in B , and vice versa. It is an *antimatching* if every vertex in A has exactly one nonneighbour in B and vice versa. Let V_8 be the graph with eight vertices a_1, \dots, a_8 , in which distinct a_i, a_j are adjacent if $j - i = 1, 4$ or 7 modulo 8 . For $i = 1, 2$, let J_i be an antisubmatching relative to (A_i, B_i) , with at least one edge; and let G be obtained from V_8 (with vertices a_1, \dots, a_8 as above) by substituting (A_1, B_1) for $a_1 a_5$, substituting (A_2, B_2) for $a_3 a_7$, and possibly deleting some of a_2, a_4, a_6, a_8 . We say G is a V_8 -*expansion*. Again, it is easy to check that every V_8 -expansion is $\{P_6, K_3\}$ -free.

Together, these two will suffice to account for the $\{P_6, K_3\}$ -free graphs that contain Q_5 . We will show the following (in the next section).

3.2 *Let G be a connected $\{P_6, K_3\}$ -free graph without twins that contains Q_5 . Then either*

- *G is a V_8 -expansion; or*
- *G admits a nontrivial simplicial homogeneous pair.*

4 Graphs that contain Q_5

We recall that Q_5 is the disjoint union of a P_3 and a P_2 , that is, the graph obtained from P_6 by deleting one of its two middle vertices. Certainly Q_5 -free graphs are P_6 -free, and it will turn out

that the converse is “nearly” true; and in particular the two classes described earlier (Clebschian graphs and climbable graphs) are Q_5 -free. In this section we describe all triangle-free graphs that are P_6 -free and not Q_5 -free. Some notation: we write $p_1-p_2-\cdots-p_k$ to denote a path with vertices p_1, \dots, p_k in order.

Let Q_7 be the graph obtained from P_7 by adding an edge between the second and fifth vertex.

4.1 *Let G be a connected $\{P_6, K_3\}$ -free graph without twins that contains Q_5 . Then G contains Q_7 .*

Proof. Let R_6 be obtained from P_5 by adding a new vertex adjacent to the second vertex of P_5 ; and let S_6 be obtained from P_5 by adding a new vertex adjacent to the first and third vertices of P_5 .

(1) *G contains one of R_6, S_6 .*

Since G contains Q_5 , there is a 3-path P and a 2-path Q in G that are anticomplete. Since G is connected, there is a path R of $G \setminus V(P \cup Q)$ that misses neither of P, Q . By choosing R minimal, it follows that $G[V(Q) \cup V(R)]$ is a path, and contains a 3-path S of G such that one of its ends, s say, has a neighbour in P , and its other vertices do not. If s is adjacent to the middle vertex of P then G contains R_6 ; if s is adjacent to both ends of P then G contains S_6 , and if s is adjacent to just one end of P , then G contains P_6 , which is impossible. This proves (1).

(2) *If G contains R_6 then G contains Q_7 .*

Since G contains R_6 , there is a 4-path $p_1-\cdots-p_4$, and two vertices a_1, a_2 both adjacent to p_1 and not to p_2, p_3, p_4 . Since a_1, a_2 are not twins, there is a vertex v adjacent to one of them and not the other, say v is adjacent to a_1 and not to a_2 . If v is adjacent to either of p_3, p_4 , then the path $a_2-p_1-a_1-v$ can be extended by $v-p_3-p_4$ or $v-p_4-p_3$, giving P_6 , a contradiction. Thus v is nonadjacent to both p_3, p_4 . If it is also nonadjacent to p_2 , then $v-a_1-p_1-p_2-p_3-p_4$ is a 6-path, a contradiction. So v is adjacent to p_2 , and so G contains Q_7 . This proves (2).

(3) *If G contains S_6 then G contains Q_7 .*

Since G contains S_6 , there is a 3-path $p_1-p_2-p_3$, two more vertices a_1, a_2 adjacent to p_1 and not to p_2, p_3 , and a vertex p_0 adjacent to a_1, a_2 and not to p_1, p_2, p_3 . Since a_1, a_2 are not twins, we may assume there is a vertex v adjacent to a_1 and not to a_2 . Thus v is nonadjacent to p_0, p_1 . If v is adjacent to one of p_2, p_3 , then the path $a_2-p_0-a_1-v$ can be extended to a 6-path via one of $v-p_2-p_3$ or $v-p_3-p_2$, a contradiction; so v is nonadjacent to p_2, p_3 , and G contains Q_7 . This proves (3).

From (1), (2) and (3), the result follows. ■

Now we can prove 3.2, which we restate:

4.2 *Let G be a connected $\{P_6, K_3\}$ -free graph without twins that contains Q_5 . Then either*

- *G is a V_8 -expansion; or*
- *G admits a nontrivial simplicial homogeneous pair.*

Proof. By 4.1, G contains Q_7 , and so there is a pair of disjoint sets $A_1, A_5 \subseteq V(G)$, both stable, with $|A_1|, |A_5| \geq 2$, such that $G[A_1 \cup A_5]$ is a connected bipartite graph and A_1 is not complete to A_5 ; and a 3-path with vertices $b_6-b_7-b_3$, such that b_6 is complete to A_5 (and consequently has no neighbours in A_1 , since G is triangle-free), and b_7, b_3 have no neighbours in A_1, A_5 . (The reason for the subscript numbering will emerge later. Throughout we read subscripts modulo 8.) Choose A_1, A_5 maximal with this property. Define A_2, A_4, A_6, A_8 by:

- let A_2 be the set of all vertices of G that are complete to $A_1 \cup \{b_3, b_6\}$;
- let A_4 be the set complete to $A_5 \cup \{b_3\}$;
- let A_6 be the set complete to $A_5 \cup \{b_7\}$ (thus, $b_6 \in A_6$); and
- let A_8 be the set complete to $A_1 \cup \{b_7\}$.

(1) *Every vertex of G with a neighbour in $A_1 \cup A_5$ belongs to one of $A_1, A_2, A_4, A_5, A_6, A_8$.*

Let $v \in V(G) \setminus (A_1 \cup A_5)$ with a neighbour in $A_1 \cup A_5$. For $i = 1, 5$ let A'_i be the set of neighbours of v in A_i . Suppose first that $A_1 \neq A'_1 \neq \emptyset$. Since $G[A_1 \cup A_5]$ is connected, there is a vertex $a_5 \in A_5$ with a neighbour $a'_1 \in A'_1$ and a neighbour $a_1 \in A_1 \setminus A'_1$. Since the path $a_1-a_5-a'_1-v$ cannot be extended to a 6-path, it follows that v is nonadjacent to b_3, b_7 ; and since $v-a'_1-a_5-b_6-b_7-b_3$ is not a 6-path, v is adjacent to b_6 . Consequently $A'_5 = \emptyset$, and v can be added to A_5 , contradicting the maximality of A_5 . This proves that either $A'_1 = A_1$ or $A'_1 = \emptyset$.

Now suppose that $A_5 \neq A'_5 \neq \emptyset$, and consequently $A'_1 = \emptyset$ and v is nonadjacent to b_6 . From the maximality of A_1 , v is adjacent to one of b_3, b_7 ; but there exists $a_1 \in A_1$ with a neighbour $a'_5 \in A'_5$ and a neighbour $a_5 \in A_5 \setminus A'_5$, and the path $a_5-a_1-a'_5-v$ can be extended to a 6-path via $v-b_3-b_7$ or $v-b_7-b_3$, a contradiction. This proves that either $A'_5 = A_5$ or $A'_5 = \emptyset$.

Suppose that $A'_1 \neq \emptyset$, and so $A'_1 = A_1$ and $A'_5 = \emptyset$. If v is adjacent to b_7 then $v \in A_8$, so we assume not. Now we must decide its adjacency to b_6 and b_3 . If v is nonadjacent to both b_6, b_3 , there is a 6-path $v-a_1-a_5-b_6-b_7-b_3$ (where $a_1 \in A_1$ and $a_5 \in A_5$ are adjacent), a contradiction. If v is adjacent to b_6 and not to b_3 , then v can be added to A_5 , contrary to the maximality of A_5 . If v is adjacent to b_3 and not to b_6 , choose $a_1 \in A_1$ and $a_5 \in A_5$ nonadjacent; then there is a 6-path $a_1-v-b_3-b_7-b_6-a_5$, a contradiction. Thus v is adjacent to both b_3, b_6 , and so $v \in A_2$, as required.

Finally suppose that $A'_5 \neq \emptyset$, and so $A'_5 = A_5$ and $A'_1 = \emptyset$, and v is nonadjacent to b_6 . From the maximality of A_1 , v is adjacent to one of b_3, b_7 , and hence belongs to one of A_4, A_6 . This proves (1).

Choose $A_3, A_7 \subseteq V(G)$ disjoint from $A_1, A_2, A_4, A_5, A_6, A_8$ and from each other, with $b_3 \in A_3$ and $b_7 \in A_7$, maximal such that

- $A_3 \cup A_7$ is anticomplete to $A_1 \cup A_5$;
- A_7 is complete to b_6 ; and
- the graph $G[A_3 \cup A_7]$ is connected.

(2) *For $1 \leq i \leq 8$, A_i is stable, complete to A_{i+1} , and anticomplete to A_{i+2}, A_{i+3} .*

There are many pairs to check, and first we check the sets and pairs not involving A_3, A_7 . A_1

is stable and complete to A_2 by definition of A_2 , and anticomplete to A_4 since every vertex in A_1 has a neighbour in A_5 which is complete to A_4 . A_2 is stable and anticomplete to A_4 since $A_2 \cup A_4$ is complete to b_3 , and anticomplete to A_5 since A_2 is complete to b_6 . A_4 is complete to A_5 by definition, and stable and anticomplete to A_6 since $A_4 \cup A_6$ is complete to $A_5 \neq \emptyset$. A_5 is stable and complete to A_6 by definition, and anticomplete to A_8 since A_8 is complete to A_1 and every vertex in A_5 has a neighbour in A_1 . A_6 is stable and anticomplete to A_8 since $A_6 \cup A_8$ is complete to b_7 ; and A_6 is anticomplete to A_1 since every vertex in A_1 has a neighbour in A_5 . A_8 is complete to A_1 by definition, and so stable and anticomplete to A_2 . That does the sets and pairs not involving A_3, A_7 .

Now A_7 is anticomplete to A_5, A_1 by definition, and stable since A_7 is complete to b_6 . Also A_3 is anticomplete to A_1, A_5 by definition. Suppose that A_3 is not stable, let $u, v \in A_3$ be adjacent, and choose $a_7 \in A_7$ adjacent to u (and hence not to v). Since u, v have neighbours in A_7 , they are nonadjacent to b_6 . Choose $a_5 \in A_5$ and $a_1 \in A_1$ adjacent; then $v-u-a_7-b_6-a_5-a_1$ is a 6-path, a contradiction. Thus A_3 is stable.

Suppose that A_6 is not complete to A_7 , and choose $a_6 \in A_6$ with a nonneighbour in A_7 . Since a_6 is adjacent to b_7 , and the graph $G[A_3 \cup A_7]$ is connected and A_3 is stable, there exists $a_3 \in A_3$ adjacent to a neighbour a_7 and a nonneighbour a'_7 of a_6 in A_7 . Choose $a_1 \in A_1$ and $a_5 \in A_5$, adjacent; then $a_1-a_5-a_6-a_7-a_3-a'_7$ is a 6-path, a contradiction. Thus A_6 is complete to A_7 . Similarly, since A_8 is complete to b_7 , it follows that A_8 is complete to A_7 ; and similarly A_3 is complete to A_2, A_4 . Since every vertex in A_3 has a neighbour in A_7 , it follows that A_3 is anticomplete to $A_6 \cup A_8$, and similarly A_7 is anticomplete to $A_2 \cup A_4$. This proves (2).

(3) A_2 is complete to A_6 , and A_4 is complete to A_8 .

Choose $a_1 \in A_1$ and $a_5 \in A_5$, nonadjacent. If $a_2 \in A_2$ is nonadjacent to $a_6 \in A_6$, then $a_1-a_2-b_3-b_7-a_6-a_5$ is a 6-path, a contradiction. If $a_4 \in A_4$ is nonadjacent to $a_8 \in A_8$, $a_1-a_8-b_7-b_3-a_4-a_5$ is a 6-path, a contradiction. This proves (3).

Thus we have a certain symmetry between the sets A_1, \dots, A_8 . It is not perfect, because A_2, A_4, A_8 might be empty (the other five sets are nonempty); and the bipartite graph between the pair A_1, A_5 contains a P_4 , and between A_2, A_6 and A_4, A_8 it is complete. Thus for instance, if we prove something about A_8 , we cannot deduce the same for A_2 (using the symmetry that exchanges A_i with A_{10-i} for $i = 2, 3, 4$) until we have proved that $A_4 \neq \emptyset$. Let A_0 be the set of vertices of G not in any of the sets A_1, \dots, A_8 .

(4) A_0 is anticomplete to A_1, A_3, A_5, A_7 .

Let $v \in A_0$. It is anticomplete to $A_1 \cup A_5$ by (1); and anticomplete to A_7 since if not it could be added to A_3 , contrary to the maximality of A_3 . If v has a neighbour $a_3 \in A_3$, choose $a_7 \in A_7$ adjacent to a_3 , and choose $a_1 \in A_1$ and $a_5 \in A_5$, adjacent. From the maximality of A_7 , v is nonadjacent to b_6 ; and so $v-a_3-a_7-b_6-a_5-a_1$ is a 6-path, a contradiction. This proves (4).

(5) We may assume that $A_8 \neq \emptyset$, and consequently the graph $G[A_1 \cup A_5]$ is an antismatching.

If A_2, A_8 are both empty, then (A_5, A_1) is a nontrivial pendant homogeneous pair. If A_4, A_8 are both empty, then $(A_1 \cup A_3, A_5 \cup A_7)$ is a nontrivial simplicial homogeneous pair. Thus we may

assume that either $A_8 \neq \emptyset$ or both $A_2, A_4 \neq \emptyset$.

Suppose that $A_8 = \emptyset$; then $A_2, A_4 \neq \emptyset$, and we may relabel the sets, exchanging A_i with A_{10-i} for $i = 2, 3, 4$. Thus we may assume that $A_8 \neq \emptyset$. Suppose that there exist $a_1, a'_1 \in A_1$ and $a_5, a'_5 \in A_5$ such that a_1, a_5 are adjacent, but the other three pairs are nonadjacent. Choose $a_8 \in A_8$. Then $a'_1-a_8-a_1-a_5-b_6-a'_5$ is a 6-path, a contradiction. Now suppose that some $a'_1 \in A_1$ is nonadjacent to two vertices $a_5, a'_5 \in A_5$. Since a_5, a'_5 are not twins, there is a vertex adjacent to exactly one of them, and by (1) it belongs to A_1 ; so we may assume there exists $a_1 \in A_1$ adjacent to a_5 and not to a'_5 , contrary to what we just proved. Similarly every vertex in A_5 is nonadjacent to at most one vertex in A_1 , and so $G[A_1 \cup A_5]$ is an antisubmatching. This proves (5).

(6) For $i = 1, 3, 5, 7$, if A_i is not complete to A_{i+4} , and $A_{i+1}, A_{i+3} \neq \emptyset$, then every vertex in A_0 is complete or anticomplete to $A_{i+1} \cup A_{i+3}$, and consequently $|A_{i+1}|, |A_{i+3}| = 1$.

Suppose that $a_{i+1} \in A_{i+1}$ and $a_{i+3} \in A_{i+3}$, and $v \in A_0$ is adjacent to a_{i+1} and not to a_{i+3} . Choose $a_{i+4}, a'_{i+4} \in A_{i+4}$ and $a_i \in A_i$ such that a_i is adjacent to a_{i+4} and not to a'_{i+4} . (This is possible since A_i is not complete to A_{i+4} and $G[A_i \cup A_{i+4}]$ is connected.) Then $v-a_{i+1}-a_i-a_{i+4}-a_{i+3}-a'_{i+4}$ is a 6-path, a contradiction. Similarly if $v \in A_0$ has a neighbour in A_{i+3} then it is complete to A_{i+1} . Thus v is complete or anticomplete to $A_{i+1} \cup A_{i+3}$. In particular, A_{i+1} is a homogeneous set, and so $|A_{i+1}| = 1$, and the same for A_{i+3} . This proves (6).

In particular, from (6) it follows that $|A_6| = |A_8| = 1$, and every vertex in A_0 is complete or anticomplete to $A_6 \cup A_8$. Let $A_8 = \{b_8\}$. Let C_7 be the set of vertices in A_0 that are complete to $A_6 \cup A_8$, let C_3 be the set of vertices in A_0 with a neighbour in $A_2 \cup A_4$, and let $C_0 = A_0 \setminus (C_3 \cup C_7)$. (It follows that $C_3 \cap C_7 = \emptyset$, since every vertex in $A_2 \cup A_4$ has a neighbour in $A_6 \cup A_8$.) Once again, we mention that A_6, A_8 are nonempty, but A_2, A_4 may be empty.

(7) We may assume that $A_4 \neq \emptyset$, and consequently $C_0 = \emptyset$, and $|A_3| = |A_7| = 1$.

For suppose that $A_2, A_4 = \emptyset$, and so $C_3 = \emptyset$. We claim that C_0 is stable; for if not, there is a 3-path $p_1-p_2-p_3$ where $p_3 \in C_7$ and $p_1, p_2 \in C_0$, and this extends to a 6-path via $p_3-b_6-a_5-a_1$, choosing $a_5 \in A_5$ and $a_1 \in A_1$ adjacent. Thus C_0 is stable, and so $(C_7 \cup A_7, C_0 \cup A_3)$ is a pendant homogeneous pair; we may assume it is not nontrivial, and hence $A_0 = \emptyset$, and $|A_3| = |A_7| = 1$. But then G is a V_8 -expansion.

Thus we may assume that one of $A_2, A_4 \neq \emptyset$, and from the symmetry (exchanging A_8 with A_6 , and exchanging A_i with A_{6-i} for $i = 1, 2$) we may assume that $A_4 \neq \emptyset$. If there is a vertex in C_0 , then since G is connected, there is an edge uv with $u \in C_0$ and $v \in C_3 \cup C_7$. But this extends to a 6-path, as follows. Choose $w \in A_2 \cup A_4 \cup A_6 \cup A_8$ adjacent to v , with $w = b_6$ if possible. It follows that $w \neq b_8$, by (6). If $w = b_6$, choose $a_4 \in A_4$; then $u-v-w-b_7-b_3-a_4$ is a 6-path. Similarly if $w \in A_4$, $u-v-w-b_3-b_7-b_6$ is a 6-path; and if $w \in A_2$, $u-v-w-b_3-b_7-b_8$ is a 6-path. Thus $C_0 = \emptyset$.

Now suppose that A_3 is not complete to A_7 . Then by (6), $|A_4| = 1$, and every vertex in A_0 is complete or anticomplete to $A_4 \cup A_6$, and therefore to $A_4 \cup A_6 \cup A_8$. No vertex is complete to $A_4 \cup A_8$ since this set is not stable, and therefore every vertex in A_0 is anticomplete to $A_4 \cup A_6 \cup A_8$. Similarly they are all anticomplete to A_2 , so $A_0 = \emptyset$. Moreover $|A_2| \leq 1$ (because there are no twins), and so G is a V_8 -expansion. We may therefore assume that A_3 is complete to A_7 . Consequently A_3 is a homogeneous set, so $|A_3| = 1$, and the same for A_7 . This proves (7).

(8) C_3 is complete to C_7 , and $|C_7| \leq 1$.

Suppose that $c_3 \in C_3$ and $c_7 \in C_7$ are nonadjacent. If there exists $a_4 \in A_4$ adjacent to c_3 , then $c_3-a_4-b_3-b_7-b_6-c_7$ is a 6-path; and similarly if there exists $a_2 \in A_2$ adjacent to c_3 then $c_3-a_2-b_3-b_7-b_8-c_7$ is a 6-path, in both cases a contradiction. Thus C_7 is complete to C_3 , so C_7 is a homogeneous set, and therefore $|C_7| \leq 1$. This proves (8).

If C_3 is not complete to A_4 , then $A_2 = \emptyset$ by (6); choose $c_3 \in C_3$ and $a_4 \in A_4$, nonadjacent. Since we may assume that (A_4, C_3) is not a nontrivial pendant homogeneous pair, it follows that $C_7 \neq \emptyset$; let $C_7 = \{c_7\}$. But then $c_3-c_7-b_6-b_7-b_3-a_4$ is a 6-path. Thus C_3 is complete to A_4 . Hence $|C_3| \leq 1$, and $|A_4| = 1$, and similarly $|A_2| \leq 1$. But then the graph $G[C_3 \cup C_7 \cup \{b_3, b_7\}]$ is an antisubmatching, relative to the bipartition $(C_7 \cup \{b_7\}, A_3 \cup C_3)$; and so G is a V_8 -expansion. This proves 4.2. \blacksquare

5 Bipartite Q_5 -free graphs

In view of 4.2 and 4.1, it suffices to understand $\{Q_5, K_3\}$ -free graphs, and that is the content of the remainder of the paper. In this section we handle the bipartite case, which is very different from the non-bipartite case. In particular, Q_5 is isomorphic to its own ‘‘bipartite complement’’, and so we should expect to prove something invariant under taking bipartite complements. (The *bipartite complement* of a graph G with a given bipartition (A, B) is the graph H with the same bipartition (A, B) , in which for $a \in A$ and $b \in B$, a, b are adjacent in exactly one of G, H .)

An *induced matching* in G is a set M of edges of G , pairwise vertex-disjoint, such that for all distinct $e, f \in M$, no end of e is adjacent in G to an end of f . We denote by $V(M)$ the set of vertices incident with edges in M , and we denote by $\mu(G)$ the cardinality of the largest induced matching in G . We remark, and leave the reader to prove, that:

5.1 *If G is $\{Q_5, K_3\}$ -free, and M is an induced matching in G , then every vertex in $V(G) \setminus V(M)$ with a neighbour in $V(M)$ has a neighbour in each edge of M .*

Let (A, B) be a bipartition of a graph G , and let $A' \subseteq A$ and $B' \subseteq B$. We say the pair (A', B') is *matched* if every vertex in A' has exactly one neighbour in B' and vice versa; and *antimatched* if every vertex in A' has exactly one nonneighbour in B' and vice versa. If (A', B') is matched or antimatched, it follows that $|A'| = |B'|$.

A *half-graph* is a bipartite graph with no induced two-edge matching; or equivalently, a graph that admits a bipartition $(\{a_1, \dots, a_m\}, \{b_1, \dots, b_n\})$ such that for all i, i' with $1 \leq i \leq i' \leq m$ and all j, j' with $1 \leq j \leq j' \leq n$, if a_i, b_j are adjacent then $a_{i'}, b_{j'}$ are adjacent. (This equivalence is well-known, but here is a proof that the first implies the second. Let (A, B) be a bipartition of a bipartite graph with no induced two-edge matching, and choose $a \in A$ with maximum degree; then all its nonneighbours in B have degree zero. This proves that if $A \neq \emptyset$ then either some vertex in B has degree zero, or some vertex in A is adjacent to all of B ; and the claim follows by induction on $|V(G)|$.) We need an extension of this.

We say a graph G is a *half-graph expansion* if there is a bipartition (A, B) of G , and partitions (A_1, \dots, A_n) of A and (B_1, \dots, B_n) of B (where some of $A_1, \dots, A_n, B_1, \dots, B_n$ may be empty), with the following properties:

- for $1 \leq i < j \leq n$, A_i is complete to B_j , and A_j is anticomplete to B_i ;
- for $1 \leq i \leq n$, either $|A_i \cup B_i| = 1$ or (A_i, B_i) is matched or antimatched.

It is easy to see that every half-graph expansion is Q_5 -free. We prove the converse:

5.2 *Every Q_5 -free bipartite graph is a half-graph expansion.*

Proof. Let G be Q_5 -free, and let (A, B) be a bipartition of G . Let us say a *box* is a maximal subset $C \subseteq V(G)$ such that either $|C| = 1$, or $|C| \geq 4$ and $(A \cap C, B \cap C)$ is matched or antimatched.

(1) *Every two distinct boxes are disjoint.*

Let C, C' be distinct boxes, and suppose that $C \cap C' \neq \emptyset$. If $|C| = 1$ then $C \subseteq C'$, and so $C = C'$ from the maximality of C , a contradiction. Consequently $|C|, |C'| \geq 4$, and we may assume that $|C| \leq |C'|$. Since G is Q_5 -free, so is its bipartite complement, and since what we want to prove is invariant under taking bipartite complements, we may assume (taking bipartite complements if necessary) that $(A \cap C, B \cap C)$ is matched. Let $A \cap C = \{a_1, \dots, a_k\}$ and $B \cap C = \{b_1, \dots, b_k\}$ where a_i, b_j are adjacent if and only if $i = j$. It follows that $k \geq 2$.

We may assume that $C' \cap C \cap B \neq \emptyset$; let $b_k \in C'$ say. Suppose that $(C' \setminus C) \cap B$ is also nonempty, and choose $d \in (C' \setminus C) \cap B$.

Suppose first that d is nonadjacent to a_k . Since $d \notin C$, 5.1 implies that d is anticomplete to $\{a_1, \dots, a_k\}$. Since $(A \cap C', B \cap C')$ is matched or antimatched, there exists $c \in A \cap C'$ adjacent to d and not to b_k . Consequently $c \notin \{a_1, \dots, a_k\}$, and so again 5.1 implies that c is anticomplete to $\{b_1, \dots, b_k\}$. But then $((A \cap C) \cup \{c\}, (B \cap C) \cup \{d\})$ is matched, contradicting that C is a box.

This proves that d, a_k are adjacent. By 5.1 d is complete to $\{a_1, \dots, a_k\}$. Since $(A \cap C', B \cap C')$ is matched or antimatched, there exists $c \in A \cap C'$ adjacent to b_k and not to d . Consequently $c \notin \{a_1, \dots, a_k\}$, and 5.1 implies that c is complete to $\{b_1, \dots, b_k\}$. If $k \geq 3$, the 3-path a_1-d-a_2 misses the edge cb_3 , a contradiction. So $k = 2$, and $(\{a_1, a_2, c\}, \{b_1, b_2, d\})$ is antimatched, contradicting that C is a box.

We have shown that $(C' \setminus C) \cap B = \emptyset$; and since $|C| \leq |C'|$, it follows that $C \cap B = C' \cap B$. Since $C \neq C'$, some vertex $v \in C' \cap A$ does not belong to C ; but then by 5.1 v is complete or anticomplete to $C \cap B$, contradicting that $(A \cap C', B \cap C')$ is matched or antimatched and $|B \cap C'| = |C'|/2 \geq 2$. This proves (1).

(2) *If C, C' are distinct boxes, then either*

- $C \cap A$ is complete to $C' \cap B$ and $C \cap B$ is anticomplete to $C' \cap A$, or
- $C \cap A$ is anticomplete to $C' \cap B$ and $C \cap B$ is complete to $C' \cap A$.

By 5.1, applied to the bipartite complement of G if necessary (and trivially if $|C'| = 1$), it follows that every vertex in $A \cap C$ is complete or anticomplete to $B \cap C'$; and similarly every vertex in $B \cap C'$ is complete or anticomplete to $A \cap C$. Consequently $A \cap C$ is complete or anticomplete to $B \cap C'$, and similarly $B \cap C$ is complete or anticomplete to $A \cap C'$. Thus, by taking bipartite complements if necessary, we may assume that $A \cap C$ is complete to $B \cap C'$, and $|C|, |C'| \geq 4$. Suppose that $B \cap C$ is complete to $A \cap C'$. If both $(A \cap C, B \cap C)$ and $(A \cap C', B \cap C')$ are antimatched, then

$(A \cap (C \cup C'), B \cap (C \cup C'))$ is antimatched, contradicting that C is a box. Thus, exchanging C, C' if necessary, we may assume that $(A \cap C, B \cap C)$ is not antimatched, and so $|C| \geq 6$ and $(A \cap C, B \cap C)$ is matched. Choose $a' \in A \cap C'$ and $b' \in B \cap C'$, nonadjacent. Then there is a 3-path in $G[(A \cap C) \cup \{b'\}]$ that misses an edge of $G[(B \cap C) \cup \{a'\}]$, a contradiction. This proves (2).

Let \mathcal{C} be the set of all boxes. Let H be the digraph with vertex set \mathcal{C} , in which for all distinct $C, C' \in \mathcal{C}$, C is adjacent from C' (that is, $C'C \in E(H)$) if $C \cap A$ and $C' \cap B$ are complete to each other, and $C \cap B$ and $C' \cap A$ are anticomplete to each other, and either

- $C \cap A$ and $C' \cap B$ are both nonempty, or
- $C \cap B$ and $C' \cap A$ are both nonempty.

From (2), if $C, C' \in \mathcal{C}$ are distinct, and $C \cup C'$ intersects both A, B , then either $CC' \in E(H)$ or $C'C \in E(H)$.

(3) H has no directed cycles.

Suppose it has, and take the shortest directed cycle D , with vertices $C_1-C_2-\dots-C_k-C_1$ say, where $C_i C_{i+1} \in E(D)$ for $1 \leq i \leq k$ (reading subscripts modulo k). It follows from the minimality of k that for all distinct $u, v \in V(D)$, if $uv, vu \notin E(D)$, then $uv, vu \notin E(H)$. Suppose first that $C_1 \cap A, C_1 \cap B \neq \emptyset$. Then by (2), one of $C_1 C_3, C_3 C_1 \in E(H)$, and so $k = 3$. Since $C_2 C_3 \in E(H)$, either $C_2 \cap A$ and $C_3 \cap B$ are both nonempty, or $C_2 \cap B$ and $C_3 \cap A$ are both nonempty.

Suppose the first. Now $C_2 \cap A$ is complete to $C_3 \cap B$, since $C_2 C_3 \in E(H)$; $C_1 \cap B$ is anticomplete to $C_2 \cap A$, since $C_1 C_2 \in E(H)$; and $C_1 \cap A$ is anticomplete to $C_3 \cap B$, since $C_3 C_1 \in E(H)$. But then an edge of G between $C_1 \cap A$ and $C_1 \cap B$ misses an edge between $C_2 \cap A$ and $C_3 \cap B$, and so these two edges make an induced matching in G whose vertex set is not included in a box, a contradiction.

Now suppose that $C_2 \cap B$ and $C_3 \cap A$ are both nonempty. Then $C_2 \cap B$ is anticomplete to $C_3 \cap A$, since $C_2 C_3 \in E(H)$; $C_1 \cap A$ is complete to $C_2 \cap B$, since $C_1 C_2 \in E(H)$; and $C_1 \cap B$ is complete to $C_3 \cap A$, since $C_3 C_1 \in E(H)$. But then an edge of G between $C_1 \cap A$ and $C_2 \cap B$ misses a (carefully chosen) edge between $C_3 \cap A$ and $C_1 \cap B$, again a contradiction. This proves that each of C_1, \dots, C_k is a subset of one of A, B .

Consequently k is even, and we may assume that $C_i \subseteq A$ for all odd i , and $C_i \subseteq B$ for all even i . Hence one of $C_1 C_4, C_4 C_1 \in E(H)$, from (2), and so $k = 4$. Since $C_1 C_2 \in E(H)$, C_1 is complete to C_2 , and similarly C_3 is complete to C_4 . Since $C_2 C_3 \in E(H)$, C_2 is anticomplete to C_3 , and similarly C_4 is anticomplete to C_1 . But then an edge of G between C_1, C_2 misses an edge between C_3, C_4 , a contradiction. This proves (3).

From (3), we may number \mathcal{C} as $\{C_1, \dots, C_n\}$ such that for all distinct $i, j \in \{1, \dots, n\}$ with $i < j$, $C_i \cap A$ is complete to $C_j \cap B$, and $C_i \cap B$ is anticomplete to $C_j \cap A$. Setting $A_i = C_i \cap A$ and $B_i = C_i \cap B$ for each i , we deduce that G is a half-graph expansion. This proves 5.2. ■

What about bipartite P_6 -free graphs? Let G be a bipartite graph. We say G is an *antimatching recursion* if G has no twins, and either

- G is an antimatching; or

- G is disconnected, and each of its components is an antimatching recursion; or
- there is a vertex v such that the set of nonneighbours of v is stable, and the graph obtained by deleting v is an antimatching recursion; or
- G admits a nontrivial simplicial homogeneous pair (A, B) , such that if J denotes $G[A \cup B]$ and H is the graph obtained from G by identifying the vertices in A and identifying the vertices in B , then both H, J are antimatching recursions.

We have:

5.3 *Let G be a bipartite graph without twins. Then G is P_6 -free if and only if G is an antimatching recursion.*

Proof. It is easy to check that all antimatching recursions are P_6 -free, and so we just need to prove the converse. Let G be a P_6 -free bipartite graph without twins. We prove by induction on $|V(G)|$ that G is an antimatching recursion. We may assume that $|V(G)| \geq 3$, and G is connected.

If there is a vertex v such that set of nonneighbours of v is stable, then the graph obtained by deleting v has no twins and so is an antimatching recursion from the inductive hypothesis, and hence so is G . We assume then that there is no such vertex v .

Suppose that G admits a nontrivial simplicial homogeneous pair (A, B) , and choose (A, B) with $|A \cup B|$ maximal. Let J denote $G[A \cup B]$ and let H be the graph obtained from G by identifying the vertices in A into one vertex a say, and identifying the vertices in B into one vertex b . Let C, D be respectively the sets of vertices in G that are not in $A \cup B$ but are complete to A , and complete to B , respectively. Then J has no twins, and so is an antimatching recursion from the inductive hypothesis. If the same holds for H then the result follows, so we may assume that H has twins, and consequently we may assume that there is some vertex $c \in C$, with neighbour set $B \cup D$ (so c and a are twins in H). From the maximality of $A \cup B$, $(A \cup \{c\}, B)$ is not a nontrivial simplicial homogeneous pair, and so $V(G) = A \cup B \cup \{c\}$; but then the set of nonneighbours of c in G is stable, a contradiction.

Thus G has no nontrivial simplicial homogeneous pair. Let (A, B) be a bipartition. We have shown that every vertex in A has a nonneighbour in B , and vice versa.

Suppose that G is a V_8 -expansion. With a_2, a_4, a_6, a_8 as in the definition of V_8 -expansion, it follows since G is bipartite that either a_2, a_6 are both deleted, or a_4, a_8 are both deleted; and we assume the first without loss of generality. Since G is connected, not both a_4, a_8 are deleted; so we may assume that $a_4 \in A$ say. But then a_4 is complete to B , a contradiction.

Thus G is not a V_8 -expansion, and so G is Q_5 -free, by 4.2. By 5.2, there are partitions (A_1, \dots, A_n) of A and (B_1, \dots, B_n) of B (where some of $A_1, \dots, A_n, B_1, \dots, B_n$ may be empty), such that:

- for $1 \leq i < j \leq n$, A_i is complete to B_j , and A_j is anticomplete to B_i ;
- for $1 \leq i \leq n$, either $|A_i \cup B_i| = 1$ or (A_i, B_i) is matched or antimatched.

We may assume that $A_i \cup B_i \neq \emptyset$ for $1 \leq i \leq n$.

If $B_1 = \emptyset$ then $A_1 \neq \emptyset$, and every vertex in A_1 is complete to B , a contradiction. Thus $B_1 \neq \emptyset$. Since $|V(G)| \geq 2$ and G is connected, it follows that $A_1 \neq \emptyset$. Every vertex in A_1 has a nonneighbour in B_1 , and every vertex in B_1 has a neighbour in A_1 ; and so $|A_1|, |B_1| \geq 2$. If $A_1 \cup B_1 \neq V(G)$,

then (A_1, B_1) is a nontrivial simplicial homogeneous pair, a contradiction; so $A_1 \cup B_1 = V(G)$. Now (A_1, B_1) is matched or antimatched, and not matched since G is connected. Consequently G is an antimatching, and hence an antimatching recursion. This proves 5.3. \blacksquare

6 Using a maximum induced matching

The goal of the remainder of the paper is to prove the following:

6.1 *Let G be connected and $\{Q_5, K_3\}$ -free, with no twins. Then either:*

- *G is Clebschian, climbable, or bipartite; or*
- *G admits a nontrivial simplicial homogeneous pair.*

Incidentally, in the remainder of the paper, there are a number of places where we claim that an explicitly-given graph is Clebschian, and we leave the reader to verify this. Here is the method we used, which we found quite practical. To verify that some triangle-free graph G (with a 5-cycle) is Clebschian, it suffices to do the following:

- select any 5-cycle C of G ; label its vertices v_1, \dots, v_5 in order (we read subscripts modulo 5);
- for each $v \in V(G) \setminus V(C)$, if v has a unique neighbour $v_i \in V(C)$, label $v = v'_i$;
- if some vertex of G has no neighbour in C , label it ∞ ;
- check that at most one vertex receives any label;
- check that if v'_i, v'_j both exist then they are adjacent if and only if $j - i \in \{2, 3\}$ (modulo 5);
- check that the unlabelled vertices are pairwise nonadjacent, and each is adjacent to the vertex labelled ∞ if it exists;
- check that every unlabelled vertex is adjacent to two nonconsecutive vertices v_{i-1}, v_{i+1} of C and to v'_{i-2}, v'_{i+2} if they exist, and not to v'_i if that exists; and that no two unlabelled vertices are adjacent to the same pair of vertices of C .

If all these checks succeed then G is Clebschian, and otherwise it is not. The point is, because of the symmetry of the Clebsch graph, it is not necessary to repeat this for every 5-cycle; doing it for one 5-cycle is already enough.

Let us set up some notation before we go on. Let $M = \{a_1b_1, \dots, a_nb_n\}$ be an induced matching in a graph G . We denote by $V(M)$ the set $\{a_1, \dots, a_n, b_1, \dots, b_n\}$, and by N_M the set of vertices of G not in $V(M)$ but with a neighbour in $V(M)$. We recall the key observation that, by 5.1, if G is $\{Q_5, K_3\}$ -free, then every vertex in $V(G) \setminus V(M)$ with a neighbour in $V(M)$ has a neighbour in each edge of M . If $u, v \in N_M$, $\delta_M(u, v)$ denotes the number of $i \in \{1, \dots, n\}$ such that u is adjacent to one of a_i, b_i and v is adjacent to the other. There is a second key observation:

6.2 *Let G be $\{Q_5, K_3\}$ -free, let M be an induced matching in G , and let $u, v \in N_M$ be distinct.*

- If u, v are adjacent then $\delta_M(u, v) = |M|$.
- If u, v are nonadjacent then $\delta_M(u, v) \leq 2$.

Proof. Each of u, v has a neighbour in every edge of M , by 5.1. If u, v are adjacent then they have no common neighbour in $V(M)$ since G is triangle-free, so the first statement is true. If u, v are nonadjacent, let $M = \{a_1b_1, \dots, a_nb_n\}$, and suppose that u is adjacent to a_1, a_2, a_3 and v to b_1, b_2, b_3 . Then a_1u-a_2 misses vb_3 , a contradiction. This proves 6.2. ■

The case when $\mu(G) \leq 1$ is exceptional (and easy) so let us get that out of the way.

6.3 *Let G be connected and K_3 -free, with no twins and with $\mu(G) \leq 1$. Then either G is a half-graph, or G is a cycle of length five.*

Proof. If G is bipartite, it is a half-graph and the theorem holds, so we assume not; and consequently G has an induced 5-cycle C , with vertices $v_1-v_2-\dots-v_5-v_1$ in order. Let $W = V(G) \setminus V(C)$. Let P be the set of vertices in W with a neighbour in $V(C)$, and $Q = W \setminus P$. Let $v \in P$; then v has at most two neighbours in $V(C)$, since G is triangle-free; and v does not have just one neighbour in $V(C)$ since $\mu(G) = 1$. Consequently v has exactly two neighbours in $V(C)$, and they are nonadjacent. For $1 \leq i \leq 5$ let P_i be the set of vertices in P adjacent to v_{i-1} and v_{i+1} (reading subscripts modulo 5). For $1 \leq i \leq 5$, P_i is complete to P_{i+1} (since $\mu(G) = 1$), and P_i is anticomplete to P_{i+2} (since G is triangle-free). Moreover, no vertex in Q has a neighbour in P (since $\mu(G) = 1$), and so $Q = \emptyset$ (since G is connected). Since G has no twins, it follows that $P = \emptyset$. This proves 6.3. ■

Incidentally, this yields a result we mentioned before – how to construct all $\{P_5, K_3\}$ -free graphs.

6.4 *A graph is $\{P_5, K_3\}$ -free if and only if each of its components is either a half-graph or obtained from a cycle of length five by adding twins.*

Proof. It is easy to see that a graph G is $\{P_5, K_3\}$ -free if and only if each of its components G' is K_3 -free and satisfies $\mu(G') \leq 1$; and by 6.3, this is true if and only if every component is either a half-graph or obtained from a cycle of length five by adding twins. ■

7 Matchings with stable neighbour sets

The goal of this section is to handle the case when there is a maximum induced matching M such that N_M is stable.

7.1 *Let G be $\{Q_5, K_3\}$ -free, and let M be a maximum induced matching in G . If N_M is stable, then M can be labelled $\{a_1b_1, \dots, a_nb_n\}$ such that either:*

- every member of N_M is adjacent to at least $n - 1$ of a_1, \dots, a_n ; or
- $n \geq 2$, and N_M is complete to $\{a_3, \dots, a_n\}$, and each of the pairs $(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2)$ has a common neighbour in N_M ; or

- $n \geq 3$, and N_M is complete to $\{a_1, \dots, a_n\}$, and every member of N_M is adjacent to an odd number of a_1, a_2, a_3 ; and each of the triples $(a_1, a_2, a_3), (a_1, b_2, b_3), (b_1, a_2, b_3), (b_1, b_2, a_3)$ has a common neighbour in N_M .

Proof. By 6.2, $\delta_M(u, v) \leq 2$ for all $u, v \in N_M$. If $N_M = \emptyset$ the claim is trivial, so we assume there exists $w \in N_M$. Choose $w \in N_M$ such that $\max_{v \in N_M} \delta_M(w, v)$ is as small as possible. Let $M = \{a_1 b_1, \dots, a_n b_n\}$, labelled such that w is adjacent to a_i for each i .

(1) For each $u \in N_M$ there exists $v \in N_M$ with $\delta_M(u, v) = 2$.

If $\max_{v \in N_M} \delta_M(w, v) \leq 1$ then the first bullet holds, so we assume this maximum equals two. Hence, from the choice of w , this proves (1).

Suppose that there exists $u \in N_M$ with $\delta_M(u, w) = 1$. From (1), there exists $v \in N_M$ with $\delta_M(u, v) = 2$, and hence with $\delta_M(u, w)$ odd, and therefore with $\delta_M(u, w) = 1$; and by (1) again, there exists $x \in N_M$ with $\delta_M(w, x) = 2$ and hence with $\delta_M(u, x) = \delta_M(v, x) = 1$. If $y \in N_M$, then since $\delta_M(y, z) \leq 2$ for each $z \in \{u, v, w, x\}$, it follows that $\delta_M(y, z) = 0$ for some $z \in \{u, v, w, x\}$ and the second bullet holds.

So we may assume that $\delta_M(u, w) = 0$ or 2 , for each $u \in N_M$. Let H be the graph with vertex set $\{1, \dots, n\}$, in which distinct i, j are adjacent if there exists $u \in N_M$ adjacent to b_i and to b_j . Every two edges of H have a common end in H . If some vertex of H belongs to all edges of H , then the second bullet holds, so we may assume that H has a triangle, and hence H has only three edges. But then the third bullet holds. This proves 7.1. ■

Let us say a *leaf* is a vertex of degree one; and an edge e such that neither end of e is a leaf is *internal*. In the remainder of this section we handle separately the three cases of the output of 7.1. The first case itself devolves into three subcases: when $\mu(G) \leq 1$ (which we have already handled), when $\mu(G) \geq 2$ but at most one edge of M is internal, and when $\mu(G) \geq 2$ and at least two of its edges are internal.

7.2 Let G be connected and $\{Q_5, K_3\}$ -free, with $\mu(G) \geq 2$, and let M be a maximum induced matching in G ; moreover, if $\mu(G) = 2$, assume that G has no twins and M is chosen such that, if possible, both its edges are internal. If there is at most one edge in M that is internal, then either

- G is Clebschian; or
- G admits a nontrivial pendant homogeneous pair.

Proof. Let $M = \{a_1 b_1, \dots, a_n b_n\}$, where $n = \mu(G) \geq 2$ and b_2, \dots, b_n are leaves of G . If either $n \geq 3$ or one of a_1, b_1 is a leaf then G admits a nontrivial pendant homogeneous pair. Consequently we may assume that $n = 2$ and $a_1 b_1$ is internal; and from the choice of M , there is no induced matching M' in G of cardinality two such that both its edges are internal. Let A, B be the sets of all vertices in N_M adjacent to a_1 and to b_1 respectively. Thus $A, B \neq \emptyset$, and by 5.1 $N_M = A \cup B$ and $A \cap B = \emptyset$, and a_2 is complete to $A \cup B$, and so $A \cup B$ is stable. Let $R = V(G) \setminus (A \cup B \cup V(M))$. Thus R is anticomplete to $V(M)$, and hence stable, from the maximality of M . Since G is connected, every vertex in R has a neighbour in $A \cup B$.

Let D be the set of all vertices in R with neighbours in both A and B , and suppose that $v \in D$. If v is not complete to one of A, B , say B , choose $b \in B$ nonadjacent to v , and choose $a \in A$ adjacent to v ; then the edges va, b_1b form a 2-edge induced matching M' and no vertex in $V(M')$ is a leaf of G , a contradiction. So v is complete to $A \cup B$; and so $(\{v, a_2\}, \{b_2\})$ is a nontrivial pendant homogeneous pair. Hence we may assume that $D = \emptyset$.

Let A' be the set of all vertices in R with neighbours in A and with none in B , and define B' similarly. If both $A', B' \neq \emptyset$, there is a 3-path of $G[A' \cup A \cup \{a_1\}]$ that is anticomplete to an edge of $G[B' \cup B]$, a contradiction. Thus we may assume that $B' = \emptyset$, and so $|B| = 1$, since G has no twins. If $A' = \emptyset$, then $|A| = 1$ since G has no twins, and so G is Clebschian. Thus we assume that $A' \neq \emptyset$, and so we may assume that $|A| = |A'| = 1$, since (A, A') is a pendant homogeneous pair, and again G is Clebschian. This proves 7.2. \blacksquare

7.3 *Let G be $\{Q_5, K_3\}$ -free, let M be a maximum induced matching in G , and suppose that at least two edges of M are internal. Suppose that N_M is stable, and M is labelled $\{a_i c_i : 1 \leq i \leq n\}$ such that every member of N_M is adjacent to at least $n - 1$ of a_1, \dots, a_n . Then G is climbable.*

Proof. First we prove:

(1) *For $1 \leq i \leq n$, there is at most one vertex in N_M nonadjacent to a_i .*

For suppose that $u, v \in N_M$ are both nonadjacent to a_1 say. Since some edge of M different from $a_1 c_1$, say $a_2 c_2$, is internal, there exists $w \in N_M$ adjacent to c_2 , and hence nonadjacent to c_1 , since w is adjacent to at least $n - 1$ vertices in $\{a_1, \dots, a_n\}$. For the same reason u, v are nonadjacent to c_2 . But $u-c_1-v$ is a 3-path by 5.1, and it misses the edge wc_2 , a contradiction. This proves (1).

For each $v \in N_M$, if there exists $i \in \{1, \dots, n\}$ such that v is nonadjacent to a_i , define $b_i = v$. (This is well-defined by (1).) Thus b_i is defined if and only if c_i is not a leaf of G . We may assume that b_i is defined for $1 \leq i \leq m$ and not defined for $m + 1 \leq i \leq n$.

Let P be the set of vertices in N_M that are complete to $\{a_1, \dots, a_n\}$, that is, $N_M \setminus \{b_1, \dots, b_m\}$. Number P as b_{n+1}, \dots, b_t say. Let Q be the set of vertices in $V(G) \setminus (V(M) \cup N_M)$ that have a neighbour in $\{b_1, \dots, b_m\}$, and R the remainder, that is, $R = V(G) \setminus (V(M) \cup N_N \cup Q)$.

(2) *$Q \cup R$ is stable, and Q is complete to $\{b_1, \dots, b_m\}$.*

Certainly $Q \cup R$ is stable, from the maximality of M . Suppose that $v \in Q$ is nonadjacent to b_1 say. Since v has a neighbour in $\{b_1, \dots, b_m\}$, we may assume that v is adjacent to b_2 . But then $v-b_2-a_1$ is a 3-path that misses the edge $a_2 b_1$, a contradiction. This proves (2).

(3) *Every vertex in Q has at most one nonneighbour in P , and no two vertices in Q have the same nonneighbour in P .*

Suppose that $v \in Q$ is nonadjacent to $x, y \in P$; then $x-a_1-y$ is a 3-path missing the edge $v-b_1$. Now suppose that $u, v \in Q$ are both nonadjacent to $x \in P$; then $u-b_1-v$ is a 3-path missing the edge xa_1 . This proves (3).

For each $v \in Q$, if v is nonadjacent to some (unique by (3)) $b_i \in P$, define $a_i = v$, and if v is complete to P , choose some (distinct for all $v \in Q$) integer $i > t$, and define $a_i = v$.

(4) *Every vertex in R has at most one neighbour in P ; and no two vertices in R have the same neighbour in P .*

Suppose that $v \in R$ is adjacent to $x, y \in P$, where $x \neq y$. Now x, y are nonadjacent to c_1 since they are adjacent to a_1 , and they are nonadjacent to b_1 since N_M is stable. Moreover, v is nonadjacent to c_1 since $v \notin N_M$, and v is nonadjacent to b_1 since $v \notin Q$. Hence $x-v-y$ is a 3-path, and misses the edge b_1c_1 , a contradiction. Now suppose that $u, v \in R$ are adjacent to $x \in P$. Then similarly $u-x-v$ is a 3-path missing the edge b_1c_1 , a contradiction. This proves (4).

(5) *For $i > m$, if $v \in R$ is adjacent to b_i then a_i is not defined.*

Suppose that a_i, b_i are defined and $v \in R$ is adjacent to b_i . Then v is nonadjacent to b_1, b_2 , since $v \notin Q$, and so $b_1-a_i-b_2$ is a 3-path missing the edge vb_i . This proves (5).

For each $v \in R$ with a (unique) neighbour in P , choose i such that this neighbour is b_i , and define $c_i = v$. (This is well-defined, from (4).) For each $v \in R$ with no neighbour in P choose an integer $i > n$ (distinct for all $v \in R$) such that b_i is not defined, and define $c_i = v$. Thus for each $c_i \in R$, a_i is not defined, and for each j such that b_j is defined, c_i is adjacent to b_j if and only if $i = j$. This proves that G is climbable, and so proves 7.3. ■

For the second outcome of 7.1 we use:

7.4 *Let G be $\{Q_5, K_3\}$ -free with $\mu(G) \geq 2$, and with no twins if $\mu(G) = 2$; and let M be a maximum induced matching in G . Suppose that N_M is stable, and M can be labelled $\{c_i d_i : 1 \leq i \leq n\}$ such that N_M is complete to $\{c_3, \dots, c_n\}$, and each of the pairs $(c_1, c_2), (c_1, d_2), (d_1, c_2), (d_1, d_2)$ has a common neighbour in N_M . Then either G is Clebschian, or G admits a nontrivial pendant homogeneous pair.*

Proof. Define $a_1 = c_1, a_3 = c_2, a_5 = d_1, a_7 = d_2$ and let $A_i = \{a_i\}$ for $i = 1, 3, 5, 7$; and for $i = 2, 4, 6, 8$, let A_i be the set of vertices in N_M complete to $\{a_{i-1}, a_{i+1}\}$ (reading subscripts modulo 8 throughout). Thus $N_M = A_2 \cup A_4 \cup A_6 \cup A_8$, and A_2 is anticomplete to A_6 , and A_4 is anticomplete to A_8 (since by hypothesis, N_M is stable). Also A_2, A_4, A_6, A_8 are all nonempty, by hypothesis. Any 3-path in $G[A_2 \cup \{a_1\}]$ misses an edge of $G[A_6 \cup \{a_7\}]$, so $|A_2| = 1$, and similarly $|A_i| = 1$ for $i = 2, 4, 6, 8$; let $A_i = \{a_i\}$ for $i = 2, 4, 6, 8$. Let $R = V(G) \setminus (A_1 \cup \dots \cup A_8)$; thus $c_i, d_i \in R$ for $3 \leq i \leq n$. If $v \in R$ is adjacent to a_2 , then since $v-a_2-a_3$ does not miss $a_5 a_6$, it follows that v is adjacent to a_6 ; and similarly for the other vertices in N_M . Define R_0, \dots, R_3 as follows.

- R_0 is the set of vertices in R with no neighbour among a_2, a_4, a_6, a_8
- R_1 is the set adjacent to a_2, a_6 and not to a_4, a_8 ;
- R_2 is the set adjacent to a_4, a_8 and not to a_2, a_6 ; and
- R_3 is the set adjacent to all of a_2, a_4, a_6, a_8 .

It follows that $R = R_0 \cup \dots \cup R_3$, and $c_3, \dots, c_n \in R_3$, and $d_3, \dots, d_n \in R_0$. Now $R_0 \cup R_1$ is stable, since it misses $a_3-a_4-a_5$, and similarly $R_0 \cup R_2$ is stable. There are no edges between R_3 and $R_1 \cup R_2$ since G is triangle-free; and so $R_1 \cup R_2$ misses $R_0 \cup R_3$, and hence does not have any neighbour in $V(M)$. Since M is a maximum induced matching it follows that $R_1 \cup R_2$ is stable. So the only edges in $G[R]$ are edges between R_0 and R_3 . Now $|R_1| \leq 1$ since any 3-path in $G[R_1 \cup \{a_2\}]$ misses a_4a_5 ; and similarly $|R_2| \leq 1$. Not both R_1, R_2 are nonempty, since otherwise a 3-path in $G[R_1 \cup \{a_2, a_6\}]$ would miss an edge of $G[R_2 \cup \{a_4\}]$; so we may assume that $R_2 = \emptyset$. If $R_0 \neq \emptyset$, then since every vertex in R_0 has a neighbour in R_3 (because G is connected), it follows that (R_3, R_0) is a pendant homogeneous pair, so we may assume that $|R_3|, |R_0| = 1$, and therefore G is Clebschian. Thus we may assume that $R_0 = \emptyset$. Hence $\mu(G) = n = 2$, and so by hypothesis G has no twins; and again it follows that $|R_3| \leq 1$ and G is Clebschian. This proves 7.4. \blacksquare

For the third outcome of 7.1 we use:

7.5 *Let G be connected and $\{Q_5, K_3\}$ -free, containing the Petersen graph. Then either G is Clebschian, or G admits a nontrivial pendant homogeneous pair.*

Proof. Let us label the vertices of the Petersen subgraph H say, as follows: there is an induced 5-cycle C with vertices $v_1-v_2-\dots-v_5-v_1$ in order; for $1 \leq i \leq 5$ there is a vertex v'_i adjacent to v_i and to no other vertex in C ; and for all distinct $i, j \in \{1, \dots, 5\}$, v'_i, v'_j are adjacent if and only if $j - i = 2$ or 3 modulo 5. (We read subscripts modulo 5 throughout.) Let $R = V(G) \setminus V(H)$. Let R_0 be the set of vertices in R with no neighbour in $V(H)$, and for $1 \leq i \leq 5$, let R_i be the set of vertices in R whose neighbour set in $V(H)$ is $\{v_{i-1}, v_{i+1}, v'_{i-2}, v'_{i+2}\}$. (The five quadruples of $V(H)$ just listed are the stable subsets of H of cardinality four, and so this list is invariant under the symmetries of the Petersen graph.)

(1) $R = R_0 \cup R_1 \cup \dots \cup R_5$.

For let $v \in R$, and let N_M be the set of its neighbours in $V(H)$. We may assume that $N_M \neq \emptyset$, and so from the symmetry we may assume that $v_1 \in N_M$. Consequently $v_2, v_5, v'_1 \notin N_M$. The path $v-v_1-v'_1$ hits v_3v_4 , so from the symmetry we may assume that $v_3 \in N_M$. Hence $v_4, v'_3 \notin N_M$. Since $v-v_1-v_2$ hits $v_4-v'_4$, it follows that $v'_4 \in N_M$, and so $v'_2 \notin N_M$. Since $v-v_3-v_4$ hits $v'_2v'_5$, it follows that $v'_5 \in N_M$; and so $N_M = \{v_1, v_3, v'_4, v'_5\}$, and hence $v \in R_2$. This proves (1).

Since every two vertices in $R_1 \cup \dots \cup R_5$ have a common neighbour, it follows that $R_1 \cup \dots \cup R_5$ is stable. Thus if $R_0 = \emptyset$, G is Clebschian, so we assume that $R_0 \neq \emptyset$. Also, R_0 is stable, since R_0 misses $v_1-v_2-v_3$.

(2) *We may assume that R_0 is complete to $R_1 \cup \dots \cup R_5$.*

Let $v \in R_0$. Since R_0 is stable and G is connected, v has a neighbour in $R_1 \cup \dots \cup R_5$, say $r_1 \in R_1$. We claim that v is complete to $R_2 \cup \dots \cup R_5$. To see this, let $r_2 \in R_2$ (from the symmetry of the Petersen graph, it is enough to show that a vertex with a neighbour in R_1 is complete to R_2). Since $v-r_1-v'_3$ hits r_2v_1 , it follows that v, r_2 are adjacent. So v is complete to $R_2 \cup \dots \cup R_5$. If one of R_2, \dots, R_5 is nonempty, the same argument shows that v is complete to R_1 as required; so we assume that $R_2, \dots, R_5 = \emptyset$. Now any 3-path in $G[R_0 \cup R_1]$ misses $v_1v'_1$, so every vertex in R_0 has

at most one neighbour in R_1 and vice versa; and since (R_1, R_0) is a pendant homogeneous pair, we may assume that $|R_0| = |R_1| = 1$, and so R_1 is complete to R_0 . This proves (2).

Any 3-path in $G[R_0 \cup R_1]$ misses $v_1v'_1$, and since R_0 is complete to R_1 and $R_0 \neq \emptyset$ it follows that $|R_1| \leq 1$, and similarly $|R_i| \leq 1$ for $1 \leq i \leq 5$. We may assume that $|R_1| = 1$, since $R_0 \neq \emptyset$, and so by the same argument $|R_0| = 1$. Hence G is Clebschian. This proves 7.5. ■

We deduce:

7.6 *Let G be connected and $\{Q_5, K_3\}$ -free, with $\mu(G) \geq 2$ and with no twins if $\mu(G) \leq 2$, and let M be a maximum induced matching in G . Suppose that N_M is stable. Then either:*

- G is Clebschian or climbable; or
- G admits a nontrivial pendant homogeneous pair.

Proof. By 7.1, M can be labelled $\{a_i b_i : 1 \leq i \leq n\}$ such that one of the following holds.

- Every member of N_M is adjacent to at least $n - 1$ of a_1, \dots, a_n . In this case, if at most one of the edges $a_i b_i$ is internal, then the theorem holds by 7.2, and if at least two edges of M are internal, it holds by 7.3.
- N_M is complete to $\{a_3, \dots, a_n\}$, and each of the pairs $(a_1, a_2), (a_1, b_2), (b_1, a_2), (b_1, b_2)$ has a common neighbour in N_M . In this case the theorem holds by 7.4.
- $n \geq 3$, N_M is complete to $\{a_4, \dots, a_n\}$, and every member of N_M is adjacent to an odd number of a_1, a_2, a_3 ; and each of the triples $(a_1, a_2, a_3), (a_1, b_2, b_3), (b_1, a_2, b_3), (b_1, b_2, a_3)$ has a common neighbour in N_M . In this case, the six vertices $a_1, b_1, a_2, b_2, a_3, b_3$, together with four common neighbours in N_M of the four triples listed above, induce a subgraph of G isomorphic to the Petersen graph; and so the theorem holds by 7.5.

This proves 7.6. ■

8 Maximum matching of cardinality two

We already handled the case when $\mu(G) \leq 1$, in 6.3; now we are ready to handle $\mu(G) = 2$, and this section is devoted to it. We recall that an antimatching is a bipartite graph, the bipartite complement of a perfect matching. Let $\nu(G)$ be the largest cardinality of an antimatching in G (more precisely, $\nu(G)$ means the largest k such that there is an antimatching with bipartition (A, B) where $|A| = |B| = k$). When $\mu(G) = 2$, the maximum induced matching induces a subgraph that is an antimatching, and it might be possible to extend this to a larger antimatching; and we find it advantageous to work with the largest antimatching.

8.1 *Let G be a connected $\{Q_5, K_3\}$ -free graph with no twins and with $\mu(G) = 2$ and $\nu(G) \geq 3$. Then either*

- G is Clebschian, climbable, or bipartite; or

- G admits a nontrivial simplicial homogeneous pair.

Proof. Let $(A, B) = (\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\})$ be the bipartition of the largest antimatching in G , where a_i, b_i are nonadjacent for $1 \leq i \leq n$. Thus $n = \nu(G) \geq 3$. For $1 \leq i \leq n$ let C_i be the set of all vertices in $V(G) \setminus (A \cup B)$ adjacent to both a_i and b_i and nonadjacent to every other vertex in $A \cup B$; let $C = C_1 \cup \dots \cup C_n$; let P be the set of vertices in $V(G) \setminus (A \cup B)$ complete to A and anticomplete to B , and let Q be the set complete to B and anticomplete to A .

(1) *Every vertex in $V(G) \setminus (A \cup B)$ with a neighbour in $A \cup B$ belongs to one of P, Q, C . Moreover, P is complete to Q , and $P \cup Q$ is anticomplete to C .*

Let $v \in V(G) \setminus (A \cup B)$ be adjacent to a_1 say. If v is complete to A then it is anticomplete to B and $v \in P$; so we may assume that v is nonadjacent to a_2 . Since the 3-path $v-a_1-b_2$ does not miss the edge a_2b_1 , it follows that v is adjacent to b_1 ; and so v has no more neighbours in $A \cup B$ and $v \in C_1$. This proves the first assertion. Now P is complete to Q by the maximality of the antimatching. Finally, $P \cup Q$ is anticomplete to C since G is triangle-free. This proves (1).

(2) *For all distinct $i, j \in \{1, \dots, n\}$, every vertex in C_i has at most one nonneighbour in C_j .*

For otherwise there is a 3-path in $G[C_j \cup \{a_j\}]$ that misses an edge of $G[C_i \cup \{a_i\}]$.

(3) *We may assume that $A \cup B \cup P \cup Q \cup C = V(G)$.*

Let $R = V(G) \setminus (A \cup B \cup P \cup Q \cup C)$, and suppose that there exists $r \in R$. Let rs be an edge incident with r . By (1), $s \notin A \cup B$. If $s \notin P \cup Q$, we may assume that $s \notin C_1 \cup C_2$ since $n \geq 3$, and then $\{rs, a_1b_2, a_2b_1\}$ is an induced matching, a contradiction. Thus $s \in P \cup Q$. Since G is triangle-free, it follows that for every vertex in R , either all its neighbours belong to P or they all belong to Q ; let R_1 be the set of vertices in R such that all their neighbours belong to P , and $R_2 = R \setminus R_1$. With r, s as before, we may assume from the symmetry that $r \in R_1$ and $s \in P$. Suppose that at least two of C_1, \dots, C_n are nonempty, say C_1, C_2 , and choose $c_i \in C_i$ ($i = 1, 2$). If c_1, c_2 are nonadjacent, then $\{rs, b_1c_1, b_2c_2\}$ is an induced matching, a contradiction. If c_1, c_2 are adjacent, the 3-path $r-s-a_3$ misses c_1c_2 , a contradiction. Thus at most one of C_1, \dots, C_n is nonempty. We may assume that G is not bipartite, and so we may assume that $C_1 \neq \emptyset$, and $C_2, \dots, C_n = \emptyset$. Since (P, R_1) is a pendant homogeneous pair, we may assume that $R_1 = \{r\}$ and $P = \{s\}$. Also either $R_2 = \emptyset$ (and hence $|Q| = 1$ since G has no twins) or (Q, R_2) is also a pendant homogeneous pair (and so we may assume that $|R_2| = |Q| = 1$). Since any 3-path in $G[C_1 \cup \{b_1\}]$ would miss the edge of $G[P \cup R_1]$, it follows that $|C_1| = 1$; but then G is climbable. This proves (3).

(4) *For $1 \leq i \leq n$, each vertex in C_i has at most one nonneighbour in $C \setminus C_i$.*

Let $i = 1$ say, and suppose that $c_1 \in C_1$ has at least two nonneighbours in $C \setminus C_1$. By (3) we may assume that c_1 is nonadjacent to $c_2 \in C_2$ and $c_3 \in C_3$. Since $\{a_1c_1, a_2c_2, a_3c_3\}$ is not an induced matching, it follows that c_2, c_3 are adjacent. But then $a_1-c_1-b_1$ misses c_2c_3 . This proves (4).

(5) *At most two of C_1, \dots, C_n are nonempty.*

Suppose C_1, C_2, C_3 are nonempty. If $n \geq 4$, then by (4) there is an edge joining two of C_1, C_2, C_3 , say c_1c_2 where $c_i \in C_i$ for $i = 1, 2$; but then $\{c_1c_2, a_3b_4, a_4b_3\}$ is an induced matching, a contradiction. Thus $n = 3$. Also $|P| \leq 1$, since otherwise for some $i \in \{1, 2, 3\}$ a 3-path in $G[P \cup \{a_i\}]$ would miss an edge of $G[C_1 \cup C_2 \cup C_3]$. Similarly $|Q| \leq 1$.

Suppose that $c_1 \in C_1$ has at least two neighbours in C_2 , say c_2, c'_2 . If c_1 is adjacent to $c_3 \in C_3$, then c_3 has two nonneighbours in C_2 , namely c_2, c'_2 , contrary to (2). So c_1 is anticomplete to C_3 . Hence by (4) some vertex (indeed, every vertex) of C_3 is adjacent to c_2, c'_2 , and so by the same argument c_2, c'_2 are complete to C_1 . In particular, there are no edges between C_1 and C_3 , so C_1, C_3 are complete to C_2 by (4), and $|C_1| = |C_3| = 1$ by (2). If $p \in P$, let $c_3 \in C_3$; then the 3-path $c_2-c_3-c'_2$ misses pa_1 , a contradiction. So $P, Q = \emptyset$, and c_2, c'_2 are twins, a contradiction.

We may assume therefore that for all distinct $i, j \in \{1, 2, 3\}$, every vertex in C_i has at most one neighbour and at most one nonneighbour in C_j . Let $c_2 \in C_2$ be adjacent to $c_1 \in C_1$ and $c_3 \in C_3$. Then no other vertex $c'_2 \in C_2$ is adjacent to either of c_1, c_3 , and therefore $\{c_1, c'_2, c_3\}$ is stable, contrary to (4), for each such vertex c'_2 ; and hence $C_2 = \{c_2\}$. Since c_2 has at most one neighbour and at most one nonneighbour in C_1 , it follows that $|C_1| \leq 2$, and similarly $|C_3| \leq 2$; and not both $|C_1|, |C_3| = 2$, since for every $c'_1 \in C_1$ and $c'_3 \in C_3$, c_2 is adjacent to one of them. So we may assume that $|C_3| = 1$, and $C_3 = \{c_3\}$. Also either $C_1 = \{c_1\}$, or $C_1 = \{c_1, c'_1\}$, for some c'_1 adjacent to c_3 . Now P is complete to Q by (1), and so one of P, Q is empty, since the 3-path $c_1-c_2-c_3$ misses every edge between P, Q . Since $|P|, |Q| \leq 1$, it follows that G is Clebschian. This proves (5).

From (5), we may assume that $C_3, \dots, C_n = \emptyset$.

(6) *We may assume that there is no edge between C_1, C_2 .*

Suppose that $c_1 \in C_1$ is adjacent to $c_2 \in C_2$. Not both P, Q are nonempty; because P is complete to Q , by (1), and if $p \in P$ is adjacent to $q \in Q$ then $q-p-a_3$ misses c_1c_2 . So we may assume that $Q = \emptyset$. If $|P| \geq 2$, a 3-path in $G[P \cup \{a_3\}]$ misses c_1c_2 ; so $|P| \leq 1$. Suppose that there exists $p \in P$. Then pa_3 misses any 3-path in $G[C_1 \cup C_2]$; so every vertex in C_1 has at most one neighbour in C_2 , and vice versa. By (2), $|C_1|, |C_2| \leq 2$. Moreover, not both $|C_1|, |C_2| = 2$, since if so there would be an induced matching consisting of two edges between C_1, C_2 and the edge pa_3 . But then G is Clebschian. We may therefore assume that $P = Q = \emptyset$. But then G is climbable, by 7.3 applied to the induced matching $\{a_1b_3, a_3b_1\}$. This proves (6).

Since G has no twins, (5) and (6) imply that $|C_i| \leq 1$ for $i = 1, 2$, and $|P|, |Q| \leq 1$. But then G is climbable. This proves 8.1. ■

8.2 *Let G be a connected $\{Q_5, K_3\}$ -free graph with no twins and with $\mu(G) = \nu(G) = 2$. Then either*

- *G is Clebschian; or*
- *G admits a nontrivial simplicial homogeneous pair.*

Proof. Let $M = \{a_1a_5, a_3a_7\}$ be a maximum induced matching in G , chosen such that no vertices in $V(M)$ are leaves if possible; and for $i = 2, 4, 6, 8$ let A_i be the set of all vertices that are complete to $\{a_{i-1}, a_{i+1}\}$. (Throughout we read subscript modulo 8.) Thus $N_M = A_2 \cup A_4 \cup A_6 \cup A_8$. For $i = 1, 3, 5, 7$ let $A_i = \{a_i\}$. If there exist $a_2 \in A_2$ and $a_6 \in A_6$, nonadjacent, then $G[\{a_1, a_2, a_3, a_5, a_6, a_7\}]$ is an antimatching, contradicting that $\nu(G) = 2$; so A_2 is complete to A_6 , and similarly A_4 to A_8 . So for all distinct i, j with $1 \leq i, j \leq 8$, if $j - i = 1, 4$ or 7 then A_i is complete to A_j and otherwise A_i is anticomplete to A_j . If some vertex of $V(M)$ is a leaf then the theorem holds by 7.2, so we may assume that both edges of M are internal. Hence either A_2, A_6 are both nonempty, or A_4, A_8 are both nonempty, and we assume the first without loss of generality.

Let R be the set of vertices of G not in A_1, \dots, A_8 . Since R is stable (because M is a maximum induced matching), it follows that every vertex in R has a neighbour in $A_2 \cup A_4 \cup A_6 \cup A_8$.

Suppose first that A_4, A_8 are both nonempty. Since any 3-path in $G[A_2 \cup A_6]$ would miss an edge of $G[A_4 \cup A_8]$, it follows that $|A_2| = |A_6| = 1$, and similarly $|A_4| = |A_8| = 1$. Let $A_i = \{a_i\}$ for $i = 2, 4, 6, 8$. If $v \in R$ is adjacent to a_2 say, then since the 3-path $v-a_2-a_6$ does not miss a_4a_8 , it follows that v is adjacent to one of a_4, a_8 ; so every vertex in R belongs to one of the four disjoint sets R_1, R_3, R_5, R_7 , where R_i is the set of vertices in R adjacent to a_{i-1} and a_{i+1} for $i = 1, 3, 5, 7$. Now R_1 is anticomplete to R_5 , since R is stable, and so if they are both nonempty, a 3-path of $G[R_1 \cup \{a_2, a_1\}]$ misses an edge of $G[R_5 \cup \{a_4\}]$. Hence one of $R_1, R_5 = \emptyset$, and similarly one of $R_3, R_7 = \emptyset$, so we may assume that $R_5, R_7 = \emptyset$. If $|R_1| > 1$ then a 3-path of $G[R_1 \cup \{a_2\}]$ misses a_4a_5 , so $|R_1|, |R_3| \leq 1$. But then G is Clebschian.

We may therefore assume that $A_8 = \emptyset$. If $A_4 = \emptyset$ then $(\{a_1, a_3\}, \{a_5, a_7\})$ is a nontrivial simplicial homogeneous pair, so we may assume that $A_4 \neq \emptyset$. Define:

- R_1 is the set of vertices in R with a neighbour in A_2 and no neighbour in A_4 ;
- R_3 is the set complete to A_2 with a neighbour in A_4 ;
- R_5 is the set complete to A_6 with a neighbour in A_4 ; and
- R_7 is the set with a neighbour in A_6 and no neighbour in A_4 .

(1) *Every vertex in R belongs to one of R_1, R_3, R_5, R_7 .*

If $v \in R$ has no neighbour in A_4 then $v \in R_1 \cup R_7$ since v has a neighbour in $A_2 \cup A_4 \cup A_6 \cup A_8$. Thus we may assume v is adjacent to $a_4 \in A_4$. Since every 3-path in $G[A_2 \cup A_6 \cup \{a_7\}]$ hits va_4 , v is complete to one of A_2, A_6 and so $v \in R_3 \cup R_5$. This proves (1).

Suppose that $R_3 = R_5 = \emptyset$. Then we may assume that $|R_1| \leq 1$, and if $|R_1| = 1$ then $|A_2| \leq 1$, because otherwise G admits a nontrivial pendant homogeneous pair. Similarly $|R_7| \leq 1$, and if $|R_7| = 1$ then $|A_6| \leq 1$. Since any 3-path of $G[A_4 \cup \{a_5\}]$ would miss any edge of $G[R_1 \cup A_2]$, it follows that if $R_1 \neq \emptyset$ then $|A_4| = 1$, and similarly if $R_7 \neq \emptyset$ then $|A_4| = 1$. Consequently G is Clebschian.

Thus we may assume that $R_3 \neq \emptyset$. Hence there is a 3-path in $G[R_3 \cup A_4 \cup \{a_3\}]$, and so there is no edge in $G[A_6 \cup R_7]$, and hence $R_7 = \emptyset$. Moreover, any 3-path in $A_6 \cup \{a_7\}$ misses an edge of $G[R_3 \cup A_4]$, so $|A_6| = 1$. Let $A_6 = \{a_6\}$. If $|R_3| > 1$, there is either a 3-path or an induced 2-edge matching in $G[R_3 \cup A_4]$, and in either case it misses a_6a_7 , a contradiction. So $|R_3| = 1$; let $R_3 = \{r_3\}$.

By the same argument, r_3 has only one neighbour in A_4 , say a_4 . Every 3-path in $G[(A_4 \setminus \{a_4\}) \cup \{a_5\}]$ misses the edges of $G[A_2 \cup \{r_3\}]$; so $|A_4 \setminus \{a_4\}| \leq 1$.

Suppose that $R_5 \neq \emptyset$. Then by the same argument, $|A_2| = 1$, $R_1 = \emptyset$, and $|R_5| = 1$, and the vertex $r_5 \in R_5$ has a unique neighbour in A_4 . Since no 3-path in $G[A_4 \cup \{r_5\} \cup A_6]$ misses r_3a_4 , it follows that a_4, r_5 are adjacent. But then G is Clebschian.

We may assume therefore that $R_5 = \emptyset$. Suppose that $R_1 \neq \emptyset$. Since (A_2, R_1) is a pendant (and hence simplicial) homogeneous pair, we may assume that $|R_1| = |A_2| = 1$; let $R_1 = \{r_1\}$ and $A_2 = \{a_2\}$. Any 3-path in $G[A_4 \cup \{a_5\}]$ misses r_1a_2 , so $A_4 = \{a_4\}$. But then G is Clebschian.

Thus we may assume that $R_1 = \emptyset$. If $A_4 \neq \{a_4\}$, then any 3-path in $G[A_2 \cup \{r_3\}]$ misses an edge of $G[(A_4 \setminus \{a_4\}) \cup \{a_5\}]$, so $|A_2| = 1$ and G is Clebschian. Thus we may assume that $A_4 = \{a_4\}$, and so again G is Clebschian. This proves 8.2. \blacksquare

Combining 8.1 and 8.2, we obtain:

8.3 *Let G be a connected $\{Q_5, K_3\}$ -free graph with $\mu(G) = 2$ and with no twins. Then either*

- *G is Clebschian, climbable, or bipartite; or*
- *G admits a nontrivial simplicial homogeneous pair.*

9 Edges in the neighbourhood of the matching

In this section we handle the final case, when $\mu(G) \geq 3$ and N_M is not stable. We divide it into three subcases, when $\mu(G) \geq 5$, when $\mu(G) = 4$, and when $\mu(G) = 3$, in increasing order of difficulty.

9.1 *Let G be connected and $\{Q_5, K_3\}$ -free, with $\mu(G) \geq 5$, and let M be a maximum induced matching in G . Suppose that N_M is not stable. Then G admits a nontrivial simplicial homogeneous pair.*

Proof. Let $M = \{a_i b_i : 1 \leq i \leq n\}$, and let $u, v \in N_M$ be adjacent. By 6.2 we may assume that u is complete to $\{a_1, \dots, a_n\}$ and v to $\{b_1, \dots, b_n\}$. Let N_1, N_2 be the sets of vertices in $V(G) \setminus V(M)$ that are complete to $\{a_1, \dots, a_n\}$ and complete to $\{b_1, \dots, b_n\}$ respectively. Thus $u \in N_1$ and $v \in N_2$. We claim that $N_M = N_1 \cup N_2$; for let $w \in N_M$. If say w is adjacent to a_1 and to b_2 , then it shares a neighbour with each of u, v and so is nonadjacent to them both. There are at most two values of i such that w is adjacent to b_i , by 6.2 applied to w, u , and also only two such that w is adjacent to a_i , by 6.2 applied to w, v . But this is impossible since $n \geq 5$. Consequently $N_M = N_1 \cup N_2$. For each $u' \in N_1$ and $v' \in N_2$, $a_1 u' a_2$ meets $b_3 v'$, and so u', v' are adjacent; so N_1 is complete to N_2 . Hence $(\{a_1, \dots, a_n\}, \{b_1, \dots, b_n\})$ is a nontrivial simplicial homogeneous pair, and the theorem holds. This proves 9.1. \blacksquare

9.2 *Let G be connected and $\{Q_5, K_3\}$ -free, with $\mu(G) = 4$, and let M be a maximum induced matching in G . Suppose that N_M is not stable. Then either G is Clebschian or G admits a nontrivial simplicial homogeneous pair.*

Proof. Let $M = \{a_i b_i : 1 \leq i \leq 4\}$. By hypothesis, there exist adjacent $p_1, p_2 \in N_M$; and by 6.2, $\delta_M(p_1, p_2) = 4$. If each $v \in N_M$ satisfies $\delta_M(p_1, v) = 0$ or $\delta_M(p_2, v) = 0$ then G admits a nontrivial simplicial homogeneous pair, so we may assume there exists $p_3 \in N_M$ with $\delta_M(p_1, p_3), \delta_M(p_2, p_3) \geq 1$. By 6.2, p_3 is nonadjacent to p_1, p_2 , so $\delta_M(p_1, p_3), \delta_M(p_2, p_3) \leq 2$; and since δ_M satisfies the triangle inequality, it follows that $\delta_M(p_1, q), \delta_M(p_2, q) = 2$, for each $q \in N_M$ with $\delta_M(p_1, q), \delta_M(p_2, q) \neq 0$.

(1) *Every vertex of G belongs to $V(M) \cup N_M$.*

Let $R = V(G) \setminus (V(M) \cup N_M)$, and suppose that $r \in R$. Thus R is stable, from the maximality of M , and so r in R has a neighbour in N_M . Let Q' be the set of $q \in N_M$ such that r is adjacent to q . Not both $p_1, p_2 \in Q'$ since p_1 is adjacent to p_2 ; and from the existence of p_1, p_2, p_3 , it follows that there exist nonadjacent $q, q' \in P$, with $q \notin Q'$ and $q' \in Q'$. We may assume that q is adjacent to a_1, a_2 and q' is adjacent to b_1, b_2 . But then the 3-path a_1-q-a_2 misses the edge rq' , a contradiction. This proves (1).

From 6.2 and (1), the neighbour set of each $v \in N_M$ is determined by the set of neighbours of v in $V(M)$, and since G has no twins, there do not exist distinct $u, v \in N_M$ with $\delta_M(u, v) = 0$. But then G is Clebschian. This proves 9.2. ■

9.3 *Let G be connected and $\{Q_5, K_3\}$ -free, with no twins and with $\mu(G) = 3$, and let M be a maximum induced matching in G . Suppose that N_M is not stable. Then either G is Clebschian, or G admits a nontrivial simplicial homogeneous pair.*

Proof. Let $M = \{a_1 b_1, a_2 b_2, a_3 b_3\}$. By hypothesis, there exist $p_1, p_2 \in N_M$ with $\delta_M(p_1, p_2) = 3$. For $i = 1, 2$, let P_i be the set of $v \in N_M$ such that $\delta_M(p_i, v) = 0$. If each $v \in N_M$ satisfies $\delta_M(p_1, v) = 0$ or $\delta_M(p_2, v) = 0$ then G admits a nontrivial simplicial homogeneous pair, so we may assume there exists $p_3 \in N_M$ with $\delta_M(p_1, p_3), \delta_M(p_2, p_3) \geq 1$. Let N_1 be the set of $v \in N_M$ adjacent to an odd number of a_1, a_2, a_3 , and $N_2 = N_M \setminus N_1$. We may assume that $p_i \in N_i$ for $i = 1, 2$.

Let $R = V(G) \setminus (V(M) \cup N_M)$. Thus R is stable, from the maximality of M , and every vertex in R has a neighbour in N_M . For $i = 1, 2$, every two vertices in N_i have a common neighbour in $V(M)$ and so N_i is stable.

(1) *If $r \in R$, and $q_1, q_2 \in N_M$ with $\delta_M(q_1, q_2) = 2$, and r is adjacent to q_1 , then r is adjacent to q_2 .*

We may assume that q_1 is adjacent to a_1, a_2, a_3 , and q_2 is adjacent to b_1, b_2, a_3 . Since the 3-path $r-q_1-a_1$ meets the edge $q_2 b_2$, it follows that r is adjacent to q_2 . This proves (1).

(2) *For $i = 1, 2$, if $r \in R$ has a neighbour in N_i , then r has a neighbour in P_i .*

Let r be adjacent to $v \in N_1$ say. If $\delta_M(v, p_1) = 0$ the claim is true, and if $\delta_M(v, p_1) = 2$ the claim follows from (1). This proves (2).

For $i = 1, 2$ let R_i be the set of vertices in R with a neighbour in N_i .

(3) *Not both R_1, R_2 are nonempty.*

If $r \in R_1 \cap R_2$, then by (2) r has neighbours in both P_1, P_2 , which is impossible since they are complete to each other by 6.2. So $R_1 \cap R_2 = \emptyset$. Suppose that there exist $r_1 \in R_1$ and $r_2 \in R_2$. We may assume that $\delta_M(p_3, p_1) = 1$; and p_1 is adjacent to a_1, a_2, a_3 and p_3 to a_1, a_2, b_3 . By (2), r_2 has a neighbour in P_2 , and hence is adjacent to p_3 by (1). Choose $v_1 \in P_1$ adjacent to r_1 . Then $r_2-p_3-b_3$ misses r_1v_1 , a contradiction. This proves (3).

(4) *We may assume that for $i = 1, 2$, R_i is complete to N_i .*

Let $i = 1$ say, and let $r_1 \in R_1$. By (2), r_1 has a neighbour in P_1 , and hence r_1 is adjacent to q for every $q \in N_i$ with $\delta_M(p_1, q) = 2$, by (1). Thus if there exists $q \in N_i$ with $\delta_M(p_1, q) = 2$, then by (1) again, r_1 is complete to P_1 and hence to N_1 ; so we may assume that $N_1 = P_1$. Thus (R_1, P_1) is a pendant homogeneous pair, and we may assume that $|R_1| = |N_1| = 1$, and once again the claim holds. This proves (4).

(5) *For $i = 1, 2$, $|R_i| \leq 1$.*

Suppose that $|R_1| \geq 2$ say. From the existence of p_3 , there exist $q_1 \in N_1$ and $q_2 \in N_2$ with $\delta_M(q_1, q_2) = 1$. Let q_1 be adjacent to a_1, a_2, a_3 , and q_2 to a_1, a_2, b_3 say. By (3) and (4), R_1 is complete to q_1 and anticomplete to q_2 . Then a 3-path in $G[R_1 \cup \{q_1\}]$ misses the edge q_2b_3 . This proves (5).

If $R = \emptyset$, G is Clebschian. So we may assume that $|R_1| = \{r_1\}$ say, by (5), and hence $R_2 = \emptyset$, by (3). Again G is Clebschian (to see this, add another vertex complete to $\{r_1\} \cup N_2$; if all the sets $N(q)$ ($q \in Q$) are singletons, this is the Clebsch graph). This proves 9.3. ■

In summary, from 9.1, 9.2 and 9.3, we have shown that:

9.4 *Let G be connected and $\{Q_5, K_3\}$ -free, with no twins and with $\mu(G) \geq 3$, and let M be a maximum induced matching in G . If N_M is not stable, then either G is Clebschian or G admits a nontrivial simplicial homogeneous pair.*

Combining with 6.3, 8.3, and 7.6, we obtain 6.1, which we restate:

9.5 *Let G be connected and $\{Q_5, K_3\}$ -free, with no twins. Then either:*

- *G is Clebschian, climbable, or bipartite; or*
- *G admits a nontrivial simplicial homogeneous pair.*

And finally we obtain our main result 1.1, which we restate:

9.6 *Let G be a connected $\{P_6, K_3\}$ -free graph without twins. Then either*

- *G is Clebschian, climbable, or a V_8 -expansion; or*
- *G admits a nontrivial simplicial homogeneous pair.*

Proof. From 9.5 and 4.2, it suffices to check that G satisfies the theorem when G is a half-graph expansion. Let G be a connected half-graph expansion without twins. If $|V(G)| \leq 4$ then G is climbable, so we assume that $|V(G)| \geq 5$. Let (A, B) be a bipartition of G , and let $A_1, \dots, A_n, B_1, \dots, B_n$ be as in the definition of half-graph expansion, where $A_i \cup B_i \neq \emptyset$ for $1 \leq i \leq n$. Since $A_1 \cup B_1 \neq \emptyset$ and G is connected, it follows that $A_1 \neq \emptyset$.

Suppose that $n = 1$. Since $|A_1 \cup B_1| = |V(G)| \geq 5$, it follows that (A_1, B_1) is matched or antimatched; and it is not matched since G is connected. Thus if $n = 1$ then G is an antimatching and so climbable, and therefore we may assume that $n \geq 2$.

If $B_1 = \emptyset$ then $|A_1| = 1$ since G has no twins; and if $B_1 \neq \emptyset$ then (A_1, B_1) is a simplicial homogeneous pair (since $n > 1$), so we may assume that $|A_1| = 1$ and $|B_1| \leq 1$. Thus in either case we have $|A_1| = 1$ and $|B_1| \leq 1$, and consequently $|V(G) \setminus (A_1 \cup B_1)| \geq 3$. Since G has no twins, it follows that $A_2 \cup \dots \cup A_n \neq \emptyset$, and so $(A_2 \cup \dots \cup A_n, B_2 \cup \dots \cup B_n)$ is a nontrivial simplicial homogeneous pair. This proves 9.6. ■

10 A structure theorem for $\{Q_5, K_3\}$ -free graphs

Finally, let us see how to convert 6.1 to a construction for all $\{Q_5, K_3\}$ -free graphs. There are two problems with 6.1:

- adding twins without restriction might introduce Q_5 ;
- if G admits a simplicial homogeneous pair (A, B) , and the corresponding graphs H, J (defined as usual) are $\{Q_5, K_3\}$ -free, that does not guarantee that G is $\{Q_5, K_3\}$ -free.

But both can easily be remedied. Let v be a vertex of G such that no edge incident with v belongs to an induced matching with cardinality two. Such a vertex v is called a *twinnable vertex* of G . If G is $\{Q_5, K_3\}$ -free, then adding twins to a twinnable vertex v results in a larger $\{Q_5, K_3\}$ -free graph.

Let H be an antisubmatching, with bipartition (A, B) , and let $a \in A$ and $b \in B$ be nonadjacent. Add a new vertex c adjacent to a, b , forming a graph G ; then c is a twinnable vertex of G . Any graph obtained from G by adding twins of c is called an *extended antisubmatching*. (This is the only time when we need to add twins to a climbable graph, so we gave such graphs a name.)

Let (A, B) be a nontrivial simplicial homogeneous pair of G , and correspondingly let G be obtained from H by substituting $J = G[A \cup B]$ for some edge ab of H . Let us say that (A, B) is *submatched* if every vertex in A has at most one neighbour in B and vice versa. This makes it safe again; if H, J are $\{Q_5, K_3\}$ -free, and (A, B) is a nontrivial submatched simplicial homogeneous pair of G , then G is $\{Q_5, K_3\}$ -free. Then we have a construction for all connected $\{Q_5, K_3\}$ -free graphs, as follows:

10.1 *Let G be connected and $\{Q_5, K_3\}$ -free. Then either:*

- G may be obtained from a Clebschian graph by adding twins of some of its twinnable vertices; or
- G is climbable, or an extended antisubmatching; or
- G is a half-graph expansion; or

- G admits a nontrivial submatched simplicial homogeneous pair.

Proof. Let G be connected and $\{Q_5, K_3\}$ -free. If G is bipartite, then G is a half-graph expansion by 5.2, and the theorem holds. Thus we may assume that G is not bipartite.

Suppose that G admits a nontrivial simplicial homogeneous pair (A, B) . If (A, B) is submatched then the theorem holds, so we assume not; and so we may assume that some vertex $a \in A$ has two neighbours $b_1, b_2 \in B$. Let C be the set of vertices in $V(G) \setminus (A \cup B)$ that are complete to A and anticomplete to B , and define D similarly with A, B exchanged. Let $R = V(G) \setminus (A \cup B \cup C \cup D)$. Since (A, B) is simplicial, C is complete to D ; and since G is Q_5 -free, the 3-path b_1-a-b_2 meets every edge, and so R is stable. Since no vertex in R has a neighbour in C and another in D , it follows that G is bipartite, a contradiction. Thus we may assume that G does not admit a nontrivial simplicial homogeneous pair.

Now G can be obtained from a graph G' by adding twins, where G' has no twins, and G' is also connected and $\{Q_5, K_3\}$ -free. It follows that G' has no nontrivial simplicial homogeneous pair. From 9.5 G' is Clebschian, climbable, or bipartite. Since G is not bipartite, neither is G' . If G' is Clebschian, then, since adding twins of a non-twinable vertex will introduce Q_5 , it follows that G may be obtained from a Clebschian graph by safely adding twins, and the theorem holds. If G' is climbable and not Clebschian then (since G' is not bipartite) it is easy to check that either G is climbable or G is an extended antisubmatching (we omit the details, which are straightforward). This proves 10.1. ■

One could, if necessary, be more exact about safely adding twins to a Clebschian graph; one could list exactly the subgraphs of the Clebsch graph that have twinable vertices, and list those vertices. (There are not many.)

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