Excluding paths and antipaths

Maria Chudnovsky*
Columbia University, New York, NY 10027

Paul Seymour†
Princeton University, Princeton, NJ 08540

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Abstract

The Erdös-Hajnal conjecture states that for every graph H, there exists a constant $\delta(H) > 0$, such that if a graph G has no induced subgraph isomorphic to H, then G contains a clique or a stable set of size at least $|V(G)|^{\delta(H)}$. This conjecture is still open. We consider a variant of the conjecture, where instead of excluding H as an induced subgraph, both H and H^c are excluded. We prove this modified conjecture for the case when H is the five-edge path. Our second main result is an asymmetric version of this: we prove that for every graph G such that G contains no induced six-edge path, and G^c contains no induced four-edge path, G contains a polynomial-size clique or stable set.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. The *complement* G^c of G is the graph with vertex set $V(G)$, such that two vertices are adjacent in G if and only if they are non-adjacent in G^c. A *clique* in G is a set of vertices all pairwise adjacent. A *stable set* in G is a set of vertices all pairwise non-adjacent (thus a stable set in G is a clique in G^c.) Given a graph H, we say that G is *H-free* if G has no induced subgraph isomorphic to H. If G is not H-free, we say that G *contains* H. For a family $\mathcal{F}$ of graphs, we say that G is $\mathcal{F}$-free is G is F-free for every $F \in \mathcal{F}$.

It is a well-known theorem of Erdös [5] that for all n there exist graphs on n vertices, with no clique or stable set of size larger than $O(\log n)$. However, in 1989 Erdös and Hajnal [6] conjectured that the situation is very different for graphs that are H-free for some fixed graph H, the following (this is the *Erdös-Hajnal conjecture*):

1.1 *For every graph H, there exists a constant $\delta(H) > 0$, such that every H-free graph G has either a clique or a stable set of size at least $O(|V(G)|^{\delta(H)})$.*

We say that a graph H has the *Erdös-Hajnal property* if there exists a constant $\delta(H) > 0$, such that every H-free graph G has either a clique or a stable set of size at least $O(|V(G)|^{\delta(H)})$.

Here we consider a variant of 1.1, the following:

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1.2 *For every graph H, there exists a constant $\delta(H) > 0$, such that every $\{H, H^c\}$-free graph G has either a clique or a stable set of size at least $O(|V(G)|^{\delta(H)})$.*

Our first main result is that 1.2 holds if H is the five-edge-path. Let us say that a graph G is *pure* if no induced subgraph of G or G^c is isomorphic to the five-edge path. We prove:

1.3 *There exists $\delta > 0$ such that every pure graph G has either a clique or a stable set of size at least $O(|V(G)|^{\delta})$.*

We also prove an asymmetric version of this result. Let us call a graph G *pristine* if no induced subgraph of G is isomorphic to the six-edge path, and no induced subgraph of G^c is isomorphic to the four-edge path. We prove:

1.4 *There exists $\delta > 0$ such that every pristine graph G has either a clique or a stable set of size at least $O(|V(G)|^{\delta})$.*

Let G be a graph. For $X \subseteq V(G)$, we denote by $G|X$ the subgraph of G induced by X. We write $G \setminus X$ for $G|(V(G) \setminus X)$, and $G \setminus v$ for $G \setminus \{v\}$, where $v \in V(G)$. For two disjoint subsets A and B of $V(G)$, we say that A is *complete* to B if every vertex of A is adjacent to every vertex of B, and that A is *anticomplete* to B if every vertex of A is non-adjacent to every vertex of B. If $A = \{a\}$ for some $a \in V(G)$, we write "a is complete (anticomplete) to B" instead of "$\{a\}$ is complete (anticomplete) to B". If $b \in V(G) \setminus A$ is neither complete nor anticomplete to A, we say that b is *mixed* on A. For $v \in V(G)$ we denote by $N_G(v)$ (or $N(v)$ when there is no risk of confusion) the set of neighbors of v in G (in particular, $v \notin N_G(v)$).

We denote by $\omega(G)$ the largest size of a clique in G, by $\alpha(G)$ the largest size of a stable set in G, and by $\chi(G)$ the chromatic number of G. The graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G. The Strong Perfect Graph Theorem [2] characterizes perfect graphs by forbidden induced subgraphs:

1.5 *A graph G is perfect if and only if no induced subgraph of G or G^c is an odd cycle of length at least five.*

Let us say that a function $f : V(G) \to [0, 1]$ is *good* if for every perfect induced subgraph P of G

$$\Sigma_{v \in V(P)} f(v) \leq 1.$$

For $\alpha \geq 1$, the graph G is *α-narrow* if for every good function f

$$\Sigma_{v \in V(G)} f(v)^{\alpha} \leq 1.$$

Thus perfect graphs are 1-narrow. The following was shown in [3], and then again with a much easier proof in [4]:

1.6 *If a graph G is α-narrow for some $\alpha > 1$, then G contains a clique or a stable set of size at least $|V(G)|^{\frac{1}{2\alpha}}$.*

Consequently, in order to prove that a certain graph H has the Erdös-Hajnal property, it is enough to show that there exists $\alpha \geq 1$ such that all H-free graphs are α-narrow. This conjecture was formally stated in [4]:

1.7 *For every graph H, there exists a constant $\alpha(H) \geq 1$, such that every H-free graph G is α-narrow.*

In fact, in order to prove 1.3, we show that

1.8 *There exists $\alpha > 1$ such that every pure graph is α-narrow.*

Similarly, in order to prove 1.4, we show that

1.9 *There exists $\alpha > 1$ such that every pristine graph is α-narrow.*

Fox [7] proved that 1.6 is in fact equivalent to 1.1, more precisely, he showed:

1.10 *Let H be a graph for which there exists a constant $\delta(H) > 0$ such for every H-free graph G either $\omega(G) \geq |V(G)|^{\delta(H)}$ or $\alpha(G) \geq |V(G)|^{\delta(H)}$. Then every H-free graph G is $\frac{3}{\delta(H)}$-narrow.*

This paper is organized as follows. In Section 2 we discuss the tools used in the proofs of 1.8 and 1.9, and prove 1.8 assuming and additional result, 2.5. In Section 3 we prove 2.5. Sections 4 and 5 are devoted to results similar to 2.5, needed for the proof of 1.9. The proof of 1.9 assuming the results of Section 4 and Section 5 is at the end of Section 4. Finally, in Section 6 we include a proof of 1.10.

2 The power of substitution

Given graphs H_1 and H_2, on disjoint vertex sets, each with at least two vertices, and $v \in V(H_1)$, we say that H is *obtained from H_1 by substituting H_2 for v*, or *obtained from H_1 and H_2 by substitution* (when the details are not important) if:

- $V(H) = (V(H_1) \cup V(H_2)) \setminus \{v\}$,
- $H|V(H_2) = H_2$,
- $H|(V(H_1) \setminus \{v\}) = H_1 \setminus v$, and
- $u \in V(H_1)$ is adjacent in H to $w \in V(H_2)$ if and only if w is adjacent to v in H_1.

A related notion is that of a "homogeneous set" in a graph. Given a graph G, a subset $X \subseteq V(G)$ is a *homogeneous set* in G if

- $1 < |X| < |V(G)|$, and
- every vertex of $V(G) \setminus X$ with a neighbor in X is complete to X.

We say that G admits a *homogeneous set decomposition* if there is a homogeneous set in G. Thus a graph admits a homogeneous set decomposition if and only if it is obtained from smaller graphs by substitution. Finally, we say that a graph is *prime* if it is not obtained from smaller graphs by substitution.

There are three main ingredients in our proof of 1.8. The first is a theorem of Alon, Pach and Solymosi [1], stating that the Erdös-Hajnal property is preserved under substitution:

2.1 *Let H_1 and H_2 be graphs, and let $0 < \delta_1, \delta_2 \le 1$ such that for $i = 1, 2$, every H_i-free graph G satisfies* $\max(\alpha(G), \omega(G)) \ge O(|V(H)|^{\delta_i})$. *Let $|V(H_1)| = k$, and let H be obtained by substitution H_2 for a vertex of H_1. Then for every δ such that*

$$\delta \le \frac{\delta_1 \delta_2}{\delta_1 + k\delta_2},$$

every H-free graph G satisfies $\max(\alpha(G), \omega(G)) \ge O(|V(H)|^{\delta})$.

A class $\mathcal{G}$ of graphs is *hereditary* if for every $G \in \mathcal{C}$, all induced subgraphs of G belong to $\mathcal{C}$. In fact, we need a slight strengthening of 2.1.

2.2 *Let $\mathcal{C}$ be a hereditary class of graphs. Let $\mathcal{H}_1$ be a finite family of graphs, let H_2 be a graph, and write $\mathcal{H}_2 = \{H_2\}$. Let $0 < \delta_1, \delta_2 \le 1$ such that for $i = 1, 2$, every $\mathcal{H}_i$-free graph $G \in \mathcal{C}$ satisfies* $\max(\alpha(G), \omega(G)) \ge O(|V(H)|^{\delta_i})$. *Let $k = \max_{H_1 \in \mathcal{H}_1} |V(H_1)|$, and for every $H_1 \in \mathcal{H}_1$, let $v(H_1) \in V(H_1)$. Define $\mathcal{H}$ to be the family of graphs obtained by substituting H_2 for $v(H_1)$ in H_1 for every $H_1 \in \mathcal{H}_1$. Then for every δ such that*

$$\delta \le \frac{\delta_1 \delta_2}{\delta_1 + k\delta_2},$$

every $\mathcal{H}$-free graph $G \in \mathcal{C}$ satisfies $\max(\alpha(G), \omega(G)) \ge O(|V(G)|^{\delta})$.

The proof of 2.2 is essentially the same as that of 2.1, and we omit it here. Given a hereditary graph class $\mathcal{C}$, we say that a family of graphs $\mathcal{H}$ has *the Erdös-Hajnal property for $\mathcal{C}$* if there exists a constant $\delta(\mathcal{H})$ such that every H-free graph $G \in \mathcal{C}$ satisfies $\max(\alpha(G), \omega(G)) \ge O(|V(G)|^{\delta(H)})$. A graph H has the the Erdös-Hajnal property for $\mathcal{C}$ if the family $\{H\}$ does.

The second ingredient also deals with substitutions, but this time we take advantage of the fact that the graph G, rather than H, from 1.1 is not prime. First, let us generalize the notion of a homogeneous set a little. Let $\mathcal{C}$ be a hereditary class of graphs, let $G \in \mathcal{C}$, and let (X, A, C) be a partition of $V(G)$, where $1 < |X| < |V(G)|$. Let G' be the graph obtained from $G \setminus X$ by adding a new vertex x, complete to C and anticomplete to A. Then (X, A, C) is a *$\mathcal{C}$-quasi-homogeneous set* in G if

- $G' \in \mathcal{C}$, and
- If P is a perfect induced subgraph of G' with $x \in V(P)$, and Q is a perfect induced subgraph of $G|X$, then $G|((V(P) \setminus \{x\}) \cup V(Q))$ is perfect.

We say that G admits a *$\mathcal{C}$-quasi-homogeneous set decomposition* if there is a $\mathcal{C}$-quasi-homogeneous set in G.

If $\mathcal{C}$ is a hereditary class of graphs, $G \in \mathcal{C}$, X is a homogeneous set in G, C is the set of vertices of $G \setminus X$ complete to X, and A is the set of vertices of $G \setminus X$ anticomplete to X, then [8] implies that (X, A, C) is a $\mathcal{C}$-quasi-homogeneous set in G.

The following was essentially proved in [4]:

2.3 *Let $\mathcal{C}$ be a hereditary class of graphs, let $G \in \mathcal{C}$, and let $\alpha > 1$. Let (X, A, C) be a $\mathcal{C}$-quasi-homogeneous set in G, and let G' be the graph obtained from $G \setminus X$ by adding a new vertex x complete to C and anticomplete to A. If the graphs G' and $G|X$ are α-narrow, then G is α-narrow.*

2.3 has the following immediate corollary:

2.4 *Let $\alpha > 1$, and G_1, G_2 be α-narrow graphs. If G is obtained from G_1 and G_2 by substitution, then G is α-narrow.*

Finally, the third ingredient of the proof of 1.8 is a structural result that we prove in the next section, as follows. Let C_5 denote the cycle of length five. Let Q be the graph obtained from C_5 by substituting a copy of C_5 for each of its vertices. More precisely,

- $V(Q) = \bigcup_{i=1}^{5} V^i$, where $V^i = \{v_1^i, v_2^i, v_3^i, v_4^i, v_5^i\}$ for every $i \in \{1, \ldots, 5\}$
- $Q|V^i$ is isomorphic to C_5 for every $i \in \{1, \ldots, 5\}$, and
- for $1 \leq i < j \leq 5$, V^i is complete to V^j if $j - i \in \{1, 4\}$, and V^i is anticomplete to V^j if $j - i \in \{2, 3\}$.

We prove:

2.5 *If a pure graph G contains Q, then G admits a homogeneous set decomposition.*

We can now prove 1.8 assuming 2.5.

Proof of 1.8. Let $\mathcal{C}$ be the class of pure graphs. Since by 1.5 every C_5-free pure graph is perfect, and therefore 1-narrow, 1.6 implies that C_5 has the Erdös-Hajnal property for $\mathcal{C}$. Therefore, by 2.2, Q has the Erdös-Hajnal property for $\mathcal{C}$. Let δ be such that every Q-free graph $G \in \mathcal{C}$ has a clique or a stable set of size at least $|V(G)|^{\delta}$, and let c be as in 1.10. Let $\alpha = \frac{3}{\delta}$.

Let $G \in \mathcal{C}$ be such that G is not α-narrow, and subject to that with $|V(G)|$ minimum. By 1.10, G is not Q-free. By 2.5, G is obtained from smaller graphs, G_1 and G_2, by substitution; and since $\mathcal{C}$ is hereditary, $G_1, G_2 \in \mathcal{C}$. But now, by the minimality of $|V(G)|$, each of G_1, G_2 is α-narrow, contrary to 2.4. This proves 1.8. ∎

The proof of 1.4 is similar, but has more steps, and we postpone it until later.

3 The proof of 2.5

Let G be a graph. A *path* P in G is an induced subgraph with vertices $p_1, \ldots, p_k$ such that either $k = 1$, or for $i, j \in \{1, \ldots, k\}$, p_i is adjacent to p_j if $|i - j| = 1$ and p_i is non-adjacent to p_j if $|i - j| > 1$. Under these circumstances we say that P is a path *from* p_1 *to* p_k, its *interior* is the set $P^* = V(P) \setminus \{p_1, p_k\}$, and the *length* of P is $k - 1$. We also say that P is a $(k-1)$-*edge path*. Sometimes, we denote P by p_1-$\ldots$-p_k. A *cycle* C in G is an induced subgraph with vertices $c_1, \ldots, c_k$ where $k \geq 3$, such that for $i, j \in \{1, \ldots, k\}$, c_i is adjacent to c_j if and only if $|i - j| = 1$ or $|i - j| = k - 1$. Under these circumstances we call k the *length* of the cycle. Sometimes, we denote C by c_1-$\ldots$-c_k-c_1.

Given a graph G and $X \subseteq V(G)$, we say that X is *connected* if $X \neq \emptyset$ and the graph $G|X$ is connected, and *anticonnected* if $X \neq \emptyset$ and the graph $G^c|X$ is connected. We say that X is *tough* if $|X| \geq 3$ and for every partition (A, B) of X with $A, B \neq \emptyset$ either

- there exist $a \in A$ and $b_1, b_2 \in B$ such that a-b_1-b_2 is a path in G, or
- there exist $a_1, a_2 \in A$ and $b \in B$ such that a_1-a_2-b is a path in G^c.

We start with a few easy lemmas.

3.1 *Let G be a graph, and let $X \subseteq V(G)$. If X is tough, then X is both connected and anticonnected.*

Proof. It is enough to prove that X is connected; the fact that X is anticonnected follows by taking complements. Thus it is enough to show that Y is not anticomplete to Z for every partition (Y, Z) of X. But this follows immediately from the definition of a tough set. This proves 3.1. ∎

3.2 *Let G be a graph, and let $X \subseteq V(G)$. Let $v \in V(G) \setminus X$ be mixed on X. Then*

1. *If X is connected, then there exist $x, y \in X$ such that v is adjacent to x and non-adjacent to y, and x is adjacent to y.*
2. *If X is anticonnected, then there exist $x, y \in X$ such that v is adjacent to x and non-adjacent to y, and x is non-adjacent to y.*

Proof. By passing to G^c if necessary, it is enough to prove 3.2.1. Since v is mixed on X, both $N(v) \cap X$ and $X \setminus N(v)$ are non-empty. Now, since X is connected it follows that $N(v) \cap X$ is not anticomplete to $X \setminus N(v)$ and 3.2.1 follows. This proves 3.2. ∎

3.3 *$V(C_5)$ is tough.*

Proof. Let $v_1, \dots, v_5$ be the vertices of C_5, such that for $1 \leq i < j \leq 5$, v_i is adjacent to v_j if and only if $j - i \in \{1, 4\}$. Let (A, B) be a partition of $\{v_1, \dots, v_5\}$ with $A, B \neq \emptyset$. Passing to the complement if necessary, we may assume that $|A| \leq 2$. This implies that some edge of C_5 has both its ends in B, say $v_1, v_2 \in B$; and since $A \neq \emptyset$, we may assume that $v_5 \in A$. But now setting $a = v_5$, $b_1 = v_1$ and $b_2 = v_2$, the first statement of the definition of a tough set holds. This proves 3.3. ∎

We now prove 2.5 that we restate:

3.4 *If a pure graph G contains Q, then G admits a homogeneous set decomposition.*

Proof. Suppose not, and let G be a pure graph that has an induced subgraph isomorphic to Q, and such that G does not admit a homogeneous set decomposition. A *Q-structure* in G consists of disjoint subsets $V_1, \dots, V_5$ such that

- for $1 \leq i < j \leq 5$, V_i is complete to V_j if $j - i \in \{1, 4\}$, and V_i is anticomplete to V_j if $j - i \in \{2, 3\}$, and
- V_i is tough for $i \in \{1, \dots, 5\}$.

We denote this Q-structure by $(V_1, V_2, V_3, V_4, V_5)$. Since G contains Q, it follows that G contains a Q-structure. Let $(V_1, V_2, V_3, V_4, V_5)$ be a Q-structure in G with $W = \bigcup_{i=1}^{5} V_i$ maximal.

We remark that both the hypotheses and the conclusion of 3.4 are invariant under taking complements, and a Q-structure in G is also a Q-structure in G^c (after re-ordering). We will use this symmetry between G and G^c in the course of the proof. For $i \in \{1, \dots, 5\}$, let X_i be the set of all vertices of $V(G) \setminus V_i$ that are mixed on V_i. Since G has no homogeneous set, $X_i \neq \emptyset$ for all $i \in \{1, \dots, 5\}$. From the definition of a Q-structure, we deduce that $X_i \cap W = \emptyset$ for all $i \in \{1, \dots, 5\}$. Let $X = \bigcup_{i=1}^{5} X_i$. For $i \in \{1, \dots, 5\}$ and $v \in V(G) \setminus W$, let $A_i(v) = N(v) \cap V_i$, and $B_i(v) = V_i \setminus A_i(v)$.

(1) No $v \in X_1$ is complete to $V_2 \cup V_5$, and anticomplete to $V_3 \cup V_4$.

Suppose such a vertex v exists. We claim that $V_1 \cup \{v\}$ is tough. Let $A = A_1(v)$, and $B = B_1(v)$. Since V_1 is tough, by taking complements if necessary, we may assume that there exist $a \in A$ and $b_1, b_2 \in B$ such that a-b_1-b_2 is a path in G. Let (A', B') be a partition of $V_1 \cup \{v\}$ with $A', B' \neq \emptyset$. We need to prove that one of the statements of the definition of a tough set holds for (A', B'). If both $A' \cap V_1 \neq \emptyset$ and $B' \cap V_1 \neq \emptyset$, then the result follows from the fact that V_1 is tough, so we may assume that either $A' = \{v\}$, or $A' = V_1$. If $A' = \{v\}$, then v-a-b_1 is a path in G, and the first statement in the definition of a tough set is satisfied; and if $A' = V_1$, then a-b_2-v is a path in G^c, and the second statement in the definition of a tough set is satisfied. This proves the claim that $V_1 \cup \{v\}$ is tough. But now $(V_1 \cup \{v\}, V_2, V_3, V_4, V_5)$ is a Q-structure, contrary to the maximality of W. This proves (1).

We say that $v \in X_i$ is a *path vertex for V_i* if there exist $a \in A_i(v)$ and $b_1, b_2 \in B_i(v)$ such that a-b_1-b_2 is a path in G; and that $v \in X_i$ is an *antipath vertex for V_i* if there exist $a_1, a_2 \in A_i(v)$ and $b \in B_i(v)$ such that b-a_1-a_2 is a path in G^c.

(2) If $v \in X_1$ is a path vertex for V_1, then v is not mixed on $V_3 \cup V_4$; and if $v \in X_1$ is an antipath vertex for V_1, then v is not mixed on $V_2 \cup V_5$. Consequently, no $v \in X_1$ is mixed on both $V_2 \cup V_5$ and $V_3 \cup V_4$.

Let $v \in X_1$. By taking complements if necessary, we may assume that v is a path vertex for V_1 and there exist $a \in A_1(v)$ and $b_1, b_2 \in B_1(v)$ such that a-b_1-b_2 is a path in G. If v is mixed on $V_3 \cup V_4$, then, since $V_3 \cup V_4$ is connected, there exist $x, y \in V_3 \cup V_4$ as in 3.2.1. But now b_2-b_1-a-v-x-y is a five-edge path in G, contrary to the fact that G is pure. Since V_1 is tough, it follows that every vertex of X_1 is either a path or an antipath vertex for V_1, and so no $v \in X_1$ is mixed on both $V_2 \cup V_5$, and $V_3 \cup V_4$. This proves (2).

(3) If $v \in X_1 \cap X_2$, then v is anticomplete to $V_3 \cup V_4 \cup V_5$; and if $v \in X_1 \cap X_3$, then v is complete to $V_2 \cup V_4 \cup V_5$.

By taking complements, it is enough to prove the first statement of (3). By 3.1 and 3.2.1, there exist $a_1 \in A_1(v)$ and $b_1 \in B_1(v)$ such that a_1 is adjacent to b_1. By 3.1 and 3.2.2, there exist $a_2 \in A_2(v)$ and $b_2 \in B_2(v)$ such that a_2 is non-adjacent to b_2. If there exists $a_3 \in A_3(v)$, then a_1-a_3-b_1-v-b_2-a_2 is a five-edge path in G^c, a contradiction. So $A_3(v) = \emptyset$, and v is anticomplete to V_3. Similarly, v is anticomplete to V_5. Since $v \in X_1$, and v is mixed on $X_2 \cup X_5$, (2) implies that v is not mixed on

$V_3 \cup V_4$, and so v is anticomplete to V_4. Consequently v is anticomplete to $V_3 \cup V_4 \cup V_5$, and (3) follows.

We say that $v \in \bigcup_{i=1}^{5} X_i$ is *minor* if it is anticomplete to at least three of the sets sets $V_1, \ldots, V_5$, *major* if it is complete to at least three of the sets $X_1, \ldots, X_5$, and *intermediate* otherwise. Observe that passing to G^c switches minor vertices with major, and leaves the set of intermediate vertices unchanged.

(4) If $v \in X_1$ and v is intermediate, $v \notin \bigcup_{i=2}^{5} X_i$, and v is complete to $V_{i-2} \cup V_{i+2}$, and anticomplete to $V_{i-1} \cup V_{i+1}$ (here the index arithmetic is mod 5).

By (2) and passing to the complement if necessary, we may assume that v is not mixed on $V_3 \cup V_4$. If v is complete to $V_3 \cup V_4$, then by (3) $v \notin X_2 \cup X_5$, and since v is intermediate, it follows that v is anticomplete to $V_2 \cup V_5$. If v is anticomplete to $V_3 \cup V_4$, then since v is intermediate, v has neighbors in each of V_2, V_5; now by (3) v is complete to $V_2 \cup V_5$, contrary to (1). This proves (4).

(5) If $x_1 \in X_1$ and $x_2 \in X_2$ are intermediate, then x_1 is adjacent to x_2; and if $x_1 \in X_1$ and $x_3 \in X_3$ are intermediate, then x_1 is non-adjacent to x_3.

By taking complements, it is enough to prove the first statement of (5). Suppose x_1 is non-adjacent to x_2. Let $v_1 \in B_1(x_1), v_2 \in B_2(x_2), v_3 \in V_3$ and $v_5 \in V_5$. Then x_1-v_3-v_2-v_1-v_5-x_2 is a five-edge path in G, a contradiction. This proves (5).

(6) At most two of the sets $X_1, \ldots, X_5$ contain intermediate vertices.

Suppose at least three of the sets $X_1, \ldots, X_5$ contain intermediate vertices. By taking complements if necessary, we may assume that $x_1 \in X_1$, $x_2 \in X_2$ and $x_3 \in X_3$ are intermediate. By (5), the pairs x_1x_2, x_2x_3 are adjacent, and the pair x_1x_3 is non-adjacent. Let $v_1 \in A_1(x_1), v_4 \in V_4$, and $v_5 \in V_5$. Then v_5-x_1-x_3-v_4-v_1-x_2 is a five-edge path in G^c, a contradiction. This proves (6).

(7) At most one of X_1, X_3 contains a minor vertex.

Suppose $x_1 \in X_1$ and $x_3 \in X_3$ are both minor. By (3), $x_1 \notin X_3 \cup X_4$, and $x_3 \notin X_1 \cup X_5$, and in particular, $x_1 \neq x_3$. By (2), if x_1 is a path vertex for V_1, then x_1 is anticomplete to $V_3 \cup V_4$, and if x_1 is an antipath vertex for V_1, then x_1 is anticomplete to $V_2 \cup V_5$. Similarly, if x_3 is a path vertex for V_3, then x_3 is anticomplete to $V_1 \cup V_5$, and if x_3 is an antipath vertex for V_3, then x_3 is anticomplete to $V_2 \cup V_4$. Since V_1, V_3 are tough, 3.1 and 3.2.1 imply that there exist $a_1 \in A_1(x_1), b_1 \in B_1(x_1), a_3 \in A_3(x_3), b_3 \in B_3(x_3)$ such that a_1b_1 and a_3b_3 are edges of G. By 3.1 and 3.2.2, there exist $a_3' \in A_3(x_3), b_3' \in B_3(x_3)$ such that a_3' is non-adjacent to b_3'.

Suppose first that x_1 is adjacent to x_3. Since b_1-a_1-x_1-x_3-a_3-b_3 is not a five-edge path in G, we may assume using symmetry that x_3 is complete to V_1. Since x_3 is minor, this implies that x_3 is anticomplete to $V_2 \cup V_4 \cup V_5$. Suppose that exists $a_5 \in A_5(x_1)$. Then x_1 is anticomplete to $V_2 \cup V_3 \cup V_4$ (since x_1 is minor). Let $v_2 \in V_2$. Then b_3'-v_2-a_3'-x_3-x_1-a_5 is a five-edge path in G, a contradiction. This proves that x_1 is anticomplete to V_5. If there exist $u, v \in A_1(x_1)$ and $w \in B_1(x_1)$ such that w-v-u is a path in G^c, then u-v-w-x_1-v_5-x_3 is a five-edge path in G^c for every $v_5 \in V_5$, a contradiction. So no such u, v, w exist. Since V_1 is tough, it follows that x_1 is a path vertex for V_1, and x_1 is anticomplete to $V_3 \cup V_4$. But now x_1-x_3-b_1-v_5-v_4-b_3 is a five-edge path in G for every $v_4 \in V_4$, a contradiction. This proves that x_1 is non-adjacent to x_3.

If x_1 is anticomplete to $V_3 \cup V_4 \cup V_5$, and x_3 is anticomplete to $V_1 \cup V_4 \cup V_5$, then x_1-a_1-v_5-v_4-a_3-x_3 is a five-edge path in G for every $v_4 \in V_4$ and $v_5 \in V_5$, a contradiction. So either x_1 has a neighbor in $V_3 \cup V_4 \cup V_5$, or x_3 has a neighbor in $V_1 \cup V_4 \cup V_5$.

Suppose first that x_1 is anticomplete to V_3, and x_3 is anticomplete to V_1. From the symmetry, we may assume that there exists $v_5 \in V_5$, adjacent to at least one of x_1, x_3. If x_3 is adjacent to v_5, and x_1 is non-adjacent to V_5, then b_3-a_3-x_3-v_5-a_1-x_1 is a path in G. If x_1 is adjacent to v_5, and x_3 is non-adjacent to v_5, then, since both x_1 and x_3 are minor, x_1-v_5-b_1-v_2-a_3-x_3 is a path in G for every $v_2 \in B_2(x_3)$, and x_1-v_5-v_4-b_3-v_2-x_3 is a path in G for every $v_4 \in V_4$ and $v_2 \in A_2(x_3)$. Finally, if x_1 and x_3 are both adjacent to v_5, then since x_1 and x_3 are both minor, b'_3-v_2-a'_3-x_3-v_5-x_1 is a path in G for every $v_2 \in V_2$. We get a contradiction in all cases, and so we may assume that x_1 is complete to V_3.

Since x_1 is minor, it follows that x_1 is anticomplete to $V_2 \cup V_4 \cup V_5$. Recall that x_3 is either a path vertex for V_3 and is anticomplete to $V_1 \cup V_5$, or an antipath vertex for V_3 and is anticomplete to $V_2 \cup V_4$. If v_3 is anticomplete to $V_1 \cup V_5$, then choosing $a'_1 \in A_1(x_1)$ and $b'_1 \in B_1(x_1)$ non-adjacent (such a'_1 and b'_1 exist by 3.1 and 3.2.2), and $v_5 \in V_5$, we get that b'_1-v_5-a'_1-x_1-a_3-x_3 is a path in G, a contradiction. So x_3 is an antipath vertex, and x_3 is anticomplete to $V_2 \cup V_4$; and since $x_3 \notin X_1 \cup X_5$, we deduce that x_3 is complete to at least, and therefore exactly, one of V_1 and V_5. If x_3 is complete to V_1, then, since both x_1 and x_3 are minor, x_1-b_3-v_4-v_5-b_1-x_3 is a path in G for every $v_4 \in V_4$ and $v_5 \in V_5$. If x_3 is complete to V_5, then, since x_3 is minor, x_3-v_5-b_1-v_2-b_3-x_1 is a path in G for every $v_5 \in V_5$ and $v_2 \in V_2$; in both cases a contradiction. This proves (7).

(8) If $x_1 \in X_1$ is minor, and $x_2 \in X_2$ is intermediate, then x_1 is anticomplete to $V_3 \cup V_4 \cup V_5 \cup \{x_2\}$, and complete to $B_2(v_2)$.

Since $x_2 \in X_2$ is intermediate, by (4) x_2 is complete to $V_4 \cup V_5$, and anticomplete to $V_1 \cup V_3$. By 3.1 and 3.2 there exist $a_1 \in A_1(x_1)$ and $b_1 \in B_1(x_1)$ adjacent to each other, and $a'_1 \in A_1(x_1)$ and $b'_1 \in B_1(x_1)$ non-adjacent to each other. Let $b_2 \in B_2(x_2)$.

Assume first that x_1 is adjacent to x_2. If x_1 is anticomplete to $V_3 \cup V_4$, then b_1-a_1-x_1-x_2-v_4-v_3 is a path in G for every $v_3 \in V_3$ and $v_4 \in V_4$. So x_1 has neighbors in at least, and therefore exactly, one of V_3, V_4. Consequently, by (2), x_1 is an antipath vertex and x_1 is anticomplete to $V_2 \cup V_5$. If x_1 is anticomplete to V_4, then b'_1-b_2-a'_1-x_1-x_2-v_4 is a path in G for every $v_4 \in V_4$, a contradiction; therefore x_1 has a neighbor in V_4 and is anticomplete to V_3. But now x_1-x_2-v_5-b_1-b_2-v_3 is a path in G for every $v_3 \in V_3$ and $v_5 \in V_5$. This proves that x_1 is non-adjacent to x_2.

Since x_1-a_1-b_2-v_3-v_4-x_2 is not a path in G for any $v_3 \in V_3, v_4 \in V_4$, it follows that x_1 is complete to at least, and therefore exactly, one of $B_2(x_2), V_3, V_4$. If x_1 is complete to V_4, then b'_1-b_2-a'_1-x_1-v_4-x_2 is a path in G for every $v_4 \in V_4$; and if x_1 is complete to V_3, then b_1-a_1-x_1-v_3-v_4-x_2 is a path in G for every $v_3 \in V_3$ and $v_4 \in V_4$, in both cases a contradiction. This proves that x_1 is complete to $B_2(x_2)$. Since x_1 is minor, it follows that x_1 is anticomplete to $V_3 \cup V_4 \cup V_5$, and (8) follows.

(9) If $x_1 \in X_1$ is minor and $x_3 \in X_3$ is intermediate, then x_1 is anticomplete to $V_4 \cup V_5$, and either

- *x_1 is anticomplete to V_3 and complete to $V_2 \cup \{x_3\}$, or*
- *x_1 is anticomplete to $V_2 \cup \{x_3\}$, and complete to V_3.*

Since $x_3 \in X_3$ is intermediate, by (4) x_3 is complete to $V_1 \cup V_5$ and anticomplete to $V_2 \cup V_5$. Assume first that x_1 is adjacent to x_3. Suppose that x_1 is an antipath vertex for V_1; and let $p \in B_1(x_1)$ and $q, r \in A_1(x_1)$ such that p-q-r is a path in G^c. Since x_1 is minor, it follows that x_1 is anticomplete to $V_2 \cup V_4$. But now r-q-p-x_1-v_2-x_3 is a path in G^c for every $v_2 \in V_2$, a contradiction. This proves that x_1 is a path vertex for V_1, and therefore, since x_1 is minor, x_1 is anticomplete to $V_3 \cup V_4$. If x_1 has a non-neighbor $v_2 \in V_2$, then x_1-x_3-b_1-v_2-b_3-v_4 is a path in G for every $b_1 \in B_1(x_1)$, $b_3 \in B_3(x_3)$ and $v_4 \in V_4$, a contradiction; so x_1 is complete to V_2. Since x_1 is minor, it is anticomplete to V_5, and the first outcome of (9) holds.

We may therefore assume that x_1 is non-adjacent to x_3. We may assume that x_1 is anticomplete to V_3, for otherwise, since x_1 is minor and by (3), the second outcome of (9) holds. Now, if x_1 has a non-neighbor $v_4 \in V_4$, then choosing $a'_3 \in A_3(x_3)$ and $b'_3 \in B_3(x_3)$ non-adjacent (by 3.1 and 3.2.2), and $a_1 \in A_1(x_1)$, we get that b'_3-v_4-a'_3-x_3-a_1-x_1 is a path in G, a contradiction. So x_1 is complete to V_4. Since x_1 is minor, x_1 is anticomplete to $V_2 \cup V_3 \cup V_5$. Let $b_1 \in B_1(x_1), b_3 \in B_3(x_3), v_2 \in V_2$ and $v_4 \in V_4$. Then x_1-v_4-b_3-v_2-b_1-x_3 is a path in G, again a contradiction. This proves (9).

By (6) and taking complements if necessary, since $X_i \neq \emptyset$ for every $i \in \{1, \dots, 5\}$, we may assume that at least two of the sets $X_1, \dots, X_5$ contain minor vertices. By (7), it follows that there are exactly two such sets, and we may assume that $x_1 \in X_1$ and $x_2 \in X_2$ are minor, and none of X_3, X_4, X_5 contain minor vertices.

(10) There are no intermediate vertices in $X_3 \cup X_5$.

From symmetry, it is enough to prove that no vertex of X_3 is intermediate. Suppose $x_3 \in X_3$ is intermediate. By (8) applied with all indices shifted by one, we deduce that x_2 is complete to $B_3(x_3)$, and anticomplete to $V_1 \cup V_4 \cup V_5 \cup \{x_3\}$. By 3.1 and 3.2.2 there exist $a_1 \in A_1(x_1)$ and $b_1 \in B_1(x_1)$ non-adjacent to each other. Let $b_3 \in B_3(x_3)$, and $v_i \in V_i$ for $i = 4, 5$.

Assume first that x_1 is adjacent to x_3. Then, by (9), x_1 is complete to V_2 and anticomplete to $V_3 \cup V_4 \cup V_5$. Now, if x_1 adjacent to x_2, then b_1-x_3-x_1-x_2-b_3-v_4 is a path in G, and if x_1 is non-adjacent to x_2, then x_1-x_3-v_5-v_4-b_3-x_2 is a path in G; in both cases a contradiction. This proves that x_1 is non-adjacent to x_3.

Consequently, by (9), x_1 is complete to V_3, and anticomplete to $V_2 \cup V_4 \cup V_5$. Now, if x_1 is non-adjacent to x_2, then b_1-v_5-a_1-x_1-b_3-x_2 is a path in G; and if x_1 is adjacent to x_2, then choosing $a_2 \in A_2(x_2)$, we get that x_1-x_2-a_2-b_1-v_5-v_4 is a path in G; in both cases a contradiction. This proves (10).

Using symmetry, it follows from (7) applied in G^c and (10) that every vertex of $X_3 \cup X_5$ is major, every vertex of $X_1 \cup X_2$ is minor, and every vertex of X_4 is intermediate. Thus the symmetry between G and G^c is restored. For $i \in \{3, 4, 5\}$, let $x_i \in X_i$.

(11) x_4 is non-adjacent to both x_1, x_2; and x_1 is adjacent to x_2.

By (9), exchanging V_3 and V_4, x_1 is anticomplete to $V_2 \cup V_3$; and similarly x_2 is anticomplete to $V_1 \cup V_5$. By 3.1 and 3.2.2, there exist $a_1 \in A_1(x_1)$ and $b_1 \in B_1(x_1)$ non-adjacent to each other. For $i \in \{2, 4\}$, let $b_i \in B_i(x_i)$.

Suppose x_1 is adjacent to x_2. Assume that x_2 has a neighbor $v_3 \in V_3$. Then by (2) x_2 is a path vertex for V_2, and so there exist $p, q, r \in V_2$ such that x_2-p-q-r is a path in G. If x_1 has a non-neighbor $v_5 \in V_5$, then b_1-v_5-a_1-x_1-x_2-v_3 is a path in G, and if x_1 is complete to V_5, then r-q-p-x_2-x_1-v_5 is a path in G for every $v_5 \in V_5$; in both cases a contradiction. So x_2 is anticomplete to V_3, and similarly x_1 is anticomplete to V_5. Now by (9), x_4 is non-adjacent to both x_1, x_2, and (11) follows. So we may assume that x_1 is non-adjacent to x_2.

Suppose that x_4 is adjacent to both x_1 and x_2. By (9) and symmetry, this implies that x_2 is complete to V_3 and anticomplete to $V_1 \cup V_4 \cup V_5$, and x_1 is complete to V_5 and anticomplete to $V_2 \cup V_3 \cup V_4$. Now x_1-v_5-b_1-b_2-v_3-x_2 is a path in G for every $v_3 \in V_3$ and $v_5 \in V_5$, a contradiction. This proves that x_4 is non-adjacent to at least one of x_1, x_2.

From the symmetry, we may assume that x_4 is non-adjacent to x_1. By (9) and symmetry, x_1 is complete to V_4 and anticomplete to $V_2 \cup V_3 \cup V_5$. Suppose x_4 is adjacent to x_2. Then by (9) and symmetry, x_2 is complete to V_3 and anticomplete to $V_1 \cup V_4 \cup V_5$. But now b_1-a_1-x_1-b_4-v_3-x_2 is a path in G for every $v_3 \in V_3$, a contradiction. So x_4 is non-adjacent to x_2. By (9) and symmetry, x_2 is complete to V_4 and anticomplete to $V_1 \cup V_3 \cup V_5$. But now b_1-b_2-a_1-x_1-b_4-x_2 is a path in G, again a contradiction. This proves (11).

By (11) and (9), x_1 and x_2 are complete to V_4, x_1 is anticomplete to $V_2 \cup V_3 \cup V_5$, and x_2 is anticomplete to $V_1 \cup V_3 \cup V_5$. Applying (11) and (9) in G^c, we deduce that x_4 is adjacent to both x_3 and x_5, and x_3 is non-adjacent to x_5; x_3 and x_5 are anti-complete to V_4, x_3 is complete to $V_1 \cup V_2 \cup V_5$, and x_5 is complete to $V_1 \cup V_2 \cup V_3$.

(12) x_3 is adjacent to x_1.

Suppose not. By 3.1 and 3.2.2, there exist $a_1 \in A_1(x_1)$ and $b_1 \in B_1(x_1)$ non-adjacent to each other. Let $b_3 \in B_3(x_3)$ and $v_4 \in V_4$. Then b_1-x_3-a_1-x_1-v_4-b_3 is a path in G, a contradiction.

By (12) applied in G^c, it follows that x_2 is non-adjacent to x_3. Since x_3 is mixed on $V_2 \cup V_4$, (2) implies that x_3 is a path vertex. Let $p \in A_3(x_3)$ and $q, r \in B_3(x_3)$ such that p-q-r is a path in G. Now r-q-p-x_3-x_1-x_2 is a path in G, contrary to the fact that G is pure. This proves 2.5. ∎

4 Pristine graphs

Let $\mathcal{C}_0$ be the class of pristine graphs. First we define a few pristine graphs that will be important in the proof of 1.9.

- Let S_0 be the three-edge path.
- Let $S_1 = C_7$.
- Let S_2^1 be the graph with vertex set $\{a_1, a_2, a_3, a_4, a_5, a_6, b\}$ such that a_1-a_2-$\ldots$-a_6-a_1 is a cycle, b is adjacent to a_3, and there are no other edges in S_2^1.

- Let S_2^2 be the graph with vertex set $\{a_1, a_2, a_3, a_4, a_5, a_6, b\}$ such that a_1-a_2-$\dots$-a_6-a_1 is a cycle, b is adjacent to a_2 and to a_3, and there are no other edges in S_2^2.
- Let S_3 be the graph with vertex set $\{a_1, a_2, a_3, a_4, a_5, b, c\}$ such that a_1-a_2-$\dots$-a_5-a_1 is a cycle, b is adjacent to a_3 and c, and there are no other edges in S_3.
- Let S_4 be the graph with vertex set $\{a_1, a_2, a_3, a_4, a_5, b, c, d\}$ such that a_1-a_2-$\dots$-a_5-a_1 is a cycle, the pairs a_1b, a_5b, a_3c, a_4d and bc are adjacent, and all other pairs are non-adjacent.
- Let S_5 be the graph with vertex set $\{a_1, a_2, a_3, a_4, a_5, b\}$ such that a_1-a_2-$\dots$-a_5-a_1 is a cycle, b is adjacent to a_2, and there are no other edges in S_5.
- Let $S_6 = C_5$.

It is easy to check that all the graphs above are pristine. We need the following subclasses of $\mathcal{C}_0$.

- Let $\mathcal{C}_1$ be the class of S_1-free graphs in $\mathcal{C}_0$.
- Let $\mathcal{C}_2$ be the class of $\{S_2^1, S_2^2\}$-free graphs in $\mathcal{C}_1$.
- Let $\mathcal{C}_3$ be the class of S_3-free graphs in $\mathcal{C}_2$.
- Let $\mathcal{C}_4$ be the class of S_4-free graphs in $\mathcal{C}_3$.
- Let $\mathcal{C}_5$ be the class of S_5-free graphs in $\mathcal{C}_4$.
- Let $\mathcal{C}_6$ be the class of S_6-free graphs in $\mathcal{C}_5$.

In the next section, we will prove a number of structural results concerning pristine graphs, namely 5.1, 5.2, 5.3, 5.4, 5.5, and 5.6. Let us now prove 1.9, that we restate, assuming these results.

4.1 *There exists $\alpha > 1$ such that every pristine graph is α-narrow.*

Proof. For $i \in \{1, 3, 4, 5, 6\}$, let S_i' be the graph obtained from S_i by substituting S_0 for a_1. For $i \in \{1, 2\}$ let $S_2^{i\prime}$ be the graph obtained from S_2^i by substituting S_0 for a_1. For $i \in \{0, \dots, 6\}$ we will show that:

- (P_i) There exists $\alpha_i \geq 1$ such that all graphs in $\mathcal{C}_i$ are α_i-narrow.

For $i \in \{0, \dots, 5\}$ we will show that:

- (Q_i) If $G \in \mathcal{C}_i$ contains S_{i+1}' (or a member of $\{S_2^{1\prime}, S_2^{2\prime}\}$ in the case when $i = 1$), then G admits a $\mathcal{C}_i$-quasi-homogeneous set decomposition.

The validity of (Q_5), ..., (Q_0) is established in 5.1, 5.2, 5.3, 5.4, 5.5, and 5.6, respectively.

(1) For $i \in \{1, \dots, 5\}$, if (P_i) holds, then (P_{i-1}) holds.

We need to show that there exists $\alpha_{i-1} \geq 1$ such that every graph in $\mathcal{C}_{i-1}$ is α_{i-1}-narrow. Since by (P_i) there exists α_i such that every graph in $\mathcal{C}_i$ is α_i-narrow, it follows from 1.6 that S_i has the Erdös-Hajnal property for $\mathcal{C}_{i-1}$ (and $\{S_2^1, S_2^2\}$ has the Erdös-Hajnal property for $\mathcal{C}_1$, in the case when $i = 2$). Since all S_0-free graphs are perfect and therefore 1-narrow, 1.6 implies that S_0 has the Erdös-Hajnal property for class of all graphs, and in particular for $\mathcal{C}_{i-1}$. Now by 2.2, S_i' has the Erdös-Hajnal property for $\mathcal{C}_{i-1}$ (and $\{S_2^{1'}, S_2^{2'}\}$ has the Erdös-Hajnal property for $\mathcal{C}_1$, in the case when $i = 2$). Therefore, by 1.10 that there exists $\alpha_{i-1} \geq 1$ such that all $\{S_i'\}$-free graphs in $\mathcal{C}_{i-1}$ (and $\{S_2^{1'}, S_2^{2'}\}$-free graphs in $\mathcal{C}_1$ in the case when $i = 2$) are α_{i-1}-narrow.

Let G be a graph in $\mathcal{C}_{i-1}$ that is not α_{i-1}-narrow with $|V(G)|$ minimum. By (Q_{i-1}), G admits a $\mathcal{C}_{i-1}$-quasi-homogeneous set decomposition. But then G is α_{i-1}-narrow by 2.3 and the minimality of $|V(G)|$, a contradiction. This proves (1).

Next we observe that 4.1 follows immediately from from (P_0). By (1), in order to prove 4.1, it is enough to prove that (P_6) holds; and since all S_6-free graphs in $\mathcal{C}_5$ are perfect by 1.5, (P_6) follows. This proves 4.1. ∎

We conclude this section with a few technical lemmas about pristine graphs.

4.2 *Let $G \in \mathcal{C}_0$, and let $X_1, X_2 \in V(G)$ be disjoint anticonnected sets complete to each other. Then no vertex of $V(G) \setminus (X_1 \cup X_2)$ is mixed on both X_1 and X_2.*

Proof. Suppose $v \in V(G) \setminus (X_1 \cup X_2)$ is mixed on both X_1 and X_2. Let $a_i, b_i \in X_i$ be such that v is adjacent to a_i and non-adjacent to b_i, and a_i is non-adjacent to b_i (such a_i, b_i exist by 3.2.2). Now a_1-b_1-v-b_2-a_2 is a four-edge path in G^c, a contradiction. This proves 4.2. ∎

Let G be a graph, H an induced subgraph of G, and $h \in V(H)$. Let $X \subseteq \{h\} \cup (V(G) \setminus V(H))$ be such that $H' = G|(X \cup (V(H) \setminus \{h\}))$ is the graph obtained from H by substituting $G|X$ for h. (This implies that $G|(V(H) \setminus \{h\} \cup \{x\})$ is isomorphic to H for every $x \in X$.) In this case we say that H' *is obtained from H by expanding h to X*. An *(H, h)-structure* in G is a set X such that

- $H' = G|(X \cup (V(H) \setminus \{h\}))$ is obtained from H by expanding h to X,
- X is both connected and anticonnected in G, and
- $|X| \geq 4$.

An (H, h)-structure X is *maximal* if X is maximal (under subset inclusion) subject to X being an (H, h)-structure.

4.3 *Let $G \in \mathcal{C}_0$, and let a-b-c-d be a path in G, say P. Let $X \subseteq V(G) \setminus \{a, b, d\}$ and let X be a (P, c)-structure in G. Let $v \in V(G) \setminus (X \cup \{a, b, d\})$ be mixed on X. Then either*

1. *v is complete to $\{b, d\}$ and non-adjacent to a, or*
2. *v is anticomplete to $\{a, b, d\}$.*

Proof. Since X and $\{b, d\}$ are anticonnected subsets of $V(G)$ complete to each other, 4.2 implies that v is either complete or anticomplete to $\{b, d\}$. If v is complete to $\{b, d\}$, then since b-d-a-x-v is not a path in G^c for any $x \in X \setminus N(v)$, it follows that v is non-adjacent to a, and 4.3.1 holds. So we may assume that v is anticomplete to $\{b, d\}$, and adjacent to a. Let $x, y \in X$ as in 3.2.1. Now b-v-y-a-x is a path in G^c, a contradiction. This proves 4.3. ∎

4.4 *Let $G \in \mathcal{C}_0$, and let e-a-b-c-d be a path in G, say P. Let $X \subseteq V(G) \setminus \{e, a, b, d\}$, and let X be a (P, c)-structure in G. Let $v \in V(G) \setminus (X \cup \{e, a, b, d\})$ be mixed on X. If v is complete to $\{b, d\}$, then v is anticomplete to $\{e, a\}$.*

Proof. By 4.3, v is non-adjacent to a. Let $x \in X$ be adjacent to v. Now since b-e-x-a-v is not a path in G^c, it follows that v is non-adjacent to e, and 4.4 holds. This proves 4.4. ∎

4.5 *Let $G \in \mathcal{C}_0$, and let a_1-a_2-a_3-a_4-a_5-a_1 be a cycle in G, say C. Let $X \subseteq V(G) \setminus \{a_2, \ldots, a_5\}$, and let X be a (C, a_1)-structure in G. Let $v \in V(G) \setminus (X \cup \{a_2, \ldots, a_5\}$ be mixed on X. Then either*

1. *v is complete to $\{a_2, a_5\}$ and anticomplete to $\{a_3, a_4\}$, or*
2. *v is anticomplete to $\{a_2, \ldots, a_5\}$.*

Proof. Apply 4.3 to a_4-a_5-a_1-a_2 and a_3-a_2-a_1-a_5. It follows that v is anticomplete to $\{a_3, a_4\}$, and either complete or anticomplete to $\{a_2, a_5\}$. This proves 4.5. ∎

4.6 *Let G be a graph, H an induced subgraph of G, and $h \in V(H)$. Let X be a maximal (H, h)-structure in G. Let $v \in V(G) \setminus (X \cup (V(H) \setminus \{h\}))$ be such that every $u \in V(H) \setminus \{h\}$ is adjacent to v if and only if u is adjacent to h. Then v is not mixed on H.*

Proof. Suppose v is mixed on X. Then $X \cup \{v\}$ is both connected and anticonnected, and so $X \cup \{v\}$ is an (H, h)-structure in G, contrary to the maximality of X. This proves 4.6. ∎

5 Decomposing pristine graphs

In this section we prove a number of structural results for pristine graphs. We remind the reader that for a hereditary class of graphs $\mathcal{C}$, if a graph $G \in \mathcal{C}$ is not prime, then G admits a homogeneous set decomposition, and therefore $\mathcal{C}$-quasi-homogeneous set decomposition, and so the results of this section are sufficient for the proof of 4.1.

5.1 *If $G \in \mathcal{C}_5$ contains S_6', then G is not prime.*

Proof. Since G contains S_6', there exists a maximal (S_6, a_1)-structure X in G. We may assume that G is prime, and so X is not a homogeneous set in G. Consequently, there exists $v \in V(G) \setminus (X \cup \{a_2, \ldots, a_5\})$ such that v is mixed on X. Apply 4.5 to C. By 4.6 and the maximality of X, 4.5.1 does not hold, and so 4.5.2 holds. But then $G|\{y, a_2, \ldots, a_5, v\}$ is isomorphic to S_5 for every $y \in X \cap N(v)$, contrary to the fact that $G \in \mathcal{C}_5$. This proves 5.1. ∎

5.2 *If $G \in \mathcal{C}_4$ contains S_5', then G admits a $\mathcal{C}_4$-quasi-homogeneous set decomposition.*

Proof. Since G contains S_5', there exists a maximal (S_5, a_1)-structure X in G. Let V be the set of vertices of $V(G) \setminus X$ that are mixed on X. Then $V \subseteq V(G) \setminus (X \cup \{a_2, \ldots, a_5, b\})$. We may assume that G is prime, and so X is not a homogeneous set in G. Consequently, $V \neq \emptyset$.

(1) V is anticomplete to $\{a_2, \ldots, a_5, b\}$.

Let $v \in V$. By 4.5 applied to a_1-a_2-a_3-a_4-a_5-a_1, it follows that v is anticomplete to $\{a_3, a_4\}$ and either complete or anticomplete to $\{a_2, a_5\}$. By 4.3 applied to b-a_2-a_1-a_5, we deduce that v is non-adjacent to b. By 4.6 and the maximality of X, v is not complete to $\{a_2, a_5\}$, and so (1) follows.

Let C be the set of vertices complete to X, and let $A = V(G) \setminus (X \cup C)$. We will show that (X, A, C) is a $\mathcal{C}_4$-quasi-homogeneous set in G. Let A' be the set of vertices in A that are anticomplete to X. Then $A = A' \cup V$.

(2) If $x \in X$ and $s, t \in A$ are adjacent, then x is not mixed on $\{s, t\}$. Consequently, V is anticomplete to A'.

Suppose x is adjacent to s and non-adjacent to t. Since X is anticomplete to A', it follows that $s \in V$. By (1), s is anticomplete to $\{a_2, \ldots, a_5, b\}$. Since $G|\{a_2, \ldots, a_5, x, s, t\}$ is not isomorphic to S_3 (because $G \in \mathcal{C}_4$), it follows that t has a neighbor in $\{a_2, \ldots, a_5\}$. Therefore, by (1), $t \notin V$, and thus $t \in A'$. Let $x', y' \in X$ be as in 3.2.1 (applied with $v = s$). Since x'-t-y'-s-a_2 and x'-t-y'-s-a_5 are not paths in G^c, it follows that t is anticomplete to $\{a_2, a_5\}$, and therefore t has a neighbor in $\{a_3, a_4\}$.

If t is adjacent to both a_3 and a_4, then t is non-adjacent to b (since t-a_2-a_4-b-a_3 is not a path in G^c), and so $G|\{a_2, \ldots, a_5, x, s, t, b\}$ is isomorphic to S_4, a contradiction. So t is adjacent to exactly one of $\{a_3, a_4\}$. Let $x'', y'' \in X$ be as in 3.2.2 (applied with $v = s$). But now if t is adjacent to a_4, then $G|\{x'', a_2, a_3, a_4, t, s, y''\}$ is isomorphic to S_2^1, and if t is adjacent to a_3 then $G|\{x'', a_5, a_4, a_3, t, s, y''\}$ is isomorphic to S_2^1; both contrary to the fact that $G \in \mathcal{C}_4$. This proves (2).

(3) There do not exist non-adjacent $c_1, c_2 \in C$ and $v \in V$ such that v is mixed on $\{c_1, c_2\}$.

(3) follows immediately from 4.2.

Let G' be obtained from $G \setminus X$ by adding a new vertex x complete to C and anticomplete to A.

(4) $G' \in \mathcal{C}_4$.

Let $\mathcal{F}$ be the set of graphs consisting of the six-edge path, the complement of the four-edge path, S_1, S_2^1, S_2^2, S_3, and S_4. Assume that G' has an induced subgraph B, isomorphic to a member of $\mathcal{F}$. Since B is not an induced subgraph of G, it follows that $x \in V(B)$, and $V(B) \cap V \neq \emptyset$. Let b be the number of components of $B|V$.

Suppose first that $b = 1$. Let $v \in V(B) \cap V$, and let $y \in X$ be non-adjacent to v. By (2), and since X is anticomplete to A', it follows that y is anticomplete to $V(B) \cap A$, and so $G|((V(B) \setminus \{x\}) \cup \{y\})$ is an induced subgraph of G isomorphic to B, contrary to the fact that $G \in \mathcal{C}_4$. This proves that $b \geq 2$.

Since by (2) A' is anticomplete to V, it follows that no component of $B|A$ meets both V and A'. Since for every $F \in \mathcal{F}$ and $w \in V(F)$, the graph $F \setminus (\{w\} \cup N_F(w))$ has at most two components, we deduce that $B|A$ has at most two components, and therefore $b = 2$, $V(B) \cap A' = \emptyset$ and $F \setminus (\{w\} \cup N_F(w))$ has at most two components. Checking the graphs of $\mathcal{F}$ one by one, we deduce that B is isomorphic either to the six-edge path, S_2^1, S_3, or S_4, and $N_B(x)$ is not a clique. The last implies that there exists a component C' of $B^c|C$ with $|V(C')| > 1$. Since no member of $\mathcal{F}$ has a homogeneous set, there exists a vertex $v \in V(B) \setminus C'$ that is mixed on C'. Then $v \neq x$, and $v \notin C \setminus C'$, and therefore $v \in V$. By 3.2.2, we get a contradiction to (3). This proves (4).

(5) If P' is a perfect induced subgraph of G' with $x \in V(P')$, and Q is a perfect induced subgraph of $G|X$, then $P = G|((V(P') \cup V(Q)) \setminus \{x\})$ is perfect.

Suppose P is not perfect. Since P is an induced subgraph of G, and $G \in \mathcal{C}_4$, it follows that P contains an induced cycle of length five, say D, with vertices d_1-d_2-d_3-d_4-d_5 in order.

We claim that some vertex of $V(D) \cap X$ is adjacent to a vertex of $V(D) \cap V$. Suppose not. Since Q contains no induced cycle of length five, $V(D) \setminus X \neq \emptyset$. Since $V(D) \cap X$ is not a homogeneous set in D, it follows that $|V(D) \cap X| = 1$. But now $P'|((V(D) \setminus X) \cup \{x\})$ is a cycle of length five, contrary to the fact that P' is perfect. This proves the claim that some vertex of $V(D) \cap X$ is adjacent to a vertex of $V(D) \cap V$.

We may assume that $d_1 \in X$ and $d_2 \in V$. By (2), $d_3 \notin A$. Since d_3 is non-adjacent to d_1, it follows that $d_3 \notin C$, and therefore $d_3 \in X$. If d_4 is in X, then, by (1), a_2-d_2-d_4-d_1-d_3 is a path in G^c, a contradiction; thus $d_4 \notin X$. Since d_4 is not adjacent to d_1, it follows that $d_4 \notin C$, and so $d_4 \in A$. Similarly, $d_5 \in A$. But now d_1 is mixed on $\{d_4, d_5\}$, contrary to (2). This proves (5).

Now (4) and (5) imply that (X, A, C) is a $\mathcal{C}_4$-quasi-homogeneous set in G. This proves 5.2. ∎

5.3 *If $G \in \mathcal{C}_3$ contains S_4', then G is not prime.*

Proof. Since G contains S_4', there exists a maximal (S_4, a_1)-structure X in G. We may assume that G is prime, and so X is not a homogeneous set in G. Consequently, there exists $v \in V(G) \setminus (X \cup \{a_2, \ldots, a_5, b, c, d\})$ such that v is mixed on X. By 4.5 applied to a_1-a_2-a_3-a_4-a_5-a_1 and a_1-a_2-a_3-c-b-a_1, it follows that v is anticomplete to $\{a_3, a_4, c\}$ and either complete or anticomplete to $\{a_2, a_5, b\}$. By 3.2.2 there exist $x \in N(v) \cap X$ and $y \in X \setminus N(v)$ non-adjacent to each other.

Suppose first that v is complete to $\{a_2, a_5, b\}$. Since $G \in \mathcal{C}_3$, it follows that $G|\{b, c, a_3, a_4, d, v, x\}$ is not isomorphic to S_2^2, and therefore v is non-adjacent to d, contrary to 4.6. This proves that v is anticomplete to $\{a_2, a_5, b\}$. Since $G \in \mathcal{C}_3$, it follows that $G|\{a_2, \ldots, a_5, y, d, v\}$ is not isomorphic to S_3, and so v is non-adjacent to d. Now v-x-b-c-a_3-a_4-d is a path of length six in G, a contradiction. This proves 5.3. ∎

5.4 *If $G \in \mathcal{C}_2$ contains S_3', then G is not prime.*

Proof. Since G contains S_3', there exists a maximal (S_3, a_1)-structure X in G. We may assume that G is prime, and so X is not a homogeneous set in G. Consequently, there exists $v \in V(G) \setminus (X \cup \{a_2, \dots, a_5, b, c\})$ such that v is mixed on X. By 4.5, v is anticomplete to $\{a_3, a_4\}$ and either complete or anticomplete to $\{a_2, a_5\}$. Let $x \in X \cap N(v)$.

Suppose first that v is complete to $\{a_2, a_5\}$. By 4.4 applied to b-a_3-a_2-a_1-a_5 we deduce that v is non-adjacent to b. Now 4.6 implies that v is adjacent to c, and $G|\{a_3, a_4, a_5, v, c, b, x\}$ is isomorphic to S_2^2 for every $y \in X \setminus N(v)$, contrary to the fact that $G \in \mathcal{C}_2$. This proves that v is anticomplete to $\{a_2, a_5\}$.

If v is non-adjacent to b, then $G|\{v, x, a_5, a_4, a_3, b, c\}$ is either a path of length six, or a cycle of length seven in G, in both cases a contradiction. So v is adjacent to b. But now $G|\{v, x, a_5, a_4, a_3, b, c\}$ is isomorphic to S_2^1 if v is non-adjacent to c, and to S_2^2 if v is adjacent to c, contrary to the fact that $G \in \mathcal{C}_2$. This proves 5.4. ∎

5.5 *If $G \in \mathcal{C}_1$ contains a member of $\{S_2^{1\prime}, S_2^{2\prime}\}$, then G is not prime.*

Proof. Since G contains a member of $\{S_2^{1\prime}, S_2^{2\prime}\}$, there exists either a maximal (S_2^1, a_1) or a maximal (S_2^2, a_1) structure in G. Denote it by X. We may assume that G is prime, and so X is not a homogeneous set in G. Consequently, there exists $v \in V(G) \setminus (X \cup \{a_2, \dots, a_6, b\})$ such that v is mixed on X.

Applying 4.3 to the paths a_3-a_2-a_1-a_6 and a_5-a_6-a_1-a_2, we deduce that either 4.3.1 holds for both paths, or 4.3.2 holds for both paths.

Assume first that 4.3.1 holds. Then v is complete to $\{a_2, a_6\}$ and anticomplete to $\{a_3, a_5\}$. Now applying 4.4 to a_4-a_3-a_2-a_1-a_6, we deduce that v is non-adjacent to a_4. We claim that v is non-adjacent to b. This follows applying 4.3 to b-a_2-a_1-a_6 if b is adjacent to a_2 (and X is an (S_2^2, a_1) structure), and applying 4.4 to b-a_3-a_2-a_1-a_6 if b is non-adjacent to a_2 (and X is an (S_2^1, a_1) structure). But now we get a contradiction to 4.6. This proves that 4.3.1 does not hold, and therefore 4.3.2 holds.

Consequently, v is anticomplete to $\{a_2, a_3, a_5, a_6\}$. Let $x, y \in X$ be as in 3.2.2. If v is non-adjacent to a_4, then either b-a_3-a_4-a_5-a_6-x-v is a path of length six in G (if v is non-adjacent to b), or b-a_3-a_4-a_5-a_6-x-v-b is a cycle of length seven in G (if v is adjacent to b); in both cases contrary to the fact that $G \in \mathcal{C}_1$. This proves that v is adjacent to a_4. If v is non-adjacent to b, then b-a_3-a_4-v-x-a_6-y is a path of length six in G, a contradiction; thus v is adjacent to b. This implies that b is non-adjacent to a_2, (for otherwise we get a contradiction applying 4.3 to a_6-a_1-a_2-b), and so X is an (S_2^1, a_1)-structure. Now b-v-a_4-a_5-a_6-y-a_2 is a path of length six in G, again a contradiction. This proves 5.5. ∎

5.6 *If $G \in \mathcal{C}_0$ contains S_1', then G is not prime.*

Proof. Since G contains S_1', there exists a maximal (S_1, a_1)-structure X in G. We may assume that G is prime, and so X is not a homogeneous set in G. Consequently, there exists $v \in V(G) \setminus (X \cup \{a_2, \dots, a_7\})$ such that v is mixed on X. Applying 4.3 to the paths a_3-a_2-a_1-a_7 and a_6-a_7-a_1-a_2, we deduce that either 4.3.1 holds for both paths, or 4.3.2 holds for both paths.

Assume first that 4.3.1 holds. Then v is complete to $\{a_2, a_7\}$ and anticomplete to $\{a_3, a_6\}$. Now applying 4.4 to a_4-a_3-a_2-a_1-a_7 and a_5-a_6-a_7-a_1-a_2, we deduce that v is anticomplete to $\{a_4, a_5\}$, contrary to 4.6. This proves that 4.3.1 does not hold, and therefore 4.3.2 holds.

It follows that v is anticomplete to $\{a_6, a_7, a_2, a_3\}$. Let $x \in X$ be adjacent to v, and $y \in X$ non-adjacent to v. If v is adjacent to a_5, then v-a_5-a_6-a_7-y-a_2-a_3 is a path of length six in G, contrary to the fact that $G \in \mathcal{C}_0$. But now, by symmetry, v is anticomplete to $\{a_4, a_5\}$, and v-x-a_2-a_3-a_4-a_5-a_6 is a path of length six in G, again a contradiction. This proves 5.6. ∎

6 The proof of 1.10

In this section we prove 1.10. This is a result of Fox [7], but we include a proof for completeness. Let us start by restating the theorem:

6.1 *Let H be a graph for which there exists a constant $\delta(H) > 0$ such for every H-free graph G either $\omega(G) \geq |V(G)|^{\delta(H)}$ or $\alpha(G) \geq |V(G)|^{\delta(H)}$. Then every H-free graph G is $\frac{3}{\delta(H)}$-narrow.*

Proof. The proof is by induction on $|V(G)|$. Let G be an H-free graph, and let $f : V(G) \to [0, 1]$ be a good function. Write $t = \frac{1}{\delta(H)}$. We need to show that:

(1) $\Sigma_{v \in V(G)} f(v)^{3t} \leq 1$.

For every integer $i \geq 0$ define:

$$V_i = \{v \in V(G) \ : \ \frac{1}{2^i} \leq f(v) < \frac{1}{2^{i-1}}\}.$$

Let $G_i = G|V_i$, and let

$$V^+ = \{v \in V(G) \ : \ f(v) > 0\}.$$

Since (1) clearly holds if $f(v) = 1$ for some $v \in V(G)$, we may henceforth assume that $V^+ = \bigcup_{i \geq 1} V_i$.

(2) $|V_i| \leq 2^{it}$.

Let $i \geq 1$ be an integer. Recall that $f(v) \geq \frac{1}{2^i}$ for every $v \in V_i$. Since f is good, this implies that if P is a perfect induced subgraph of G_i, then $|V(P)| \leq 2^i$. In particular, both $\alpha(G_i) \leq 2^i$ and $\omega(G_i) \leq 2^i$. On the other hand, since G_i is H-free, it follows that either $\alpha(G_i) \geq |V_i|^{\frac{1}{t}}$ or $\omega(G_i) \geq |V_i|^{\frac{1}{t}}$. Thus

$$2^i \geq |V_i|^{\frac{1}{t}},$$

and therefore $|V_i| \leq 2^{it}$. This proves (2).

(3) If $V_1 = \emptyset$, then the theorem holds.

Since $V_1 = \emptyset$, it follows that

$$\Sigma_{v \in V(G)} f(v)^{3t} = \Sigma_{v \in V^+} f(v)^{3t} = \Sigma_{i \geq 2} \Sigma_{v \in V_i} f(v)^{3t}.$$

Since for $i \geq 1$, $f(v) < \frac{1}{2^{i-1}}$ for every $v \in V_i$, it follows that

$$\Sigma_{i\geq 2}\Sigma_{v\in V_i} f(v)^{3t} \leq \Sigma_{i\geq 2}\Sigma_{v\in V_i} \frac{1}{2^{3t(i-1)}}.$$

By (2), for fixed $i \geq 2$,

$$\Sigma_{v\in V_i} \frac{1}{2^{3t(i-1)}} \leq \frac{2^{it}}{2^{3t(i-1)}} = \frac{2^{3t}}{2^{2it}}.$$

Now, exchanging variables,

$$\Sigma_{i\geq 2} \frac{2^{3t}}{2^{2it}} = \Sigma_{j\geq 0} \frac{2^{3t}}{2^{2(j+2)t}} = 2^{-t}\Sigma_{j\geq 0}(\frac{1}{2^{2t}})^j = \frac{2^t}{2^{2t}-1} \leq 1.$$

This proves that

$$\Sigma_{v\in V(G)} f(v)^{3t} \leq 1,$$

and threfore proves (3).

By (3) we may assume that for some $v_0 \in V(G)$, $f(v_0) \geq \frac{1}{2}$. Let $N = N(v_0)$ and $M = V(G) \setminus (N \cup \{v_0\})$. Since if P is a perfect induced subgraph of $G|N$, then $G|(V(P) \cup \{v_0\})$ is perfect, it follows that

$$\Sigma_{v\in V(P)} f(v) \leq 1 - f(v_0)$$

for every perfect induced subgraph P of of $G|N$. Consequently, $g(v) = \frac{f(v)}{1-f(v_0)}$ is a good function on $G|N$. Inductively, this implies that

$$\Sigma_{v\in N} g(v)^{3t} \leq 1,$$

and thus

$$\Sigma_{v\in N} f(v)^{3t} \leq (1 - f(v_0))^{3t}.$$

Similarly,

$$\Sigma_{v\in M} f(v)^{3t} \leq (1 - f(v_0))^{3t}.$$

Therefore,

$$\Sigma_{v\in V(G)} f(v)^{3t} \leq f(v_0)^{3t} + 2(1 - f(v_0))^{3t}.$$

Let $q = 3t$ and let

$$F(x) = x^q + 2(1-x)^q$$

Then $F(x)$ is convex for $x \in [\frac{1}{2}, 1]$. Consequently, $F(x) \leq \max(F(\frac{1}{2}), F(1))$ for every $x \in [\frac{1}{2}, 1]$. Thus $F(x) \leq \max(\frac{3}{2^q}, 1)$, and since $q > 2$, it follows that $F(x) \leq 1$ for all $x \in [\frac{1}{2}, 1]$. Now, setting $x = f(v_0)$, we obtain (1). This proves 6.1. ∎

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