Graph Minors. XXII. Irrelevant vertices in linkage problems

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Abstract

In the algorithm for the disjoint paths problem given in Graph Minors XIII, we used without proof a lemma that, in solving such a problem, a vertex which was sufficiently "insulated" from the rest of the graph by a large planar piece of the graph was irrelevant, and could be deleted without changing the problem. In this paper we prove the lemma.

1 Introduction

Let Γ be a graph drawn in a plane, let v be a vertex of Γ, and suppose that there are many (h, say) vertex-disjoint circuits of Γ, all surrounding v. Suppose also that Γ is a subgraph of a larger graph G, which is not necessarily planar, and the only vertices of Γ incident with edges of G not in Γ lie in the plane outside the outermost of the h circuits. Finally, suppose that $s_1, t_1, \dots, s_p, t_p$ are vertices of G but not of Γ, and we are concerned with the existence of p disjoint paths $P_1, \dots, P_p$ of G, where P_i has ends s_i and t_i ($1 \leq i \leq p$). It is intuitively plausible, and indeed true, that if h is large enough as a function of p, then if $P_1, \dots, P_p$ exist at all they can be chosen so that none of them uses v. This fact, and a generalization of it, was used in theorem (10.2) of [5] as a lemma to prove the correctness of an algorithm to decide whether $P_1, \dots, P_p$ do exist. However, the proof of that lemma was postponed to the present, because it seems to need some of the main results of this series. Proving the lemma is the main goal of this paper.

We shall derive it from the result about "vital linkages" proved in [7]. A *linkage* in a graph G is a subgraph of G, every component of which is a path. (*Paths* have at least one vertex, and have no "repeated" vertices.) If L is a linkage in G, a vertex $v \in V(G)$ is a *terminal* of L if $v \in V(L)$ and v has degree at most one in L. We say a linkage L is a *p-linkage* if it has at most p terminals. The *pattern* of a linkage L is the partition of its set of terminals determined by the components of L; that is, two terminals belong to the same block of the pattern if and only if they are the ends of some component of L. W say a linkage L in G is *vital* if $V(L) = V(G)$ and there is no linkage $L' \neq L$ in G with the same pattern as L.

A *tree-decomposition* of a graph G is a pair (T, W), where T is a tree and $W = (W_t : t \in V(T))$ is a family of subgraphs of G, such that

1. $\bigcup(W_t : t \in V(T)) = G$, and
2. if $t, t', t'' \in V(T)$ and t' lies on the path of T between t and t'' then $W_t \cap W_{t''} \subseteq W_{t'}$.

Its *width* is $\max(|V(W_t)| - 1 : t \in V(T))$, and the *tree-width* of G is the minimum width of a tree-decomposition of G. The following is theorem (1.1) of [7], and in this paper we derive the unproved lemma of [5] from it.

1.1 *For every integer $p \geq 0$ there exists $w \geq 0$ such that every graph with a vital p-linkage has tree-width $\leq w$.*

2 Vital subgraphs

We need to extend (1.1) from linkages to general subgraphs. If L is a subgraph of G we write $L \subseteq G$. If also $Z \subseteq V(G)$, we define the *effect of L on Z* to be the partition of $V(L) \cap Z$ in which two vertices belong to the same block if and only if they belong to the same component of L. If two subgraphs L_1, L_2 have the same effect on Z then necessarily $V(L_1) \cap Z = V(L_2) \cap Z$. We say that a subgraph L is *vital* for Z in G if $Z \subseteq V(L)$ and no subgraph $L' \neq L$ in G has the same effect on Z as L. We shall show

2.1 *For every integer $p \geq 0$, there exists $w \geq 0$ such that, if a graph G has a subgraph which is vital for some $Z \subseteq V(G)$ with $|Z| \leq p$, then G has tree-width $\leq w$.*

We begin with the following.

2.2 *If L is a subgraph of G, and L is vital for $Z \subseteq V(G)$, then L is a forest, $V(L) = V(G)$, and every vertex of L not in Z has degree at least 2 in L.*

Proof. If L has a circuit C, let $e \in E(C)$; then L and $L \setminus \{e\}$ have the same effect on Z, a contradiction. Thus L is a forest. If $v \in V(G) \setminus V(L)$, then $v \notin Z$ since $Z \subseteq V(L)$; let L' be the forest obtained from L by adding v. Then L and L' have the same effect on Z, a contradiction. Thus $V(L) = V(G)$. If $v \in V(L) \setminus Z$ has degree at most 1 in L, then $L \setminus \{v\}$ has the same effect on Z as L, again a contradiction. The result follows. ■

Secondly, we need

2.3 *Let L be a forest, and for each $v \in V(L)$ let $d(v)$ be the degree of v in L. Suppose that there are at most p vertices of L with $d(v) \leq 1$. Then for all $Y \subseteq V(L)$, $\sum_{y \in Y} d(y) \leq 2|Y| + p$.*

Proof. Let $L_1, \ldots, L_t$ be the components of L, for $1 \leq i \leq t$ let L_i have p_i vertices of degree at most one, and let $Y_i = Y \cap V(L_i)$.

(1) *For $1 \leq i \leq t$, $\sum_{y \in Y_i} d(y) \leq 2|Y_i| + p_i$.*

Subproof. This is true if $|V(L_i)| = 1$, and so we may assume that $d(v) \geq 1$ for each $v \in V(L_i)$. Since L_i is a tree,

$$0 \leq 2|V(L_i)| - 2|E(L_i)| = \sum_{v \in V(L_i)} (2 - d(v)) = \sum_{v \in Y_i} (2 - d(v)) + \sum_{v \in V(L_i) \setminus Y_i} (2 - d(v)).$$

But $2 - d(v) \leq 1$ for all $v \in V(L_i) \setminus Y_i$, with equality for at most p_i vertices v; and so the last term above is at most p_i. Hence $\sum_{v \in Y_i} (2 - d(v)) + p_i \geq 0$ and so (1) holds.

From (1), the result follows by summing over i ($1 \leq i \leq t$). ■

We also need the following, and we leave its proof to the reader.

2.4 *Let L be a subgraph of G, vital for $Z \subseteq V(G)$, and let $e \in E(L)$ with both ends in Z. Then $L \setminus \{e\}$ is vital for Z in G.*

Proof of (2.1). Choose $w \geq 0$ so that (1.1) is satisfied with p replaced by $7p$. We claim that w satisfies (2.1). For let L be a subgraph of a graph G, vital for $Z \subseteq V(G)$, where $|Z| \leq p$. From (2.2), L is a forest, $V(L) = V(G)$ and every vertex of L not in Z has degree at least 2 in L. Consequently, L has $\leq p$ vertices with degree at most 1. Let Y be the set of vertices of L with degree at least 3. From (2.3),

$$3|Y| \leq \sum_{y \in Y} d(y) \leq 2|Y| + p$$

where $d(y)$ denotes the degree of y in L; and hence $|Y| \le p$. Let $X = Y \cup Z$; then $|X| \le 2p$, and so from (2.3) again,

$$\sum_{y \in X} d(y) \le 2|X| + p \le 5p.$$

Let Z' be the set of all vertices in X and all their neighbours in L. Then

$$|Z'| \le |X| + \sum_{y \in X} d(y) \le 7p.$$

Since $Z \subseteq Z'$ it follows that L is vital for Z' in G. Let F be the set of all edges in L with both ends in Z'. Then by (2.4), $L \setminus F$ is vital for Z' in G.

(1) *$L \setminus F$ is a linkage in G with set of terminals Z'.*

Subproof. If $v \in V(L)$ has degree at least 3 in L then $v \in Y \subseteq X$ and so all edges of L incident with v are in F; and hence v has degree 0 in $L \setminus F$. Consequently, every vertex of $L \setminus F$ has degree at most 2. If $v \in Z'$, then either $v \in X$ and hence v has degree 0 in $L \setminus F$, or $v \notin X$ and v has a neighbour in X in L, which implies that v has degree at least 2 in L and at most 1 in $L \setminus F$. Thus each vertex in Z' is a terminal of $L \setminus F$. Conversely, let $v \in V(G) \setminus Z'$. Then $v \notin X = Y \cup Z$, and so v has degree 2 in L (for by (2.2), Z contains every vertex of L with degree at most 1). Since $v \notin X$, no edge incident with v is in F, and so v has degree 2 in $L \setminus F$, and hence is not a terminal of $L \setminus F$. This proves (1).

It follows from (1) that $L \setminus F$ is a vital $7p$-linkage in G. By (1.1), G has tree-width $\le w$, as required. ■

If G is a graph and $Z \subseteq V(G)$, a *Z-division* of G is a set $\{A_1, \ldots, A_k\}$ of subgraphs of G, such that $A_1 \cup \cdots \cup A_k = G$, and $E(A_i \cap A_j) = \emptyset$ and $V(A_i \cap A_j) \subseteq Z$ for $1 \le i < j \le k$. If $L \subseteq G$, we say $u, v \in V(G)$ are *L-connected* if $u, v \in V(L)$ and u, v belong to the same component of L.

2.5 *Let L be a subgraph of a graph G, let $Z \subseteq V(G)$, and let $\{A_1, \ldots, A_k\}$ be a Z-division of G. Let G' be a graph, let $Z' \subseteq V(G')$, and let $\{A'_1, \ldots, A'_k\}$ be a Z'-division of G'. Let $\alpha : Z' \to Z$ be a function, and for $1 \le i \le k$ let $L'_i \subseteq A'_i$, such that*

- (a) *for $1 \le i \le k$, α maps $Z' \cap V(A'_i)$ onto $Z \cap V(A_i)$, and*
- (b) *if $u, v \in Z'$ are distinct and $\alpha(u) = \alpha(v)$ then $u, v \in V(A'_i)$ for some i $(1 \le i \le k)$*
- (c) *for $1 \le i \le k$, $u, v \in Z' \cap V(A'_i)$ are L'_i-connected if and only if $\alpha(u), \alpha(v)$ are $L \cap A_i$-connected.*

Let $L' = L'_1 \cup \cdots \cup L'_k$. Then $L'_i = L' \cap A'_i$ for $1 \le i \le k$, and $u, v \in Z'$ are L'-connected if and only if $\alpha(u), \alpha(v)$ are L-connected.

Proof. For $1 \le i \le k$, let $Z_i = Z \cap V(A_i)$ and $Z'_i = Z' \cap V(A'_i)$. Hypothesis (c) implies (taking $u = v$) that

(1) *For* $1 \le i \le k$, *if* $v \in Z'_i$, *then* $v \in V(L'_i)$ *if and only if* $\alpha(v) \in V(L)$.

(2) *For* $1 \le i \le k$, $L'_i = L' \cap A'_i$.

Subproof. Certainly $E(L'_i) = E(L' \cap A'_i)$ and $V(L'_i) \subseteq V(L' \cap A'_i)$. Suppose that $v \in V(L' \cap A'_i)$. Since $v \in V(L')$ there exists j with $1 \le j \le k$ so that $v \in V(L'_j)$. If $j = i$ then $v \in V(L'_i)$ as required, and so we assume that $j \ne i$. Since $L'_j \subseteq A'_j$ it follows that $v \in V(A'_i \cap A'_j) \subseteq Z'$. Since $v \in V(L'_j)$ and $v \in Z'_j$, it follows from (1) that $\alpha(v) \in V(L)$. Since $v \in Z'_i$, it follows from (1) that $v \in V(L'_i)$. This proves (2).

(3) *If* $u, v \in Z'$ *are* L'*-connected then* $\alpha(u)$ *and* $\alpha(v)$ *are* L*-connected.*

Subproof. Let P be a path of L' with ends $u, v \in Z'$. Let us number the vertices of P in Z' as $v_1, \dots, v_n$, in order on P, where $u = v_1$ and $v = v_n$. We may assume that $n > 1$. Let $1 \le j < n$, and let P_j be the path in P with ends v_j, v_{j+1}. Since no internal vertex of P_j is in Z', there exists i with $1 \le i \le k$ such that $P_j \subseteq A'_i$. Since $P_j \subseteq P \cap A'_i \subseteq L' \cap A'_i = L'_i$, it follows that v_j, v_{j+1} are L'_i-connected. By hypothesis (c), $\alpha(v_j)$ and $\alpha(v_{j+1})$ are $L \cap A_i$-connected and hence L-connected. We have proved then that for $1 \le j < n, \alpha(v_j)$ and $\alpha(v_{j+1})$ are L-connected. Consequently $\alpha(u) = \alpha(v_1)$ and $\alpha(v) = \alpha(v_n)$ are L-connected. This proves (3).

(4) *If* $u, v \in Z'$ *and* $\alpha(u) = \alpha(v) \in V(L)$ *then* u, v *are* L'*-connected.*

Subproof. By hypothesis (b), there exists i $(1 \le i \le k)$ such that $u, v \in V(A'_i)$, and hence $u, v \in Z'_i$, and so $\alpha(u) = \alpha(v) \in Z_i \subseteq V(A_i)$ by hypothesis (a). Since $\alpha(u) = \alpha(v) \in V(L \cap A_i)$ and hence $\alpha(u), \alpha(v)$ are $L \cap A$-connected, it follows from hypothesis (c) that u, v are L'_i-connected and hence L'-connected. This proves (4).

(5) *If* $u, v \in Z'$ *and there is a path* P *of* L *with ends* $\alpha(u), \alpha(v)$ *and with no internal vertex in* Z*, then* u, v *are* L'*-connected.*

Subproof. Since no internal vertex of P is in Z, and $V(A_i \cap A_j) \subseteq Z$ for $1 \le i < j \le k$, it follows that $P \subseteq A_i$ for some i, and $\alpha(u), \alpha(v) \in Z_i$. By hypothesis (a), there exist $u', v' \in Z'_i$ such that $\alpha(u) = \alpha(u')$ and $\alpha(v) = \alpha(v')$. By hypothesis (c), u' and v' are L'_i-connected and hence L'-connected, and by (4) so are u and u', and so are v and v'. Consequently u and v are L'-connected. This proves (5).

(6) *If* $u, v \in Z'$ *and* $\alpha(u), \alpha(v)$ *are* L*-connected then* u, v *are* L'*-connected.*

Subproof. Let P be a path of L with ends $\alpha(u), \alpha(v)$, and let $V(P) \cap Z = \{z_1, \dots, z_n\}$ in order, where $z_1 = \alpha(u)$ and $z_n = \alpha(v)$. For $1 \le i \le n$, choose $v_i \in Z'$ with $\alpha(v_i) = z_i$, with $v_1 = u$ and $v_n = v$. (This is possible by hypothesis (a).) By (5), for $1 \le i < n$, v_i and v_{i+1} are L'-connected. Hence u, v are L'-connected. This proves (6).

From (2), (3) and (6), the result follows. This completes the proof of (2.5). ■

Here is a corollary of (2.5). A *separation* of G is a pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$.

2.6 *Let (A, B) be a separation of a graph G, let $Z \subseteq V(G)$ with $V(A \cap B) \subseteq Z$, and let $L \subseteq G$. Let $L' \subseteq A$ with the same effect on $Z \cap V(A)$ as $L \cap A$. Then $L' \cup (L \cap B) \subseteq G$ has the same effect on Z as L.*

Proof. Let $A_1 = A'_1 = A, A_2 = A'_2 = B, G = G'$, and $Z = Z'$, and let $\alpha : Z' \to Z$ be the identity. Let $L_1 = L \cap A, L'_1 = L', L_2 = L'_2 = L \cap B$. The result follows from (2.5). ■

From (2.6) we deduce

2.7 *Let L be a subgraph of G, vital for $Z \subseteq V(G)$, and let (A, B) be a separation of G. Then $L \cap A$ is vital for $(Z \cap V(A)) \cup V(A \cap B)$ in A.*

Proof. Let $Z' = Z \cup V(A \cap B)$. Then L is vital for Z' in G, and so by (2.6), $L \cap A$ is vital for Z' in A, as required. ■

3 Drawings in a disc

In this section we prove the result outlined in the first paragraph of section 1. A *surface* is a connected compact 2-manifold, possibly with boundary. If Σ is a surface, a subset $X \subseteq \Sigma$ is an *O-arc* if it is homeomorphic to a circle, and a *line* if it is homeomorphic to the unit interval $[0, 1]$. The boundary of Σ is denoted by $bd(\Sigma)$, and the components of $bd(\Sigma)$ are called the *cuffs* of Σ; each cuff is an O-arc. If $X \subseteq \Sigma$, its topological closure is denoted by $\bar{X}$.

A *drawing* in Σ is a pair (U, V), where $U \subseteq \Sigma$ is closed, $V \subseteq U$ is finite, $U \cap bd(\Sigma) \subseteq V$, $U \setminus V$ has only finitely many arc-wise connected components, called *edges*, and for each edge e, either $\bar{e}$ is an O-arc and $|\bar{e} \cap V| = 1$, or $\bar{e}$ is a line and $\bar{e} \cap V$ is the set of ends of $\bar{e}$. If $\Gamma = (U, V)$ is a drawing in Σ, we write $U(\Gamma) = U$ and $V(\Gamma) = V$. We use graph-theoretic terminology for drawings in the natural way. If Γ is a drawing in Σ, we say $X \subseteq \Sigma$ is Γ-*normal* if $X \cap U(\Gamma) \subseteq V(\Gamma)$. The *regions* of Γ in Σ are the components of $\Sigma \setminus U(\Gamma)$. Note that in this paper, we do not insist that $V(\Gamma)$ meets every cuff.

If Γ is a drawing in Σ, and $T \subseteq \Sigma$ has the property that either $e \cap T = \emptyset$ or $\bar{e} \subseteq T$ for every $e \in E(\Gamma)$, we define $\Gamma \cap T$ to be the subdrawing $(U(\Gamma) \cap T, V(\Gamma) \cap T)$ of Γ. Let Γ be a drawing in a surface Σ, and let $Y \subseteq \Sigma$. We say $x \in \Sigma$ is *h-insulated (in Σ) from Y (by Γ)* if there are h disjoint circuits of Γ, all bounding discs in Σ containing x in their interiors and with no point of Y in their interiors; or more precisely, there are h closed discs $\Delta_1, \ldots, \Delta_h \subseteq \Sigma$ such that

- $x \in \Delta_h \setminus bd(\Delta_h)$, and $Y \cap \Delta_1 = \emptyset$
- for $1 \leq i < h$, $\Delta_{i+1} \subseteq \Delta_i \setminus bd(\Delta_i)$
- for $1 \leq i \leq h$, $bd(\Delta_i) \subseteq U(\Gamma)$.

The main result of this section is the following.

3.1 *For every integer $p \geq 0$ there exists $h \geq 1$ with the following property. Let Γ, K be subgraphs of a graph G, and let Γ be a drawing in a surface Σ. Let $v \in V(\Gamma)$ be h-insulated from $V(\Gamma \cap K)$ by Γ, let $Z \subseteq V(K)$ with $|Z| \leq p$, and let $L \subseteq G$. Then there is a subgraph L' of $G \setminus \{v\}$ with the same effect on Z as L and with $L' \cap K \subseteq L$.*

To prove (3.1) we need two lemmas.

3.2 *Let $C_1, \ldots, C_h$ be mutually vertex-disjoint connected subgraphs of a graph G, and also let $D_1, \ldots, D_h$ be mutually vertex-disjoint connected subgraphs of G. Suppose that $C_i \cap D_j$ is non-null for $1 \leq i, j \leq h$. Then G has tree-width at least $h - 1$.*

Proof. For each $X \subseteq V(G)$ with $|X| < h$, there exists i with $1 \leq i \leq h$ such that $X \cap V(C_i) = \emptyset$, and hence there is a component H of $G \setminus X$ with $C_i \subseteq H$. Since $C_i \cap D_j$ is non-null for each j, it follows that $D_j \subseteq H$ for every j with $X \cap V(D_j) = \emptyset$, and there is such a j. By the same argument, H includes every one of $C_1, \ldots, C_h$ which is disjoint from X. Define $\beta(X) = V(H)$. Then $\beta(X) \subseteq \beta(Y)$ if $Y \subseteq X \subseteq V(G)$ and $|X| < h$, that is, β is a "haven of order h in G" in the terminology of [8], and by theorem (1.4) of [8], G has tree-width at least $h - 1$, as required. ■

A line F in a surface Σ is *proper* if its ends are in $bd(\Sigma)$ and no other point of F is in $bd(\Sigma)$. The second lemma we need is as follows.

3.3 *Let Γ be a drawing in a closed disc Δ, and let $L \subseteq \Gamma$ with $V(\Gamma) \cap bd(\Delta) \subseteq V(L)$. Let $v \in V(\Gamma) \setminus bd(\Delta)$, and suppose that there is no subgraph of $\Gamma \setminus \{v\}$ with the same effect on $V(\Gamma) \cap bd(\Delta)$ as L. Then there is a Γ-normal proper line $F \subseteq \Delta$ with $v \in F \cap V(\Gamma)$, such that there are $F \cap V(\Gamma)$ components of L with a vertex in $F \cap V(\Gamma)$.*

Proof. Let the effect of L on $V(\Gamma) \cap bd(\Delta)$ be $\{Z_i : 1 \leq i \leq k\}$ say. By theorem (3.6) of [1], there is a $(\Gamma \setminus \{v\})$-normal proper line $F \subseteq \Delta$ such that

$$|F \cap V(\Gamma \setminus \{v\})| < |\{i : 1 \leq i \leq k, F_1 \cap Z_i \neq \emptyset \neq F_2 \cap Z_i\}|$$

where F_1 and F_2 are the two lines in $bd(\Delta)$ with the same ends as F. Let r be the region of $\Gamma \setminus \{v\}$ containing v. We may choose F so that it is Γ-normal; for if $F \cap r = \emptyset$ then F is already Γ-normal, and if $F \cap r \neq \emptyset$, choose a maximal line $F' \subseteq F$ with both ends in $\bar{r}$, and replace F' in F by a Γ-normal line in $\bar{r}$, with no point in $\bar{r}$ except its ends.

Let us renumber $Z_1, \ldots, Z_k$ so that for $1 \leq i \leq k$, Z_i meets both F_1 and F_2 if and only if $i \leq j$. For $1 \leq i \leq k$, let L_i be the component of L with $V(L_i) \cap bd(\Delta) = Z_i$. Since for $1 \leq i \leq j$, $U(L_i)$ meets both F_1 and F_2, it follows that $F \cap U(L_i) = \emptyset$, and since F is Γ-normal, there exists $v_i \in F \cap V(L_i)$. Now $L_1, \ldots, L_j$ are mutually vertex-disjoint, and so $v_1, \ldots, v_j$ are all distinct. But

$$\{v_1, \ldots, v_j\} \subseteq F \cap V(\Gamma) \subseteq (F \cap V(\Gamma \setminus \{v\})) \cup \{v\}$$

and $|F \cap V(\Gamma \setminus \{v\})| < j$, from the choice of j. Consequently, we have equality throughout, and so $v \in F \cap V(\Gamma)$, and $j = |F \cap V(\Gamma)|$, and $L_1, \ldots, L_j$ all have a vertex in F. The result follows. ■

Proof of (3.1). Let w be as in (2.1), and let $h = \lceil 5w/4 \rceil + 2$. We claim that h satisfies (3.1). For suppose not; then we can choose a graph G satisfying (1) and (2) below.

(1) *For some Γ, K, v, Z, L as in the theorem, with $\Gamma \cup K \subseteq G$, no subgraph L' of $G \setminus \{v\}$ with $L' \cap K \subseteq L$ has the same effect on Z as L.*

(2) *Subject to (1), $|V(G)| + |E(G)|$ is minimum.*

Choose Γ, K, v, Z, L as in (1), and let $\Delta_1, \ldots, \Delta_h$ be as in the definition of "h-insulated". Then we see that

(3) *$V(K) \cap \Delta_1 = \emptyset$ and hence $Z \cap \Delta_1 = \emptyset$.*

It follows that

(4) *$Z \subseteq V(L)$, and $K \subseteq L$; and no subgraph of $G \setminus \{v\}$ has the same effect on Z as L.*

Subproof. If there exists $z \in Z \setminus V(L)$, let $G' = G \setminus \{z\}$, and let $\Gamma' = \Gamma \setminus \{z\}$ if $z \in V(\Gamma)$ and $\Gamma' = \Gamma$ otherwise; then $L, \Gamma' \subseteq G'$ and $Z' \subseteq V(G')$ where $Z' = Z \setminus \{z\}$, and no subgraph of $G' \setminus \{v\}$ has the same effect on Z' as L, contrary to (2). Thus $Z \subseteq V(L)$. Suppose next that there exists $e \in E(K) \setminus E(L)$. If $e \in E(\Gamma)$ then $e \cap \Delta_1 = \emptyset$, and moreover $L, \Gamma' \subseteq G \setminus \{e\}$ (where $\Gamma' = \Gamma \setminus \{e\}$ if $e \in E(\Gamma)$, and $\Gamma' = \Gamma$ otherwise), contrary to (2). Thus $E(K) \subseteq E(L)$, and similarly $V(K) \subseteq V(L)$. The last claim follows from (1). This proves (4).

Let C_i be the circuit of Γ with $U(C_i) = bd(\Delta_i)$ $(1 \leq i \leq h)$. Let $C_1 \cup \cdots \cup C_h = M$.

(5) *$|E(C_i)| \geq 2$ for $1 \leq i \leq h$.*

Subproof. If $|E(C_i)| = 1$, let $X = V(L) \cap (\Delta_i \setminus bd\Delta_i)$; then $L \setminus X$ has the same effect on Z as L, and $v \notin V(L \setminus X)$, contrary to (1). This proves (5).

(6) *L is vital for Z in G.*

Subproof. Let $L' \subseteq G$ have the same effect on Z as L. By (2), $L' \cup M = G$. Suppose that there exists $e \in E(L' \cap M)$. By (5), e is not a loop, and e is not incident with v, since $v \notin V(M)$. No end of e is in Z, by (3). Hence no subgraph of $(G/e) \setminus \{v\}$ has the same effect on Z as L/e (we denote the contraction operation by /), if we interpret Z as a subset of $V(G/e)$ in the natural way. But this contradicts (2). Consequently $E(L' \cap M) = \emptyset$, and so $E(L') = E(G) \setminus E(M)$. Since the same holds for L, we deduce that $E(L') = E(L)$.

Suppose that there exists $u \in V(G) \setminus V(L')$. Since $L' \cup M = G$, it follows that $u \in V(M)$, and by (5), there is a non-loop edge e of M incident with u. Let L'' be obtained from L' by adding e and its ends u, u' say. Now $u, u' \notin Z$ by (3), and so L'' has the same effect on Z as L' and hence as L. Yet $E(L'' \cap M) \neq \emptyset$, contrary to what we just proved. This shows that $V(L') = V(G)$, and hence $L' = L$, and therefore L is vital. This proves (6).

Let $\Gamma_1 = \Gamma \cap \Delta_1$.

(7) *At most* $\frac{1}{2}(w+1)$ *components of* $L \cap \Gamma_1$ *meet* Δ_{w+3}.

Subproof. Let $L_1, \ldots, L_t$ be components of $L \cap \Gamma_1$ meeting Δ_{w+3}, and for $1 \leq i \leq t$ let $v_i \in V(L_i) \cap \Delta_{w+3}$. Let $1 \leq i \leq t$. Since L is a forest there is a path of L passing through v_i with both ends of degree at most 1 in L, and hence with both ends in Z, by (6) and (2.2). Since $Z \subseteq V(K)$, it follows that there is a path P of $L \cap \Gamma_1$ with $v_i \in V(P)$ and with both ends in $V(C_1)$. Since both subpaths of P from v_i to its ends meet $V(C_{w+2})$, P contains two vertex-disjoint paths between $V(C_{w+2})$ and $V(C_1)$. Since this holds for all i with $1 \leq i \leq t$, there are $2t$ mutually vertex-disjoint paths of $L \cap \Gamma_1$, each meeting $V(C_{w+2})$ and $V(C_1)$ and hence meeting all of $V(C_1), V(C_2), \ldots, V(C_{w+2})$. If $2t \geq w+2$ then by (3.2) G has tree-width $\geq w+1$ contrary to (6) and (4.1). Thus $2t \leq \omega + 1$. This proves (7).

Let $\Delta \subseteq \Sigma$ be a closed disc with $\Delta_1 \subseteq \Delta$, $U(\Gamma) \cap \Delta_1 = U(\Gamma) \cap \Delta$, and $U(\Gamma) \cap bd(\Delta) = V(C_1)$.

(8) *If* F *is a* Γ*-normal proper line in* Δ *with* $v \in F \cap V(\Gamma)$, *there are fewer than* $|F \cap V(\Gamma)|$ *components of* $L \cap \Gamma_1$ *which meet* $F \cap V(\Gamma)$.

Subproof. Suppose that there are $|F \cap V(\Gamma)|$ such components. Then each vertex of $F \cap V(\Gamma)$ belongs to a different component of $L \cap \Gamma_1$. But there are $\geq 2h - 3 - 2w \geq \frac{1}{2}w + 1$ vertices of $F \cap V(\Gamma)$ in Δ_{w+3}, because $v \in F \cap V(\Gamma) \cap \Delta_{w+3}$, and

$$|V(C_i) \cap (F \cap V(\Gamma) \cap \Delta_{w+3})| = |F \cap U(C_i)| \geq 2$$

for $w+3 \leq i \leq h$. Hence there are $\geq \frac{1}{2}w + 1$ components of $L \cap \Gamma_1$ meeting Δ_{w+3}, contrary to (7). This proves (8).

Now Γ_1 is a drawing in Δ, and $L \cap \Gamma_1 \subseteq \Gamma_1$ with $V(\Gamma_1) \cap bd(\Delta) \subseteq V(L \cap \Gamma_1)$. By (3.3) and (8), there is a subgraph L'' of $\Gamma_1 \setminus \{v\}$ with the same effect on $V(\Gamma_1) \cap bd(\Delta) = V(C_1)$ as $L \cap \Gamma_1$. Consequently $L \cap \Gamma_1$ is not vital for $V(C_1)$ in Γ_1, because $L'' \neq L \cap \Gamma_1$ since $v \in V(L \cap \Gamma_1)$ by (6).

Let Γ_2 be the drawing formed by the edges of Γ not in Δ_1, and the vertices of Γ not in $\Delta_1 \setminus bd(\Delta_1)$. Then (Γ_1, Γ_2) is a separation of Γ with $V(\Gamma_1 \cap \Gamma_2) = V(C_1)$. Let $K_1 = \Gamma_2 \cup K$. Since $V(\Gamma \cap K) \subseteq V(\Gamma_2)$, it follows that (Γ_1, K_1) is a separation of G, and $V(\Gamma_1 \cap K_1) = V(C_1)$. But $Z \subseteq V(K_1)$, and L is vital for Z in G, and $L \cap \Gamma_1$ is not vital for $(Z \cup V(\Gamma_1)) \cup V(\Gamma_1 \cap K_1) = V(C_1)$ in Γ_1, contrary to (2.7). The result follows. ■

4 Changing the drawing

(3.1) allows us to delete vertices of Γ without changing whether a subgraph exists with a desired effect on Z. But it can also be used in reverse, for it allows us to introduce new vertices into Γ without changing whether the desired subgraph exists. By doing both, we can replace parts of Γ by completely different drawings. This is quite powerful, as we shall see in this section and the next.

We have a pair of subgraphs Γ, K of a graph G with $\Gamma \cup K = G$, where Γ is a drawing in a surface Σ; and we wish to consider the effect of replacing Γ by a new drawing Γ' in Σ with $\Gamma \cap K \subseteq \Gamma'$. We would like there to be a graph G' with $\Gamma', K \subseteq G'$ and with $\Gamma' \cap K = \Gamma \cap K$, and if this is so we write $\Gamma' \cap K = \Gamma \cap K$ for brevity.

Let Γ and Γ' be drawings in a surface Σ, and let $T \subseteq \Sigma$. We say that Γ' is a *T-variant* of Γ in Σ if

- $V(\Gamma) \setminus T = V(\Gamma') \setminus T$ and
- if $e \in E(\Gamma) \setminus E(\Gamma')$ or $e \in E(\Gamma') \setminus E(\Gamma)$, then $\bar{e} \subseteq T$.

From (3.1) we deduce the following.

4.1 *For every integer $p \geq 0$ there exists $h \geq 1$ with the following property. Let Γ, K be subgraphs of a graph $\Gamma \cup K$, let Γ be a drawing in a surface Σ and let T be the set of all points of Σ that are h-insulated from $V(\Gamma \cap K)$ by Γ. Let Γ' be a T-variant of Γ in Σ with $\Gamma' \cap K = \Gamma \cap K$, let $L' \subseteq \Gamma' \cup K$, and let $Z \subseteq V(K)$ with $|Z| \leq p$. Then there exists a subgraph L of $\Gamma \cup K$ with the same effect on Z as L', with $L \cap K \subseteq L'$.*

Proof. Now T is open, for it is the union of the interiors of finitely many closed discs (namely, those discs bounded by circuits of Γ which are "surrounded" by $h-1$ other circuits). For each edge $e' \in E(\Gamma') \setminus E(\Gamma)$ we may therefore perturb e' slightly (since $\bar{e}' \subseteq T$) so that $e' \cap U(\Gamma)$ is finite, preserving the property that $\Gamma' \cap K = \Gamma \cap K$. Consequently, we may assume that there is a drawing Γ^* in Σ with $\Gamma^* \cap K = \Gamma \cap K$, which is a T-variant of Γ such that $U(\Gamma^*) = U(\Gamma) \cup U(\Gamma')$ and $V(\Gamma) \cup V(\Gamma') \subseteq V(\Gamma^*)$. (The second inclusion may not be an equality since to make Γ^* a drawing it must have a vertex wherever an edge e of Γ meets an edge $e' \neq e$ of Γ'.) Let $L^* \subseteq \Gamma^*$ with $U(L^*) = U(\Gamma' \cap L')$. Then by (2.6), $(K \cap L) \cup L^* \subseteq K \cup \Gamma^*$ has the same effect on Z as L'. Consequently, Γ^* has all the defining properties of Γ', and we may therefore assume that $\Gamma^* = \Gamma'$, that is,

(1) $U(\Gamma) \subseteq U(\Gamma')$ *and* $V(\Gamma) \subseteq V(\Gamma')$.

Under condition (1), we proceed by induction on $|V(\Gamma')| + |E(\Gamma')|$. Suppose first that $U(\Gamma') = U(\Gamma)$. Since $V(\Gamma) \subseteq V(\Gamma')$ it follows from (2.6) (as above) that there is a subgraph $L \subseteq \Gamma \cup K$ with $U(L \cap \Gamma) = U(L' \cap \Gamma')$ and $L \cap K = L' \cap K$, with the same effect on Z as L'; but then the theorem is true.

We may therefore assume that $U(\Gamma') \neq U(\Gamma)$. Choose $x \in U(\Gamma') \setminus U(\Gamma)$. Choose $v \in V(\Gamma')$ so that $x = v$ if $x \in V(\Gamma')$, and v is an end of e if $x \in e$ for some $e \in E(\Gamma')$. We claim that $v \in T$. For $x \in T$, so if $x = v$ this is true. If $x \in e \in E(\Gamma')$ and v is an end of e, then $e \notin E(\Gamma)$ since $x \notin U(\Gamma)$, and $v \in \bar{e} \subseteq T$ since Γ' is a T-variant of Γ. This proves that $v \in T$, and hence v is h-insulated by Γ and hence by Γ' from $V(\Gamma \cap K) = V(\Gamma' \cap K)$. By (3.1) with Γ replaced by Γ', there exists $L'' \subseteq \Gamma' \cup K$ with $v \notin V(L)$ and $L'' \cap K \subseteq L' \cap K$, such that L'' has the same effect on Z as L'. Let Γ'' be the T-variant of Γ' (and hence of Γ) obtained from Γ' by deleting x if $x \in V(\Gamma')$, and deleting e if $x \in e \in E(\Gamma')$; then $U(\Gamma) \subseteq U(\Gamma''), V(\Gamma) \subseteq V(\Gamma'')$, and

$$|V(\Gamma'')| + |E(\Gamma'')| < |V(\Gamma')| + |E(\Gamma')|.$$

Moreover, $L'' \subseteq \Gamma'' \cup K$, and so from the inductive hypothesis, there exists $L \subseteq \Gamma \cup K$ with the same effect on Z as L'' and hence as L', and with $L \cap K \subseteq L'' \cap K \subseteq L' \cap K$, as required. ∎

Let Σ be a surface. We denote by $\hat{\Sigma}$ the surface obtained from Σ by pasting an open disc onto each cuff of Σ. Let Γ be a drawing in Σ. If C is a cuff of Σ, a *sleeve for C in Γ* is a closed disc $\Delta \subseteq \hat{\Sigma}$ such that

- $bd(\Delta) \subseteq U(\Gamma)$
- Δ includes the open disc pasted onto C in forming $\hat{\Sigma}$
- $\Delta \cap bd(\Sigma) = C$.

4.2 *For every integer $p \geq 0$ there exists $h \geq 1$ with the following property. Let Γ, K be subgraphs of a graph $\Gamma \cup K$, let Γ be a drawing in a surface Σ, and let $Z \subseteq V(K) \cup (V(\Gamma) \cap bd(\Sigma))$, with $|Z| \leq p$. For each cuff C of Σ let $S(C)$ be a sleeve for C in Γ, so that $S(C_1) \cap S(C_2) = \emptyset$ for all distinct cuffs C_1, C_2. Let S be the union of $\Sigma \cap S(C)$ over all cuffs C, and let T be the set of all points of Σ that are h-insulated in $\hat{\Sigma}$ from $V(\Gamma \cap K) \cup (V(\Gamma) \cap bd(\Sigma))$ by Γ. Suppose that*

(i) *for each cuff C there are $|V(\Gamma) \cap C|$ mutually vertex-disjoint paths of Γ between $V(\Gamma) \cap C$ and $V(\Gamma) \cap bd(S(C))$*

(ii) *for each cuff C, $bd(S(C)) \subseteq T$ and $S(C) \cap V(\Gamma \cap K) = \emptyset$*

(iii) *Γ' is an $(S \cup T)$-variant of Γ with $\Gamma' \cap K = \Gamma \cap K$ and $V(\Gamma') \cap bd(\Sigma) = V(\Gamma) \cap bd(\Sigma)$, and $L' \subseteq \Gamma' \cup K$.*

Then there exists $L \subseteq \Gamma \cup K$ with the same effect on Z as L', with $L \cap K \subseteq L'$.

Proof. Let h be as in (4.1), and let Γ, K etc. be as in the theorem. Since $\Gamma' \cap K = \Gamma \cap K$, we may assume for convenience that $V(K) \cap \Sigma \subseteq V(\Gamma)$. Let $C_1, \ldots, C_r$ be the cuffs of Σ. For $1 \leq i \leq r$, let C_i' be the circuit of Γ with $U(C_i') = bd(S(C_i))$, let $|V(\Gamma) \cap C_i| = k_i$, and let M_i be a minimal linkage in Γ with k_i components, each with one end in $V(\Gamma) \cap C_i$ and the other end in $V(C_i')$. For each component P of M_i, let the ends of P be $s(P) \in V(\Gamma) \cap C_i$ and $s'(P) \subseteq V(C_i')$. From the minimality of M_i it follows that $U(M_i) \subseteq S(C_i)$, and for each component P of M_i, $U(P) \cap bd(S(C_i)) = \{s'(P)\}$.

Let Σ_0 be the surface obtained from Σ by deleting $\Sigma \cap (S(C) \setminus bd(S(C)))$, for each cuff C. Then Σ_0 is homeomorphic to Σ. Since T is open and $bd(S(C)) \subseteq T$ for each cuff C, there is a homeomorphism $\alpha : \Sigma \to \Sigma_0$ fixing $\Sigma \setminus (S \cup T)$ pointwise, such that for $1 \leq i \leq r$, α maps $U(C_i)$ onto $U(C_i')$, and for each component P of M_i, α maps $s(P)$ to $s'(P)$. Let Γ_0 be the image of Γ' under α. Then Γ_0 is a drawing in Σ_0. Since Γ' is an $(S \cup T)$-variant of Γ and α fixes $\Sigma \setminus (S \cup T)$ pointwise, it follows that Γ_0 is an $(S \cup T)$-variant of Γ. Moreover, for $1 \leq i \leq r$,

$$U(\Gamma_0) \cap U(C_i') = V(\Gamma_0) \cap U(C_i') = V(M_i \cap C_i').$$

Let $\Gamma'' = \Gamma_0 \cup \bigcup (\Gamma \cap S(C_i) : 1 \leq i \leq r)$. Then Γ'' is a drawing in Σ, and $V(\Gamma'') \cap bd(\Sigma) = V(\Gamma) \cap bd(\Sigma)$. Moreover, Γ'' is a T-variant of Γ, for it is an $(S \cup T)$-variant of Γ (since Γ_0 is) and for each cuff C, $\Gamma'' \cap S(C) = \Gamma \cap S(C)$.

(1) *$V(\Gamma'' \cap K) = V(\Gamma \cap K) \subseteq \Sigma \setminus (S \cup T)$.*

Subproof. Certainly $V(\Gamma'' \cap K) \subseteq V(\Gamma \cap K)$, since $V(K) \cap \Sigma \subset V(\Gamma)$. Let $v \in V(\Gamma \cap K)$. Since $v \notin T$

by definition of T, and $v \in V(\Gamma)$, it follows that $v \in V(\Gamma'')$, and so $v \in V(\Gamma'' \cap K)$. Also, $v \notin S$, by hypothesis (ii), and so $v \in \Sigma \setminus (S \cup T)$. This proves (1).

For $1 \le i \le r$, let M_i' be the union of the components P of M_i such that $s(P) \in V(L')$. Let L_0 be the image of $L' \cap \Gamma'$ under α, and let $L'' = L_0 \cup M_1' \cup \cdots \cup M_r'$. Then $L'' \subseteq \Gamma''$, with the same effect on $V(\Gamma \cap K) \cup (V(\Gamma) \cap bd(\Sigma))$ as $L' \cap \Gamma'$, since $\alpha(v) = v$ for each $v \in V(\Gamma \cap K)$ by (1). By (2.6), $L'' \cup (L' \cap K) \subseteq \Gamma'' \cup K$ has the same effect on Z as L'. By (4.1) applied to $\hat{\Sigma}$, there exists $L \subseteq \Gamma \cup K$ with the same effect on Z as $L'' \cup (L' \cup K)$ and hence as L', and with

$$L \cap K \subseteq (L'' \cup (L' \cap K)) \cap K \subseteq L',$$

as required. ■

For our applications of (4.2) in this paper, we only really need (4.2) when $\hat{\Sigma}$ is a sphere. But for general surfaces it is still of some interest. For instance, the special case of (4.2) when K is null, $Z = V(\Gamma) \cap bd(\Sigma)$ and $S \cup T = \Sigma$ is still powerful, for it readily implies the main theorem of [2], indeed in a strengthened form (it shows that the lower bound on $\alpha(G)$ discussed in theorem (7.5) of [2] can be replaced by one independent of the surface). This would therefore give a new and virtually painless proof of the result of [2], if only an easy proof of (1.1) could be found.

5 Tangles

If (A, B) is a separation of G, its *order* is $|V(A \cap B)|$. A *tangle of order* $\theta \ge 1$ in a graph G is a set $\mathcal{T}$ of separations of G, all of order $< \theta$, such that

- for every separation (A, B) of G of order $< \theta$, $\mathcal{T}$ contains one of $(A, B), (B, A)$
- if $(A_i, B_i) \in \mathcal{T}$ $(i = 1, 2, 3)$ then $A_1 \cup A_2 \cup A_3 \neq G$
- if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

We write $ord(\mathcal{T}) = \theta$. If Γ is a drawing in a surface Σ with $bd(\Sigma) = \emptyset$, a tangle $\mathcal{T}$ in Γ is *respectful* if for every Γ-normal O-arc $F \subseteq \Sigma$ with $|F \cap V(\Gamma)| < ord(\mathcal{T})$ there is a closed disc $\Delta \subseteq \Sigma$ bounded by F with

$$(\Gamma \cap \Delta, \Sigma \cap \overline{\Sigma \setminus \Delta}) \in \mathcal{T}.$$

In this case, we write $\Delta = ins(F)$. We say Γ is 2-*cell* if every region is homeomorphic to an open disc. Every connected drawing with a respectful tangle is 2-cell. The *atoms* of Γ are sets r where r is a region of Σ in Σ, the sets $e \in E(\Gamma)$ and the sets $\{v\}$ where $v \in V(\Gamma)$. The set of atoms of Γ is denoted by $A(\Gamma)$. If Γ is 2-cell, and $\mathcal{T}$ is a respectful tangle in Γ, we define a metric on $A(\Gamma)$ as discussed in [4]; this is called the *metric of* $\mathcal{T}$, and denoted by d. If $X, Y \subseteq \Sigma$, we define $d(X, Y)$ to be the minimum of $d(a, b)$, taken over all atoms a, b with $a \cap X \neq \emptyset$ and $b \cap Y \neq \emptyset$, or $d(X, Y) = ord(\mathcal{T})$ if one of X, Y is empty. We need the following, from theorem (9.2) of [6].

5.1 *Let Γ be a 2-cell drawing in a surface Σ with $bd(\Sigma) = \emptyset$, and let $\mathcal{T}$ be a respectful tangle in Γ, with metric d. Let $z \in A(H)$, and let κ be an integer with $2 \le \kappa \le ord(\mathcal{T}) - 3$. Then there is a closed disc $\Delta \subseteq \Sigma$ satisfying*

(i) $bd(\Delta) \subseteq U(\Gamma)$

(ii) $d(z,x) \leq \kappa + 2$ *for all* $x \in A(\Gamma)$ *with* $x \cap \Delta \neq \emptyset$

(iii) $d(z,x) \geq \kappa$ *for all* $x \in A(\Gamma)$ *with* $x \not\subseteq \Delta \setminus bd(\Delta)$ *(and in particular,* $z \subseteq \Delta \setminus bd(\Delta)$*).*

We deduce

5.2 *Let* $h \geq 1$ *be an integer, let* Γ *be a 2-cell drawing in a surface* Σ *with* $bd(\Sigma) = \emptyset$*, and let* $\mathcal{T}$ *be a respectful tangle in* Γ *of order* $\geq 2h+5$*, with metric* d*. Let* $x \in \Sigma$*, and let* Y *be the union of all atoms* $y \in A(\Gamma)$ *with* $d(y,z) \geq 2h+5$*, where* z *is the atom of* Γ *with* $x \in z$*. Then* x *is* h*-insulated from* Y *by* Γ*.*

Proof. Let $\kappa = 2h+2$, and let Δ be as in (5.1).

(1) *If* $r_1, \ldots, r_t$ *is a sequence of regions of* Γ *with* $z \subseteq \bar{r}_1$*,* $\bar{r}_t \cap bd(\Delta) \neq \emptyset$*, and* $\bar{r}_i \cap \bar{r}_{i+1} \neq \emptyset$ *for* $1 \leq i < t$*, then* $t \geq h$*.*

Subproof. Let $z' \in A(\Gamma)$ with $\bar{r}_t \cap bd(\Delta) \cap z' \neq \emptyset$. Then

$$d(z,z') \leq d(z,r_1) + \sum_{1 \leq i \leq t-1} d(r_i, r_{i+1}) + d(r_t, z');$$

but $d(z,r_1) \leq 2$, $d(r_i,r_{i+1}) \leq 2$ for $1 \leq i \leq t-1$, and $d(r_t,z') \leq 2$, and so $d(z,z') \leq 2t+2$. But from (5.1)(iii), $d(z,z') \geq 2h+2$ since $z' \not\subseteq \Delta \setminus bd(\Delta)$. Hence $h \leq t$. This proves (1).

Let C_1 be the circuit of Γ with $U(C_1) = bd(\Delta)$. From (1) and theorem (5.5) of [6], there are circuits $C_2, \ldots, C_h$ of Γ, mutually vertex-disjoint and with $U(C_i) \subseteq \Delta \setminus bd(\Delta)$ $(2 \leq i \leq h)$, such that $\Delta_2 \supseteq \Delta_3 \supseteq \cdots \supseteq \Delta_h$ and $z \subseteq \Delta_h \setminus bd(\Delta_h)$, where Δ_i is the closed disc in Δ bounded by $U(C_i)$ $(2 \leq i \leq h)$. But if $y \in A(\Gamma)$ with $y \cap \Delta \neq \emptyset$, then $y \subseteq \Delta$; and so by (5.1)(ii), $d(z,y) \leq 2h+4$. Consequently, $Y \cap \Delta = \emptyset$, and so x is h-insulated from Y by Γ, as required. ■

The main result of this section is the following.

5.3 *For every integer* $p \geq 0$ *there exists* $\theta > p$ *with the following property. Let* Γ, K *be subgraphs of a graph* $\Gamma \cup K$*, let* Γ *be a 2-cell drawing in a surface* Σ *with* $bd(\Sigma) = \emptyset$*, and let* $\mathcal{T}$ *be a respectful tangle in* Γ *of order* $\geq \theta$*, with metric* d*. Let* $Z \subseteq V(\Gamma \cup K)$ *with* $|Z| \leq p$*, and let* $F_1, \ldots, F_t$ *be* Γ*-normal O-arcs, such that*

$$(F_1 \cup \cdots \cup F_t) \cap V(\Gamma) \subseteq Z \subseteq ((F_1 \cup \cdots \cup F_t) \cap V(\Gamma)) \cup V(K)$$

and $ins(F_1), \ldots, ins(F_t)$ *are mutually disjoint. Suppose that*

(i) *for* $1 \leq i \leq t$*, there is no* Γ*-normal O-arc* $F \subseteq \Sigma$ *with* $|F \cap V(\Gamma)| < |F_i \cap V(\Gamma)|$ *and* $ins(F_i) \subseteq ins(F)$

(ii) *for* $1 \leq i < j \leq t$*,* $d(ins(F_i), ins(F_j)) \geq \theta$

(iii) *for* $1 \leq i \leq k$, $d(ins(F_i), v) \geq \theta$ *for every* $v \in V(\Gamma \cap K)$.

Let R *be the union of all atoms* z *of* Γ *with* $d(z, v) \geq \theta$ *for all* $v \in V(\Gamma \cap K)$*, and let*

$$\Sigma' = \Sigma \setminus \bigcup_{1 \leq i \leq t} (ins(F_i) \setminus F_i).$$

Let Γ' *be an* $(R \cap \Sigma')$*-variant of* Γ *in* Σ *with* $\Gamma' \cap K = \Gamma \cap K$ *and with* $\Gamma \cap ins(F_i) = \Gamma' \cap ins(F_i)$ $(1 \leq i \leq t)$*, and let* $L' \subseteq (\Gamma' \cap \Sigma') \cup K$*. Then there exists* $L \subseteq (\Gamma \cap \Sigma') \cup K$ *with the same effect on* Z *as* L'*, such that* $L \cap K \subseteq L'$.

Proof. Let $h \geq 1$ be as in (4.2), and let $\theta = 2p + 4h + 15$. We claim that θ satisfies the theorem. For let Γ, K etc. be as in the theorem. Let r_i be a region of Γ in Σ with $r_i \cap F_i \neq \emptyset$, for $1 \leq i \leq t$. Since $|F_i \cap V(\Gamma)| \leq |Z| \leq p$, we have

(1) *For* $1 \leq i \leq t$*, if* $z \in A(\Gamma)$ *and* $z \cap ins(F_i) \neq \emptyset$ *then* $d(z, r_i) \leq p$.

By (5.1), we deduce

(2) *For* $1 \leq i \leq t$ *there is a closed disc* $S_i \subseteq \Sigma$ *such that*

(i) $bd(S_i) \subseteq U(\Gamma)$,

(ii) $d(r_i, x) \leq p + 2h + 7$ *for all* $x \in A(\Gamma)$ *with* $x \cap S_i \neq \emptyset$*, and*

(iii) $d(r_i, x) \geq p + 2h + 5$ *for all* $x \in A(\Gamma)$ *with* $x \not\subseteq S_i \setminus bd(S_i)$.

(3) *For* $1 \leq i < j \leq t$*,* $S_i \cap S_j = \emptyset$.

Subproof. If x is an atom with $x \cap S(C_i) \cap S(C_j) \neq \emptyset$, then by (2)(ii), $d(r_i, x), d(r_j, x) \leq p + 2h + 7$, and so $d(r_i, r_j) \leq 2p + 4h + 14 < \theta$. Consequently, $d(ins(F_i), ins(F_j)) < \theta$ contrary to hypothesis (ii). This proves (3).

(4) *For* $1 \leq i \leq t$*,* $S_i \cap V(\Gamma \cap K) = \emptyset$ *and* $ins(F_i) \subseteq S_i$.

Subproof. If $v \in V(\Gamma \cap K)$ then $d(v, ins(F_i)) \geq \theta$ by hypothesis (iii), and in particular $d(v, r_i) \geq \theta$. Consequently, $v \notin S_i$ by (2)(ii), and so $S_i \cap V(\Gamma \cap K) = \emptyset$. Let z be an atom with $z \subseteq ins(F_i)$. By (1), $d(r_i, z) \leq p$, and so $z \subseteq S_i$ by (2)(iii). This proves (4).

(5) *For* $1 \leq i \leq t$ *there are* $|F_i \cap V(\Gamma)|$ *mutually disjoint paths of* $\Gamma \cap \Sigma'$ *between* $V(\Gamma) \cap F_i$ *and* $V(\Gamma) \cap bd(S_i)$.

Subproof. If not, then by a form of Menger's theorem applied to $\Gamma \cap S_i$, there is a Γ-normal O-arc $F \subseteq \Sigma$ with $|F \cap V(\Gamma)| < |F_i \cap V(\Gamma)|$, bounding a closed disc $\Delta \subseteq S_i$ with $F_i \subseteq \Delta$. By theorem (7.5) of [6], with $H, \Sigma, \theta, \lambda$ replaced by $\Gamma, \Sigma, ord(\mathcal{T}), p + 2h + 7$, it follows that $\Delta = ins(F)$, since

$$2|F \cap V(\Gamma)| < 2|F_i \cap V(\Gamma)| \leq 2p \leq 2(ord(\mathcal{T}) - (2h + 8)).$$

This contradicts hypothesis (i), and therefore proves (5).

Let $X = V(\Gamma \cap K) \cup ins(F_1) \cup \cdots \cup ins(F_t)$. Let $S = (S_1 \cup \cdots \cup S_t) \cap \Sigma'$, and let T be the set of all points of Σ' that are h-insulated in Σ from X by $\Gamma \cap \Sigma'$.

(6) *$R \cap \Sigma' \subseteq S \cup T$.*

Subproof. Let $z \in A(\Gamma)$ such that $d(z, v) \geq \theta$ for all $v \in V(\Gamma \cap K)$. If $d(z, ins(F_i)) \leq 2h + 4$ for some i ($1 \leq i \leq t$) then $d(z, r_i) \leq 2h + 4 + p$ by (1), and so $z \subseteq S_i$ by (2)(iii). We assume then that $d(z, ins(F_i)) \geq 2h + 5$ for $1 \leq i \leq t$. Hence $d(z, X) \geq 2h + 5$, since $\theta \geq 2h + 5$. By (5.2), v is h-insulated in Σ from X by Γ and hence by $\Gamma \cap \Sigma'$ (since $\Sigma \setminus \Sigma' \subseteq X$), and so $z \subseteq T$. This proves (6).

(7) *For $1 \leq i \leq t$, $bd(S_i) \subseteq T$.*

Subproof. Let $z \in A(\Gamma)$ with $z \subseteq bd(S_i)$. By (1)(i) and (1)(ii),

$$p + 2h + 5 \leq d(r_i, z) \leq p + 2h + 7.$$

We claim that $d(z, X) \geq 2h + 5$. For let $x \in A(\Gamma)$ with $x \cap X \neq \emptyset$. If $x \cap ins(F_i) \neq \emptyset$, then by (1),

$$p + 2h + 5 \leq d(r_i, z) \leq d(r_i, x) + d(x, z) \leq p + d(x, z)$$

and so $d(x, z) \geq 2h + 5$. If $x \cap ins(F_j) \neq \emptyset$ for some $j \neq i$ with $1 \leq j \leq t$, then by hypothesis (ii),

$$\theta \leq d(ins(F_i), ins(F_j)) \leq d(x, r_i) \leq d(x, z) + d(r_i, z) \leq d(x, z) + p + 2h + 7$$

and so $d(x, z) \geq 2h + 5$. Finally, if $x \in V(\Gamma \cap K)$, then by hypothesis (iii),

$$\theta \leq d(ins(F_i), x) \leq d(r_i, x) \leq d(r_i, z) + d(x, z) \leq d(x, z) + p + 2h + 7$$

and again $d(x, z) \geq 2h + 5$. This proves that $d(z, X) \geq 2h + 5$. Consequently z is h-insulated in Σ from X by Γ and hence by $\Gamma \cap \Sigma'$, and so $z \subseteq T$. This proves (7).

From (5), $\Gamma' \cap \Sigma'$ is an $(S \cup T)$-variant of $\Gamma \cap \Sigma'$. By (2), (3), (5), (7) and (4.2) (applied to $\Gamma' \cap \Sigma'$), the result follows. ■

We observe that the special case of (5.3) when K is null is precisely theorem (3.2) of [4], except that now θ does not depend on Σ.

6 Rooted digraphs

A *digraph* is a directed graph. When without explanation we use graph-theoretic terms for digraphs, such as "connected", "path", "separation", "subgraph", these should be taken to refer to the undirected graph underlying the digraph.

A *rooted digraph* $(G, u_1, \ldots, u_q)$ consists of a digraph G and a sequence $u_1, \ldots, u_q$ of vertices of G, not necessarily distinct. A rooted digraph $(G, u_1, \ldots, u_q)$ has *detail* $\leq \delta$, where $\delta \geq 0$ is an integer, if $|E(G)| \leq \delta$ and $|V(G) \setminus \{u, \ldots, u_q\}| \leq \delta$. If $(G, u_1, \ldots, u_q)$ and $(H, v_1, \ldots, v_q)$ are rooted digraphs, both with q roots, a *model* of the second in the first is a function ϕ with domain $V(H) \cup E(H)$, such that

(i) for each $v \in V(H), \phi(v)$ is a non-null connected subgraph of G; for all distinct $v, v' \in V(H)$, $\phi(v) \cap \phi(v')$ is null; and for $1 \leq i \leq q$, $u_i \in V(\phi(v_i))$

(ii) for each $e \in E(H)$, $\phi(e)$ is an edge of G; for all distinct $e, e' \in E(H)$, $\phi(e) \neq \phi(e')$; for all $e \in E(H)$ and $v \in V(H)$, $\phi(e) \notin E(\phi(v))$; and if $e \in E(H)$ has head $v \in V(H)$ and tail $v' \in V(H)$ then $\phi(e)$ has head in $V(\phi(v))$ and tail in $V(\phi(v'))$.

For $\delta \geq 0$, the *δ-folio* of $(G, u_1, \ldots, u_q)$ is the class of all rooted digraphs with detail $\leq \delta$ of which there is a model in $(G, u_1, \ldots, u_q)$. In [5] we gave an algorithm to compute the δ-folio of a rooted digraph $(G, u_1, \ldots, u_q)$; it had running time $O(|V(G)|^3)$ for fixed q and δ. However, the proof of its correctness used a result (theorem (10.2) of [5]) which was not proved in [5], and proving it is the objective of this paper.

Let ϕ be a model of $(H, v_1, \ldots, v_q)$ in $(G, u_1, \ldots, u_q)$. A *basis* for ϕ is a subset $Z \subseteq V(G)$ such that $u_1, \ldots, u_q \in Z$, both ends of $\phi(e)$ belong to Z for every $e \in E(H)$, and $Z \cap V(\phi(v)) \neq \emptyset$ for every $v \in V(H)$. (The third condition is implied by the first two except for vertices v of H different from $v_1, \ldots, v_q$ and not incident with any edge of H.) We observe that, obviously,

6.1 *If H has detail $\leq \delta$, every basis for ϕ includes a basis of cardinality $\leq q + 3\delta$.*

6.2 *Let ϕ be a model of $(H, v_1, \ldots, v_q)$ in $(G, u_1, \ldots, u_q)$, let Z be a basis for ϕ, let $L = \bigcup(\phi(v) : v \in V(H))$, and let $L' \subseteq G \setminus \phi(E(H))$ with the same effect on Z as L. Define $\phi'(e) = \phi(e)$ $(e \in E(H))$, and for $v \in V(H)$ let $\phi'(v)$ be the component T of L' with $V(T) \cap Z = V(\phi(v)) \cap Z$. Then ϕ' is a model of $(H, v_1, \ldots, v_q)$ in $(G, u_1, \ldots, u_q)$.*

Proof. For distinct $v_1, v_2 \in V(H)$, there is a vertex z of Z in $V(\phi(v_1))$ and hence not in $V(\phi(v_2))$ since Z is a basis; consequently, $z \in V(\phi'(v_1)) \setminus V(\phi'(v_2))$, and so $\phi'(v_1) \neq \phi'(v_2)$. Since $\phi'(v_1)$ and $\phi'(v_2)$ are both components of L' it follows that $\phi'(v_1) \cap \phi'(v_2)$ is null. For $1 \leq i \leq q$, $u_i \in Z \cap V(\phi(v_i))$, and hence $u_i \in V(\phi'(v_i))$. This proves condition (i) in the definition of "model".

For condition (ii), the first three statements are clear. For the fourth, let $e \in E(H)$ have head v and tail v', and let $\phi(e)$ have head u and tail u'. Then $u, u' \in Z$, and $u \in V(\phi(v))$, and $u' \in V(\phi(v'))$. Consequently, $u \in V(\phi'(v))$ and $u' \in V(\phi'(v'))$. This proves (ii), and so completes the proof of (6.1). ■

If G is a digraph and $Z \subseteq V(G)$, a *Z-division* of G is a set $\{A_1, \ldots, A_k\}$ of subdigraphs of G such that $A_1 \cup \cdots \cup A_k = G$, and $E(A_i \cap A_j) = \emptyset$ and $V(A_i \cap A_j) \subseteq Z$ for $1 \leq i < j \leq k$. If X is a finite set, an *ordering* of X is a sequence $x_1, \ldots, x_n$ such that $x_1, \ldots, x_n$ are all distinct and $X = \{x_1, \ldots, x_n\}$. We shall need the following lemma.

6.3 *Suppose that the following hold:*

- *$(G, u_1, \ldots, u_q)$, $(G', u'_1, \ldots, u'_q)$ and $(H, v_1, \ldots, v_q)$ are rooted digraphs;*
- *$Z \subseteq V(G)$ with $u_1, \ldots, u_q \in Z$; $\{A_0, A_1, \ldots, A_k\}$ is a Z-division of G with $Z \subseteq V(A_0)$, and for $1 \leq i \leq k$, π_i is an ordering of $Z \cap V(A_i)$;*
- *$Z' \subseteq V(G')$ with $u'_1, \ldots, u'_q \in Z'$; $\{A'_0, A'_1, \ldots, A'_k\}$ is a Z'-division of G' with $Z' \subseteq V(A'_0)$; and for $1 \leq i \leq k$, π'_i is an ordering of $Z' \cap V(A'_i)$;*

- *$\delta \geq 0$ is an integer such that $(H, v_1, \ldots, v_q)$ has detail $\leq \delta$, and for $1 \leq i \leq k$, (A'_i, π'_i) has the same δ-folio as (A_i, π_i);*
- *$\alpha : Z' \to Z$ is a function mapping Z' onto Z and for $1 \leq i \leq k$ mapping π'_i to π_i;*
- *ϕ is a model of $(H, v_1, \ldots, v_q)$ in $(G, u_1, \ldots, u_q)$ such that $\phi(E(H)) \subseteq E(A_1 \cup \cdots \cup A_k)$ and $\phi(v) \cap (A_1 \cup \cdots \cup A_k)$ is non-null for each $v \in V(H)$;*
- *$L \subseteq G$ is minimal such that $Z \subseteq V(L)$ and $\phi(v) \subseteq L$ for each $v \in V(H)$; and*
- *$L'_0 \subseteq A'_0$ is such that $u, v \in Z'$ are L'_0-connected if and only if $\alpha(u), \alpha(v)$ are $(L \cap A_0)$-connected.*

Then there is a model ϕ' of $(H, v_1, \ldots, v_q)$ in $(G', u'_1, \ldots, u'_q)$ such that $\phi'(E(H)) \subseteq E(A'_1 \cup \cdots \cup A'_k)$ and $\phi'(v) \cap (A'_1 \cup \cdots \cup A'_k)$ is non-null for each $v \in V(H)$.

Proof. For $0 \leq i \leq k$, let $L_i = L \cap A_i$, let $Z_i = Z \cap V(A_i)$, and let $Z'_i = Z' \cap V(A'_i)$. From the definition of L, we see

(1) *For each $v \in V(H), \phi(v)$ is a component of L, and every other component of L is an isolated vertex in Z.*

For the moment, fix i with $1 \leq i \leq k$. Let J be the digraph with vertex set the set of components of L_i, and edge set $\phi(E(H)) \cap E(A_i)$, where for $e \in \phi(E(H)) \cap E(A_i)$, if in A_i, e has head (respectively, tail) u, then in J, e has head (respectively, tail) the component of L_i containing u. This exists, for if $e = \phi(f)$ where $f \in E(H)$ and f has head (respectively, tail) v, then $u \in V(\phi(v)) \subseteq V(L)$. Let π_i be the sequence $p_1, \ldots, p_t$, and for $1 \leq j \leq t$ let P_i be the component of L_i with $p_i \in V(P_i)$. (This exists since $p_1, \ldots, p_t \in Z \subseteq V(L)$.) Then $(J, P_1, \ldots, P_t)$ is a rooted digraph.

(2) *$(J, P_1, \ldots, P_t)$ has detail $\leq \delta$, and there is a model of it in (A_i, π_i).*

Subproof. Certainly

$$|E(J)| = |\phi(E(H)) \cap E(A_i)| \leq |\phi(E(H))| = |E(H)| \leq \delta.$$

If $P \in V(J)$ and $P \neq P_1, \ldots, P_t$, then $p_1, \ldots, p_t \notin V(P)$, and so $V(P) \cap Z = \emptyset$. Consequently, every edge of G incident with a vertex in P is an edge of A_i, since $V(A_i \cap A_j) \subseteq Z$ for $j \neq i$, and so every edge of L incident with a vertex in P is an edge of L_i, and hence belongs to $E(P)$. We deduce that P is a component of L with $u_1, \ldots, u_q \notin V(P)$. Let $v \in V(H)$ with $P = \phi(v)$; then $v \neq v_1, \ldots, v_q$, since $u_1, \ldots, u_q \notin V(P)$. But since $(H, v_1, \ldots, v_q)$ has detail at most δ, there are at most δ such vertices v in H, and consequently at most δ such vertices P of J. This proves that $(J, P_1, \ldots, P_t)$ has detail at most δ. Define $\psi(e) = e$ for $e \in E(J)$, and $\psi(P) = P$ for $P \in V(J)$; then ψ is a model of $(J, P_1, \ldots, P_t)$ in (A_i, π_i). This proves (2).

Since (A'_i, π'_i) has the same δ-folio as (A_i, π_i), it follows from (2) that there is a model of $(J, P_1, \ldots, P_t)$ in (A'_i, π'_i). In other words,

(3) *For each component P of L_i there is a non-null connected subgraph $\psi_i(P) \subseteq A'_i$, and for each $e \in \phi(E(H)) \cap E(A_i)$ there is an edge $\psi_i(e) \in E(A'_i)$, with the following properties:*

- *for distinct components P_1, P_2 of L_i, $\psi_i(P_1) \cap \psi_i(P_2)$ is null; and if P is a component of L_i, then P contains the jth term of π_i if and only if $\psi_i(P)$ contains the jth term of π'_i*
- *for distinct edges $e_1, e_2 \in \phi(E(H)) \cap E(A_i)$, $\psi_i(e) \neq \psi_i(e')$; for $e \in \phi(E(H)) \cap E(A_i)$, $\psi_i(e) \notin E(\psi_i(P))$ for each component P of L_i; and if in A_i, $e \in \phi(E(H)) \cap E(A_i)$ has head (respectively, tail) u, then in A'_i, $\psi_i(e)$ has head (respectively, tail) in $V(\psi_i(P))$, where P is the component of L_i containing u.*

For each $e \in E(H)$, let $\phi'(e) = \psi_i(\phi(e))$, where $\phi(e) \in E(A_i)$ and $1 \leq i \leq k$ (such an i exists and is unique, from the hypothesis). For $1 \leq i \leq k$, let

$$L'_i = \bigcup(\psi_i(P) : P \text{ is a component of } L_i).$$

Then L'_i is a subgraph of A'_i. Let $L' = L'_0 \cup L'_1 \cup \cdots \cup L'_k$, where L'_0 is as in the theorem.

(4) *For $0 \leq i \leq k$, $Z_i \subseteq V(L_i)$ and $Z'_i \subseteq V(L'_i)$.*

Subproof. From the choice of L it follows that $Z_0 = Z \subseteq V(L_0)$. If $u' \in Z'_0 = Z'$, then $\alpha(u') \in Z_0 \subseteq V(L_0)$, and so $u' \in V(L'_0)$ from the hypothesis about L'_0 (with $u = v$). Thus (4) holds if $i = 0$, and we assume that $i \geq 1$. Again $Z_i \subseteq V(L_i)$ since $Z \subseteq V(L)$. If $u' \in Z'_i$, let u' be the jth term of π'_i, let u be the jth term of π_i, and let P be the component of L_i with $u \in V(P)$. By (3)(i), $u' \in V(\psi_i(P)) \subseteq V(L')$. Hence $Z'_i \subseteq V(L'_i)$, as required. This proves (4).

(5) *For $0 \leq i \leq k$, $u, v \in Z'_i$ are L'_i-connected if and only if $\alpha(u), \alpha(v)$ are L_i-connected.*

Subproof. For $i = 0$ this is a hypothesis of the theorem, and so we assume that $1 \leq i \leq k$. Let $u, v \in Z'_i$. Let π_i be the sequence $p_1, \ldots, p_t$, let π'_i be $p'_1, \ldots, p'_t$, and for $1 \leq j \leq t$ let P_i be the component of L_i containing p_i. Let $u = p'_r$, $v = p'_s$ say. Now $\psi_i(P_r)$ is the component of L'_i containing p'_r, by (3)(i), and so u, v are L'_i-connected if and only if $\psi_i(P_r) = \psi_i(P_s)$. By (3)(i), $\psi_i(P_r) = \psi_i(P_s)$ if and only if $P_r = P_s$. But $P_r = P_s$ if and only if $\alpha(u), \alpha(v)$ are L_i-connected, for $\alpha(u) = p_r \in V(P_r)$ and $\alpha(v) = p_s \in V(P_s)$. This proves (5).

(6) *$L'_i = L' \cap A'_i$ for $0 \leq i \leq k$, and $u, v \in Z'$ are L'-connected if and only if $\alpha(u), \alpha(v)$ are L-connected.*

Subproof. This follows from (5) and (2.5).

For $v \in V(H)$ we define $\phi'(v)$ to be a component of L', as follows. If $V(\phi(v)) \cap Z \neq \emptyset$, choose $z' \in Z'$ such that $\alpha(z') \in V(\phi(v)) \cap Z$, and let $\phi'(v)$ be the component of L containing z'. (This exists, by (4).) If $V(\phi(v)) \cap Z = \emptyset$, then since $\phi(v) \cap (A_1 \cup \cdots \cup A_k)$ is non-null by hypothesis, there is a unique i ($1 \leq i \leq k$) with $\phi(v) \subseteq A_i$. Then $\phi(v)$ is a component of L_i; let $\phi'(v) = \psi_i(\phi(v))$. Since $\phi'(v) \subseteq A'_i$ and by (3)(i), $V(\phi'(v)) \cap Z'_i = \emptyset$, it follows that $\phi'(v)$ is a component of L'.

(7) *For $v \in V(H)$, if $z \in Z'$, then $z \in V(\phi'(v))$ if and only if $\alpha(z) \in V(\phi(v))$.*

Subproof. Suppose that $z \in Z'$ and $\alpha(z) \in V(\phi(v))$. Then $V(\phi(v)) \cap Z \neq \emptyset$, and so there exists $z' \in Z'$ with $\alpha(z') \in V(\phi(v))$, such that $z' \in V(\phi'(v))$. Thus $\alpha(z)$ and $\alpha(z')$ are L-connected, and so by (6), z and z' are L'-connected, that is, $z \in V(\phi'(v))$, as required. Conversely, suppose that $z \in Z' \cap V(\phi'(v))$. If $V(\phi(v)) \cap Z = \emptyset$ then $V(\phi'(v)) \cap Z' = \emptyset$ from the definition of $\phi'(v)$, a contradiction. Thus $V(\phi(v)) \cap Z \neq \emptyset$, and so there exists $z' \in V(\phi'(v)) \cap Z'$ such that $\alpha(z') \in V(\phi(v))$. Then z and z' are L'-connected, and so by (6), $\alpha(z)$ and $\alpha(z')$ are L-connected, that is, by (1), $\alpha(z) \in V(\phi(v))$. This proves (7).

(8) *If $v_1, v_2 \in V(H)$ are distinct then $\phi'(v_1) \cap \phi'(v_2)$ is null.*

Subproof. Suppose that $\phi'(v_1) \cap \phi'(v_2)$ is non-null. Since $\phi'(v_1)$ and $\phi'(v_2)$ are both components of L', it follows that $\phi'(v_1) = \phi'(v_2)$. If $V(\phi'(v_1)) \cap Z' = \emptyset$, then $V(\phi(v_1)) \cap Z = \emptyset = V(\phi(v_2) \cap Z)$, and so there exists i with $1 \leq i \leq k$ such that $\phi'(v_1) \subseteq A'_i \setminus Z'_i$; and hence $\phi(v_1), \phi(v_2) \subseteq A_i$. Then

$$\psi_i(\phi(v_1)) = \phi'(v_1) = \phi'(v_2) = \psi_i(\phi(v_2))$$

and so by (3)(i), $\phi(v_1) = \phi(v_2)$; and hence $v_1 = v_2$ since ϕ is a model. This is a contradiction.

It follows that there exists $z \in V(\phi'(v_1)) \cap Z' = V(\phi'(v_2)) \cap Z'$. By (7), $\alpha(z) \in V(\phi(v_1)$ and $\alpha(z) \in V(\phi(v_2))$, and so $\phi(v_1) = \phi(v_2)$ and $v_1 = v_2$, again a contradiction. This proves (8).

(9) *For $1 \leq i \leq q$, $u'_i \in V(\phi'(v_i))$.*

Subproof. For $u'_i \in Z'$ and $\alpha(u'_i) = u_i \in V(\phi(v_i))$, and so by (7), $u'_i \in V(\phi'(v_i))$, as required. This proves (9).

(10) *If $e \in E(H)$ has head (respectively, tail) $v \in V(H)$, then $\phi'(e)$ has head (respectively, tail) in $V(\phi'(v))$.*

Subproof. We assume without loss of generality that v is the head of e. Choose i with $1 \leq i \leq k$ such that $\phi(e) \in E(A_i)$, and let u be the head of $\phi(e)$ in A_i. Then $u \in V(\phi(v))$. Let u' be the head of $\phi'(e)$ in A'_i; we must show that $u' \in V(\phi'(v))$. Let P be the component of L_i containing u. By (3)(ii), $u' \in V(\psi_i(P))$. Since by (1), $\phi(v)$ is the component of L containing u, it follows that $P \subseteq \phi(v)$. Now there are two cases. If $V(P) \cap Z_i = \emptyset$, then P is a component of L, and so by (1), $P = \phi(v)$ and

$$u' \in V(\psi_i(P)) = V(\psi_i(\phi(v))) = V(\phi'(v))$$

as required. If $V(P) \cap Z_i \neq \emptyset$, choose $z \in Z'_i$ with $\alpha(z) \in V(P) \cap Z_i$. By (3)(i), $z \in V(\psi_i(P))$ since α maps π'_i to π_i. But $\alpha(z) \in V(P) \subseteq V(\phi(v))$, and so $z \in V(\phi'(v))$ by (7). Since $\psi_i(P)$ is a connected subgraph of L', and $\phi'(v)$ is a component of L', and $\psi_i(P) \cap \phi'(v)$ is non-null, it follows that $\psi_i(P) \subseteq \phi'(v)$, and hence

$$u' \in V(\psi_i(P)) \subseteq V(\phi'(v))$$

as required. This proves (10).

Since $L' \subseteq G' \setminus \phi'(E(H))$, it follows from (8), (9), (10) that ϕ' is a model of $(H, v_1, \ldots, v_q)$ in $(G', u'_1, \ldots, u'_q)$. Since $\phi'(E(H)) \subseteq E(A'_1 \cup \cdots \cup A'_k)$ by the definition of ϕ', it remains to show that if $v \in V(H)$ then $\phi'(v) \cap (A'_1 \cup \cdots \cup A'_k)$ is non-null. Let $v \in V(H)$, and choose $i \geq 1$ so that $\phi(v) \cap A_i$ is non-null. If $V(\phi(v)) \cap Z_i = \emptyset$ then $\phi'(v) = \psi_i(\phi(v))$ and so $\phi'(v) \cap (A'_1 \cup \cdots \cup A'_k)$ is non-null. If $z \in V(\phi(v)) \cap Z_i$, choose $z' \in Z'_i$ with $\alpha(z') = z$; then $z' \in V(\phi'(v))$ by (7), and so again $\phi'(v) \cap (A'_1 \cup \cdots \cup A'_k)$ is non-null. This completes the proof. ■

7 A generalization

As we said, the objective of this paper is to prove theorem (10.2) of [5]. Now (3.1) is already a rudimentary version of what we need, but it has to be "bootstrapped" up into a more general, and unfortunately much more complicated, result. That is the goal of this section. We need several results about a system of subgraphs of a graph with the following properties (J1)–(J6).

(J1) (G, ω) is a rooted digraph where ω is the sequence $w_1, \ldots, w_q$; $w_1, \ldots, w_q$ are all distinct and $W = \{w_1, \ldots, w_q\}$; and N_W is the graph with vertex set W and no edges.

(J2) $\mathcal{A}$ is a set of subdigraphs of G; for all distinct $A, A' \in \mathcal{A}$, $E(A \cap A') = \emptyset$; for all $A \in \mathcal{A}$, $W \subseteq V(A)$ and $\pi(A)$ is a sequence of distinct vertices of A not in W, with one, two or three terms, and $\bar{\pi}(A)$ is the set of terms of $\pi(A)$; and for all distinct $A, A' \in \mathcal{A}$,

$$V(A \cap A') = (\bar{\pi}(A) \cap \bar{\pi}(A')) \cup W.$$

(J3) $\Gamma \subseteq G \setminus W$ is a directed 2-cell drawing in a sphere Σ; an orientation of Σ is specified, called "clockwise"; $\mathcal{T}$ is a tangle in Γ of order $\geq \theta \geq 4$, ins is defined by $\mathcal{T}$, and d is the metric of $\mathcal{T}$.

(J4) For each $A \in \mathcal{A}$, $D(A) \subseteq \Sigma$ is a closed disc such that $bd(D(A))$ is Γ-normal, $D(A) = ins(bd(D(A)))$, $\Gamma \cap D(A) = \Gamma \cap A$, $\bar{\pi}(A) = bd(D(A)) \cap V(G)$, and if $|\bar{\pi}(A)| = 3$ then $\pi(A)$ enumerates $\bar{\pi}(A)$ in clockwise order around $D(A)$; and for all distinct $A, A' \in \mathcal{A}$, $D(A) \cap D(A') = \bar{\pi}(A) \cap \bar{\pi}(A')$.

(J5) $N = \Gamma \cup N_W \cup \bigcup(A : A \in \mathcal{A})$; (N, K) is a separation of G and $W \subseteq V(N \cap K)$; $\Delta \subseteq \Sigma$ is a closed disc with $bd(\Delta) \subseteq U(\Gamma)$; $d(v, \Sigma \setminus \Delta) \geq \theta$ for all $v \in V(\Gamma \cap K)$; $d(D(A), \Sigma \setminus \Delta) \geq \theta$ for all $A \in \mathcal{A}$ with $A \cap K \neq N_W$; and $v^* \in V(\Gamma)$ with $v^* \notin \Delta$.

(J6) $\delta \geq 0$ is an integer; (H, χ) is a rooted digraph with detail $\leq \delta$; ϕ is a model of (H, χ) in (G, ω); for each $v \in V(H)$, $\phi(v) \cap (K \cup \bigcup(A : A \in \mathcal{A}))$ is non-null; and for each $e \in E(H)$, $\phi(e) \in E(K \cup \bigcup(A : A \in \mathcal{A}))$.

There are (at least) two points that need clarification. First, Γ is a drawing, but it is also a subgraph of the digraph G, and so its edges inherit directions from G. We therefore regard Γ both as a drawing and as a digraph. Secondly, in general there are vertices of G in Σ that are not in $V(\Gamma)$, for (J4) implies that $\bar{\pi}(A) \subseteq \Sigma$ for each $A \in \mathcal{A}$, and yet $\bar{\pi}(A)$ is not necessarily a subset of $V(\Gamma)$.

7.1 *Let (J1)–(J6) hold, and let* $K_1 = K \cup \bigcup(A \in \mathcal{A} : d(D(A), \Sigma \setminus \Delta) \geq \theta)$. *Then*

(i) $d(V(\Gamma \cap K_1), \Sigma \setminus \Delta) \geq \theta$

(ii) *for* $A \in \mathcal{A}$, *if* $d(D(A), \Sigma \setminus \Delta) < \theta$ *then* $E(A \cap K_1) = \emptyset$ *and* $V(A \cap K_1) \subseteq \bar{\pi}(A) \cup W$

(iii) *for* $A \in \mathcal{A}$*, if* $d(D(A), \Sigma \setminus \Delta) < \theta - 3$ *then* $A \cap K_1 = N_W$*.*

Proof. To prove (i), let $v \in V(\Gamma \cap K_1)$. If $v \in V(K)$, then $v \in V(\Gamma \cap K)$, and so $d(v, \Sigma \setminus \Delta) \geq \theta$ by (J5). If $v \notin V(K)$, then $v \in V(A)$ for some $A \in \mathcal{A}$ with $d(D(A), \Sigma \setminus \Delta) \geq \theta$; but then $v \in V(A \cap \Gamma) \subseteq D(A)$ by (J4), and so

$$d(v, \Sigma \setminus \Delta) \geq d(D(A), \Sigma \setminus \Delta) \geq \theta$$

as required. This proves (i).

For (ii), let $A \in \mathcal{A}$ with $d(D(A), \Sigma \setminus \Delta) < \theta$. By (J5), $A \cap K = N_W$; and for all $A' \in \mathcal{A}$ with $d(D(A), \Sigma \setminus \Delta) \geq \theta$, since $A \neq A'$ it follows from (J2) that $E(A \cap A') = \emptyset$ and $V(A \cap A') \subseteq \bar{\pi}(A) \cup W$. This proves (ii).

For (iii), let $A \in \mathcal{A}$ with $d(D(A), \Sigma \setminus \Delta) < \theta - 3$, and suppose that $A \cap K_1 \neq N_W$. By the argument of (ii), $A \cap K$ is null, and so there exists $A' \in \mathcal{A}$ with $d(D(A'), \Sigma \setminus \Delta) \geq \theta$ such that $A \cap A' \neq N_W$. Since $A \neq A'$, by (J2), $\bar{\pi}(A) \cap \bar{\pi}(A') \neq \emptyset$. By (J4), $D(A) \cap D(A') \neq \emptyset$. Choose $z \in Z(\Gamma)$ with $D(A) \cap D(A') \cap z \neq \emptyset$. Since $d(D(A), \Sigma \setminus \Delta) < \theta - 3$, there exists $y \in A(\Gamma)$ with $y \cap D(A) \neq \emptyset$ such that $d(y, \Sigma \setminus \Delta) < \theta - 3$. Now y, z both intersect $D(A)$, and $bd(D(A))$ is a Γ-normal O-arc with $|bd(D(A)) \cap V(\Gamma)| \leq 3$ and $ins(D(A))) = D(A)$, by (J4). Consequently $d(y, z) \leq 3$. But

$$\theta \leq d(D(A'), \Sigma \setminus \Delta) \leq d(z, \Sigma \setminus \Delta) \leq d(y, z) + d(y, \Sigma \setminus \Delta) \leq 3 + (\theta - 4),$$

a contradiction. This proves (iii). ■

Let (J1)–(J6) hold, and let ϕ' be a model of (H, χ) in (G, ω). We say that $\mathcal{A}' \subseteq \mathcal{A}$ is *adequate for* ϕ' if

(i) for each $v \in V(H)$ and $A \in \mathcal{A}$, if $A \cap \phi'(v) \not\subseteq \Gamma \cup N_W$ then $A \in \mathcal{A}'$

(ii) for each $v \in V(H)$, $\phi'(v) \cap (K \cup \bigcup(A \,:\, A \in \mathcal{A}'))$ is non-null,

(iii) for each $e \in E(H)$, $\phi'(e) \in E(K \cup \bigcup(A \,:\, A \in \mathcal{A}'))$, and

(iv) for each $A \in \mathcal{A}$, if $A \cap K \neq N_W$ then $A \in \mathcal{A}'$.

This implies that, if we define $N' = \Gamma \cup N_W \cup \bigcup(A \,:\, A \in \mathcal{A}')$ and $G' = N' \cup K$, then (J1)–(J6) remain true with $G, \mathcal{A}, N, \phi$ replaced by $G', \mathcal{A}', N', \phi'$ respectively.

7.2 *For all* $q, \delta \geq 0$ *there exists* $\theta \geq 4$ *with the following property. Let (J1)–(J6) hold, and let* $\mathcal{A}' \subseteq \mathcal{A}$ *be adequate for some model* ϕ' *of* (H, χ) *in* (G, ω)*, where* $d(v^*, D(A)) \geq \theta$ *for all* $A \in \mathcal{A}'$*. Then there is a model of* (H, χ) *in* $(G \setminus \{v^*\}, \omega)$*.*

Proof. Let $p = q + 3\delta$, choose $h \geq 1$ so that (3.1) holds, and let $\theta = 2h + 5$. We claim that θ satisfies (7.2). For let the hypotheses of (7.2) hold. Let $K' = K \cup \bigcup(A : A \in \mathcal{A}')$.

(1) v^* *is* h*-insulated from* $V(\Gamma \cap K')$ *by* Γ*.*

Subproof. Let $v \in V(\Gamma \cap K')$; we claim that $d(v^*, v) \geq \theta$. If $v \in V(K)$ this follows from (J5). If $v \notin V(K)$ then $v \in V(A)$ for some $A \in \mathcal{A}'$; but then $v \in V(A \cap \Gamma) \subseteq D(A)$ by (J4), and so

$d(v^*, v) \geq d(v^*, D(A)) \geq \theta$. This proves that $d(v^*, V(\Gamma \cap K')) \geq \theta$. By (5.2), this proves (1).

(2) *There is a basis Z for ϕ' with $Z \subseteq V(K')$.*

Subproof. For $W \subseteq V(K) \subseteq V(K')$, and $\phi(e) \in E(K')$ for each $e \in E(H)$, by statement (iii) in the definition of "adequate"; and $\phi'(v) \cap K'$ is non-null for each $v \in V(H)$, by statement (ii) in the definition of "adequate". This proves (2).

Choose Z as in (2), minimal. Then $|Z| \leq q + 3\delta = p$, by (6.1). Let $L = \bigcup(\phi'(v) : v \in V(H))$. Since $|Z| \leq p$ and $Z \subseteq V(K')$, it follows from (1) and (3.1) (with K, v replaced by K', v^*) that there exists $L' \subseteq (\Gamma \cup K') \setminus \{v^*\}$ with the same effect on Z as L, such that $L' \cap K' \subseteq L$. Now $\phi'(E(H)) \subseteq E(K')$, and $\phi'(E(H)) \cap E(L) = \emptyset$, and so $\phi'(E(H)) \cap E(L') = \emptyset$, since $L' \cap K' \subseteq L$. By (6.2), there is a model of (H, χ) in $(G \setminus \{v^*\}, \omega)$, as required. ■

If π and ω are the finite sequences $v_1, \dots, v_p$ and $w_1, \dots, w_q$, we denote their concatenation $v_1, \dots, v_p, w_1, \dots, w_q$ by $\pi + \omega$.

7.3 *For all integers $q, \delta, \tau \geq 0$ there exists $\theta \geq 5$ with the following property. Let (J1)–(J6) hold, and let $\mathcal{B} \subseteq \mathcal{A}$ be adequate for ϕ. Let $A_1, \dots, A_t \in \mathcal{B}$ where $t \leq \tau$, and let $d(D(A), \Sigma \setminus \Delta) \geq \theta$ for every $A \in \mathcal{B} \setminus \{A_1, \dots, A_t\}$. Let $A'_1, \dots, A'_t \in \mathcal{A}$, and suppose that*

- (i) *for $1 \leq i \leq t$, $(A'_i, \pi(A'_i) + \omega)$ has the same δ-folio as $(A_i, \pi(A_i) + \omega)$*
- (ii) *for $1 \leq i \leq t$, $D(A_i) \cap \Delta = \emptyset$*
- (iii) *for $1 \leq i \leq t$, $d(v^*, D(A'_i)) \geq \theta$ and $D(A'_i) \cap \Delta = \emptyset$*
- (iv) *for $1 \leq i < j \leq t$, $d(D(A'_i), D(A'_j)) \geq \theta$*
- (v) *for $1 \leq i \leq t$, there is no Γ-normal O-arc $F \subseteq \Sigma$ with $|F \cap V(\Gamma)| < |\bar{\pi}(A'_i)|$ and with $D(A'_i) \subseteq ins(F)$.*

Then there is a model of (H, χ) in $(G \setminus \{v^*\}, \omega)$.

Proof. Let $p = q + 3\delta + 3\tau$. Choose $\theta' \geq \max(p, 4)$ so that (7.2) holds with θ replaced by θ' and so that (5.3) holds with θ replaced by θ'. Let $\theta = \theta' + 3$. We claim that θ satisfies (7.3). For let the hypothesis of (7.3) hold. Let

$$K_1 = K \cup \bigcup(A \in \mathcal{A} : d(D(A), \Sigma \setminus \Delta) \geq \theta).$$

Let $L \subseteq G$ be minimal such that $\phi(v) \subseteq L$ for each $v \in V(H)$ and $\bar{\pi}(A_i) \subseteq V(L)$ for $1 \leq i \leq t$.

(1) *$L \subseteq \Gamma \cup K_1 \cup A_1 \cup \cdots \cup A_t$, and $\phi(E(H)) \subseteq E(K_1 \cup A_1 \cup \cdots \cup A_t)$.*

Subproof. Now $\phi(v) \subseteq \Gamma \cup K_1 \cup A_1 \cup \cdots \cup A_t$ for all $v \in V(H)$ since $\mathcal{B}$ is adequate for ϕ; and $\bar{\pi}(A_i) \subseteq V(A_i)$ for $1 \leq i \leq t$. Hence the first inclusion holds, and the second also holds since $\mathcal{B}$ is adequate for ϕ. This proves (1).

(2) *We may assume that* $d(V(K_1) \cap \Sigma, \Sigma \setminus \Delta) \geq \theta$.

Subproof. We may assume that no vertex of G is in Σ except for the vertices of Γ and the vertices of $\bigcup(\bar{\pi}(A) : A \in \mathcal{A})$. Let $v \in V(K_1) \cap \Sigma$. If $v \in V(\Gamma)$ then by (7.1)(i), $d(v, \Sigma \setminus \Delta) \geq \theta$ as required. We assume then that $v \in V(A)$ for some $A \in \mathcal{A}$ with $d(D(A), \Sigma \setminus \Delta) \geq \theta$. Since $v \in \Sigma \cap V(G)$ and $v \notin V(\Gamma)$, there exists $A' \in \mathcal{A}$ with $v \in \bar{\pi}(A')$, by our assumption. We claim that $v \in \bar{\pi}(A)$; for if $A = A'$ this is true since $v \in \bar{\pi}(A')$, and if $A \neq A'$ it follows from (J2). Thus $v \in \bar{\pi}(A) \subseteq D(A)$, and so

$$d(v, \Sigma \setminus \Delta) \geq d(D(A), \Sigma \setminus \Delta) \geq \theta$$

as required. This proves (2).

For $1 \leq i \leq t$, let $bd(D(A'_i)) = F_i$.

(3) *For* $1 \leq i \leq t$, $F_i \cap V(\Gamma) = \bar{\pi}(A'_i)$

Subproof. By hypothesis (v), $|F \cap V(\Gamma)| \geq |\bar{\pi}(A'_i)|$. But $F \cap V(\Gamma) \subseteq \bar{\pi}(A'_i)$ by (J4), and so there is equality. This proves (3).

For $1 \leq i \leq t$, let $D_i \subseteq D(A_i) \setminus bd(D(A_i))$ be a closed disc.

(4) *There is a homeomorphism* $\beta : \Sigma \to \Sigma$ *fixing* Δ *pointwise and mapping* D_i *to* $D(A'_i)$ *for* $1 \leq i \leq t$.

Subproof. For $D(A_1), \ldots, D(A_t)$ are disjoint from Δ by hypothesis (ii). Hence $D_1, \ldots, D_t, \Delta$ are mutually disjoint closed discs. But $D(A'_1), \ldots, D(A'_t)$ are also disjoint from Δ, by hypothesis (iii), and from each other, also by (iii). This proves (4).

For $1 \leq i \leq t$, let π_i be the sequence of points of $bd(D_i)$ mapped by β to $\pi(A'_i)$ and let $\bar{\pi}_i$ be the set of terms of π_i.

(5) *For* $1 \leq i \leq t$, $|\bar{\pi}_i| = |\bar{\pi}(A_i)| = |\bar{\pi}(A'_i)|$.

Subproof. Since β is a homeomorphism, it follows that $|\bar{\pi}_i| = |\bar{\pi}(A'_i)|$. But $|\bar{\pi}(A_i)| = |\bar{\pi}(A'_i)|$ since $(A_i, \pi(A_i) + \omega)$ and $(A'_i, \pi(A'_i) + \omega)$ have the same δ-folio. This proves (5).

For $1 \leq i \leq t$, let M_i be a drawing in $D(A_i)$ with vertex set $\bar{\pi}(A_i) \cup \bar{\pi}_i$ and with $|\pi_i|$ edges e_j $(1 \leq j \leq |\pi_i|)$, where e_j has ends the jth term of $\pi(A_i)$ and the jth term of π_i, and $e_j \cap D_i = \emptyset$. This exists, because $|\bar{\pi}_i| \leq 3$ by (J2), and if $|\bar{\pi}_i| = 3$ then the circular orders of π_i around D_i and $\pi(A_i)$ around $D(A_i)$ agree (by (J4), and since β preserves the orientation of Σ, because it fixes Δ pointwise). Let

$$\Gamma_0 = \Gamma \cap (\Sigma \setminus \bigcup(D(A_i) \setminus bd(D(A_i)) \ : \ 1 \leq i \leq t));$$

then Γ_0 is a drawing in Σ. Let $\Gamma_1 = \Gamma_0 \cup M_1 \cup \cdots \cup M_t$; this is a drawing in Σ. Let Γ_2 be the image of Γ_1 under β, and let Γ_3 be the union of Γ_2 and $\Gamma \cap ins(F_i)$ for $1 \leq i \leq t$. Let R be the union of all

$z \in A(\Gamma)$ with $d(z, v) \geq \theta'$ for all $v \in V(\Gamma \cap K_1)$. Let

$$\Sigma' = \Sigma \setminus \bigcup (ins(F_i) \setminus F_i \ : \ 1 \leq i \leq t).$$

Thus, $\Gamma_2 = \Gamma_3 \cap \Sigma'$.

(6) Γ_3 *is a* $(\Sigma \setminus \Delta)$*-variant of* Γ*, and hence an* $(R \cap \Sigma')$*-variant of* Γ*, and* $\Gamma \cap ins(F_i) = \Gamma_3 \cap ins(F_i)$ *for* $1 \leq i \leq t$*.*

Subproof. Now $\Gamma_3, \Gamma_2, \Gamma_1, \Gamma_0, \Gamma$ each differ from the next only in $\Sigma \setminus \Delta$, by hypotheses (ii) and (iii), and since β fixes Δ pointwise. Thus Γ_3 is a $(\Sigma \setminus \Delta)$-variant of Γ. Since $\Sigma \setminus \Delta \subseteq R$ by (2), it follows that Γ_3 is an R-variant of Γ. For $1 \leq i \leq t$, $\Gamma \cap ins(F_i) = \Gamma_3 \cap ins(F_i)$, and the result follows. This proves (6).

(7) *We may assume that* $\Gamma_3 \cap K_1 = \Gamma \cap K_1$*.*

Subproof. For we may assume that no edge of G is in Σ except for the edges of Γ. Now $\Gamma \cap K_1$ is a subgraph of Γ_3 by (6) and (2), and so it suffices to show that

$$V(\Gamma_3) \cap V(K_1) \subseteq V(\Gamma)$$
$$E(\Gamma_3) \cap E(K_1) \subseteq E(\Gamma).$$

The second inclusion is true since $E(\Gamma_3) \cap E(G) \subseteq E(\Gamma)$. For the first inclusion, let $v \in V(\Gamma_3) \cap V(K_1)$. By (2), $v \in \Delta$, and since Γ_3 is a $(\Sigma \setminus \Delta)$-variant of Γ, it follows that $v \in V(\Gamma)$ as required. This proves (7).

Let $L_0 = L \cap \Gamma_0$, let $L_1 = L_0 \cup M_1 \cup \cdots \cup M_t$, and let L_2 be the image of L_1 under β. Then $L_2 \cap K_1 = L \cap K_1$, by (2) and the argument used to prove (6); and $L_2 \cup (L \cap K_1)$ is a subgraph of $\Gamma_2 \cup K_1$. Choose $Y_1 \subseteq V(K_1)$, minimal such that $Y_1 \cap V(\phi(v)) \neq \emptyset$ for every $v \in V(H)$ with

$$V(\phi(v)) \cap (W \cup V(A_1 \cup \cdots \cup A_t)) = \emptyset.$$

This is possible by (J6), and $|Y_1| \leq \delta$ since there are $\leq \delta$ such vertices $v \in V(H)$. Let Y_2 be the set of all vertices of G incident with an edge $f \in \phi(E(H))$ where $f \in E(\Gamma_0 \cup K_1)$; then $|Y_2| \leq 2|E(H)| \leq 2\delta$. Let $Z_0 = Y_1 \cup Y_2 \cup W$; then $|Z_0| \leq q + 3\delta$, and $Z_0 \subseteq V(K_1)$. Let

$$Z' = Z_0 \cup \bar{\pi}(A_1') \cup \cdots \cup \bar{\pi}(A_t').$$

Then $Z' \leq q + 3\delta + 3\tau = p$, and $Z' \subseteq V(\Gamma \cup K_1)$.

(8) *There is a subgraph* L_0' *of* $(\Gamma \cap \Sigma') \cup K_1$ *with the same effect on* Z' *as* $L_2 \cup (L \cap K_1)$ *and with* $E(L_0') \cap \phi(E(H)) \subseteq E(A_1 \cup \cdots \cup A_t)$*.*

Subproof. Let us apply (5.3), with

$$p, \theta, \Gamma, K, \Sigma, \mathcal{T}, d, Z, F_1, \ldots, F_t, R, \Sigma', \Gamma', L'$$

replaced by

$$p, \theta', \Gamma, K_1, \Sigma, \mathcal{T}, d, Z', F_1, \dots, F_t, R, \Sigma', \Gamma_3, L_2 \cup (L \cap K_1)$$

respectively. We recall that θ' was chosen so that (5.3) holds with p, θ replaced by p, θ'. To verify the hypotheses of (5.3) is straightforward. (5.3)(i) follows from (7.3)(v); (5.3)(ii) from (7.3)(iv); (5.3)(iii) from (2); and the other hypotheses follow from (6) and (7). Consequently, by (5.3), there exists $L'_0 \subseteq (\Gamma \cap \Sigma') \cup K_1$ with the same effect on Z' as $L_2 \cup (L \cap K_1)$ and with $L'_0 \cap K_1 \subseteq L_2 \cup (L \cap K_1)$. Now $L_2 \cap K_1 = L \cap K_1$, and so $L'_0 \cap K_1 \subseteq L$. Let $f \in \phi(E(H)) \cap E(L'_0)$. Since $\phi(E(H)) \cap E(L) = \emptyset$, it follows that $f \notin E(L)$, and hence $f \notin E(K_1)$. By (1), $f \in E(A_1 \cup \cdots \cup A_t)$. Consequently,

$$\phi(E(H)) \cap E(L'_0) \subseteq E(A_1 \cup \cdots \cup A_t).$$

This proves (8).

Let $Z = Z_0 \cup \bar{\pi}(A_1) \cup \cdots \cup \bar{\pi}(A_t)$, and define $\alpha : Z' \to Z$ as follows: if $v \in Z_0$, let $\alpha(v) = v$, and if $v \in \bar{\pi}(A'_i)$ where $1 \le i \le t$, and v is the jth term of $\pi(A'_i)$ say where $1 \le j \le |\bar{\pi}_i|$, let $\alpha(v)$ be the jth term of $\pi(A_i)$. This defines a function since the sets $Z_0, \bar{\pi}(A'_1), \dots, \bar{\pi}(A'_t)$ are mutually disjoint. Similarly, for $v \in Z'$ define $\mu(v) = v$ if $v \in Z_0$, and if v is the jth term of $\pi(A'_i)$ let $\mu(v)$ be the jth term of π_i. Thus, if $v \in \bar{\pi}(A'_i)$, $\beta(\mu(v)) = v$.

(9) *$u, v \in Z'$ are L'_0-connected if and only if $\alpha(u), \alpha(v)$ are $L_0 \cup (L \cap K_1)$-connected.*

Subproof. To show this we make a sequence of equivalent statements, starting with:

(a) $\alpha(u), \alpha(v)$ are $L_0 \cup (L \cap K_1)$-connected.

Since $\alpha(u), \alpha(v) \in Z \subseteq V(L_0)$, (a) is equivalent to

(b) $\alpha(u), \alpha(v)$ are $L_1 \cup (L \cap K_1)$-connected,

because $L_1 \cup (L \cap K_1)$ is obtained from $L_0 \cup (L \cap K_1)$ by adding vertices of degree 1. Now $\alpha(u)$ and $\mu(u)$ are either equal or are adjacent in L_1; and similarly for $\alpha(v), \mu(v)$. Consequently, (b) is equivalent to

(c) $\mu(u), \mu(v)$ are $L_1 \cup (L \cap K_1)$-connected.

There is an isomorphism between $L_1 \cup (L \cap K_1)$ and $L_2 \cup (L \cap K_1)$ (since β fixes $U(L_1 \cap (L \cap K_1))$ pointwise), mapping each vertex x to $\beta(x)$ if $x \in \Sigma$ and mapping x to itself otherwise. Since $\beta(\mu(v)) = v$ for $v \in \bar{\pi}(A'_1) \cup \cdots \cup \bar{\pi}(A'_t)$, this isomorphism maps $\mu(u)$ to u and $\mu(v)$ to v. Consequently, (c) is equivalent to

(d) u, v are $L_2 \cup (L \cap K_1)$-connected.

But by (8), (d) is equivalent to

(e) u, v are L_0'-connected.

Hence (a) is equivalent to (e). This proves (9).

Let A_{t+1} be the subdigraph of G with vertex set Z_0 and edge set $\phi(E(H)) \cap E(K_1)$, let $\pi(A_{t+1})$ be some ordering of $Z_0 \setminus W$ and let $\bar{\pi}(A_{t+1}) = Z_0 \setminus W$. Let $A_0 = (\Gamma_0 \cup K_1) \setminus (\phi(E(H)) \cap E(K_1))$.

(10) *$\{A_0, A_1, \ldots, A_{t+1}\}$ is a Z-division of $\Gamma \cup K_1 \cup A_1 \cup \cdots \cup A_t$, and $Z \subseteq V(A_0)$, and for $1 \leq i \leq t+1, \pi(A_i) + \omega$ is an ordering of $Z \cap V(A_i)$.*

Subproof. Now $A_0 \cup A_{t+1} = \Gamma \cup K_1$, and so

$$A_0 \cup A_1 \cup \cdots \cup A_{t+1} = \Gamma \cup K_1 \cup A_1 \cup \cdots \cup A_t.$$

Let $0 \leq i < j \leq t+1$; we must show that $V(A_i \cap A_j) \subseteq Z$ and $E(A_i \cap A_j) = \emptyset$. If $1 \leq i < j \leq t$, this follows from (J2). If $0 = i < j \leq t$ it follows from (6.1) and the definition of Γ_0. If $1 \leq i < j = t+1$ it follows since $A_i \cap A_{t+1} = N_W$ by (7.1)(iii). Finally if $i = 0$ and $j = t+1$, then clearly $E(A_i \cap A_j) = \emptyset$, and $V(A_i \cap A_j) \subseteq V(A_{t+1}) = Z_0 \subseteq Z$. This proves (10).

Let $A'_{t+1} = A_{t+1}, \pi(A'_{t+1}) = \pi(A_{t+1})$, and $\bar{\pi}(A'_{t+1}) = \bar{\pi}(A_{t+1})$. Let

$$A'_0 = ((\Gamma \cap \Sigma') \cup K_1) \setminus (\phi(E(H)) \cap E(K_1)).$$

(11) *$\{A'_0, A'_1, \ldots, A'_{t+1}\}$ is a Z'-division of $\Gamma \cup K_1 \cup A'_1 \cup \cdots \cup A'_t$, and for $1 \leq i \leq t+1, \pi(A'_i) + \omega$ is an ordering of $Z' \cap V(A'_i)$.*

The proof is similar to that of (10).

(12) *There is a model ϕ' of (H, χ) in $(\Gamma \cup K_1 \cup A'_1 \cup \cdots \cup A'_t, \omega)$ such that $\phi'(E(H)) \subseteq E(A'_1 \cup \cdots \cup A'_{t+1})$ and $\phi'(v) \cap (A'_1 \cup \cdots \cup A'_{t+1})$ is non-null for each $v \in V(H)$.*

Subproof. Let χ be $x_1, \ldots, x_q$. Let us apply (6.3), with

$$G, u_1, \ldots, u_q, G', u'_1, \ldots, u'_q, H, v_1, \ldots, v_q$$

replaced by

$$\Gamma \cup K_1 \cup A_1 \cup \cdots \cup A_t, w_1, \ldots, w_q, \Gamma \cup K_1 \cup A'_1 \cup \cdots \cup A'_t, w_1, \ldots, w_q, H, x_1, \ldots, x_q$$

and with

$$\delta, Z, k, A_0, A_1, \ldots, A_k, \pi_i, Z', A'_0, A'_1, \ldots, A'_k, \pi'_i, \alpha, \phi, L, L'_0$$

replaced by

$$\delta, Z, t+1, A_0, A_1, \ldots, A_{t+1}, \pi(A_i), Z', A'_0, A'_1, \ldots, A'_{t+1}, \pi(A'_i), \alpha, \phi, L, L'_0$$

respectively. We must verify the hypotheses of (6.3); let us do them in order as in the statement of (6.3). The first ones are obvious, or follow from (10) and (11). For $1 \le i \le t+1$, $(A_i, \pi(A_i) + \omega)$ has the same δ-folio as $(A_i', \pi(A_i') + \omega)$, trivially if $i = t+1$, and by hypothesis (i) of (7.3) if $i \le t$. From the definition of α, it maps Z' onto Z, and maps $\pi(A_i')$ to $\pi(A_i)$ for $1 \le i \le t+1$. By (1),

$$\phi(E(H)) \subseteq E(K_1 \cup A_1 \cup \cdots \cup A_t),$$

and $\phi(E(H)) \cap E(K_1) \subseteq E(A_{t+1})$, and so $\phi(E(H)) \subseteq E(A_1 \cup \cdots \cup A_{t+1})$. For each $v \in V(H)$, if

$$V(\phi(v)) \cap (W \cup V(A_1 \cup \cdots \cup A_t)) \neq \emptyset$$

then $\phi(v) \cap A_1 \cup \cdots \cup A_{t+1}$ is non-null since $W \subseteq V(A_{t+1})$, and if

$$V(\phi(v)) \cap (W \cap V(A_1 \cup \cdots \cup A_t)) = \emptyset$$

then $Y_1 \cap V(\phi(v)) \neq \emptyset$ by definition of Y_1, and so again $\phi(v) \cap A_1 \cap \cdots \cap A_{t+1}$ is non-null, since $Y_1 \subseteq V(A_{t+1})$. Next, $L \subseteq \Gamma \cup K_1 \cup A_1 \cup \cdots \cup A_t$ by (1), and L is minimal with $Z \subseteq V(L)$ and $\phi(v) \subseteq L$ for each $v \in V(H)$ by its definition. By (8), $L_0' \subseteq (\Gamma \cap \Sigma') \cup K_1$, and by (8) again, $E(L_0') \cap \phi(E(H)) \subseteq E(A_1 \cup \cdots \cup A_t)$, and so $E(L_0') \cap \phi(E(H)) \cap E(K_1) = \emptyset$. Consequently $L_0' \subseteq A_0'$. By (9), $u, v \in Z'$ are L_0'-connected if and only if $\alpha(u), \alpha(v)$ are $L_0 \cup (L \cap K_1)$-connected, and

$$L \cap A_0 = L \cap (\Gamma_0 \cup K_1) = L_0 \cup (L \cap K_1)$$

since $\phi(E(H)) \cap E(L) = \emptyset$ and $L_0 = L \cap \Gamma_0$. Thus all the hypotheses of (6.3) hold. This proves (12).

Let $\mathcal{A}' = \{A \in \mathcal{A} : d(v^*, D(A)) \ge \theta'\}$.

(13) *$\mathcal{A}'$ is adequate for ϕ'.*

Subproof. Let us verify the four conditions in the definition of "adequate". For (i), let $v \in V(H)$ and $A \in \mathcal{A} \setminus \mathcal{A}'$; we must show that $A \cap \phi'(v) \subseteq \Gamma \cup N_W$. Now trivially $A \cap \Gamma \subseteq \Gamma \cup N_W$. Since $A \notin \mathcal{A}'$ it follows from (7.1)(iii) that $A \cap K_1$ is null. For $1 \le i \le t, A \neq A_i'$ since $A \notin \mathcal{A}'$, and so $A \cap A_i' \subseteq \Gamma \cup N_W$ by (J2) and (3). Consequently,

$$A \cap \phi'(v) \subseteq A \cap (\Gamma \cup K_1 \cup A_1' \cup \cdots \cup A_t') \subseteq \Gamma \cup N_W.$$

This proves (i).

For (ii), let $v \in V(H)$. Now

$$A_1' \cup \cdots \cup A_{t+1}' \subseteq K \cup \bigcup (A \ : \ A \in \mathcal{A}'),$$

since $A_1', \ldots, A_t' \in \mathcal{A}'$ and $A_{t+1}' \subseteq K_1 \subseteq K \cup \bigcup (A \ : \ A \in \mathcal{A}')$. But by (12), $\phi'(v) \cap (A_1' \cup \cdots \cup A_{t+1}')$ is non-null, and (ii) follows.

For (iii), let $e \in E(H)$. By (12),

$$\phi'(e) \in E(A_1' \cup \cdots \cup A_{t+1}') \subseteq E(K \cup \bigcup (A \ : \ A \in \mathcal{A}')).$$

This proves (iii).

For (iv), let $A \in \mathcal{A}$ with $A \cap K \neq N_W$. By (J5), $d(v^*, D(A)) \ge \theta$, and so $A \in \mathcal{A}'$. This proves (iv), and hence proves (13).

From (13), hypotheses (iii) and (7.2), the result follows. ■

Now we need to relax the definition of "adequate" a little. If (J1)–(J6) hold and $\mathcal{A}' \subseteq \mathcal{A}$, and ϕ' is a model of (H,χ) in (G,ω), we say that $\mathcal{A}'$ is *sufficient* for ϕ' if

- for each $v \in V(H)$ and each $A \in \mathcal{A}$, if some edge of $A \cap \phi'(v)$ is incident with a vertex in W then $A \in \mathcal{A}'$
- for each $v \in V(H)$, $\phi'(v) \cap (K \cup \bigcup(A : A \in \mathcal{A}'))$ is non-null,
- for each $e \in E(H)$, $\phi'(e) \subseteq E(K \cup \bigcup(A : A \in \mathcal{A}'))$, and
- for each $A \in \mathcal{A}$, if $A \cap K \neq N_W$ then $A \in \mathcal{A}'$.

Thus, if $\mathcal{A}'$ is adequate for ϕ' then it is sufficient for ϕ'.

Also, let us introduce another condition, the following.

(J7) For each $A \in \mathcal{A}$, if $u, v \in \bar{\pi}(A)$ there is a path of A with ends u, v and with no internal vertex in $\bar{\pi}(A) \cup W$; for each $A \in \mathcal{A}$, there is no separation (C, D) of $G \setminus W$ with order $< |\bar{\pi}(A)|$ such that $A \setminus W \subseteq C$ and $(C \cap \Gamma, D \cap \Gamma) \in \mathcal{T}$; and for each $A \in \mathcal{A}$, either

- $V(\Gamma \cap A) \subseteq \bar{\pi}(A)$ and $E(\Gamma \cap A) = \emptyset$, or
- $\Gamma \cap A$ is a path with both ends in $\bar{\pi}(A)$, or
- $|\bar{\pi}(A)| = 3$, $\bar{\pi}(A) \subseteq V(\Gamma)$, some $v \in \bar{\pi}(A)$ has degree 0 in $\Gamma \cap A$, and $(\Gamma \cap A) \setminus \{v\}$ is a path with both ends in $\bar{\pi}(A)$, or
- $|\bar{\pi}(A)| = 3$, $\bar{\pi}(A) \subseteq V(\Gamma)$, and for all $u, v \in \bar{\pi}(A)$ there is a path of $\Gamma \cap A$ with ends u, v and with no internal vertex in $\bar{\pi}(A)$.

Then (7.3) can be modified as follows.

7.4 *For all integers $q, \delta, \tau \geq 0$ there exists $\theta \geq 4$ with the following property. Let (J1)–(J7) hold, and let $\mathcal{B} \subseteq \mathcal{A}$ be sufficient for ϕ. Let $A_1, \ldots, A_t \in \mathcal{B}$ where $t \leq \tau$, and let $d(D(A), \Sigma \setminus \Delta) \geq \theta$ for every $A \in \mathcal{B} \setminus \{A_1, \ldots, A_t\}$. Let $A'_1, \ldots, A'_t \in \mathcal{A}$, and suppose that*

(i) *for $1 \leq i \leq t$, $(A'_i, \pi(A'_i) + \omega)$ has the same δ-folio as $(A_i, \pi(A_i) + \omega)$*

(ii) *for $1 \leq i \leq t$, $D(A_i) \cap \Delta = \emptyset$*

(iii) *for $1 \leq i \leq t$, $d(v^*, D(A'_i)) \geq \theta$ and $D(A'_i) \cap \Delta = \emptyset$*

(iv) *for $1 \leq i < j \leq t$, $d(D(A'_i),\ D(A'_j)) \geq \theta$.*

Then there is a model of (H, χ) in $(G \setminus \{v^*\}, \omega)$.

Proof. Choose θ so that (7.3) is satisfied. We claim that (7.4) is satisfied. For let the hypotheses of (7.4) hold, and suppose the conclusion does not hold for some G. For the given graph G, let us choose the counterexample such that

(1) $|E(\Gamma)|$ *is maximum;*

subject to (1), such that

(2) $\Gamma \cup V(\phi(v) : v \in V(H))$ *is minimal;*

and, subject to (1) and (2), such that

(3) $\bigcup(\phi(v) : v \in V(H))$ *is minimal.*

Let $K_1 = K \cup \bigcup(A \in \mathcal{A} \ : \ d(D(A), \Sigma \setminus \Delta) \geq \theta)$. Let us say $A \in \mathcal{A}$ is *good* if $\bar{\pi}(A) \subseteq V(\Gamma)$ and for all $u, v \in \bar{\pi}(A)$ there is a path of $\Gamma \cap A$ with ends u, v and with no internal vertex in $\bar{\pi}(A)$. We say $A \in \mathcal{A}$ is *bad* if it is not good.

(4) *If $A \in \mathcal{A}$ is bad then either $d(D(A), \Sigma \setminus \Delta) \geq \theta$, or $D(A) \cap U(\Gamma) = \emptyset$.*

Subproof. Suppose that $D(A) \cap U(\Gamma) \neq \emptyset$ and $d(D(A), \Sigma \setminus \Delta) < \theta$. Since $bd(D(A))$ is Γ-normal and $D(A) = ins(bd(D(A)))$ and Γ is 2-cell, it follows that $bd(D(A)) \cap V(\Gamma) \neq \emptyset$. Now by (J7), since A is bad, either

- $V(\Gamma \cap A) \subseteq \bar{\pi}(A)$ and $E(\Gamma \cap A) = \emptyset$, or
- $|\bar{\pi}(A)| = 3$, $\bar{\pi}(A) \subseteq V(\Gamma)$, some $v \in \bar{\pi}(A)$ has degree 0 in $\Gamma \cap A$, and $(\Gamma \cap A) \setminus \{v\}$ is a path with both ends in $\bar{\pi}(A)$, or
- $|\bar{\pi}(A)| = 3$ and $\Gamma \cap A$ is a path with both ends in $\bar{\pi}(A)$, possibly with an internal vertex in $\bar{\pi}(A)$.

In each case, there exist distinct $u, v \in \bar{\pi}(A)$ such that there is no path of $\Gamma \cap A$ with ends u, v and with no internal vertex in $\bar{\pi}(A)$, and since $\bar{\pi}(A) \cap V(\Gamma) \neq \emptyset$, we may choose such a pair u, v with $u \in V(\Gamma)$. But by (J7), there is a path of $A \setminus W$ with ends u, v and with no internal vertex in $\bar{\pi}(A)$; let us choose $Q \subseteq A \setminus W$ minimal such that $(\Gamma \cap A) \cup Q$ includes such a path. It follows that Q is a path with distinct ends both in $V(\Gamma \cap A) \cup \{v\}$, with no internal vertex in $V(\Gamma \cap A) \cup \bar{\pi}(A)$. By (i) and (ii) above, it follows that there is a line I in $D(A)$ with ends the ends of Q and with no internal point in $U(\Gamma) \cup bd(D(A))$. We may assume that Q is a drawing in Σ and $U(Q) = I$. Let $\Gamma' = \Gamma \cup Q$; then Γ' is 2-cell, since Γ is 2-cell and at least one end of I is in $V(\Gamma)$. Let $\mathcal{T}'$ be the set of all separations (C, D) of Γ' of order $< \theta$ such that $(C \cap \Gamma, D \cap \Gamma) \in \mathcal{T}$; then $\mathcal{T}'$ is a respectful tangle in Γ' of order θ. Let d' be its metric; then if $a, b \in A(\Gamma)$ and $a', b' \in A(\Gamma')$ and $a' \subseteq a$ and $b' \subseteq b$, then $d'(a', b') \geq d(a, b)$. If we replace Γ by Γ' and d and d' then (J1)–(J6) remain satisfied, as is easily seen. Also, (J7) remains satisfied, as we see as follows. Let $A_0 \in \mathcal{A}$, and suppose that (C, D) is a separation of $G \setminus W$ with order $< |\bar{\pi}(A_0)|$ such that $A_0 \setminus W \subseteq C$ and $(C \cap \Gamma', D \cap \Gamma') \in \mathcal{T}'$. Then $(C \cap \Gamma, D \cap \Gamma) \in \mathcal{T}$, since $(D \cap \Gamma, C \cap \Gamma) \notin \mathcal{T}$ by definition of $\mathcal{T}'$; and this contradicts the truth of (J7) for $\Gamma, \mathcal{T}$. Thus there is no such A_0, C, D. Now let $A_0 \in \mathcal{A}$. If $A_0 \neq A$ then $E(A_0 \cap \Gamma') = E(A_0 \cap \Gamma)$, and

$$V(A_0 \cap \Gamma) \subseteq V(A_0 \cap \Gamma') \subseteq V(A_0 \cap \Gamma) \cup \bar{\pi}(A_0)$$

and so $A_0 \cap \Gamma'$ satisfies (J7); while if $A = A_0$ then again $A_0 \cap \Gamma'$ satisfies (J7) by the choice of Q. This proves that (J7) remains satisfied. Now $\mathcal{B}$ remains sufficient for ϕ, since that does not depend on Γ or $\mathcal{T}$; and since all distances are increased by replacing Γ by Γ' and $\mathcal{T}$ by $\mathcal{T}'$ (more precisely, $d'(a', b') \geq d(a, b)$ as we said above), the hypotheses of (7.4) remain satisfied. But this contradicts

(1), and therefore proves (4).

(5) *If $F \subseteq \Sigma$ is an O-arc with $F \cap U(\Gamma) = \emptyset$ and $F \cap D(A) = \emptyset$ for each $A \in \mathcal{A}$, then $ins(F) \cap U(\Gamma) = \emptyset$ and $ins(F) \cap D(A) = \emptyset$ for all $A \in \mathcal{A}$ with $d(D(A), \Sigma \setminus \Delta) < \theta$.*

Subproof. Now Γ is connected since it is 2-cell, and $U(\Gamma) \not\subseteq ins(F)$ by the third axiom for tangles. Consequently $ins(F) \cap U(\Gamma) = \emptyset$. Suppose that $D(A_0) \subseteq ins(F)$ for some $A_0 \in \mathcal{A}$ with $d(D(A_0), \Sigma \setminus \Delta) < \theta$. Let C be the union of $A \setminus W$ over all $A \in \mathcal{A}$ with $D(A) \subseteq ins(F)$, and let D be the union of $K \setminus W, \Gamma$ and $A \setminus W$ over all $A \in \mathcal{A}$ with $D(A) \not\subseteq ins(F)$. Let $z \in A(\Gamma)$ with $F \subseteq z$; then $D(A_0) \subseteq z$, and so $d(z, \Sigma \setminus \Delta) = d(D(A_0), \Sigma \setminus \Delta) < \theta$. Consequently $d(D(A), \Sigma \setminus \Delta) = d(z, \Sigma \setminus \Delta) < \theta$ for all $A \in \mathcal{A}$ with $D(A) \subseteq ins(F)$, and by hypothesis, $A \cap K = N_W$ for every such A. Since $V(A \cap A') = (\bar{\pi}(A) \cap \bar{\pi}(A')) \cup W = W$ if $A, A' \in \mathcal{A}$ and $D(A) \subseteq ins(F)$ and $D(A') \not\subseteq ins(F)$, it follows that $C \cap D$ is null. But $(C \cap \Gamma, D \cap \Gamma) \in \mathcal{T}$ since $C \cap \Gamma$ is null, and this contradicts (J7) since $|\bar{\pi}(A_0)| \geq 1$ by (J2). Hence (5) holds.

(6) *Every $A \in \mathcal{A}$ with $d(D(A), \Sigma \setminus \Delta) < \theta$ is good.*

Subproof. Suppose that $A \in \mathcal{A}$ is bad and $d(D(A), \Sigma \setminus \Delta) < \theta$. By (4), $D(A) \cap U(\Gamma) = \emptyset$; let r be the region of Γ with $D(A) \subseteq r$. Since $d(D(A), \Sigma \setminus \Delta) < \theta$, it follows that $d(r, \Sigma \setminus \Delta) < \theta$. By (5), there is a sequence $A^1, A^2, \ldots, A^k$ of members of $\mathcal{A}$ such that $A^1 = A$, $D(A^k) \cap U(\Gamma) \neq \emptyset$, and for $1 \leq i < k, D(A^i) \cap D(A^{i+1}) \neq \emptyset$. By choosing k minimum, we may assume that $D(A^1), \ldots, D(A^{k-1})$ are all disjoint from $U(\Gamma)$, and hence $D(A^i) \cap r \neq \emptyset$ for $1 \leq i \leq k$. Consequently,

$$d(D(A^i), \Sigma \setminus \Delta) \leq d(r, \Sigma \setminus \Delta) < \theta$$

for $1 \leq i \leq k$. Choose i with $1 \leq i \leq k$ maximum so that A^i is bad. Since A^k is good by (4), it follows that $i < k$, and A^{i+1} is good, and so $\bar{\pi}(A^{i+1}) \subseteq V(\Gamma)$. But

$$\emptyset \neq D(A^i) \cap D(A^{i+1}) = \bar{\pi}(A^i) \cap \bar{\pi}(A^{i+1}) \subseteq V(\Gamma)$$

and so $D(A^i) \cap V(\Gamma) \neq \emptyset$, a contradiction. This proves (6).

Let Z be a basis for ϕ with $Z \subseteq V(K \cup \bigcup(A : A \in \mathcal{B}))$; this exists, since $\mathcal{B}$ is sufficient for ϕ. Let $L = \bigcup(\phi(v) : v \in V(H))$.

(7) *If L' is a subgraph of $L \cup \Gamma$ with $\phi(E(H)) \cap E(L') = \emptyset$ and with the same effect on Z as L, then $L' \cup \Gamma = L \cup \Gamma$, and if $L' \subseteq L$ then $L' = L$.*

Subproof. By (6.2) there is a model ϕ' of (H, χ) in (G, ω) such that $\phi'(e) = \phi(e)$ for all $e \in E(H)$ and $\bigcup(\phi'(v) : v \in V(H)) \subseteq L'$. Now $\mathcal{B}$ is sufficient for ϕ', from the choice of Z; and (J1)–(J7) and the other hypotheses of (7.4) remain satisfied if we replace ϕ by ϕ'. From (2), $L' \cup \Gamma = L \cup \Gamma$, and from (3), if $L' \subseteq L$ then $L' = L$. This proves (7).

(8) *L is a forest, and every vertex of L with degree at most 1 belongs to Z.*

Subproof. This follows from the second assertion of (7).

(9) *If $A \in \mathcal{A} \setminus \mathcal{B}$, then $L \cap A \subseteq \Gamma \cup N_W$.*

Subproof. Since $A \notin \mathcal{B}$, it follows that $d(D(A), \Sigma \setminus \Delta) < \theta$. Since $A \in \mathcal{A}$, we deduce from (6) that A is good, and therefore $\bar{\pi}(A) \subseteq V(\Gamma)$, and for all $u, v \in \bar{\pi}(A)$ there is a path of $\Gamma \cap A$ with ends u, v and with no internal vertex in $\bar{\pi}(A)$. Since no edge of $L \cap A$ has an end in W (because $A \notin \mathcal{B}$) there is a subgraph L' of $(\Gamma \cap A) \cup N_W$ with the same effect in $\bar{\pi}(A) \cup W$ as $L \cap A$. Since $d(D(A), \Sigma \setminus \Delta) < \theta$, it follows that $A \cap K$ is null, and so there is a subgraph B of G such that (A, B) is a separation and $V(A \cap B) = \bar{\pi}(A) \cup W$. Since $Z \subseteq V(B)$, it follows from (2.6) (with Z replaced by $Z \cup \bar{\pi}(A)$) that $L' \cup (L \cap B)$ has the same effect on Z as L. Now $L' \subseteq (\Gamma \cap A) \cup N_w \subseteq \Gamma \cup L$, and

$$\phi(E(H)) \cap E(L') \subseteq \phi(E(H)) \cap E(A) = \emptyset$$

since $A \notin \mathcal{B}$ and $\mathcal{B}$ is sufficient for ϕ. Consequently, $L' \cup (L \cap B) \subseteq \Gamma \cup L$, and $\phi(E(H)) \cap E(L' \cup (L \cup B)) = \emptyset$. By (8),

$$L' \cup (L \cap B) \cup \Gamma = L \cup \Gamma,$$

and so

$$L \cap A \subseteq (L \cup \Gamma) \cap A = (L' \cup (L \cap B) \cup \Gamma) \cap A = (L' \cap A) \cup (L \cap A \cap B) \cup (\Gamma \cap A).$$

But $L' \cap A \subseteq (\Gamma \cap A) \cup N_W$, and $L \cap A \cap B$ has no edges and has vertex set $\bar{\pi}(A) \cup W \subseteq V(\Gamma \cap A) \cup W$. Consequently,

$$L \cap A \subseteq (\Gamma \cap A) \cup N_W \subseteq \Gamma \cup N_W$$

as required. This proves (9).

(10) *$\mathcal{B}$ is adequate for ϕ.*

Subproof. Let $v \in V(H)$ and $A \in \mathcal{A}$; we must show that if $A \cap \phi(v) \not\subseteq \Gamma \cup N_W$ then $A \in \mathcal{B}$. But $\phi(v) \subseteq L$, so this follows from (9).

(11) *For $1 \leq i \leq t$, there is no separation (C, D) of G with $W \subseteq V(C \cap D)$ such that*

- *$A'_i \subseteq C$*
- *$K \subseteq D$, and $A \subseteq D$ for all $A \in \mathcal{A}$ with $d(D(A), \Sigma \setminus \Delta) \geq \theta$, and*
- *$(\Gamma \cap C, \Gamma \cap D) \in \mathcal{T}$ and has order $< |\bar{\pi}(A'_i)|$.*

Subproof. Suppose there is such a separation (C, D), and choose it of minimum order. Suppose first that it has order $\geq |\bar{\pi}(A'_i)| + |W|$. Then $V(C \cap D) \not\subseteq V(\Gamma) \cup W$; choose $v \in V(C \cap D) \setminus (V(\Gamma) \cup W)$. If there is no $A \in \mathcal{A}$ with $v \in V(A)$ such that $d(D(A), \Sigma \setminus \Delta) < \theta$, then every edge of G incident with v belongs to $E(D)$ and $v \notin V(A'_i)$; but then $(C \setminus \{v\}, D)$ is a separation of G contrary to the minimality of $|V(C \cap D)|$. Thus there exists $A \in \mathcal{A}$ with $v \in V(A)$ such that $d(D(A), \Sigma \setminus \Delta) < \theta$. Let $B \subseteq G$ be such that (A, B) is a separation of G and $V(A \cap B) = \bar{\pi}(A) \cup W$. (This exists, since $d(D(A), \Sigma \setminus \Delta) < \theta$ and so $A \cap K = N_W$, from the hypothesis.) Now $(A \cup C, B \cap D)$ is a separation of G. Moreover, $W \subseteq V((A \cup C) \cap B \cap D)$ and $A'_i \subseteq A \cup C$, and $K \subseteq B \cap D$ (because $A \cap K = N_W \subseteq B$), and $A' \subseteq B \cap D$ for each $A' \in \mathcal{A}$ with $d(D(A'), \Sigma \setminus \Delta) \geq \theta$, since $A' \neq A$. The separation $(\Gamma \cap (A \cup C),$ $\Gamma \cap (B \cap D))$ has order at most

$$|V(\Gamma \cap C \cap D)| + |V(\Gamma \cap A \cap B)| \leq |\bar{\pi}(A_i')| + |\bar{\pi}(A)| - 1 \leq 5 \leq \theta$$

and so $(\Gamma \cap (A \cup C), \Gamma \cap (B \cap D)) \in \mathcal{T}$. Since $(A \cup C, B \cap D)$ does not contradict the choice of (C, D), it follows that either

$$|V((A \cup C) \cap B \cap D \cap \Gamma)| > |V(C \cap D \cap \Gamma)|$$

or

$$|V((A \cup C) \cap B \cap D)| > |V(C \cap D)|.$$

Consequently, either

$$|V(B \cap C \cap D \cap \Gamma)| + |V(A \cap B \cap \Gamma) \setminus V(C)| > |V(B \cap C \cap D \cap \Gamma)| + |V(C \cap D \cap \Gamma) \setminus V(B)|,$$

that is,

$$|V(A \cap B \cap \Gamma) \setminus V(C)| > |V(C \cap D \cap \Gamma) \setminus V(B)|,$$

or

$$|V(A \cap B) \setminus V(C)| > |V(C \cap D) \setminus V(B)|.$$

Since $|V(A \cap B) \setminus V(C)| \geq |V(A \cap B \cap \Gamma) \setminus V(C)|$ and

$$|V(capC \cap D) \setminus V(B)| > |V(C \cap D \cap \Gamma) \setminus V(B)|,$$

it follows that, in either case,

$$|V(A \cap B) \setminus V(C)| > |V(C \cap D \cap \Gamma) \setminus V(B)|.$$

In particular, $A \not\subseteq C$, and so $A \neq A_i'$. A similar argument, using that the separation $(B \cap C, A \cup D)$ does not violate the choice of (C, D), yields that

$$|V(A \cap B) \setminus V(D)| > |V(C \cap D \cap \Gamma) \setminus V(B)|.$$

But

$$|V(A \cap B) \setminus V(C)| + |V(A \cap B) \setminus V(D)| \leq |V(A \cap B) \setminus W| = |\bar{\pi}(A)| \leq 3,$$

and so $|V(C \cap D \cap \Gamma) \setminus V(B)| = 0$, that is, $C \cap D \cap \Gamma \subseteq B$. Since $|V(A \cap B) \setminus V(C)| > 0$ and $|V(A \cap B) \setminus V(D)| > 0$, there exist $u, v \in V(A \cap B)$ with $u \in V(C) \setminus V(D)$ and $v \in V(D) \setminus V(C)$. Since $W \subseteq V(C \cap D)$ and $V(A \cap B) = \bar{\pi}(A) \cup W, u$ and v both belong to $\bar{\pi}(A)$. But A is good by (6) since $d(D(A), \Sigma \setminus \Delta) < \theta$, and so there is a path of $\Gamma \cap A$ with ends u, v and with no internal vertex in $\bar{\pi}(A)$. Consequently, it has no internal vertex in $V(B)$, but it has one in $V(C \cap D)$ since (C, D) is a separation. Hence $C \cap D \cap \Gamma \not\subseteq B$, a contradiction.

Our assumption that (C, D) has order $\geq |\bar{\pi}(A_i')| + |W|$ is therefore false. Consequently, $(C \setminus W, D \setminus W)$ is a separation of $G \setminus W$ of order $< |\bar{\pi}(A_i')|$, and $A_i' \setminus W \subseteq C \setminus W$, and $((C \setminus W) \cap \Gamma, (D \setminus W) \cap \Gamma) \in \mathcal{T}$, contrary to (J7). This proves (11).

(12) *For $1 \leq i \leq t$ there is no Γ-normal O-arc $F \subseteq \Sigma$ with $|F \cap V(\Gamma)| < |\bar{\pi}(A_i')|$ and with $D(A_i') \subseteq ins(F)$.*

Subproof. Suppose that F is such an O-arc. Let $K_2 = A'_i \cup (\Gamma \cap ins(F))$. By (7.1)(iii), $K_1 \cap A'_i = N_W$. Suppose that $v \in V(K_1 \cap (\Gamma \cap ins(F)))$. Since $D(A'_i) \subseteq ins(F)$ it follows that $d(v, D(A'_i)) \leq 3$ and hence $d(v, \Sigma \setminus \Delta) \leq 3$, contrary to (7.1)(i). We deduce that there is no such v, and so $K_1 \cap K_2 = N_W$.

It follows that there is a separation (C, D) of G with $C \cap \Gamma = \Gamma \cap ins(F)$ and $D \cap \Gamma = \Gamma \cap \overline{\Sigma \setminus \Delta_1}$, where $\Delta_1 = ins(F)$, such that $K_2 \subseteq C$ and $K_1 \subseteq D$. But this contradicts (11). Consequently (12) holds.

From (10), (12) and (7.3), the result follows. ■

8 Homogeneity

The advantage of using "sufficient" instead of "adequate" is that the following is true.

8.1 *Let (J1)–(J6) hold. Then there exists $\mathcal{A}' \subseteq \mathcal{A}$, sufficient for some model of (H, χ) in (G, ω), such that $d(D(A), \Sigma \setminus \Delta) < \theta$ for at most $3q + 5\delta$ members A of $\mathcal{A}'$.*

Proof. Let Z be a basis for ϕ with $Z \subseteq V(K \cup \bigcup(A : A \in \mathcal{A}))$ and $|Z| \leq q + 3\delta$; this exists, from (6.1) and (J6). Choose a model ϕ' of (H, χ) in (G, ω) such that $\phi'(e) = \phi(e)$ for all $e \in E(H)$, and

$$\bigcup(\phi'(v) : v \in V(H)) \subseteq \bigcup(\phi(v) : v \in V(H)),$$

with $\bigcup(\phi'(v) : v \in V(H))$ minimal. Let $L = \bigcup(\phi'(v) : v \in V(H))$. It follows that L is a forest, and Z contains every vertex of L with degree at most 1. For $v \in V(L)$, let $d(v)$ be its degree in L. By (2.3)

$$\sum_{y \in W} d(y) \leq 2|W| + |Z| \leq 3q + 3\delta.$$

Let $\mathcal{A}_1$ be the set of all $A \in \mathcal{A}$ such that some edge of $A \cap L$ has an end in W. Since the members of $\mathcal{A}$ are edge-disjoint, it follows that $|\mathcal{A}_1| \leq 3q + 3\delta$. Since $\phi'(v) \cap (K \cup \bigcup(A : A \in \mathcal{A}))$ is non-null for each $v \in V(H)$, and $\phi'(v) \cap K$ is non-null if v is a root of (H, ω), there exists $\mathcal{A}_2 \subseteq \mathcal{A}$ with $|\mathcal{A}_2| \leq \delta$ such that $\phi(v) \cap (K \cup \bigcup(A : A \in \mathcal{A}_2))$ is non-null for each $v \in V(H)$. Let

$$\mathcal{A}_3 = \{A \in \mathcal{A} : E(A) \cap \phi'(E(H)) \neq \emptyset\};$$

then $|\mathcal{A}_3| \leq |E(H)| \leq \delta$. Finally, let

$$\mathcal{A}_4 = \{A \in \mathcal{A} : d(D(A), \Sigma \setminus \Delta) \geq \theta\}.$$

Let $\mathcal{A}' = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$. Then $\mathcal{A}'$ is sufficient for ϕ', and satisfies the theorem. ■

8.2 *For all integers $q, \delta, \eta \geq 0$ there exists $\theta \geq 4$ with the following property. Let (J1)–(J7) hold, and suppose that for every $A \in \mathcal{A}$ with $D(A) \cap \Delta = \emptyset$ and every $v \in V(\Gamma)$ with $v \notin \Delta$, there exists $A' \in \mathcal{A}$ with $d(v, D(A')) \leq \eta$ such that $(A', \pi(A') + \omega)$ has the same δ-folio as $(A, \pi(A) + \omega)$. Suppose also that $d(v^*, \Delta) \geq \theta$. Then there is a model of (H, χ) in $(G \setminus \{v^*\}, \omega)$.*

Proof. Let $\tau = 3q + 5\delta$. Choose $\theta' \geq 4$ so that (7.4) holds with θ replaced by θ', and let $\theta = 2(\tau+1)(\theta' + 2\eta + 7) + 3$. We claim that θ satisfies (8.2). For let the hypotheses of (8.2) hold. By (8.1) we may assume (by replacing ϕ by the model of (8.1)) that $\mathcal{B} \subseteq \mathcal{A}$ is sufficient for ϕ, and $d(D(A), \Sigma \setminus \Delta) < \theta$ for at most τ members A of B. Let

$$\mathcal{A}' = \{A \in \mathcal{B} : d(D(A), \Sigma \setminus \Delta) < \theta\}.$$

Then $|\mathcal{A}'| \leq \tau$.

(1) *For $3 \leq n \leq \theta - 3$, there is a closed disc $\Delta_n \subseteq \Sigma$ such that $v^* \notin \Delta_n, \Delta \subseteq \Delta_n$, $bd(\Delta_n) \subseteq U(\Gamma)$, $d(v^*, \Delta_n) \geq n$, and $d(v^*, x) \leq n + 2$ for every $x \in A(\Gamma)$ with $x \not\subseteq \Delta_n \setminus bd(\Delta_n)$.*

Subproof. By (5.1) with z, κ replaced by v^*, n, there is a closed disc $\Delta' \subseteq \Sigma$ satisfying (5.1)(i), (ii), (iii) (with Δ replaced by Δ'). Since $d(v^*, \Delta) \geq \theta \geq n + 3$ it follows that $\Delta \cap \Delta' = \emptyset$. Let $\Delta_n = \overline{\Sigma \setminus \Delta'}$; then it satisfies (1).

(2) *There is a closed disc $\Delta' \subseteq \Sigma$ such that $v^* \notin \Delta'$, $\Delta \subseteq \Delta'$, $bd(\Delta') \subseteq U(\Gamma)$, $d(v^*, \Delta') \geq (\theta' + 2\eta + 7)(\tau + 1)$, and for each $A \in \mathcal{B}$, either $D(A) \cap \Delta' = \emptyset$ or $d(D(A), \Sigma \setminus \Delta') \geq \theta'$.*

Subproof. For $i = 0, 1, \ldots, \tau + 1$, define $n(i) = (\tau + i + 1)(\theta' + 2\eta + 7)$, and let

$$\mathcal{A}_i = \{A \in \mathcal{B} : D(A) \cap \Delta_n(i) = \emptyset\}.$$

Since $\Delta \subseteq \Delta_{n(\tau+1)}$, it follows that $\mathcal{A}_{\tau+1} \subseteq \mathcal{A}'$ and so $|\mathcal{A}_{\tau+1}| < \tau + 1$. Choose i with $0 \leq i \leq \tau + 1$ minimum such that $|\mathcal{A}_i| < i$. It follows that $i \geq 1$, and $|\mathcal{A}_{i-1}| \geq i - 1$. But $\mathcal{A}_{i-1} \subseteq \mathcal{A}_i$ since $\Delta_n(i) \subseteq \Delta_n(i-1)$ by (1). Consequently $\mathcal{A}_i = \mathcal{A}_{i-1}$. Let $\Delta' = \Delta_n(i-1)$; we claim it satisfies (2). Certainly $v^* \notin \Delta'$, $\Delta \subseteq \Delta'$, and $bd(\Delta') \subseteq U(\Gamma)$ from (1). Also from (1), since $i \geq 1$,

$$d(v^*, \Delta') \geq n(i-1) = (\tau + i)(\theta' + 2\eta + 7) \geq (\tau + 1)(\theta' + 2\eta + 7).$$

Let $A \in \mathcal{B}$. If $A \notin A_i$, then $D(A) \cap \Delta_n(i) \neq \emptyset$, and since $|bd(D(A)) \cap V(\Gamma)| \leq 3$ and $d(v^*, \Delta_n(i)) \geq n(i)$, it follows that $d(v^*, D(A)) \geq n(i) - 3$. But then for each $z \in A(\Gamma)$ with $z \subseteq \Sigma \setminus \Delta'$,

$$d(D(A), z) \geq d(v^*, D(A)) - d(z, v^*) \geq n(i) - 3 - (n(i-1) + 2) \geq \theta'.$$

Thus if $A \in \mathcal{B}$ and $A \notin \mathcal{A}_i$ then $d(D(A), \Sigma \setminus \Delta') \geq \theta'$. On the other hand, if $A \in \mathcal{B}$ and $A \in \mathcal{A}_i$, then $A \in \mathcal{A}_{i-1}$ and so $D(A) \cap \Delta' = \emptyset$. This proves (2).

Let Δ' be as in (2).

(3) *There are vertices $v_1, \ldots, v_\tau$ of Γ such that*

(i) *for $1 \leq i \leq \tau, d(v^*, v_i) \geq \theta' + \eta + 3$,*

(ii) *for $1 \leq i \leq \tau, d(v_i, \Delta') \geq \eta + 4$, and*

(iii) *for $1 \leq i < j \leq \tau, d(v_i, v_j) \geq \theta' + 2\eta + 6$.*

Subproof. For let P be a path of Γ from v^* to $V(\Gamma) \cap bd(\Delta')$. For $1 \leq i \leq \tau$, let v_i be the first vertex of P such that

$$d(v^*, v_i) \geq (\theta' + 2\eta + 7)i;$$

this exists, for the last vertex, u say, of P belongs to $bd(\Delta')$ and hence satisfies $d(v^*, u) \geq (\theta' + 2\eta + 7)(\tau + 1)$. We claim that $v_1, \dots, v_\tau$ satisfy (3). Certainly (i) holds.

Let $1 \leq i \leq \tau$. Since $d(v^*, v^*) = 0$ it follows that $v_i \neq v^*$, and so there is a vertex v say of P immediately preceding v_i in P. From the definition of v_i, $d(v^*, v) < (\theta' + 2\eta + 7)i$, and since v is adjacent to v_i, $d(v, v_i) \leq 2$; consequently,

$$d(v^*, v_i) \leq d(v^*, v) + d(v, v_{i-1}) \leq (\theta' + 2\eta + 7)i + 1.$$

It follows that

$$(\theta' + 2\eta + 7)(\tau + 1) \leq d(v^*, \Delta') \leq d(v^*, v_i) + d(v_i, \Delta') \leq (\theta' + 2\eta + 7)i + 1 + d(v_i, \Delta'),$$

and since $i \leq \tau$, we deduce that $d(v_i, \Delta') \geq \theta' + 2\eta + 6 \geq \eta + 4$. Hence (ii) holds.

For (iii), let $1 \leq i < j \leq \tau$. Then

$$(\theta' + 2\eta + 7)j \leq d(v^*, v_j) \leq d(v^*, v_i) + d(v_i, v_j) \leq (\theta' + 2\eta + 7)i + 1 + d(v_i, v_j),$$

and since $j \geq i + 1$, we deduce that $d(v_i, v_j) \geq \theta' + 2\eta + 6$. Hence (iii) holds. This proves (3).

Let $v_1, \dots, v_\tau$ be as in (3), and let $\{A \in \mathcal{B} : D(A) \cap \Delta' = \emptyset\} = \{A_1, \dots, A_t\}$. Then $A_1, \dots, A_t \in \mathcal{A}'$, and so $t \leq \tau$. For $1 \leq i \leq t$, choose $A'_i \in \mathcal{A}$ with $d(v_i, D(A'_i)) \leq \eta$ such that $(A'_i, \pi(A'_i))$ has the same δ-folio as $(A_i, \pi(A_i))$ (this is possible from the hypothesis). Then for $1 \leq i \leq t$, there exists $z_i \in A(\Gamma)$ such that $d(v_i, z_i) \leq \eta$ and $z_i \cap D(A'_i) \neq \emptyset$.

(4) *The following hold:*

- (i) *For* $1 \leq i \leq t$, $d(v^*, D(A'_i)) \geq \theta'$.
- (ii) *For* $1 \leq i \leq t$, $D(A'_i) \cap \Delta = \emptyset$.
- (iii) *For* $1 \leq i < j \leq t$, $d(D(A'_i),\ D(A'_j)) \geq \theta'$.

Subproof. To see (i), let $z \in A(\Gamma)$ with $z \cap D(A'_i) \neq \emptyset$. Then $d(z, z_i) \leq 3$ since z and z_i both intersect $D(A'_i)$, and so by (3)(i),

$$\theta' + \eta + 3 \leq d(v^*, v_i) \leq d(v^*, z) + d(z, z_i) + d(v_i, z_i) \leq d(v^*, z) + 3 + \eta.$$

Thus $d(v^*, z) \geq \theta'$, and so $d(v^*, D(A'_i)) \geq \theta'$. Hence (i) holds.

To see (ii), suppose that $z \in A(\Gamma)$ and $z \cap D(A'_i) \cap \Delta \neq \emptyset$. Then $d(z, z_i) \leq 3$, and so by (3)(iii),

$$\eta + 4 \leq d(v_i, \Delta') \leq d(v_i, z) \leq d(v_i, z_i) + d(z_i, z) \leq \eta + 3$$

a contradiction. Thus (ii) holds.

To see (iii), let $y, z \in A(\Gamma)$ with $y \cap D(A'_i) \neq \emptyset$ and $z \cap D(A'_j) \neq \emptyset$. Then by (3)(ii),

$$\theta' + 2\eta + 6 \leq d(v_i, v_j) \leq d(v_i, z_i) + d(z_i, y) + d(y, z) + d(z, z_j) + d(v_j, z_j) \leq \eta + 3 + d(y, z) + 3 + \eta$$

and so $d(y,z) \geq \theta'$. This proves (iii), and completes the proof of (4).

Let us apply (7.4), with Δ, θ replaced by Δ', θ' and with no other replacements. We recall that θ' was chosen to satisfy (7.4). Let us verify the hypothesis of (7.4). Now (J1)–(J4) and (J6), (J7) obviously still hold. For (J5), let $v \in V(\Gamma \cap K)$; then

$$d(v, \Sigma \setminus \Delta') \geq d(v, \Sigma \setminus \Delta) \geq \theta \geq \theta'$$

since $\Delta \subseteq \Delta'$, and similarly $d(D(A), \Sigma \setminus \Delta) \geq \theta'$ for all $A \in \mathcal{A}$ with $A \cap K \neq N_W$. Hence (J5) holds. $\mathcal{B}$ is sufficient for ϕ, and $A_1, \ldots, A_t \in \mathcal{B}$. If $A \in \mathcal{B} \setminus \{A_1, \ldots, A_t\}$, then $d(D(A), \Sigma \setminus \Delta') \geq \theta'$ by (2). Finally, hypothesis (i) of (7.4) is true by the choice of A_i'; (ii) of (7.4) holds by definition of $A_1, \ldots, A_t$; and (iii) and (iv) of (7.4) hold because of (4). Thus, all the hypotheses of (7.4) hold, and the result follows from (7.4). ■

At last we are able to formulate and prove a statement that implies theorem (10.2) of [5]. To understand the motivation of the various hypotheses of the next result, it might help to read the final paragraph of this section before the next proof.

8.3 *For all $q, \delta \geq 0$ and $h \geq 4$, there exists $\theta \geq h$ with the following property. Let G be a digraph, let $W \subseteq V(G)$ with $|W| = q$, and let ω be an ordering of W. Let $\Gamma \subseteq G \setminus W$ satisfying the following.*

(i) *Γ is a drawing in a sphere Σ, and Γ is a subdivision of a simple 3-connected graph, and there is an orientation of Σ called clockwise.*

(ii) *C_0 is a circuit of Γ, and $U(C_0)$ bounds a region of Γ.*

(iii) *$\Pi \subseteq V(C_0)$ with $|\Pi| = 4$.*

(iv) *$\mathcal{T}$ is a tangle in Γ of order $\geq \theta$, and there is no $(A,B) \in \mathcal{T}$ with order ≤ 3 such that $\Pi \subseteq V(A)$; d is the metric of $\mathcal{T}$.*

(v) *$J \subseteq G$ has vertex set the union of $W, V(\Gamma)$, and the vertex sets of all components of $G \setminus (V(C_0) \cup W)$ which meet $V(\Gamma)$, and edge set all edges of G with both ends in $V(J)$.*

(vi) *$Z \subseteq V(J) \setminus W$ with $\Pi \subseteq Z$, and $\mathcal{A}$ is a $(Z \cup W)$-division of J, such that $W \subseteq V(A)$ for all $A \in \mathcal{A}$.*

(vii) *For each $A \in \mathcal{A}$, $Z \cap V(A) = \bar{\pi}(A)$, and $|\bar{\pi}(A)| \leq 3$, and $\pi(A)$ is a linear order of $\bar{\pi}(A)$.*

(viii) *For each $A \in \mathcal{A}$, there are $\bar{\pi}(A)$ mutually vertex-disjoint paths of $J \setminus W$ between $\bar{\pi}(A)$ and Π, and if $|\bar{\pi}(A)| = 3$ and $\pi(A)$ is s_1, s_2, s_3 say, these three paths can be chosen with ends s_i, t_i $(i = 1, 2, 3)$ so that t_1, t_2, t_3 occur in clockwise order in the boundary of the disc containing $U(\Gamma)$ bounded by $U(C_0)$.*

(ix) *For each $A \in \mathcal{A}$, if $u, v \in \bar{\pi}(A)$ there is a path of $A \setminus W$ between u and v with no internal vertex in $\bar{\pi}(A)$.*

(x) *Let G' be the bipartite graph with vertex set $Z \cup \mathcal{A}$, in which $z \in Z$ and $A \in \mathcal{A}$ are adjacent if $z \in V(A)$; then G' is planar, and can be drawn in a closed disc with the vertices of Π in the boundary of the disc, in the same order in which they occur in $U(C_0)$.*

(xi) *For each* $A \in \mathcal{A}$*, there is a vertex* $v(A) \in V(\Gamma)$ *such that there is a path of* $G \setminus W$ *between* $v(A)$ *and a vertex of* $\bar{\pi}(A)$*, with no vertex in* $V(\Gamma)$ *except* $v(A)$*.*

(xii) $D \subseteq \Sigma$ *is a closed disc with* $bd(D) \subseteq U(\Gamma)$ *including the region of* Γ *bounded by* $U(C_0)$*.*

(xiii) *If* $A \in \mathcal{A}$ *and* $v(A) \in \Sigma \setminus D$ *then for every* $v \in V(\Gamma) \setminus D$*, either* $d(v, D) \leq h$ *or there exists* $A' \in \mathcal{A}$ *such that* $d(v, v(A')) \leq h$ *and* $(A', \pi(A') + \omega)$ *has the same* δ*-folio as* $(A, \pi(A) + \omega)$*.*

(xiv) $v^* \in V(\Gamma) \setminus D$*, and* $d(v^*, D) \geq \theta$*.*

Then $(G \setminus \{v^*\}, \omega)$ *has the same* δ*-folio as* (G, ω)*.*

Proof. Let $\eta = h+1$, and choose $\theta' \geq 4$ so that (8.2) holds with θ replaced by θ'. Let $\theta = 2\theta' + h + 14$. We shall show that θ satisfies (8.3).

Our method is to apply (8.2), and we must find suitable choices for $\mathcal{A}', K', N'$ etc. so that (J1)-(J7) are satisfied. Let ω be $w_1, \ldots, w_q$ and let N_W be defined as in (J1); then (J1) is satisfied. Let $\mathcal{A}'$ be the set of all $A \in \mathcal{A}$ such that $d(v(A), V(C_0)) \geq 5$. Then (J2) holds with $\mathcal{A}$ replaced by $\mathcal{A}'$, by (vi), (viii) and (xi) ((xi) implies that $\bar{\pi}(A) \neq \emptyset$). Also, (J3) holds with Γ and $\mathcal{T}$ as given, and with θ replaced by θ', since $\theta \geq \theta'$.

For (J4) we need several lemmas. The first is the following. Let $\Delta_0 \subseteq \Sigma$ be a closed disc such that $U(\Gamma) \subseteq \Delta_0$ and $bd(\Delta_0) \cap U(\Gamma) = \Pi$, obtained by deleting a suitable open disc from the region of Γ bounded by $U(C_0)$.

(1) *For each* $v \in Z$ *there exists* $\alpha(v) \in \Delta_0$*, and for each* $A \in \mathcal{A}$ *there exists a closed disc* $D(A) \subseteq \Delta_0$*, such that*

- $\alpha(v) = v$ *for all* $v \in \Pi$
- *for each* $A \in \mathcal{A}$ *and* $v \in \bar{\pi}(A)$*,* $\alpha(v) \in bd(D(A))$*; and for each* $v \in Z$ *and* $A \in \mathcal{A}$*, if* $\alpha(v) \in D(A)$ *then* $v \in \bar{\pi}(A)$
- *for all distinct* $A, A' \in \mathcal{A}$*,* $D(A) \cap D(A') = \{\alpha(v) : v \in \bar{\pi}(A) \cap \bar{\pi}(A')\}$
- *for all* $A \in \mathcal{A}$*,* $D(A) \cap bd(\Delta_0) = \{\alpha(v) : v \in \bar{\pi}(A) \cap \Pi\}$
- *for all distinct* $v, v' \in Z$*,* $\alpha(v) \neq \alpha(v')$*.*

Subproof. The graph G' of hypothesis (x) can be drawn in some closed disc with the vertices from Π in the boundary and in the right order; and hence it can be drawn in Δ_0 with each vertex in Π represented by itself, and with no other vertex in $bd(\Delta_0)$. Each vertex $A \in \mathcal{A}$ of G' has degree ≤ 3 in G', and we may replace it by a suitable closed disc $D(A)$ in its neighbourhood to satisfy (1).

(2) *We may choose the function* α *and the discs* $D(A)$ $(A \in \mathcal{A})$ *to satisfy* (1) *and in addition such that*

- $\alpha(v) = v$ *for each* $v \in Z \cap V(\Gamma)$*, and*
- *for each* $A \in \mathcal{A}$*,* $bd(D(A))$ *is* Γ*-normal, and* $\Gamma \cap D(A) = \Gamma \cap A$*, and* $bd(D(A)) \cap V(\Gamma) = \bar{\pi}(A)$*.*

Subproof. By hypothesis (i), G is a subdivision of a simple 3-connected graph, and hence for every closed disc $\Delta \subseteq \Delta_0$ with $bd(\Delta)$ Γ-normal and $|bd(\Delta) \cap V(\Gamma)| \leq 2$, either $E(\Gamma \cap \Delta) = \emptyset$ and $V(\Gamma \cap \Delta) \subseteq bd(\Delta)$ or $\Gamma \cap \Delta$ is a path with both ends in $bd(\Delta)$. Hence (2) follows from theorem (6.5) of [6].

To simplify notation we assume (for instance, by replacing G by an isomorphic digraph) that $\alpha(v) = v$ for each $v \in Z$. Then (1) and (2) can be summarized as follows:

(3) *$Z \subseteq \Delta_0$ and $Z \cap bd(\Delta_0) = \Pi$; and for each $A \in \mathcal{A}$ there exists a closed disc $D(A) \subseteq \Delta_0$, such that*

- *for each $A \in \mathcal{A}$, $bd(D(A))$ is Γ-normal and $bd(D(A)) \cap bd(\Delta_0) \subseteq \Pi$*
- *for each $A \in \mathcal{A}$, $\Gamma \cap D(A) = \Gamma \cap A$, and $bd(D(A)) \cap V(\Gamma) = \bar{\pi}(A)$*
- *for all distinct $A, A' \in \mathcal{A}$, $D(A) \cap D(A') = \bar{\pi}(A) \cap \bar{\pi}(A')$.*

To complete the verification of (J4), we need

(4) *For each $A \in \mathcal{A}$, if $|\bar{\pi}(A)| = 3$ and $\pi(A)$ is s_1, s_2, s_3 say, then s_1, s_2, s_3 determine the clockwise orientation of $D(A)$.*

Subproof. From hypothesis (viii), there are mutually vertex-disjoint paths P_1, P_2, P_3 of $J \setminus W$ with ends s_i, t_i $(1 \leq i \leq 3)$, such that $t_1, t_2, t_3 \in \Pi$ and t_1, t_2, t_3 occur in clockwise order in the boundary of Δ_0. Let $L = P_1 \cup P_2 \cup P_3$, and for each $A' \in \mathcal{A}$ with $E(P \cap A') \neq \emptyset$, choose a line $F(A') \subseteq D(A')$ with ends the two vertices in $\bar{\pi}(A')$ with degree 1 in $P \cap A'$, and with no other point in $bd(D(A'))$. Let M be the union of $F(A')$ over all such $A' \in \mathcal{A}$. Then M is the union of three mutually disjoint lines in Δ_0 with ends s_i, t_i $(1 \leq i \leq 3)$, and $M \cap D(A) = \{s_1, s_2, s_3\}$. Since t_1, t_2, t_3 occur in clockwise order in $bd(\Delta_0)$, it follows that s_1, s_2, s_3 occur in clockwise order in $bd(D(A))$. This proves (4).

(5) *For each $A \in \mathcal{A}$, $D(A) = ins(bd(D(A)))$.*

Subproof. Let $F = bd(D(A))$. Since $|F \cap V(\Gamma)| \leq 3 < \theta$, it follows that $ins(F)$ exists. But $\Pi \not\subseteq ins(F)$ by hypothesis (iv), and if D is the closed disc in Σ bounded by F with $D \neq D(A)$, then $\Pi \subseteq bd(\Delta_0) \subseteq D$. Consequently, $D \neq ins(F)$, and so $D(A) = ins(F)$. This proves (5).

From (3), (4) and (5) we see that (J4) holds.

(6) *For each $A \in \mathcal{A}$, there is a region of Γ incident with $v(A)$ having non-empty intersection with $D(A)$.*

Subproof. If $v(A) \in V(A)$ then $v(A) \in V(A \cap \Gamma) \subseteq D(A)$ and the claim is true. We assume then that $v(A) \notin V(A)$. Let P be a path of $G \setminus W$ between $v(A)$ and a vertex of $\bar{\pi}(A)$ with no vertex in $V(\Gamma)$ except $v(A)$. By hypothesis (v), the only vertices of $J \setminus W$ incident in G with edges not in J belong to $V(C_0)$, and no vertex of P belongs to $V(C_0) \subseteq V(\Gamma)$ except possibly $v(A)$. Since both ends of P belong to $V(J \setminus W)$, it follows that $P \subseteq J \setminus W$. Let the vertices of P in $Z \cup \{v\}$ be

$v_0, v_1, \ldots, v_k$ in order in P, where $v_0 \in \bar{\pi}(A)$ and $v_k = v$. For $1 \leq i \leq k$, let P_i be the subpath of P between v_{i-1} and v_i. For $0 \leq i \leq k-1$ let r_i be the region of Γ in Σ containing v_i; this exists since $v_i \in Z \subseteq \Sigma$ and $v_i \notin U(\Gamma)$. For $1 \leq i \leq k$, since no internal vertex of P_i belongs to Z and $\mathcal{A}$ is a Z-division of J, there exists $A_i \in \mathcal{A}$ such that $P_i \subseteq A_i$. For $1 \leq i < k$, both v_{i-1} and v_i belong to Z and hence to $Z \cap V(A_i) = \bar{\pi}(A_i)$. Consequently, v_{i-1} and v_i are ends of a line in $bd(D(A_i))$ with no internal point in $V(\Gamma)$, and hence $r_{i-1} = r_i$. Similarly, if $v_k \in Z$ then r_{k-1} is incident with $v_k = v$, and since $r_0 = r_1 = \ldots = r_{k-1}$ the result is true. We assume then that $v_k \notin Z$, and so $v_k \notin \bar{\pi}(A_k)$. Consequently, $v_{k-1} \in \bar{\pi}(A_k)$, and since $v_{k-1} \notin V(\Gamma)$ we deduce that $|\bar{\pi}(A_k) \cap V(\Gamma)| \leq 2$, and so $|bd(D(A_k)) \cap V(\Gamma)| \leq 2$. Since $V(\Gamma \cap D(A_k)) \not\subseteq bd(D(A_k))$, $\Gamma \cap D(A_k)$ is a path with both ends in $bd(D(A_k))$, and so $r_0 = r_1 = \ldots = r_{k-1}$ is incident with $v = v_k \in V(\Gamma \cap D(A_k))$. This proves (6).

Let $K_0 \subseteq G$ be such that (J, K_0) is a separation of G with $V(J \cap K_0) = V(C_0) \cup W$; this exists, from the definition of J.

(7) *$A \cap K_0 = N_W$ for all $A \in \mathcal{A}'$.*

Subproof. Suppose that $A \in \mathcal{A}$ and $A \cap K_0 \neq N_W$. Since $A \subseteq J$ and $E(J \cap K_0) = \emptyset$ and $V(J \cap K_0) = V(C_0) \cup W$, it follows that $V(A \cap C_0) \neq \emptyset$. Hence $d(v(A), V(C_0)) \leq 3$ by (6), and so $A \notin \mathcal{A}'$. This proves (7).

Let $N = \Gamma \cup N_W \cup \bigcup(A \,:\, A \in \mathcal{A}')$, and let $K = K_0 \cup \bigcup(A \setminus E(A \cap \Gamma) \,:\, A \in \mathcal{A} \setminus \mathcal{A}')$.

(8) *(N, K) is a separation of G and $W \subseteq V(K)$, and if $v \in V(K \cap N) \setminus W$ then $v \in \Sigma$ and $d(v, \Sigma \setminus \Delta_0) \leq 7$.*

Subproof. Now

$$\Gamma \cup \bigcup(A \setminus E(A \cap \Gamma) \,:\, A \in \mathcal{A} \setminus \mathcal{A}') \cup \bigcup(A \,:\, A \in \mathcal{A}') = \bigcup(A \,:\, A \in \mathcal{A}) = J$$

and $J \cup K_0 = G$, and so $N \cup K = G$. If $e \in E(K \cap N)$, then $e \notin E(K_0)$ since $N \subseteq J$ and $E(J \cap K_0) = \emptyset$, and so $e \in E(A \setminus E(A \cap \Gamma))$ for some $A \in \mathcal{A} \setminus \mathcal{A}'$; but then $e \notin E(\Gamma)$, and $e \notin E(\bigcup(A : A \in \mathcal{A}'))$ by hypothesis (vi), and so $e \notin E(N)$, a contradiction. Thus (N, K) is a separation of G, and $W \subseteq V(K_0) \subseteq V(K)$. Let $v \in V(K \cap N) \setminus W$. If $v \in V(K_0)$ then

$$v \in V(K_0 \cap N) \subseteq V(K_0 \cap J) = V(C_0) \cup W$$

and so $(v, \Sigma \setminus \Delta_0) \leq 1$ as required. If $v \notin V(K_0)$, let $v \in V(A \cap N)$ where $A \in \mathcal{A} \setminus \mathcal{A}'$. Either $v \in V(\Gamma)$, or $v \in V(A')$ for some $A' \in \mathcal{A}'$ and hence $v \in \bar{\pi}(A)$, and since $V(\Gamma \cap A) \subseteq D(A)$ and $\bar{\pi}(A) \subseteq D(A)$ it follows that $v \in D(A)$. By (5) and (6), $d(v, v(A)) \leq 3$ (for either $v(A) \in V(A)$ or $\bar{\pi}(A) \not\subseteq V(\Gamma)$). But $d(v(A), \Sigma \setminus \Delta_0) \leq 4$ since $A \notin \mathcal{A}'$, and so $d(v, \Sigma \setminus \Delta_0) \leq 7$. This proves (8).

(9) *There is a closed disc $\Delta \subseteq \Sigma$ with $bd(\Delta) \subseteq U(\Gamma)$ and $v^* \notin \Delta$ and $\Sigma \setminus \Delta_0 \subseteq \Delta$, such that $d(v^*, \Delta) \geq \theta'$ and $d(v^*, x) \leq \theta' + 2$ for every $x \in A(\Gamma)$ with $x \not\subseteq \Delta \setminus bd(\Delta)$.*

Subproof. This follows by (5.1) (with κ, z replaced by θ', v^*), and taking the closure of the complement of the disc given by (5.1).

(10) *$d(D, \Sigma \setminus \Delta) \geq \theta - \theta' - 2$, and in particular, $D \subseteq \Delta$.*

Subproof. Let $x \in \Sigma \setminus \Delta$. By (8), $d(v^*, x) \leq \theta' + 2$, and by hypothesis (xiv), $d(v^*, D) \geq \theta$, and so

$$\theta \leq d(v^*, D) \leq d(v^*, x) + d(x, D) \leq \theta' + 2 + d(x, D).$$

This proves (10).

(11) *$d(v, \Sigma \setminus \Delta) \geq \theta'$ for all $v \in V(\Gamma \cap K)$, and $d(D(A), \Sigma \setminus \Delta) \geq \theta'$ for all $A \in \mathcal{A}'$ with $A \cap K \neq N_W$.*

Subproof. If $v \in V(\Gamma \cap K)$, then by (7), $d(v, \Sigma \setminus \Delta_0) \leq 7$. Since $\Sigma \setminus \Delta_0 \subseteq D$, it follows that $d(v, D) \leq 7$, and so by (9),

$$\theta - \theta' - 2 \leq d(D, \Sigma \setminus \Delta) \leq d(D, v) + d(v, \Sigma \setminus \Delta) \leq 7 + d(v, \Sigma \setminus \Delta)$$

and so $d(v, \Sigma \setminus \Delta) \geq \theta - \theta' - 9 \geq \theta'$ as required. Secondly, let $A \in \mathcal{A}'$ with $A \cap K \neq N_W$, and let $z \in A(\Gamma)$ with $z \cap D(A) \neq \emptyset$. By (7), $A \cap K_0 = N_W$, and so there exists $A' \in \mathcal{A} \setminus \mathcal{A}'$ with $A \cap A' \neq N_W$. Hence $D(A) \cap D(A') \neq \emptyset$, and so by (5) and (6), $d(z, v(A')) \leq 7$. But $d(v(A'), \Sigma \setminus \Delta_0) \leq 4$ since $A' \notin \mathcal{A}'$, and so $d(z, \Sigma \setminus \Delta_0) \leq 11$. Since $\Sigma \setminus \Delta_0 \subseteq D$, it follows that $d(z, D) \leq 11$. Hence by (10),

$$\theta - \theta' - 2 \leq d(D, \Sigma \setminus \Delta) \leq d(z, D) + d(z, \Sigma \setminus \Delta) \leq 11 + d(z, \Sigma \setminus \Delta),$$

and so $d(z, \Sigma \setminus \Delta) \geq \theta - \theta' - 13 \geq \theta'$. This proves (11).

From (8)–(11), we see that (J5) holds with $\mathcal{A}, \theta$ replaced by $\mathcal{A}', \theta'$. Let (H, χ) belong to the δ-folio of (G, ω), and let ϕ be a model of (H, χ) in (G, ω). Then (J6) is satisfied with $\mathcal{A}$ replaced by $\mathcal{A}'$, since $K \cup \bigcup(A \ : \ A \in \mathcal{A}') = G$.

(12) *For each $A \in \mathcal{A}'$ there is no separation (C, D) of $G \setminus W$ of order $< |\bar{\pi}(A)|$ such that $A \setminus W \subseteq C$ and $(C \cap \Gamma, D \cap \Gamma) \in \mathcal{T}$.*

Subproof. Suppose that (C, D) is such a separation. By hypothesis (viii), there are $|\bar{\pi}(A)|$ mutually vertex-disjoint paths of $J \setminus W$ between $\bar{\pi}(A)$ and Π, and therefore there is a path P of $J \setminus W$ between $\bar{\pi}(A)$ and Π with $V(P) \subseteq V(C) \setminus V(C \cap D)$. Since $\Pi \cap V(C) \not\subseteq V(D)$ and Γ is a subdivision of a 3-connected graph, and $(C \cap \Gamma, D \cap \Gamma)$ has order $\leq |\bar{\pi}(A)| - 1 \leq 2$, it follows that $C \cap \Gamma$ is a path with both ends in $V(C \cap D \cap \Gamma)$. In particular, $C \cap \Gamma$ is connected, and $|V(C \cap D)| = 2$, and $v(A) \in V(C)$, and so there is a path of $C \cap \Gamma$ between $v(A)$ and Π. Consequently $d(v(A), \Pi) \leq 4$, and so $A \notin \mathcal{A}'$, a contradiction. This proves (12).

Now we verify (J7), with $\mathcal{A}$ replaced by $\mathcal{A}'$. The first condition of (J7) follows from hypothesis (ix), and the second from (12). For the third, let $A \in \mathcal{A}'$. By (3), $\Gamma \cap A = \Gamma \cap D(A)$. If $|bd(D(A)) \cap V(\Gamma)| \leq 2$ then by hypothesis (i) and (5), $\Gamma \cap D(A)$ is either a path with both ends in $bd(D(A))$, or $E(\Gamma \cap D(A)) = \emptyset$ and $V(\Gamma \cap D(A)) \subseteq \bar{\pi}(A)$, and (J7)(i) or (J7)(ii) is true, as required. We assume then that $|bd(D(A)) \cap V(\Gamma)| = 3$, $bd(D(A)) \cap V(\Gamma) = \{s_1, s_2, s_3\}$ say, and assume (J7)(iv) is false, and without loss of generality that every path of $\Gamma \cap A$ between s_1 and s_2 uses s_3. Since

$\Gamma \cap A$ is a drawing in $D(A)$, and $s_1, s_2, s_3 \in bd(D(A))$, there is a region of $\Gamma \cap A$ in $D(A)$ incident with s_3 and including the open line segment in $bd(D(A))$ with ends s_1, s_2. Since Γ is a subdivision of a 3-connected graph, it follows that (J7)(i), (J7)(ii) or (J7)(iii) is true, as required. Consequently (J7) holds with $\mathcal{A}$ replaced by $\mathcal{A}'$.

(13) *For every $A \in \mathcal{A}'$ with $D(A) \cap \Delta = \emptyset$ and every $v \in V(\Gamma)$ with $v \notin \Delta$, there exists $A' \in \mathcal{A}'$ with $d(v, D(A')) \leq \eta$ such that $(A', \pi(A') + \omega)$ has the same δ-folio as $(A, \pi(A) + \omega)$.*

Subproof. Since $D(A) \cap \Delta = \emptyset$ and $bd(\Delta) \subseteq U(\Gamma)$, it follows from (6) that $v(A) \notin \Delta \setminus bd(\Delta)$, and hence from (9), $d(v^*, v(A)) \leq \theta' + 2$. Hence $v(A) \in \Sigma \setminus D$, because by hypothesis (xiv), $d(v^*, D) \geq \theta > \theta' + 2$. Also, by (10), $d(v, D) \geq \theta - \theta' - 2 > h$. By hypothesis (xiii), there exists $A' \in \mathcal{A}$ such that $d(v, v(A')) \leq h$ and $(A', \pi(A') + \omega)$ has the same δ-folio as $(A, \pi(A) + \omega)$. Hence, by (6), $d(v, D(A')) \leq h + 1 = \eta$. Finally,

$$\theta - \theta' - 2 \leq d(v, D) \leq d(v, v(A')) + d(v(A'), D) \leq h + d(v(A'), V(C_0))$$

and so $d(v(A'), V(C_0)) \geq \theta - \theta' - 2 - h \geq 5$. Hence $A' \in \mathcal{A}'$. This proves (13).

Consequently, all the hypotheses of (8.2) hold with $\mathcal{A}$ and θ replaced by $\mathcal{A}'$ and θ', and it follows from (8.2) that there is a model of (H, χ) in $(G \setminus \{v^*\}, \omega)$. We deduce that the δ-folio of (G, ω) is a subset of the δ-folio of $(G \setminus \{v^*\}, \omega)$, and we therefore have equality, since the reverse inclusion is trivial. The result follows. ■

Finally, a few words on deriving theorem (10.2) of [5] from (8.3). In the language of [5] we have a wall with an h-homogeneous subwall of height θ (θ replaces the $f(h)$ of [5]). Take Γ to be the original wall, and let C_0 be its perimeter. This wall has height at least θ since it has a subwall of height θ, and hence it contains the $\theta \times \theta$ grid as a minor, and from theorems (6.1) and (7.3) of [3] it therefore has a tangle $\mathcal{T}$ of order θ. Let Π be the set of corners of Γ, and let $\mathcal{A}$ be the set of graphs called $\tilde{A}$ in the final section of [5]. Let $\pi(A) + \omega$ be the "attachment sequence" of A in the language of [5]. Let D be the closed disc with boundary the perimeter of the h-homogeneous subwall including the "infinite" region of Γ. Then hypotheses (i)–(xiv) all are satisfied (for (xiii), we use that the subwall is h-homogeneous). Consequently, theorem (10.2) of [5] is true.

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