Graph minors
XX. Wagner's conjecture

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Abstract

We prove Wagner's conjecture, that for every infinite set of finite graphs, one of its members is isomorphic to a minor of another.

1 Introduction

A famous conjecture of Wagner [6] asserts that for any infinite set of graphs, one of its members is isomorphic to a minor of another (all graphs in this paper are finite). It has been one of the main goals of this series of papers to prove the conjecture, and in this paper the proof is completed.

Our method is roughly as follows. If $\{G_1, G_2, \dots\}$ is a counterexample to Wagner's conjecture then none of $G_2, G_3, \dots$ has a minor isomorphic to G_1, and so to prove Wagner's conjecture it suffices to show the following.

1.1 *For every graph H and every infinite set of graphs each with no minor isomorphic to H, some member of the set is isomorphic to a minor of another member of the set.*

It was shown in [3] that

1.2 *For every graph H, if G has no minor isomorphic to H, then every "highly connected component" of G can "almost" be drawn on a surface on which H cannot be drawn.*

(The meanings of "highly connected component" and "almost" here are complicated and we shall postpone the exact statement of this theorem as long as possible. *Surfaces* are connected and compact.)

We may assume the surface in 1.2 is without boundary; and since up to homeomorphism there are only finitely many such surfaces in which H cannot be drawn, to prove 1.1 and hence Wagner's conjecture it suffices to show that

1.3 *If $\Sigma_1, \dots, \Sigma_n$ are surfaces then for every infinite set $\mathcal{F}$ of graphs, if every highly connected component of every member of $\mathcal{F}$ can almost be drawn in one of $\Sigma_1, \dots, \Sigma_n$, then some member of $\mathcal{F}$ is isomorphic to a minor of another member of $\mathcal{F}$.*

To prove 1.3 we use the main results of two other papers of this series [4, 5]. The main result of [4] asserts that, if $\mathcal{F}$ is an infinite set of graphs and all the highly connected components of all members of $\mathcal{F}$ have a certain "well-behaved" structure, then some member of $\mathcal{F}$ is isomorphic to a minor of another member of $\mathcal{F}$. It therefore suffices to show that the hypothesis of 1.3 implies that all these highly connected components have a well-behaved structure. To show this, we apply the main result of [5], which asserts that for any infinite set of hypergraphs all drawable in a fixed surface (where the edges of the hypergraphs all have two or three ends, and each edge is labeled from a fixed well-quasi-order), some member of the set is isomorphic to a minor of another (with an appropriate definition of "minor" for hypergraphs).

In sections 2–10 we finish the proof of Wagner's conjecture, and in section 11 we prove a slight strengthening.

2 Hypergraphs and tangles

For the purposes of this paper, a *hypergraph* G consists of a finite set $V(G)$ of *vertices*, a finite set $E(G)$ of *edges*, and an incidence relation between them. The vertices incident with an edge are the *ends* of the edge. (A hypergraph is thus a graph if every edge has one or two ends.) A hypergraph

H is a *subhypergraph* of a hypergraph G (written $H \subseteq G$) if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and for every $v \in V(G)$ and $e \in E(H)$, e is incident with v in G if and only if $v \in V(H)$ and e is incident with v in H. If G_1, G_2 are subhypergraphs of G we denote by $G_1 \cup G_2$, $G_1 \cap G_2$ the subhypergraphs with vertex sets $V(G_1) \cup V(G_2)$, $V(G_1) \cap V(G_2)$ and edge sets $E(G_1) \cup E(G_2)$, $E(G_1) \cap E(G_2)$ respectively. A *separation* of G is an ordered pair (G_1, G_2) of subhypergraphs with $G_1 \cup G_2 = G$ and $E(G_1 \cap G_2) = \emptyset$, and its *order* is $|V(G_1 \cap G_2)|$.

A central idea in our approach is that of a *tangle* in a hypergraph, which was introduced in [2]. Intuitively, a tangle of order θ is a "θ-connected component" of the hypergraph, which therefore resides on one side or the other of every separation of order $< \theta$. Formally, let G be a hypergraph and $\theta \geq 1$ an integer. A *tangle of order* θ in G is a set $\mathcal{T}$ of separations of G, each of order $< \theta$, such that

- for every separation (A, B) of G of order $< \theta$, $\mathcal{T}$ contains one of (A, B), (B, A)
- if $(A_i, B_i) \in \mathcal{T}$ $(i = 1, 2, 3)$ then $A_1 \cup A_2 \cup A_3 \neq G$
- if $(A, B) \in \mathcal{T}$ then $V(A) \neq V(G)$.

Let us mention one lemma that we shall need later.

2.1 *Let G be a hypergraph, let $G' \subseteq G$ and let $\mathcal{T}'$ be a tangle in G' of order θ. Let $\mathcal{T}$ be the set of all separations (A, B) of G of order $< \theta$ such that $(A \cap G', B \cap G') \in \mathcal{T}'$. Then $\mathcal{T}$ is a tangle in G of order θ.*

The proof is clear.

A *tie-breaker* in a hypergraph G is a function λ which maps each separation (A, B) of G to some member $\lambda(A, B)$ of a linearly ordered set $(\Lambda, \leq)$ (we call $\lambda(A, B)$ the λ-*order* of (A, B)) in such a way that for all separations (A, B), (C, D) of G,

- $\lambda(A, B) = \lambda(C, D)$ if and only if $(A, B) = (C, D)$ or $(A, B) = (D, C)$
- either $\lambda(A \cup C, B \cap D) \leq \lambda(A, B)$ or $\lambda(A \cap C, B \cup D) < \lambda(C, D)$
- if $|V(A \cap B)| < |V(C \cap D)|$ then $\lambda(A, B) < \lambda(C, D)$.

Let λ be a tie-breaker in a hypergraph G. If $\mathcal{T}_1$, $\mathcal{T}_2$ are tangles in G with $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \not\subseteq \mathcal{T}_1$, then there is a unique $(A, B) \in \mathcal{T}_1$ such that $(B, A) \in \mathcal{T}_2$ of minimum λ-order, called the $(\mathcal{T}_1, \mathcal{T}_2)$-*distinction.*

A *march* in a set V is a finite sequence of distinct elements of V; and if π is the march $v_1, \ldots, v_k$, we denote the set $\{v_1, \ldots, v_k\}$ by $\bar{\pi}$. We denote the null march by 0. A *rooted hypergraph* G is a pair $(G^-, \pi(G))$ where G^- is a hypergraph and $\pi(G)$ is a march in $V(G^-)$. We define $V(G) = V(G^-)$, $E(G) = E(G^-)$. If G is a rooted hypergraph, a *tangle* in G is a tangle in G^-, and a *tie-breaker* in G is a tie-breaker in G^-.

A *separation* of a rooted hypergraph G is a pair (A, B) of rooted hypergraphs such that (A^-, B^-) is a separation of G^-, $\bar{\pi}(A) = V(A \cap B)$, and $\pi(B) = \pi(G)$. If G, A are rooted hypergraphs, we write $A \subseteq G$ if $A^- \subseteq G^-$. If $A \subseteq G$, we say A is *complemented* if there exists $B \subseteq G$ such that (A, B) is a separation of G, and we define $G \setminus A = B$. A *rooted location* in a rooted hypergraph G is a set $\mathcal{L}$ of complemented rooted hypergraphs A with $A \subseteq G$ such that $E(A_1^- \cap A_2^-) = \emptyset$ and

$V(A_1^- \cap A_2^-) = \bar{\pi}(A_1) \cap \bar{\pi}(A_2)$ for all distinct A_1, $A_2 \in \mathcal{L}$. Its *order* is $\max(|\bar{\pi}(A)| : A \in \mathcal{L})$, or 0 if $\mathcal{L} = \emptyset$. If $\mathcal{L}$ is a rooted location in G, we define $\mathcal{L}^- = \{(A^-, (G \setminus A)^-) : A \in \mathcal{L}\}$, and we define $M(G, \mathcal{L})$ to be $\cap((G \setminus A)^- : A \in \mathcal{L})$ if $\mathcal{L} \notin \emptyset$, and to be G^- if $\mathcal{L} = \emptyset$.

Let G be a rooted hypergraph, let $\mathcal{T}$ be a tangle in G, and let λ be a tie-breaker in G. A rooted location $\mathcal{L}$ in G is said to *θ-isolate* $\mathcal{T}$ if $\theta \geq 1$, $\mathcal{L}$ has order $< \theta$, $\mathcal{L}^- \subseteq \mathcal{T}$, and for each $A \in \mathcal{L}$, and for every tangle $\mathcal{T}'$ in G of order $\geq \theta$ with $((G \setminus A)^-, A^-) \in \mathcal{T}'$, the $(\mathcal{T}, \mathcal{T}')$-distinction (C, D) satisfies $C \subseteq A^-$ and $(G \setminus A)^- \subseteq D$.

3 Patchworks

If V is a finite set we denote by K_V the complete graph on V, that is, the simple graph with vertex set V and edge set the set of all subsets of V of cardinality 2, with the natural incidence relation. A *grouping* in V is a subgraph of K_V every component of which is complete. A *pairing* in V is a grouping in V every component of which has most two vertices. A pairing K in V is said to *pair* X, Y if X, $Y \subseteq V$ are disjoint and

- every 2-vertex component of K has one vertex in X and the other in Y, and
- every vertex of $X \cup Y$ belongs to some 2-vertex component of K.

A *patch* Δ in V consists of a subset $V(\Delta) \subseteq V$, and a collection of groupings in V, each with the same vertex set $V(\Delta) \subseteq V$. We denote the collection of groupings by the same symbol Δ. A patch Δ is *free* if it contains every grouping in V with vertex set $V(\Delta)$; and it is *robust* if for every choice of X, $Y \subseteq V(\Delta)$ with $|X| = |Y|$ and $X \cap Y = \emptyset$, there is a pairing in Δ which pairs X, Y.

A *patchwork* is a triple $P = (G, \mu, \Delta)$, where

- G is a rooted hypergraph
- μ is a function with domain *dom* $(\mu) \subseteq E(G)$; and for each $e \in$ *dom*(μ), $\mu(e)$ is a march with $\bar{\mu}(e)$ the set of ends of e in G
- Δ is a function with domain $E(G)$, and for each $e \in E(G)$, $\Delta(e)$ is a patch with $V(\Delta(e))$ the set of ends of e; and for each $e \in E(G) \setminus$ *dom*(μ), $\Delta(e)$ is free.

The patchwork is *robust* if each $\Delta(e)(e \in E(G))$ is robust. (This is automatic for $e \notin$ *dom*(μ), since free patches are robust.) It is *rootless* if $\bar{\pi}(G) = \emptyset$.

A *quasi-order* Ω is a pair $(E(\Omega), \leq)$, where $E(\Omega)$ is a set and $\leq$ is a reflective transitive relation on $E(\Omega)$. It is a *well-quasi-order* if for every countable sequence x_i $(i = 1, 2, \dots)$ of elements of $E(\Omega)$ there exist $j > i \geq 1$ such that $x_i \leq x_j$. If Ω_1, Ω_2 are quasi-orders with $E(\Omega_1) \cap E(\Omega_2) = \emptyset$ we denote by $\Omega_1 \cup \Omega_2$ the quasi-order Ω with $E(\Omega) = E(\Omega_1) \cup E(\Omega_2)$ in which $x \leq y$ if for some i $(i = 1, 2) x, y \in E(\Omega_i)$ and $x \leq y$ in Ω_i. If Ω_1, Ω_2 are quasi-orders we write $\Omega_1 \subseteq \Omega_2$ if $E(\Omega_1) \subseteq E(\Omega_2)$ and for x, $y \in E(\Omega_1)$, $x \leq y$ in Ω_1 if and only if $x \leq y$ in Ω_2.

If Ω is a quasi-order, a *partial Ω-patchwork* is a quadruple (G, μ, Δ, ϕ), where (G, μ, Δ) is a patchwork and ϕ is a function from a subset *dom*(ϕ) of $E(G)$ into $E(\Omega)$. It is an *Ω-patchwork* if *dom*$(\phi) = E(G)$. It is *robust* if (G, μ, Δ) is robust. It is *rootless* if $\bar{\pi}(G) = \emptyset$.

If V is a finite set, N_V denotes the graph with vertex set V and no edges. A *realization* of a patchwork (G, μ, Δ) is a subgraph of $K_{V(G)}$ expressible in the form

$$N_{V(G)} \cup \bigcup_{e \in E(G)} \delta_e$$

where $\delta_e \in \Delta(e)$ for each $e \in E(G)$. A *realization* of a partial Ω-patchwork (G, μ, Δ, ϕ) is a realization of (G, μ, Δ). If μ_1, μ_2 are marches with the same length, we denote by $\mu_1 \to \mu_2$ the bijection from $\overline{\mu_1}$ onto $\overline{\mu_2}$ that maps μ_1 onto μ_2. Let $P = (G, \mu, \Delta, \phi)$, $P' = (G', \mu, \Delta', \phi')$ be Ω-patchworks. An *expansion* of P in P' is a function η with domain $V(G) \cup E(G)$ such that

- for each $v \in V(G)$, $\eta(v)$ is a non-empty subset of $V(G')$, and for each $e \in E(G)$, $\eta(e) \in E(G')$
- for distinct v_1, $v_2 \in V(G)$, $\eta(v_1) \cap \eta(v_2) = \emptyset$
- for distinct e_1, $e_2 \in E(G)$, $\eta(e_1) \neq \eta(e_2)$
- for each $e \in E(G)$, $e \in dom(\mu)$ if and only if $\eta(e) \in dom(\mu')$
- for each $e \in E(G) \setminus dom(\mu)$, if v is an end of e in G then $\eta(v)$ contains an end of $\eta(e)$ in G'
- for each $e \in dom(\mu)$, $\mu(e)$ and $\mu'(\eta(e))$ have the same length, k say, and for $1 \leq i \leq k$, $\eta(v)$ contains the ith term of $\mu'(\eta(e))$ where v is the ith term of $\mu(e)$
- $\pi(G)$ and $\pi(G')$ have the same length, k say, and for $1 \leq i \leq k$, $\eta(v)$ contains the ith term of $\pi(G')$ where v is the ith term of $\pi(G)$
- for each $e \in dom(\mu)$, $\Delta'(\eta(e))$ is the image of $\Delta(e)$ under $\mu(e) \to \mu'(\eta(e))$
- for each $e \in E(G)$, $\phi(e) \leq \phi'(\eta(e))$.

If G is a hypergraph and $F \subseteq E(G)$, $G \backslash F$ denotes the subhypergraph with the same vertex set and edge set $E(G) \setminus F$. If G is a rooted hypergraph, $G \backslash F$ denotes $(G^- \backslash F, \pi(G))$. If $P = (G, \mu, \Delta, \phi)$ is an Ω-patchwork and $F \subseteq E(G)$, $P \backslash F$ denotes the Ω-patchwork $(G \backslash F,\ \mu'\ \Delta',\ \phi')$ where μ', Δ', ϕ' are the restrictions of μ, Δ, ϕ to $dom(\mu) \cap E(G \backslash F)$, $E(G \backslash F)$, $E(G \backslash F)$ respectively. Let η be an expansion of $P = (G, \mu, \Delta, \phi)$ in $P' = (G', \mu', \Delta', \phi)$. A realization H of $P' \backslash \eta(E(G))$ is said to *realize* η if for every $v \in V(G)$, $\eta(v)$ is the vertex set of some component of H; and if there is such a realization, η is said to be *realizable*. Let us say that P is *simulated* in P' if there is a realizable expansion of P in P'.

If $P = (G, \mu, \Delta)$ is patchwork and A is a rooted hypergraph with $A \subseteq G$, we denote by $P|A$ the patchwork (A, μ', Δ'), where μ', Δ' are the restrictions of μ, Δ to $E(A) \cap dom(\mu)$, $E(A)$ respectively. If $P = (G, \mu, \Delta, \phi)$ is a partial Ω-patchwork, $P|A$ is the partial Ω-patchwork $(A, \mu', \Delta', \phi')$ where μ', Δ' are as before and ϕ' is the restriction of ϕ to $E(A) \cap dom(\phi)$.

Let $P = (G, \mu, \Delta)$ be a patchwork. A grouping K is *feasible* in P if $V(K) = \bar{\pi}(G)$ and there is a realization H of P such that for distinct x, $y \in V(K)$, x and y belong to the same component of H if and only if they are adjacent in K.

Let $P = (G, \mu, \Delta)$ be a patchwork and let $\mathcal{L}$ be a rooted location in G. For each $A \in \mathcal{L}$ let $e(A)$ be a new element, and let G' be the rooted hypergraph with

$$\begin{aligned} V(G') &= V(M(G, \mathcal{L})) \\ E(G') &= E(M(G, \mathcal{L})) \cup \{e(A) : A \in \mathcal{L}\} \\ \pi(G') &= \pi(G) \end{aligned}$$

where for $e \in E(M(G, \mathcal{L}))$ its ends are as in G^-, and for $A \in \mathcal{L}$ the ends of $e(A)$ are the vertices in $\bar{\pi}(A)$. For $e \in E(M(G, \mathcal{L})) \cap dom(\mu)$ let $\mu'(e) = \mu(e)$, and for $A \in \mathcal{L}$ let $\mu'(e(A)) = \pi(A)$. For $e \in E(M(G, \mathcal{L}))$ let $\Delta'(e) = \Delta(e)$, and for $A \in \mathcal{L}$ let $\Delta'(e(A))$ be the set of all groupings feasible in $P|A$, with $V(\Delta'(e(A))) = \bar{\pi}(A)$. Then (G', μ', Δ') is a patchwork which we call a *heart* of $(P, \mathcal{L})$. (It is unique up to the choice of the new elements $e(A)$.)

Now let $P' = (G, \mu, \Delta, \phi)$ be an Ω-patchwork, and let $P = (G, \mu, \Delta)$ and $\mathcal{L}$ be as before. For $e \in E(M(G, \mathcal{L}))$ let $\phi'(e) = \phi(e)$; then, with G', μ', Δ' defined as before, $(G', \mu', \Delta', \phi')$ is a partial Ω-patchwork which we call a *heart* of $(P', \mathcal{L})$.

Let $P = (G, \mu, \Delta, \phi)$ be a partial Ω-patchwork, and let Ω' be a quasi-order with $\Omega \subseteq \Omega'$. By an Ω'-*completion* of P we mean an Ω'-patchwork (G, μ, Δ, ϕ') such that $\phi'(e) = \phi(e)$ for each $e \in dom(\phi)$. A set C of partial Ω-patchworks is *well-behaved* if Ω is a well-quasi-order and for every well-quasi-order Ω' with $\Omega \subseteq \Omega'$ and every countable sequence P'_i $(i = 1, 2, \dots)$ of Ω'-completions of members of C there exist $j > i \geq 1$ such that P'_i is simulated in P'_j. Let $\Omega_1 \subseteq \Omega_2$ be well-quasi-orders, and let C be a set of partial Ω_1-patchworks. Then C is also a set of partial Ω_2-patchworks; and it is an easy exercise to show that C is well-behaved taking $\Omega = \Omega_1$, if and only if it is well-behaved with $\Omega = \Omega_2$. Thus, our terminology suppressing the dependence on Ω is not misleading.

The following is theorem 6.7 of [4].

3.1 *Let Ω be a well-quasi-order, let $\mathcal{F}$ be a well-behaved set of rootless partial Ω-patchworks, and let $\theta \geq 1$ be an integer. Let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \dots)$ be a countable sequence of rootless robust Ω-patchworks. For each $i \geq 1$ let λ_i be a tie-breaker in G_i; and suppose that for every tangle $\mathcal{T}$ in G_i of order $\geq \theta$ there is a rooted location $\mathcal{L}$ in G_i such that $\mathcal{L}$ θ-isolates $\mathcal{T}$ and $(P_i, \mathcal{L})$ has a heart in $\mathcal{F}$. Then there exist $j > i \geq 1$ such that P_i is simulated in P_j.*

4 Well-behaved sets of patchworks

The previous result 3.1, combined with the main result of [3] (see 10.3 of the present paper), almost proves Wagner's conjecture. Not quite, however; although the rooted locations provided by [3] have hearts in a well-behaved set, they do not quite θ-isolate the corresponding tangles and so 3.1 cannot be applied to them. In the next few sections we prove a strengthening 7.3 of 3.1, that bridges the gap. We show that the locations of [3] can be modified such that the new locations still have hearts in a (new) well-behaved set and do θ'-isolate the corresponding tangles, for an appropriate θ'. The main problem is that there are a bounded number of vertices that need to be removed; and in essence 7.3 addresses the problems caused by removing these vertices.

To prove 7.3 we first need to develop ways of constructing new well-behaved sets of patchworks from old ones, and that is the object of this section. Incidentally, the rooted locations $\mathcal{L}$ provided by [3] have the property that $\bigcup(A^- : A \in \mathcal{L}) = G^-$, which has two desirable consequences; that their hearts have no "isolated vertices", and that their hearts have no edges labeled from Ω, and hence are more naturally regarded as patchworks than as partial Ω-patchworks. This motivates the following.

If $P = (G, \mu, \Delta)$ is a patchwork and Ω is a quasi-order, we call every Ω-patchwork (G, μ, Δ, ϕ) an Ω-*completion of* P. A set $\mathcal{F}$ of patchworks is *well-behaved* if for every well-quasi-order Ω and every countable sequence P_i $(i = 1, 2, \dots)$ of Ω-completions of members of $\mathcal{F}$ there exist $j > i \geq 1$ such that P_i is simulated in P_j.

4.1 *If $\mathcal{F}$ is well-behaved, then there exists $N \geq 0$ such that if $(G, \mu, \Delta) \in \mathcal{F}$ and $e \in dom(\mu)$ then $|\bar{\mu}(e)| \leq N$.*

Proof. Let Ω be the well-quasi-order with $E(\Omega) = \{\omega_1, \omega_2\}$ say, where ω_1, ω_2 are incomparable (that is, $\omega_1 \not\leq \omega_2 \not\leq \omega_1$). Suppose that there is no N as in the theorem. Then there exist integers n_i and $P_i = (G_i, \mu_i, \Delta_i) \in \mathcal{F}$ and $e_i \in E(G_i) \cap dom(\mu_1)$ with $|\bar{\mu}_i(e_i)| = n_i$ for $i = 1, 2, \dots$, such that $n_1 < n_2 < \dots$. For $i \geq 1$, define $\phi_i : E(G_i) \to E(\Omega)$ by $\phi_i(e_i) = \omega_2$ and $\phi_i(e) = \omega_1 (e \neq e_i)$. Then $(G_i, \mu_i, \Delta_i, \phi_i)$ $(= Q_i$, say) is an Ω-completion of P_i. Since $\mathcal{F}$ is well-behaved, there exist $j > i \geq 1$ such that there is a realizable expansion η of Q_i in Q_j. Consequently

$$\omega_2 = \phi_i(e_i) \leq \phi_j(\eta(e_i))$$

and so $\phi_j(\eta(e_i)) = \omega_2$, that is, $\eta(e_i) = e_j$. But $e_i \in dom(\mu_i)$, and so $\mu_i(e_i)$ and $\mu_j(\eta(e_i))$ have the same length; that is,

$$n_i = |\bar{\mu}_i(e_i)| = |\bar{\mu}_j(\eta(e_i))| = |\bar{\mu}_j(e_j)| = n_j,$$

a contradiction. The result follows. ■

Let Ω_1, Ω_2 be quasi-orders, and let $\mathcal{F}_i$ be a set of Ω_i-patchworks $(i = 1, 2)$. A function $\gamma : \mathcal{F}_2 \to \mathcal{F}_1$ is an *encoding* of $\mathcal{F}_2$ in $\mathcal{F}_1$ if P is simulated in P' for all P, $P' \in \mathcal{F}_2$ such that $\gamma(P)$ is simulated in $\gamma(P')$. The following is a convenient lemma for producing new well-behaved sets of patchworks.

4.2 *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be sets of patchworks where $\mathcal{F}_1$ is well-behaved. Suppose that for every well-quasi-order Ω_2 there is a well-quasi-order Ω_1 and an encoding of the set of all Ω_2-completions of members of $\mathcal{F}_2$ in the set of all Ω_1-completions of members of $\mathcal{F}_1$. Then $\mathcal{F}_2$ is well-behaved.*

The proof is clear.

4.3 *Let $\mathcal{F}_1$ be a well-behaved set of patchworks. Let $\mathcal{F}_2$ be the set of all patchworks $P_2 = (G_2, \mu, \Delta)$ such that there exist $(G_1, \mu, \Delta) \in \mathcal{F}_1$ and $v \in V(G_1) \setminus \bar{\pi}(G_1)$ such that $G_2^- = G_1^-$ and $\pi(G_2)$ is the concatenation of $\pi(G_1)$ with a new last term v and v is incident with some edge $e \in dom(\mu)$. Then $\mathcal{F}_2$ is well-behaved.*

Proof. Choose N as in 4.1 (with $\mathcal{F}$ replaced by $\mathcal{F}_1$). For $1 \leq r \leq N$, let C^r be the set of those patchworks $P_2 = (G_2, \mu, \Delta) \in \mathcal{F}_2$ such that v, e may be chosen as above with v the rth term of $\mu(e)$. Since $\mathcal{F}_2 = \mathcal{F}^1 \cup \dots \cup \mathcal{F}^N$ and the union of finitely many well-behaved sets is well-behaved, it suffices to show that $\mathcal{F}^r$ is well-behaved for each r.

Let Ω_2 be a well-quasi-order. Let Ω_3 be an isomorphic copy of Ω_2 with $E(\Omega_2) \cap E(\Omega_3) = \emptyset$, and let $\lambda = \Omega_2 \to \Omega_3$ be an isomorphism. Let $\Omega_1 = \Omega_2 \cup \Omega_3$. Let $Q_2 = (G_2, \mu, \Delta, \phi_2)$ be an Ω_2-completion of a member $P_2 = (G_2, \mu, \Delta)$ of $\mathcal{F}^r$. Let v be the last term of $\pi(G_2)$, and let G_1 be the hypergraph with

$G_1^- = G_2^-$ and $\pi(G_1)$ the sequence obtained from $\pi(G_2)$ by deleting v. Then $P_1 = (G_1, \mu, \Delta) \in C_1$. Choose $f \in dom(\mu)$ such that v is the rth term of $\mu(f)$. Define an Ω_1-completion $Q_1 = (G_1, \mu, \Delta, \phi_1)$ of P_1 as follows:

$$\phi_1(e) = \phi_2(e) \ (e \in E(G_1) \setminus \{f\})$$
$$\phi_1(f) = \lambda(\phi_2(f)).$$

We define $\lambda(Q_2) = Q_1$, and claim that γ is an encoding. For suppose that $\gamma(Q_2') = Q_1'$, where $Q_2' = (G_2', \mu', \Delta', \phi_2')$ etc., and η is a realizable expansion of Q_1 in Q_1'. Then $\eta(f) = f'$, since f' is the only edge e of Q_2' with $\phi_1'(e) \in E(\Omega_3)$. Since $f \in dom(\mu)$ and $f' \in dom(\mu')$ it follows that $v' \in \eta(v)$, and hence η is a realizable expansion of Q_2 in Q_2', as required. Thus γ is an encoding, and the theorem follows from 4.2. ■

4.4 *Let $\mathcal{F}_1$ be a well-behaved set of patchworks, and let $\mathcal{F}_2$ be the set of all rootless patchworks $P_2 = (G_2, \mu, \Delta)$ such that there exists $(G_1, \mu, \Delta) \in \mathcal{F}_1$ with $G_1^- = G_2^-$. Then $\mathcal{F}_2$ is well-behaved.*

The proof is clear (for any realization expansion of one patchwork in another is a realizable expansion of the corresponding patchworks with roots forgotten).

A patchwork (G, μ, Δ) is *active* if every vertex of G is incident with some $e \in dom(\mu)$.

4.5 *Let $\mathcal{F}_1$ be a well-behaved set of active patchworks, let $k \geq 0$ and let $\mathcal{F}_2$ be the set of all patchworks (G_2, μ, Δ) such that $|\bar{\pi}(G_2)| \leq k$ and there exists $(G_1, \mu, \Delta) \in \mathcal{F}_1$ with $G_1^- = G_2^-$. Then $\mathcal{F}_2$ is well-behaved.*

Proof. It suffices to prove that $\{(G_2, \mu, \Delta) \in \mathcal{F}_2 : |\bar{\pi}(G_2)| = k'\}$ is well-behaved, for each k' with $0 \leq k' \leq k$. For $k' = 0$ this follows from 4.4, and in general by induction on k' from 4.3. ■

4.6 *Let $\mathcal{F}_1$ be a well-behaved set of patchworks, and let $\mathcal{F}_2$ be a set of patchworks such that for each $P_2 = (G_2, \mu_2, \Delta_2) \in \mathcal{F}_2$ there exists $f \in E(G_2)$ such that $P_2 \backslash \{f\} \in \mathcal{F}_1$ and every end of f belongs to $\bar{\pi}(G_2)$. Then $\mathcal{F}_2$ is well-behaved.*

The proof is clear.

Let $P_1 = (G_1, \mu_1, \Delta_1)$ be a patchwork and $f \in dom(\mu_1)$. Take a new vertex v and let G_2 be the rooted hypergraph with $\pi(G_2) = \pi(G_1)$, $E(G_2) = E(G_1)$, $V(G_2) = V(G_1) \cup \{v\}$ where f is incident with v but otherwise the incidence relation is the same as for G. Let $\mu_2(f)$ be an arbitrary march and let $\Delta_2(f)$ be an arbitrary patch, except that $\bar{\mu}(f)$, $V(\Delta_2(f))$ are both the set of ends of f in G_2. For $e \in dom(\mu_1) \setminus \{f\}$ let $\mu_2(e) = \mu_1(e)$, and for $e \in E(G_1) \setminus \{f\}$ let $\Delta_2(e) = \Delta_1(e)$. Then (G_2, μ_2, Δ_2) is a patchwork, which we say is a *1-vertex extension* of (G_1, μ_1, Δ_1).

4.7 *Let $\mathcal{F}_1$ be a well-behaved set of patchworks and let $\mathcal{F}_2$ be a set of patchworks each of which is a 1-vertex extension of a member of $\mathcal{F}_1$. Then $\mathcal{F}_2$ is well-behaved.*

Proof. Let Ω_2 be a well-quasi-order, and let $N \geq 0$ be an integer such that for every $(G, \mu, \Delta) \in \mathcal{F}_1$ and every $e \in dom(\mu)$, e has $\leq N$ ends. Let Ω be the well-quasi-order with $E(\Omega)$ the set of all Ω_2-patchworks (G, μ, Δ, ϕ) with $|E(G)| = 1$ and $|V(G)| \leq N+1$, ordered by simulation. (Evidently, this is indeed a well-quasi-order.) We may assume that $E(\Omega) \cap E(\Omega_2) = \emptyset$. Let $\Omega_1 = \Omega \cup \Omega_2$.

Let $Q_2 = (G_2, \mu_2, \Delta_2, \phi_2)$ be an Ω_2-completion of a member of $\mathcal{F}_2$. Choose $(G_1, \mu_1, \Delta_1) \in \mathcal{F}_1$ and $f \in dom(\mu_1)$ and $v \in V(G_2)$, as in the definition of 1-vertex extension. Let $Q_1 = (G_1, \mu_1, \Delta_1, \phi_1)$ be the Ω_1-completion of (G_1, μ_1, Δ_1) where

$$\phi_1(e) = \phi_2(e) \ (e \in E(G_1) \setminus \{f\})$$
$$\phi_1(f) = Q_2|H$$

where H is the rooted hypergraph such that $H \subseteq G_2$, $\pi(H) = \mu_1(f)$, $E(H) = \{f\}$, and $V(H)$ is the set of ends of f in G_2. Let us define $Q_1 = \lambda(Q_2)$; then it is easy to see that λ is an encoding, and the result follows from 4.2. ■

4.8 *Let $\mathcal{F}_1$ be a well-behaved set of patchworks, let $k \geq 0$, and let $\mathcal{F}_2$ be the set of all patchworks P_2 such that there exist $P_1 \in \mathcal{F}_1$ and a sequence*

$$P_1 = P^0, P^1, \ldots, P^{k'} = P_2$$

where $k' \leq k$ and for $1 \leq i \leq k'$, P^i is a 1-vertex extension of P^{i-1}. Then $\mathcal{F}_2$ is well-behaved.

Proof. Let us express $\mathcal{F}_2 = \mathcal{F}^0 \cup \mathcal{F}^1 \cup \cdots \cup \mathcal{F}^k$, where for $P_2 \in \mathcal{F}^i$ the k' above can be chosen with $k' = i$. By repeated use of 4.7, $\mathcal{F}^{k'}$ is well-behaved for each k', and hence $\mathcal{F}_2$ is well-behaved. ■

If G is a hypergraph and $W \subseteq V(G)$, G/W denotes the hypergraph G' with $V(G') = V(G) \setminus W$ and $E(G') = E(G)$, in which $v \in V(G) \setminus W$ and $e \in E(G)$ are incident if and only if they are incident in G. If π is a march in a set V and $W \subseteq V$, π/W denotes the march obtained by omitting all terms in W. If G is a rooted hypergraph and $W \subseteq V(G)$, G/W denotes $(G^-/W, \pi(G), W)$. If $P = (G, \mu, \Delta)$ is a patchwork and $W \subseteq V(G)$, P/W denotes the patchwork $(G/W, \mu', \Delta')$ where for $e \in dom(\mu)$, $\mu'(e) = \mu(e)/W$, and for $e \in E(G)$, if Z denotes the set of ends of e in G then $\Delta'(e)$ consists of all groupings K' with vertex set $Z \setminus W$ such that $K' \cup N_{W \cap Z} \in \Delta(e)$. If $P = (G, \mu, \Delta, \phi)$ is an Ω-patchwork and $W \subseteq V(G)$, P/W denotes the Ω-patchwork $(G/W, \mu', \Delta', \phi)$, where μ', Δ' are as before.

4.9 *Let $\mathcal{F}_1$ be a well-behaved set of patchworks, let $\theta \geq 0$, and let $\mathcal{F}_2$ be the set of all patchworks $P_2 = (G_2, \mu_2, \Delta_2)$ such that $dom(\mu_2) = E(G_2)$ and there exists $W \subseteq V(G_2)$ with $|W| \leq \theta$ and $P_2/W \in \mathcal{F}_1$. Then $\mathcal{F}_2$ is well-behaved.*

Proof. It suffices (by induction on $|W|$) to prove this when for each $P_2 = (G_2, \mu_2, \Delta_2) \in \mathcal{F}_2$ there exists $v \in V(G_2)$ such that $P_2/\{v\} \in \mathcal{F}_1$. Let Ω_2 be a well-quasi-order and define N, Ω, Ω_1 as in the proof of 4.7. Let $Q_2 = (G_2, \mu_2, \Delta_2, \phi_2)$ be an Ω_2-completion of a member P_2 of $\mathcal{F}_2$, and choose $v \in V(G_2)$ such that $P_2/\{v\} = P_1 \in \mathcal{F}_1$. Let $P_1 = (G_1, \mu_1, \Delta_1)$ and let Q_1 be the Ω_1-completion $(G_1, \mu_1, \Delta_1, \phi_1)$ of P_1 where

$$\phi_1(e) = \phi_2(e) \text{ if } e \in E(G_1) \text{ is not incident with } v \text{ in } G_2$$
$$\phi_1(e) = Q_2|H \text{ if } e \in E(G_1) \text{ is incident with } v \text{ in } G_2$$

where in the second case, H is the rooted hypergraph such that $H \subseteq G_2$, $\pi(H) = \mu_2(e)$, $E(H) = \{e\}$ and $V(H)$ is the set of ends of e in G_2. Let us define $\gamma(Q_2) = Q_1$; then it is easy to see that γ is an encoding and the result follows from 4.2. ■

Let $P_1 = (G_1, \mu_1, \Delta_1)$ and $P_2 = (G_2, \mu_2, \Delta_2)$ be patchworks. We say that P_1 is a *condensation* of P_2 if $V(G_1) = V(G_2)$, $\pi(G_1) = \pi(G_2)$, $dom(\mu_1) = E(G_1)$, $dom(\mu_2) = E(G_2)$, for each $e \in E(G_1)$ there is a rooted subhypergraph $A_e \subseteq G_2$ with the following properties:

- $V(A_e)$ is the set of ends of e in G_1, and $\pi(A_e) = \mu_1(e)$
- $\bigcup_{e \in E(G_1)} E(A_e) = E(G_2)$
- for distinct e, $e' \in E(G_1)$, $E(A_e) \cap E(A_{e'}) = \emptyset$
- for each $e \in E(G_1)$ and $K \in \Delta_1(e)$, K is feasible in $P_2|A_e$.

A patchwork $P = (G, \mu, \Delta)$ is *removable* if for every $e \in E(G)$, $\Delta(e)$ contains N_V where V is the set of ends of e.

4.10 *Let $\mathcal{F}_1$ be a well-behaved set of removable patchworks and let $\mathcal{F}_2$ be a set of patchworks such that for each $P_2 \in \mathcal{F}_2$ some $P_1 \in \mathcal{F}_1$ is a condensation of P_2. Then $\mathcal{F}_2$ is well-behaved.*

Proof. Choose $N \geq 0$ (by 4.1) such that for every $(G, \mu, \Delta) \in \mathcal{F}_1$ and every $e \in dom(\mu)$, e has $\leq N$ ends. Now let Ω_2 be a well-quasi-order. Let Ω_1 be the well-quasi-order with $E(\Omega_1)$ the set of all Ω_2-patchworks (G, μ, Δ, ϕ) where $|V(G)| \leq N$, and (G, μ, Δ) is removable, ordered by simulation. (That Ω_1 is a well-quasi-order is proved in the same way as theorem 8.4 of [1] and we omit the proof.)

Now let $Q_2 = (G_2, \mu_2, \Delta_2, \phi_2)$ be an Ω_2-completion of some $P_2 \in \mathcal{F}_2$. Choose $P_1 = (G_1, \mu_1, \Delta_1) \in \mathcal{F}_1$ such that P_1 is a condensation of P_2, and choose the rooted subhypergraphs A_e ($e \in E(G_1)$) as in the definition of condensation. Let $Q_1 = (G_1, \mu_1, \Delta_1, \phi_1)$ be the Ω_1-completion of P_1 where $\phi_1(e) = Q_2|A_e$ for each $e \in dom(\mu_1) = E(G_1)$. Let $Q_1 = \gamma(Q_2)$; then theorem 5.7 of [4] implies that γ is an encoding, and the result follows. ■

4.11 *Let $\mathcal{F}_1$ be a well-behaved set of active patchworks, and let $\mathcal{F}_2$ be the set of all patchworks $P_2 = (G_2, \mu_2, \Delta_2)$ such that there exists $P_1 = (G_1, \mu_1, \Delta_1) \in \mathcal{F}_1$ with $G_2 \subseteq G_1$, $\pi(G_2) = \pi(G_1)$, G_2 complemented in G_1 and $P_2 = P_1|G_2$. Then $\mathcal{F}_2$ is well-behaved.*

Proof. Let Ω_2 be a well-quasi-order. Let $* \notin E(\Omega_2)$ be a new element and let Ω_1 be the well-quasi-order with $\Omega_2 \subseteq \Omega_1$ and $E(\Omega_1) = E(\Omega_2) \cup \{*\}$, where if $x \leq *$ or $* \leq x$ then $x = *$. Now let $Q_2 = (G_2, \mu_2, \Delta_2, \phi_2)$ be an Ω_2-completion of $P_2 = (G_2, \mu_2, \Delta_2) \in \mathcal{F}_2$. Choose $P_1 = (G_1, \mu_1, \Delta_1) \in \mathcal{F}_1$ so that $G_2 \subseteq G_1$, $\pi(G_2) = \pi(G_1)$, G_2 is complemented in G_1, and $P_2 = P_1|G_2$. Let $Q_1 = (G_1, \mu_1, \Delta_1, \phi_1)$ be the Ω-completion of P_1 where

$$\begin{aligned} \phi_1(e) &= \phi_2(e) \ (e \in E(G_2)) \\ &= * \ (e \in E(G_1) \setminus E(G_2)). \end{aligned}$$

Let $\gamma(Q_2) = Q_1$; we claim that γ is an encoding.

Let $Q'_i = (G'_i, \mu'_i, \Delta'_i, \phi'_i)$ $(i = 1, 2)$, such that $\gamma(Q'_2) = Q'_1$, and let η be a realizable expansion of Q_1 in Q'_1. We shall show that there is a realizable expansion of Q_2 in Q'_2. Define η_2 by

$$\begin{aligned} \eta_2(v) &= \eta(v) \cap V(G'_2) \ (v \in V(G_2)) \\ \eta_2(e) &= \eta(e) \ (e \in E(G_2)). \end{aligned}$$

(1) *For each $e \in E(G_2)$, $\eta_2(e) \in E(G'_2)$ and $\phi_2(e) \le \phi'_2(\eta_2(e))$.*

Subproof. Certainly $\phi_1(e) \le \phi'_1(\eta(e))$ and so $\phi'_1(\eta(e)) \neq *$, since $\phi_1(e) \neq *$; hence $\eta(e) \in E(G'_2)$ and the claim follows.

(2) *For each $v \in V(G_2)$, $\eta_2(v) \neq \emptyset$.*

Subproof. If $v \in \bar{\pi}(G_1)$ and v is the ith term of $\pi(G_1)$ say, then $\eta(v)$ contains the ith term of $\pi(G'_1)$, which belongs to $V(G'_2)$ since $\pi(G'_1) = \pi(G'_2)$. Thus we may assume that $v \notin \bar{\pi}(G_1)$. Since G_1 is active, there is an edge $e \in E(G_1)$ incident with v, and then $e \in E(G_2)$ since $v \notin \bar{\pi}(G_2)$ and G_2 is complemented in G_1. Then $\eta(e)$ is incident with a vertex of $\eta(v)$; but every end of $\eta(e)$ is in $V(G'_2)$ by (1), and so $\eta_2(v) \neq \emptyset$. This proves (2).

From (1) and (2) it is easy to verify that η_2 is an expansion of Q_2 in Q'_2. Now let H_1 be a realization of $Q'_1 \backslash \eta(E(G_1))$ realizing η. Let $G'_3 = G'_1 \setminus G'_2$. Then $H_1 = H_2 \cup H_3$ where H_i is a realization of $(Q'_1 \backslash \eta(E(G_1)))|(G'_i \setminus (E(G'_i) \cap \eta(E(G_1))))$ $(i = 2, 3)$. Now for $e \in E(G_1)$

$$e \notin E(G_2) \Leftrightarrow \phi_1(e) = * \Leftrightarrow \phi_2(\eta(e)) = * \Leftrightarrow \eta(e) \notin E(G'_2)$$

and so $\eta(E(G_1)) \cap E(G'_2) = \eta(E(G_2))$. Hence

$$(Q'_1 \backslash \eta(E(G_1)))|(G'_2 \setminus (E(G'_2) \cap \eta(E(G_1)))) = Q'_2 \backslash \eta(E(G_2))$$

and so H_2 is a realization of $Q'_2 \backslash \eta(E(G_2))$. We claim that H_2 realizes η_2. For let $v \in V(G_2)$. We must show that $\eta_2(v)$ is the vertex set of a component of H_2. Let C_1 be a component of H_1 with $V(C_1) = \eta(v)$. Then $V(C_1)$ contains at most one vertex of $\bar{\pi}(G'_2)$, since $\pi(G'_2) = \pi(G'_1)$ and η is an expansion of Q_1 in Q'_1. Choose $C_2 \subseteq H_2$, $C_3 \subseteq H_3$ such that $C_1 = C_2 \cup C_3$, with $V(C_i) = V(C_1) \cap V(H_i)$ $(i = 2, 3)$. Since C_3 contains at most one vertex of $\bar{\pi}(G'_2)$ and G'_3 is a complement of G'_2, it follows that $|V(C_2 \cap C_3)| \le 1$ and hence C_2 is connected, and is therefore a component of H_2, since

$$V(C_2) = V(C_1) \cap V(H_2) = \eta_2(v) \neq \emptyset.$$

This proves that H_2 realizes η_2, and completes the proof of the theorem. ■

4.12 *Let $\mathcal{F}_1$ be a well-behaved set of active patchworks, let $k \ge 0$, and let $\mathcal{F}_2$ be a set of patchworks such that for each $P_2 = (G_2, \mu_2, \Delta_2) \in \mathcal{F}_2$ there exists $f \in dom(\mu_2)$ with $\le k$ ends and $P_1 = (G_1, \mu_1, \Delta_1) \in \mathcal{F}_1$ so that $G_2 \backslash f = G_1 \backslash A$ and $\pi(A) = \mu_2(f)$ for some complemented rooted hypergraph $A \subseteq G_1$, and $P_2 \backslash \{f\} = P_1|(G_2 \backslash \{f\})$. Then $\mathcal{F}_2$ is well-behaved.*

Proof. Let $\mathcal{F}_3$ be the set of all patchworks (G_3, μ_3, Δ_3) such that $|\bar{\pi}(G_3)| \le k$ and there exists a march π such that $((G_3^-, \pi), \mu_3, \Delta_3) \in \mathcal{F}_1$. By 4.5, $\mathcal{F}_3$ is well-behaved. Let $\mathcal{F}_4$ be related to $\mathcal{F}_3$ as $\mathcal{F}_2$ is to $\mathcal{F}_1$ in 4.11. By 4.11, $\mathcal{F}_4$ is well-behaved. Let $\mathcal{F}_5$ be related to $\mathcal{F}_4$ as $\mathcal{F}_2$ is to $\mathcal{F}_1$ in 4.6. By 4.6, $\mathcal{F}_5$ is well-behaved. We claim that $\mathcal{F}_2 \subseteq \mathcal{F}_5$; for let $P_2 = (G_2, \mu_2, \Delta_2) \in \mathcal{F}_2$, and let f, P_1 be as in the statement of the theorem. Then $((G_1^-, \mu_2(f)), \mu_1, \Delta_1) \in \mathcal{F}_3$, and so $P_2 \setminus f \in \mathcal{F}_4$, and therefore $P_2 \in \mathcal{F}_5$. This proves that $\mathcal{F}_2 \subseteq \mathcal{F}_5$, and the result follows. ■

By w applications of 4.12, we deduce

4.13 *Let $\mathcal{F}_1$ be a well-behaved set of active patchworks, let k, $w \geq 0$ and let $\mathcal{F}_2$ be a set of patchworks such that for each $P_2 = (G_2, \mu_2, \Delta_2) \in \mathcal{F}_2$ there exists $F \subseteq dom(\mu_2)$ with $|F| \leq w$ and $P_1 = (G_1, \mu_1, \Delta_1) \in \mathcal{F}_1$, and a rooted location $\mathcal{L} = \{A_f : f \in F\}$ in G_1, such that*

- $G_2^- \backslash F = G_1^- \cap \bigcap((G_1 \setminus A)^- : A \in \mathcal{L})$
- $P_2 \backslash F = P_1 | (G_2 \backslash F)$, *and*
- *for each $f \in F$, $\pi(A_f) = \mu_2(f)$ and f has $\leq k$ ends.*

Then $\mathcal{F}_2$ is well behaved.

5 Isolation modulo a subset

In the previous section we gave several ways to construct new well-behaved sets from old. Now we use these constructions to begin to bridge the gap between what is given by the theorem of [3] and what is required by 3.1.

If G is a hypergraph or rooted hypergraph, we denote $V(G) \cup E(G)$ by $Z(G)$. Let $\mathcal{T}$ be a tangle in a hypergraph G, let λ be a tie-breaker in G, let $\theta \geq 1$, and let $W \subseteq Z(G)$. We define $\mathcal{M}(\mathcal{T}, W, \theta)$ to be the set of all separations $(A, B) \in \mathcal{T}$ such that

- (A, B) has order $< \theta$ and $W \not\subseteq Z(B)$
- (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction for some tangle $\mathcal{T}'$
- there is no $(A', B') \in \mathcal{T}$ with $(A', B') \neq (A, B)$ satisfying the first two conditions with $A \subseteq A'$ and $B' \subseteq B$.

5.1 *Let $(C, D) \in \mathcal{M}(\mathcal{T}, W, \theta)$, and let (A, B) be the $(\mathcal{T}, \mathcal{T}')$-distinction for some tangle $\mathcal{T}'$. Then either $A \subseteq C$ and $D \subseteq B$, or $A \subseteq D$ and $C \subseteq B$, or $C \subseteq A$ and $B \subseteq D$, and if (A, B) has order $< \theta$ then one of the first two alternatives holds.*

Proof. By theorems 9.4 and 10.2 of [2], either one of these three alternatives holds or $D \subseteq A$ and $B \subseteq C$; and this last is impossible since (A, B), $(C, D) \in \mathcal{T}$. If (A, B) has order $< \theta$ then the third alternative also is impossible, because of the third condition in the definition of $\mathcal{M}(\mathcal{T}, W, \theta)$, unless $(A, B) = (C, D)$ when the first alternative holds as well. ■

5.2 *If (A, B), $(A', B') \in \mathcal{M}(\mathcal{T}, W, \theta)$ are distinct then $A \subseteq B'$; and $\mathcal{M}(\mathcal{T}, W, \theta)$ has cardinality $\leq |W|$.*

Proof. Suppose that $A \not\subseteq B'$. By 5.1, $A \subseteq A'$ and $B' \subseteq B$, since (A, B) has order $< \theta$ and (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction for some $\mathcal{T}'$. Similarly, with (A, B) and (A', B') exchanged, it follows that $A' \subseteq A$ and $B \subseteq B'$. But then $(A, B) = (A', B')$, a contradiction. This proves the first claim.

From this, it follows that

$$E(A) \cup (V(G) \setminus V(B)) \quad ((A, B) \in \mathcal{M}(\mathcal{T}, W, \theta))$$

are mutually disjoint, and each contains a member of W. It follows that $|\mathcal{M}(\mathcal{T}, W, \theta)| \leq |W|$, as required. ■

If $\mathcal{T}$ is a tangle in $G, \theta \geq 1$ is an integer, λ is a tie-breaker in G and $W \subseteq Z(G)$, a rooted location $\mathcal{L}$ in G is said to *θ-isolate $\mathcal{T}$ modulo W* if $\mathcal{L}$ has order $< \theta$, $\mathcal{L}^- \subseteq \mathcal{T}$, and for each $A \in \mathcal{L}$ and every tangle $\mathcal{T}'$ in G of order $\geq \theta$ with $((G \setminus A)^-, A^-) \in \mathcal{T}'$, if (C, D) is the $(\mathcal{T}, \mathcal{T}')$-distinction then either $C \subseteq A^-$ and $(G \setminus A)^- \subseteq D$, or $W \not\subseteq Z(D)$.

A rooted location $\mathcal{L}$ in a rooted hypergraph G is *fine* if $\bigcup(A^- : A \in \mathcal{L}) = G^-$. Let $\theta \geq 1$ be an integer, let $P = (G, \mu, \Delta)$ be a patchwork, let λ be a tie-breaker in G, let $\mathcal{T}$ be a tangle in G of order $\geq \theta^2$, and let $W \subseteq Z(G)$ with $|W| \leq \theta$. In these circumstances, a rooted location $\mathcal{L}$ in G is said to be *W-suitable* if

- $\mathcal{L}$ is fine, and $\mathcal{L}^- \subseteq \mathcal{T}$, and $\mathcal{L}$ has order $< \theta^2$
- for each tangle $\mathcal{T}'$ in G of order $\geq \theta^2$, if $(C, D) \in \mathcal{L}$ and $(D, C) \in \mathcal{T}'$ and (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction then either $A \subseteq C$ and $D \subseteq B$, or $A \subseteq A^*$ and $B^* \subseteq B$ for some $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$.

5.3 *Let $\mathcal{F}$ be a well-behaved set of patchworks and let $\theta \geq 1$. Then there is a well-behaved set of patchworks $\mathcal{F}'$ with the following property. Let $P = (G, \mu, \Delta)$ be a patchwork, let λ be a tie-breaker in G, let $\mathcal{T}$ be a tangle in G of order $\geq \theta^2$, let $W \subseteq Z(G)$ with $|W| \leq \theta$, let $\mathcal{L}$ be a fine rooted location in G that θ-isolates $\mathcal{T}$ modulo W, and let $\mathcal{F}$ contain a heart of $(P, \mathcal{L})$. Then there is a rooted location $\mathcal{L}'$ in G and $W' \subseteq W$ such that*

- *$\mathcal{L}'$ is W'-suitable and $\mathcal{F}'$ contains a heart of $(P, \mathcal{L}')$,*
- *for each $(A, B) \in \mathcal{M}(\mathcal{T}, W', \theta)$,*
  - *$V(A \cap B) \cap V(C) \subseteq \bar{\pi}(C)$ for each $C \in \mathcal{L}'$, and*
  - *there is no $(C, D) \in \mathcal{L}'^-$ with $A \subseteq C$ and $D \subseteq B$.*

Proof. Let $\mathcal{F}'$ be related to $\mathcal{F}$ as $\mathcal{F}_2$ is to $\mathcal{F}_1$ in 4.8, where $k = \theta^2$. By 4.8, $\mathcal{F}'$ is well-behaved, and we claim that it satisfies the theorem. For let $P = (G, \mu, \Delta), \lambda, \mathcal{T}, W \subseteq Z(G)$ and $\mathcal{L}$ satisfy the hypotheses of the theorem. Choose $W' \subseteq W$ minimal such that $\mathcal{L}$ θ-isolates $\mathcal{T}$ modulo W'.

(1) *For each $(A, B) \in \mathcal{M}(\mathcal{T}, W', \theta)$, there is no $(C, D) \in \mathcal{L}^-$ with $A \subseteq C$ and $D \subseteq B$.*

Subproof. Let $(A, B) \in \mathcal{M}(\mathcal{T}, W', \theta)$ and suppose that there is such a (C, D). Since $(A, B) \in \mathcal{M}(\mathcal{T}, W', \theta)$ there is a tangle $\mathcal{T}'$ such that (A, B) is the $(\mathcal{T}, \mathcal{T}')$-distinction, and there exists $z \in W' \setminus Z(B)$. Now from the minimality of W', $\mathcal{L}$ does not θ-isolate $\mathcal{T}$ modulo $W' \setminus \{z\}$, and so there exists $(C', D') \in \mathcal{L}^-$ and a tangle $\mathcal{T}''$ in G of order $\geq \theta$ with $(D', C') \in \mathcal{T}''$ with the property that $W' \setminus \{z\} \subseteq Z(B')$ and not both $A' \subseteq C'$ and $D' \subseteq B'$, where (A', B') is the $(\mathcal{T}, \mathcal{T}'')$-distinction. Since $\mathcal{L}$ does θ-isolate $\mathcal{T}$ modulo W', it follows that $W' \not\subseteq Z(B')$ and so $z \notin Z(B')$. Hence $B \cup B' \neq G$. Moreover, since $(A, B) \in \mathcal{M}(\mathcal{T}, W', \theta)$, it follows that not both $A \subseteq A'$ and $B' \subseteq B$, from the third condition in the definition of $\mathcal{M}(\mathcal{T}, W', \theta)$. From 5.1, $A' \subseteq A$ and $B \subseteq B'$. Now $A \subseteq C$ and $D \subseteq B$, and so $A' \subseteq C$ and $D \subseteq B'$; and hence $(C, D) \neq (C', D')$, since not both $A' \subseteq C'$ and $D' \subseteq B'$. Moreover, $(B', A') \in \mathcal{T}''$, and $A' \subseteq C$ and $D \subseteq B'$, and so $(D, C) \in \mathcal{T}''$ since (D, C) has order $< \theta$ and $\mathcal{T}''$ has order $\geq \theta$. Since $\mathcal{L}$ is a rooted location and (C, D), $(C', D') \in \mathcal{L}^-$ it follows that $D \cup D' = G^-$. But (D, C), $(D', C') \in \mathcal{T}''$ contrary to the second axiom for tangles. This proves (1).

Let $X = \bigcup(V(A \cap B) : (A, B) \in \mathcal{M}(\mathcal{T}, W', \theta))$. Since $|\mathcal{M}(\mathcal{T}, W', \theta))| \leq |W'| \leq |W| \leq \theta$ by 5.2, it follows that $|X| \leq \theta(\theta - 1)$. For each $C \in \mathcal{L}$, let $f(C)$ be a rooted hypergraph with $f(C)^- = C^-$ and $\bar{\pi}(f(C)) = \bar{\pi}(C) \cup (X \cap V(C))$, taking $f(C) = C$ if $X \cap V(C) \subseteq \bar{\pi}(C)$. Let $\mathcal{L}' = \{f(C) : C \in \mathcal{L}\}$. Then $\mathcal{L}'$ is a fine rooted location, and $\mathcal{L}'$ has order at most $\theta(\theta - 1)$ more than the order of $\mathcal{L}$, and hence at most $\theta^2 - 1$. We observe

(2) *For each $(C', D') \in \mathcal{L}'^-$ there exists $(C, D) \in \mathcal{L}^-$ with $C = C'$ and $D \subseteq D'$; and $E(D') = E(D)$, and $V(D') \setminus V(D) = X \cap (V(C) \setminus \bar{\pi}(C))$.*

Since $|X| \leq \theta(\theta - 1)$ and each $x \in X$ belongs to $V(C) \setminus \bar{\pi}(C)$ for at most one $C \in \mathcal{L}$, we see that $\mathcal{F}'$ contains a heart of $(P, \mathcal{L}')$, from the definition of $\mathcal{F}'$. Since $\mathcal{T}$ has order $\geq \theta^2$ and $\mathcal{L}^- \subseteq \mathcal{T}$ it follows from (2) that $\mathcal{L}'^- \subseteq \mathcal{T}$. To verify that $\mathcal{L}'$ is W'-suitable, let $\mathcal{T}'$ be a tangle of order $\geq \theta^2$, let $(C', D') \in \mathcal{L}'^-$ with $(D', C') \in \mathcal{T}'$, and let (A, B) be the $(\mathcal{T}, \mathcal{T}')$-distinction. We may assume that:

(3) *There is no $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W', \theta)$ such that $A \subseteq A^*$ and $B^* \subseteq B$.*

We must therefore show that $A \subseteq C'$ and $D' \subseteq B$. Choose (C, D) as in (2). Then (A, B) has order at most that of (C, D), and hence $< \theta$. If $W' \not\subseteq Z(B)$, then from the definition of $\mathcal{M}(\mathcal{T}, W', \theta)$, there exists some $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W', \theta)$ violating (3); so $W' \subseteq Z(B)$. Since $\mathcal{L}$ θ-isolates $\mathcal{T}$ modulo W' and $(D, C) \in \mathcal{T}'$, it follows that $A \subseteq C$ and $D \subseteq B$. Since $C = C'$ it remains to show that $D' \subseteq B$. Let $v \in V(D') \setminus V(D)$. Then $v \in X$, and so $v \in V(A^* \cap B^*)$ for some $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W', \theta)$. By 5.1, (3) and the third condition in the definition of $\mathcal{M}(\mathcal{T}, W', \theta)$, it follows that $A \subseteq B^*$ and $A^* \subseteq B$; and in particular $v \in V(B)$. Consequently $V(D') \setminus V(D) \subseteq V(B)$; and since $E(D') = E(D)$ and $D \subseteq B$, it follows that $D' \subseteq B$ as required. This proves that $\mathcal{L}'$ is W'-suitable. The final statement holds because of (1) and the definition of $\mathcal{L}'$. ∎

If x, y are vertices of a graph H, we say they are *connected in H* if they belong to the same connected component of H.

5.4 *Let $\mathcal{F}$ be a well-behaved set of patchworks, and let $\theta \geq 1$. Then there is a well-behaved set of patchworks $\mathcal{F}'$ with the following property. Let $P = (G, \mu, \Delta)$ be a patchwork, let λ be a tie-breaker in G, let $\mathcal{T}$ be a tangle in G of order $\geq \theta^2$, and let $W \subseteq Z(G)$ with $|W| \leq \theta$. Suppose that:*

- *P is removable,*
- *$\mathcal{L}$ is a W-suitable rooted location in G, such that $\mathcal{F}$ contains a heart of $(P, \mathcal{L})$,*
- *for each $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$,*
  - *$V(A^* \cap B^*) \cap V(C) \subseteq \bar{\pi}(C)$ for each $C \in \mathcal{L}$, and*
  - *there is no $(C, D) \in \mathcal{L}^-$ with $A^* \subseteq C$ and $D \subseteq B^*$.*

Then there is a W-suitable rooted location $\mathcal{L}'$ in G such that $\mathcal{F}'$ contains a heart of $(P, \mathcal{L}')$, and for each $C \in \mathcal{L}'$ and each $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$, either $C^- \subseteq A^*$ and $B^* \subseteq (G \setminus C)^-$, or $C^- \subseteq B^*$ and $A^* \subseteq (G \setminus C)^-$.

Proof. Let $\mathcal{F}'$ be the set of all removable patchworks P' such that some $P \in \mathcal{F}$ is a condensation of P'. By 4.10, $\mathcal{F}'$ is well-behaved, and we claim the theorem is satisfied. For let $P = (G, \mu, \Delta), \lambda, \mathcal{T}, W \subseteq Z(G), \mathcal{L}$ be as in the theorem. Let

$$\mathcal{M}(\mathcal{T}, W, \theta) = \{(A_i, B_i) : 1 \leq i \leq k\}.$$

Let $A_0 = G^- \cap B_1 \cap \cdots \cap B_k$, $B_0 = A_1 \cup \cdots \cup A_k$. Then (A_0, B_0) is a separation of G^-. For each $C \in \mathcal{L}$ and $0 \leq i \leq k$ let $f_i(C)$ be a rooted hypergraph with $f_i(C)^- = C^- \cap A_i$ and $\bar{\pi}(f_i(C)) = \bar{\pi}(C) \cap V(A_i)$.

(1) *For each $C \in \mathcal{L}$,*

- $C^- = f_0(C)^- \cup f_1(C)^- \cup \cdots \cup f_k(C)^-$,
- $\bar{\pi}(C) = \bar{\pi}(A_0) \cup \bar{\pi}(A_1) \cup \cdots \bar{\pi}(A_k)$,
- *for* $1 \leq i \leq k$, $f_i(C)^- \subseteq A_i$ *and* $B_i \subseteq (G \setminus f_i(C))^-$, *and*
- *for* $0 \leq i < j \leq k$, $V(f_i(C)) \cap V(f_j(C)) \subseteq \bar{\pi}(f_i(C)) \cap \bar{\pi}(f_j(C))$.

Subproof. The first two statements follow since $A_0^- \cup A_1^- \cup \cdots \cup A_k^- = G^-$. For the third, let $1 \leq i \leq k$. Then $(f_i(C))^- \subseteq A_i$ by definition, and so $E(B_i) \subseteq E(G \setminus f_i(C))$; it remains to prove the same inclusion for vertex sets. Let $v \in V(B_i)$, and suppose for a contradiction that $v \notin V(G \setminus f_i(C))$. Thus $v \in V(f_i(C)) \setminus \bar{\pi}(f_i(C))$. Consequently $v \in V(A_i \cap B_i) \subseteq \bar{\pi}(C)$, and yet $V(f_i(C)) \cap \bar{\pi}(C) = \bar{\pi}(f_i(C))$, a contradiction. This proves the third statement. For the fourth, let $0 \leq i < j \leq k$, and let $v \in V(f_i(C)) \cap V(f_j(C))$. Then $v \in V(C) \cap V(A_i) \cap V(A_j) \subseteq V(C) \cap V(A_j \cap B_j)$, and since $j \geq 1$ it follows from the hypothesis that $v \in \bar{\pi}(C)$. Consequently $v \in \bar{\pi}(f_i(C)) \cap \bar{\pi}(f_j(C))$. This proves (1).

(2) *Let $C \in \mathcal{L}$ and let K be a grouping feasible in $P|C$. Then there are groupings K_i feasible in $P|f_i(C)$ $(0 \leq i \leq k)$ such that for distinct x, $y \in \bar{\pi}(C)$, x and y are adjacent in K if and only if x and y are connected in $K_0 \cup K_1 \cup \cdots \cup K_k$.*

Subproof. Let H be a realization of $P|C$ such that for distinct x, $y \in \bar{\pi}(C)$, x and y are adjacent in K if and only if x and y are connected in H. Then $H = H_0 \cup H_1 \cup \cdots \cup H_k$ where H_i is a realization of $P|f_i(C)$ $(0 \leq i \leq k)$ by (1). Let K_i be the grouping with $V(K_i) = \bar{\pi}(f_i(C))$ such that distinct x, $y \in \bar{\pi}(f_i(C))$ are adjacent in K_i if and only if they are connected in H_i. By $k+1$ applications of (1) and theorem 5.1 of [4], distinct x, $y \in \bar{\pi}(C)$ are connected in H if and only if they are connected in $K_0 \cup K_1 \cup \cdots \cup K_k$. This proves (2).

Let $\mathcal{L}' = \{f_i(C) : C \in \mathcal{L}, 0 \leq i \leq k\}$. Then by (1), $\mathcal{L}'$ is a fine rooted location in G, and $\mathcal{L}' \subseteq \mathcal{T}$, and $\mathcal{L}'$ has order at most that of $\mathcal{L}$, and hence $< \theta^2$.

To verify that $\mathcal{L}'$ is W-suitable, take a member of $\mathcal{L}'^-$, say $(f_h(C), G \setminus f_h(C))$ where $C \in \mathcal{L}$ and $0 \leq h \leq k$. Let $\mathcal{T}'$ be a tangle in G of order $\geq \theta^2$ such that $(G \setminus f_h(C), f_h(C)) \in \mathcal{T}'$, and let (A', B') be the $(\mathcal{T}, \mathcal{T}')$-distinction. We will show that either

- $h = 0$ and $A' \subseteq f_0(C)$ and $G \setminus f_0(C) \subseteq B'$, or
- $A' \subseteq A_i$ and $B_i \subseteq B'$ for some i with $1 \leq i \leq k$.

Since $(G \setminus f_h(C), f_h(C)) \in \mathcal{T}'$ it follows that $((G \setminus C)^-, C^-) \in \mathcal{T}'$ since $\mathcal{T}'$ has order $\geq \theta^2$ and $|\bar{\pi}(C)| < \theta^2$. We may assume that $A' \subseteq C^-$ and $(G \setminus C)^- \subseteq B'$, since otherwise the second alternative above holds because $\mathcal{L}$ is W-suitable. For $1 \leq i \leq k$ it is not true that $A_i \subseteq A'$ and $B' \subseteq B_i$, since that would imply that $A_i \subseteq C^-$ and $(G \setminus C)^- \subseteq B_i$ contrary to the hypothesis. We may also assume it is not true that $A' \subseteq A_i$ and $B_i \subseteq B'$, since otherwise we are done. By 5.1 it follows that $A_i \subseteq B'$ and $A' \subseteq B_i$ for $1 \leq i \leq k$, and hence $A' \subseteq A_0$ and $B_0 \subseteq B'$. Since $(B', A'), (G \setminus f_h(C), f_h(C)) \in \mathcal{T}'$, it follows that $f_h(C) \not\subseteq B'$, and so $f_h(C) \not\subseteq B_0$. Consequently $h = 0$, and the first alternative above holds, as required. This proves that $\mathcal{L}'$ is W-suitable.

From (2) and the facts that P is removable and $\mathcal{L}$, $\mathcal{L}'$ are both fine (and hence their hearts (G_1, μ_1, Δ_1), (G_2, μ_2, Δ_2) satisfy $dom(\mu_i) = E(G_i)$ $(i = 1, 2)$), it follows that $\mathcal{F}'$ contains a heart of $(P, \mathcal{L}')$. Let $C \in \mathcal{L}$ and $0 \leq i \leq k$. For $1 \leq j \leq k$, if $i = j$ then $f_i(C)^- \subseteq A_i = A_j$ and $B_j = B_i \subseteq (G \setminus f_i(C))^-$; and if $i \neq j$ then $f_i(C)^- \subseteq A_i \subseteq B_j$ and $A_j \subseteq B_i \subseteq (G \setminus f_i(C))^-$. This proves 5.4. ■

5.5 *Let $\mathcal{F}$ be a well-behaved set of patchworks and let $\theta \geq 1$. Then there is a well-behaved set of patchworks $\mathcal{F}'$ with the following property. Let $P = (G, \mu, \Delta)$ be a patchwork, let λ be a tie-breaker in G, let $\mathcal{T}$ be a tangle in G of order $\geq \theta^2$, and let $W \subseteq Z(G)$ with $|W| \leq \theta$. Suppose that*

- *P is rootless*
- *$\mathcal{L}$ is a W-suitable rooted location in G such that $\mathcal{F}$ contains a heart of $(P, \mathcal{L})$, and*
- *for each $C \in \mathcal{L}$ and each $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$, either*
  - *$C^- \subseteq A^*$ and $B^* \subseteq (G \setminus C)^-$, or*
  - *$C^- \subseteq B^*$ and $A^* \subseteq (G \setminus C)^-$.*

Then there is a fine rooted location $\mathcal{L}'$ such that $\mathcal{L}'$ θ^2-isolates $\mathcal{T}$ and $\mathcal{F}'$ contains a heart of $(P, \mathcal{L}')$.

Proof. Let $\mathcal{F}_1$ be the set of active members of $\mathcal{F}$, and let $\mathcal{F}_2$ be defined as in 4.13, taking $k = w = \theta$. We claim that $\mathcal{F}_2$ satisfies the theorem. For let $P = (G, \mu, \Delta), \lambda, \mathcal{T}, W, \mathcal{L}$ be as above. Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ where $C \in \mathcal{L}$ belongs to $\mathcal{L}_2$ if and only if there exists $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$ with $C^- \subseteq A^*$ and $B^* \subseteq (G \setminus C)^-$, and $\mathcal{L}_1 = \mathcal{L} \setminus \mathcal{L}_2$. For each $(A, B) \in \mathcal{M}(\mathcal{T}, W, \theta)$, let $f(A, B)$ be a rooted hypergraph with $f(A, B)^- = A$ and $\bar{\pi}(f(A, B)) = V(A \cap B)$. Let $\mathcal{L}' = \mathcal{L}_1 \cup \{f(A^*, B^*) : (A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)\}$.

(1) *$\mathcal{L}'$ is a fine rooted location.*

Subproof. Certainly $\mathcal{L}_1$ and $\{f(A^*, B^*) : (A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)\}$ are rooted locations (by 5.2, and since P is rootless), and so to check that $\mathcal{L}'$ is a rooted location it suffices to show that for each $C \in \mathcal{L}_1$ and each $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$,

$$V(C^- \cap f(A^*, B^*)^-) \subseteq \bar{\pi}(C) \cap \bar{\pi}(f(A^*, B^*))$$
$$E(C^- \cap f(A^*, B^*)^-) = \emptyset.$$

Suppose, therefore, that $C \in \mathcal{L}_1$ and $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$. Then since $C \notin \mathcal{L}_2$ it follows that not both $C^- \subseteq A^*$ and $B^* \subseteq (G \setminus C)^-$. Hence from the hypothesis of the theorem, $C^- \subseteq B^*$ and

$A^* \subseteq (G \setminus C)^-$. Since $f(A^*, B^*)^- = A^*$, and

$$\begin{aligned} V(C^- \cap A^*) \subseteq V(C^- \cap (G \setminus C)^-) \quad &\cap \quad V(A^* \cap B^*) = \bar{\pi}(C) \cap \bar{\pi}(f(A^*, B^*)) \\ E(C^- \cap A^*) \quad &\subseteq \quad E(A^* \cap B^*) = \emptyset \end{aligned}$$

it follows that $\mathcal{L}'$ is a rooted location. To see that it is fine, we observe that

$$\begin{aligned} \bigcup(C^- : C \in \mathcal{L}') &= \bigcup(C^- : C \in \mathcal{L}_1) \cup \bigcup(f(A^*, B^*)^- : (A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)) \\ &= \bigcup(C^- : C \in \mathcal{L}_1) \cup \bigcup(A^* : (A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)) \\ &\supseteq \bigcup(C^- : C \in \mathcal{L}_1) \cup \bigcup(C^- : C \in \mathcal{L}_2) = G^- \end{aligned}$$

the inclusion holding since if $C \in \mathcal{L}_2$ then $C^- \subseteq A^*$ for some $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$. This proves (1).

(2) *$\mathcal{L}'$ θ^2-isolates $\mathcal{T}$.*

Subproof. Now $\mathcal{L}'^- \subseteq \mathcal{T}$ and its members have order $< \theta^2$. Let $\mathcal{T}'$ be a tangle of order $\geq \theta^2$, let $(A', B') \in \mathcal{L}'^-$ with $(B', A') \in \mathcal{T}'$, and let (A, B) be the $(\mathcal{T}, \mathcal{T}')$-distinction. Suppose first that $(A', B') \in \mathcal{L}_1^-$. Then since $\mathcal{L}$ is W-suitable, either $A \subseteq A'$ and $B' \subseteq B$ or $A \subseteq A^*$ and $B^* \subseteq B$ for some $(A^*, B^*) \in \mathcal{M}(\mathcal{T}, W, \theta)$. The first is the desired conclusion, and we assume the second. Then $(B^*, A^*) \in \mathcal{T}'$ since $A \subseteq A^*$ and $(B, A) \in \mathcal{T}'$ and $\mathcal{T}'$ has order $\geq \theta^2$ and (B^*, A^*) has order $< \theta \leq \theta^2$. Since $(A', B') \notin \mathcal{L}_2^-$, it follows as in the proof of (1) that $A' \subseteq B^*$, and so $B^* \cup B' = G^-$, a contradiction to the second tangle axiom since (B', A'), $(B^*, A^*) \in \mathcal{T}'$. We may assume then that $(A', B') \notin \mathcal{L}_1^-$; and so $(A', B') \in \mathcal{M}(\mathcal{T}, W, \theta)$, and therefore (A', B') has order $< \theta$. Since $(B', A') \in \mathcal{T}'$ it follows that (A, B) has order at most that of (A', B') and hence $< \theta$. From 5.1, either $A \subseteq A'$ and $B' \subseteq B$, or $A \subseteq B'$ and $A' \subseteq B$. The first is the desired conclusion and the second is impossible since (B', A'), $(B, A) \in \mathcal{T}'$. This proves (2).

Now $|\bar{\pi}(f(A, B))| < \theta$ for each $(A, B) \in \mathcal{M}(\mathcal{T}, W, \theta)$, and the heart of $(P, \mathcal{L})$ in $\mathcal{F}$ is active (since $\mathcal{L}$ is fine) and hence belongs to $\mathcal{F}_1$. Consequently, $(P, \mathcal{L}')$ has heart in $\mathcal{F}_2$. This proves 5.5. ∎

By applying 5.3, 5.4 and 5.5 in turn, we deduce:

5.6 *Let $\mathcal{F}$ be a well-behaved set of patchworks, and let $\theta \geq 1$. Then there is a well-behaved set of patchworks $\mathcal{F}'$ with the following property. Let $P = (G, \mu, \Delta)$ be a rootless removable patchwork, let λ be a tie-breaker in G, let $\mathcal{T}$ be a tangle in G of order $\geq \theta^2$, and let $W \subseteq Z(G)$ with $|W| \leq \theta$. Suppose that $\mathcal{L}$ is a fine rooted location in G such that $\mathcal{L}$ θ-isolates $\mathcal{T}$ modulo W, and $\mathcal{F}$ contains a heart of $(P, \mathcal{L})$. Then there is a fine rooted location $\mathcal{L}'$ in G such that $\mathcal{L}$ θ^2-isolates $\mathcal{T}$ and $\mathcal{F}'$ contains a heart of $(P, \mathcal{L}')$.*

From 3.1 and 5.6 we deduce the main result of this section:

5.7 *Let Ω be a well-quasi-order, let $\mathcal{F}$ be a well-behaved set of patchworks, and let $\theta \geq 1$. Let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \dots)$ be a countable sequence of rootless robust Ω-patchworks. For each $i \geq 1$ let λ_i be a tie-breaker in G_i; and suppose that for each tangle $\mathcal{T}$ in G_i of order $\geq \theta$, there exist $W \subseteq Z(G_i)$ with $|W| \leq \theta$ and a fine rooted location $\mathcal{L}$ in G_i, such that $\mathcal{L}$ θ-isolates $\mathcal{T}$ modulo W, and $\mathcal{F}$ contains a heart of $((G_i, \mu_i, \Delta_i), \mathcal{L})$. Then there exist $j > i \geq 1$ such that P_i is simulated in P_j.*

Proof. Define $\mathcal{F}'$ as in 5.6, and let $\mathcal{F}''$ be the set of all rootless partial Ω-patchworks (G, μ, Δ, ϕ) with $dom(\phi) = \emptyset$ and $(G, \mu, \Delta) \in \mathcal{F}'$. Thus $\mathcal{F}''$ is a well-behaved set of partial Ω-patchworks. We claim that the hypothesis of 3.1 are satisfied, with $\mathcal{F}$, θ replaced by $\mathcal{F}''$, θ^2. For let $i \geq 1$, let $Q = (G_i, \mu_i, \Delta_i)$, and let $\mathcal{T}$ be a tangle in G_i of order $\geq \theta^2$. Then $\mathcal{T}$ has order $\geq \theta$, and so there exist W, $\mathcal{L}$ as in the hypothesis of 5.7. Hence Q, λ_i, $\mathcal{T}_i$, W, $\mathcal{L}$ satisfy the hypothesis of 5.6 (in particular Q is removable, since it is robust), and so there is a fine rooted location $\mathcal{L}'$ in G_i which θ^2-isolates $\mathcal{T}$, such that $\mathcal{F}'$ contains a heart of $(Q, \mathcal{L}')$. Since $\mathcal{L}'$ is fine and Q is rootless, the heart of $(P_i, \mathcal{L}')$ belongs to $\mathcal{F}''$. Consequently the hypotheses of 3.1 are satisfied, and the result follows from 3.1. ■

6 Eliminating the tie-breaker

Our next objective is to prove a form of 5.7 with no tie-breakers. Let G be a hypergraph and let $f \in E(G)$. For each $x \in Z(G)$ let $\nu(x) > 0$ be a real number, such that the numbers $\nu(x)$ $(x \in Z(G))$ are rationally independent. For each separation (A, B) of G with $f \in E(A)$, we define

$$\lambda(A, B) = (|V(A \cap B)|, \Sigma(\nu(x) : x \in Z(G) \setminus Z(A)), \Sigma(\nu(x) : x \in V(A \cap B))).$$

Thus each $\lambda(A, B)$ is a triple of real numbers. We order $\mathbf{R}^3$ lexicographically, that is, $(a_1, a_2, a_3) < (b_1, b_2, b_3)$ if for some $k \in \{1, 2, 3\}$, $a_i = b_i$ for $1 \leq i < k$ and $a_k < b_k$. If (A, B) is a separation with $f \in E(B)$, we define $\lambda(A, B) = \lambda(B, A)$.

6.1 *λ is a tie-breaker.*

Proof. We must verify the three axioms. Suppose first that $(A, B), (C, D)$ are separations and $\lambda(A, B) = \lambda(C, D)$. We may assume that $f \in E(A)$ and $f \in E(C)$. Hence $|V(A \cap B)| = |V(C \cap D)|$, and $Z(G) \setminus Z(A) = Z(G) \setminus Z(C)$, that is, $A = C$, since the ν's are rationally independent; and for the same reason, $V(A \cap B) = V(C \cap D)$. Since $E(A) = E(C)$ it follows that $E(B) = E(D)$; and since $V(A \cap B) = V(C \cap D)$, it follows that $B = D$. Thus $(A, B) = (C, D)$. This proves the first axiom, for the "if" part of the first axiom is clear.

For the second axiom, let (A, B), (C, D) be separations, and suppose that $\lambda(A \cup C, B \cap D) > \lambda(A, B)$ and $\lambda(A \cap C, B \cup D) \geq \lambda(C, D)$. Now $(A \cup C, B \cap D)$ has order at least that of (A, B), and $(A \cap C, B \cup D)$ has order at least that of (C, D). But the sum of the orders of $(A \cup C, B \cap D)$ and $(A \cap C, B \cup D)$ equals the sum of the orders of (A, B) and (C, D), and so we have equality; that is, $(A \cup C, B \cap D)$ has the same order as (A, B), and $(A \cap C, B \cup D)$ has the same order as (C, D).

Suppose first that $f \in E(A)$. Since $\lambda(A \cup C, B \cap D) > \lambda(A, B)$, it follows that

$$\Sigma(\nu(x) : x \in Z(A \cup C)) \leq \Sigma(\nu(x) : x \in Z(A)),$$

and so $C \subseteq A$ (since $\nu(x) > 0$ for all x). Hence $V((A \cup C) \cap (B \cap D)) \subseteq V(A \cap B)$, and so equality holds since these two sets have the same cardinality. But then $\lambda(A \cup C, B \cap D) = \lambda(A, B)$, a contradiction.

Thus $f \in E(B)$. Suppose that $f \in E(D)$. Since $\lambda(A \cap C, B \cup D) \geq \lambda(C, D)$ we deduce, as above, that $B \subseteq D$ and $V((A \cap C) \cap (B \cup D)) = V(C \cap D)$, and so $\lambda(A \cap C, B \cup D) = \lambda(C, D)$. By the first axiom, $(A \cap C, B \cup D) = (C, D)$ or (D, C), and since $f \in E(D)$ it follows that $(A \cap C, B \cup D) = (C, D)$. Thus $C \subseteq A$ and $B \subseteq D$, and so $(A \cup C, B \cap D) = (A, B)$. But $\lambda(A \cup C, B \cap D) \neq \lambda(A, B)$, a contradiction.

We have shown then that $f \notin E(A)$ and $f \notin E(D)$, and so $f \in E(B \cap C)$. Since $\lambda(A \cup C, B \cap D) > \lambda(A, B)$ it follows that

$$\Sigma(\nu(x) : x \in Z(A \cup C)) \leq \Sigma(\nu(x) : x \in Z(B)).$$

Since $\lambda(A \cap C, B \cup D) \geq \lambda(C, D)$ it follows that

$$\Sigma(\nu(x) : x \in Z(B \cup D)) \leq \Sigma(\nu(x) : x \in Z(C)).$$

But $Z(A \cup C) \supseteq Z(C)$ and $Z(B \cup D) \supseteq Z(B)$, and so we have equality throughout, that is $Z(A \cup C) = Z(C)$ and $Z(B \cup D) = Z(B)$; and consequently $A \subseteq C$ and $D \subseteq B$. Moreover,

$$\Sigma(\nu(x) : x \in Z(B)) = \Sigma(\nu(x) : x \in Z(C))$$

and so $B = C$. Since (A, B) is a separation and $A \subseteq C = B$, it follows that $B = G$.

From comparing the third components of the tie-breaker, we deduce

$$\Sigma(\nu(x) : x \in V((A \cup C) \cap B \cap D)) > \Sigma(\nu(x) : x \in V(A \cap B)),$$

that is,

$$\Sigma(\nu(x) : x \in V(D)) > \Sigma(\nu(x) : x \in V(A)),$$

and

$$\Sigma(\nu(x) : x \in V((A \cap C) \cup (B \cup D))) \geq \Sigma(\nu(x) : x \in V(C \cap D)),$$

that is,

$$\Sigma(\nu(x) : x \in V(A)) \geq \Sigma(\nu(x) : x \in V(D)),$$

a contradiction. This proves the second axiom.

The third axiom is clear because of the lexicographical order on $\mathbf{R}^3$. This proves 6.1. ■

We call a tie-breaker λ as in 6.1 the tie-breaker *defined by* f, ν; we call tie-breakers of this form *edge-based*.

Let G be a rooted hypergraph, and let $\mathcal{T}$ be a tangle in G. A rooted location $\mathcal{L}$ is *linked* to $\mathcal{T}$ if $\mathcal{L}^- \subseteq \mathcal{T}$ and for each $A \in \mathcal{L}$ there is no $(A', B') \in \mathcal{T}$ of order less than $|\bar{\pi}(A)|$ with $A^- \subseteq A'$ and $B' \subseteq (G \setminus A)^-$. If $\mathcal{T}$ is a tangle in a hypergraph G of order θ, and $W \subseteq V(G)$ with $|W| < \theta$, we define

$$\mathcal{T}/W = \{(A/W, B/W) : (A, B) \in \mathcal{T}, W \subseteq V(A \cap B)\}.$$

It is shown in theorem 6.2 of [2] that $\mathcal{T}/W$ is a tangle in G/W of order $\theta - |W|$.

If $\mathcal{L}$ is a rooted location in a rooted hypergraph G, and $W \subseteq V(G)$, and $W \subseteq \bar{\pi}(A)$ for all $A \in \mathcal{L}$, then $\{A/W : A \in \mathcal{L}\}$ is a rooted location in G/W which we denote by $\mathcal{L}/W$.

6.2 *Let G be a rooted hypergraph, and let $\mathcal{T}$ be a tangle in G of order $\theta \geq 1$. Let λ be an edge-based tie-breaker in G defined by f, ν say. Let $\mathcal{L}$ be a rooted location in G with order $< \theta$, and let $W \subseteq V(G)$ be such that $W \subseteq \bar{\pi}(A)$ for all $A \in \mathcal{L}$. Let $\mathcal{L}/W$ be linked to $\mathcal{T}/W$. Then $\mathcal{L}$ θ-isolates $\mathcal{T}$ modulo $W \cup \{f\}$.*

Proof. Let $A \in \mathcal{L}$, and let $B = G \backslash A$. Since $\mathcal{L}/W$ is linked to $\mathcal{T}/W$, it follows that $(A^-/W, B^-/W) \in \mathcal{T}/W$, and so $(A^-, B^-) \in \mathcal{T}$. Let $\mathcal{T}'$ be a tangle in G of order $\geq \theta$ with $(B^-, A^-) \in \mathcal{T}'$, and let (C, D) be the $(\mathcal{T}, \mathcal{T}')$-distinction. We must show that either $C \subseteq A^-$ and $B^- \subseteq D$, or $W \cup \{f\} \not\subseteq Z(D)$. We assume that $W \cup \{f\} \subseteq Z(D)$, and in particular $f \in E(D)$.

(1) $\lambda(A^- \cap C, B^- \cup D) \geq \lambda(C, D)$

Subproof. We may assume that the separation $(A^- \cap C, B^- \cup D)$ has order at most that of (C, D), for otherwise the desired inequality holds. But (C, D) has order at most the order of (A^-, B^-), since $(A^-, B^-) \in \mathcal{T}$ and $(B^-, A^-) \in \mathcal{T}'$, and hence (C, D) has order $< \theta$. Consequently $(A^- \cap C, B^- \cup D)$ has order $< \theta$, and so $(A^- \cap C, B^- \cup D) \in \mathcal{T}$ since $(A^-, B^-) \in \mathcal{T}$. But $(A^- \cap C, B^- \cup D) \notin \mathcal{T}'$ since (B^-, A^-), $(D, C) \in \mathcal{T}'$ and $(A^- \cap C) \cup B^- \cup D = G^-$. Consequently $(B^- \cup D, A^- \cap C) \in \mathcal{T}'$. Since (C, D) is the $(\mathcal{T}, \mathcal{T}')$-distinction it follows that $\lambda(A^- \cap C, B^- \cup D) \geq \lambda(C, D)$. This proves (1).

By (1) and the second tie-breaker axiom and 6.1, $\lambda(A^- \cup C, B^- \cap D) \leq \lambda(A^-, B^-)$. In particular, $(A^- \cup C, B^- \cap D)$ has order $< \theta$, and so $(A^- \cup C, B^- \cap D) \in \mathcal{T}$ (because $(B^- \cap D, A^- \cup C) \notin \mathcal{T}$ by the second tangle axiom, since (A^-, B^-), $(C, D) \in \mathcal{T}$). But $W \subseteq V(A^- \cap B^-)$ since $W \subseteq \bar{\pi}(A)$; and $W \subseteq V((A^- \cup C) \cap (B^- \cap D))$ since $W \subseteq V(D)$ by our previous assumption. Since $\mathcal{L}/W$ is linked to $\mathcal{T}/W$, and $A/W \in \mathcal{L}/W$, and $((A^- \cup C)/W,\ (B^- \cap D)/W) \in \mathcal{T}/W$, it follows that the order of $(A^-/W, B^-/W)$ is at most that of $((A^- \cup C)/W,\ (B^- \cap D)/W)$; that is, the order of (A^-, B^-) is at most that of $(A^- \cup C, B^- \cap D)$. Since $\lambda(A^- \cup C, B^- \cap D) \leq \lambda(A^-, B^-)$, it follows that (A^-, B^-) has the same order as $(A^- \cup C, B^- \cap D)$.

Now the sum of the orders of $(A^- \cup C, B^- \cap D)$ and $(A^- \cap C, B^- \cup D)$ equals the sum of the orders of (A^-, B^-) and (C, D); and so $(A^- \cap C, B^- \cup D)$ has the same order as (C, D). Since $\lambda(A^- \cap C, B^- \cup D) \geq \lambda(C, D)$, and $f \in E(D)$, it follows that

$$\Sigma(\nu(x) : x \in Z(B^- \cup D)) \leq \Sigma(\nu(x) : x \in Z(D))$$

and so $B^- \subseteq D$. Hence

$$V((A^- \cap C) \cup (B^- \cup D)) \subseteq V(C \cap D);$$

but these two sets have the same cardinality, and so equality holds. Consequently $\lambda(A^- \cap C, B^- \cup D) = \lambda(C, D)$, and so $A^- \cap C = C$ by the first tie-breaker axiom (for $A^- \cap C \neq D$ since $f \in E(D)$). Hence $C \subseteq A^-$. This proves 6.2. ■

By combining 6.2 and 5.7 we obtain a form of 5.7 which does not involve tie-breakers, the following.

6.3 *Let Ω be a well-quasi-order, let $\mathcal{F}$ be a well-behaved set of patchworks, and let $\theta \geq 1$. Let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \dots)$ be a countable sequence of rootless robust Ω-patchworks. Suppose that for each tangle $\mathcal{T}$ in G_i of order $\geq \theta$, there exist $W \subseteq V(G_i)$ with $|W| < \theta$ and a fine rooted location $\mathcal{L}$ in G_i, such that*

- *$W \subseteq \bar{\pi}(A)$ for all $A \in \mathcal{L}$*
- *$\mathcal{L}/W$ is linked to $\mathcal{T}/W$, and*
- *$\mathcal{F}$ contains a heart of $((G_i, \mu_i, \Delta_i), \mathcal{L})$.*

Then there exist $j > i \geq 1$ such that P_i is simulated in P_j.

Proof. If P, P' are two rootless Ω-patchworks with $E(P) = E(P') = \emptyset$, then one of P, P' is simulated in the other. We may therefore assume that $E(G_i) \neq \emptyset$ for each $i \geq 1$. For $i \geq 1$, let λ_i be an edge-based tie-breaker in G_i defined by f_i, ν_i say. We claim that the hypotheses of 5.7 are satisfied. For let $\mathcal{T}$ be a tangle in G_i of order $\geq \theta$, and let $\mathcal{T}'$ be the set of all $(A, B) \in \mathcal{T}$ of order $< \theta$. Then $\mathcal{T}'$ is a tangle in G_i of order θ. Choose W, $\mathcal{L}$ as in 6.3 (with $\mathcal{T}$ replaced by $\mathcal{T}'$). Since $\mathcal{L}/W$ is linked to $\mathcal{T}'/W$, it follows that $\mathcal{L}/W$ has order $< \theta - |W|$, and so $\mathcal{L}$ has order $< \theta$. Since $\mathcal{L}/W$ is linked to $\mathcal{T}'/W$, it follows that $\mathcal{L}/W$ is linked to $\mathcal{T}/W$. By 6.2, $\mathcal{L}$ θ-isolates $\mathcal{T}$ modulo $W \cup \{f_i\}$. Since $|W \cup \{f_i\}| \leq \theta$, the hypotheses of 5.7 are satisfied. The result follows from 5.7. ■

7 Another adjustment

Before we apply 6.3 to Wagner's conjecture, it is convenient to make one further small adjustment to it. We begin with the following lemma. A patchwork (G, μ, Δ) or Ω-patchwork (G, μ, Δ, ϕ) is *free* if $\Delta(e)$ is free for all $e \in E(G)$.

7.1 *Let $P = (G, \mu, \Delta)$ be a free patchwork and $W \subseteq \bar{\pi}(G)$. Let K be a grouping with $V(K) = \bar{\pi}(G) \setminus W$. Then K is feasible in P/W if and only if $K \cup N_W$ is feasible in P.*

Proof. If K is feasible in P/W, let

$$H' = N_{V(G)\setminus W} \cup \bigcup(\delta'_e : e \in E(G))$$

be a realization of P/W such that for distinct x, $y \in \bar{\pi}(G) \setminus W$, x and y are connected in H if and only if they are adjacent in K. For each $e \in E(G)$ there exists $\delta_e \in \Delta(e)$ such that the vertices of δ_e in W are isolated vertices of δ_e and their removal yields δ'_e. Let

$$H = N_{V(G)} \cup \bigcup(\delta_e : e \in E(G)).$$

Then for distinct x $y \in \bar{\pi}(G)$, x and y are connected in H if and only if they are adjacent in $K \cup N_W$, as required.

For the converse, let $K \cup N_W$ be feasible in P, and choose a corresponding realization

$$H = N_{V(G)} \cup \bigcup(\delta_e \in E(G)).$$

Since P is free, we may choose H and the δ_e's such that for each $e \in E(G))$ every vertex of W in $V(\delta_e)$ is an isolated vertex of δ_e. Then H/W is a realization of P/W with the required properties. This proves 7.1. ■

7.2 *Let $\mathcal{F}$ be a well-behaved set of patchworks and let $\theta \geq 1$. Then there is a well-behaved set of patchworks $\mathcal{F}'$ with the following property. Let $P = (G, \mu, \Delta)$ be a free patchwork, let $\mathcal{T}$ be a tangle in G of order $\geq \theta$, let $W \subseteq V(G)$ with $|W| < \theta$, and let $\mathcal{L}$ be a fine rooted location in G/W such that $\mathcal{L}$ is linked to $\mathcal{T}/W$, and $\mathcal{F}$ contains a heart of $(P/W, \mathcal{L})$. Then there is a fine rooted location $\mathcal{L}'$ in G such that*

- $W \subseteq \bar{\pi}(A)$ *for all* $A \in \mathcal{L}'$

- $\mathcal{L}'/W = \mathcal{L}$ *and hence is linked to* $\mathcal{T}/W$*, and*

- $\mathcal{F}'$ *contains a heart of* $(P, \mathcal{L}')$.

Proof. Let $\mathcal{F}'$ be related to $\mathcal{F}$ as $\mathcal{F}_2$ is related to $\mathcal{F}_1$ in 4.9. By 4.9, $\mathcal{F}'$ is well-behaved, and we claim it satisfies the theorem. For let P, $\mathcal{T}$, W, $\mathcal{L}$ be as above. Let $\mathcal{L}'$ be the rooted location in G such that $W \subseteq \bar{\pi}(A)$ for every $A \in \mathcal{L}'$ and $\mathcal{L}'/W = \mathcal{L}$. We claim that $\mathcal{L}'$ has the desired properties. Certainly the first two statements holds. To see the third, let $P' = (G', \mu', \Delta')$ be a heart of $(P, \mathcal{L}')$. Then P'/W is defined. We claim that P'/W is a heart of $(P/W, \mathcal{L})$. To show this, it suffices to show that if $A \in \mathcal{L}'$ and K is a grouping with $V(K) = \bar{\pi}(A) \setminus W$, then K is feasible in $(P/W)|(A/W)$ if and only if $K \cup N_W$ is feasible in $P|A$. But this follows from 7.1, since $(P/W)|(A/W) = (P|A)/W$, and $P|A$ is free. Hence P'/W is a heart of $(P/W, \mathcal{L})$ as claimed. Since $\mathcal{F}$ contains a heart of $(P/W, \mathcal{L})$, we may choose P' such that $P'/W \in \mathcal{F}$. But $dom(\mu') = E(G')$ since no edge of G' is an edge of G, and so $P' \in \mathcal{F}'$. This proves that the third statement holds, as required. ■

Incidentally, the hypothesis that P be free in 7.2 is not really necessary, but it makes the proof slightly easier, and our only application is to a free patchwork anyway. From 7.2 and 6.3 we obtain another variant of 3.1, as follows.

7.3 *Let* Ω *be a well-quasi-order, let* $\mathcal{F}$ *be a well-behaved set of patchworks, and let* $\theta \geq 1$*. Let* $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \dots)$ *be a countable sequence of free rootless* Ω*-patchworks. Suppose that for each tangle* $\mathcal{T}$ *in* G_i *of order* $\geq \theta$*, there exist* $W \subseteq V(G_i)$ *with* $|W| < \theta$ *and a fine rooted location* $\mathcal{L}$ *in* G/W*, such that* $\mathcal{L}$ *is linked to* $\mathcal{T}/W$*, and* $\mathcal{F}$ *contains a heart of* $((G_i, \mu_i, \Delta_i)/W, \mathcal{L})$*. Then there exist* $j > i \geq 1$ *such that* P_i *is simulated in* P_j*.*

Proof. Let $\mathcal{F}'$ be as in 7.2. We claim that the hypotheses of 6.3 are satisfied (with $\mathcal{F}$ replaced by $\mathcal{F}'$). For let $\mathcal{T}$ be a tangle in G_i of order $\geq \theta$. Let W, $\mathcal{L}$ be as in the hypotheses of 7.3, and choose $\mathcal{L}'$ as in the proof of 7.2. Thus the hypotheses of 6.3 hold (with $\mathcal{L}$ replaced by $\mathcal{L}'$) and the result follows from 6.3. ■

8 Surfaces and paintings

Now we come to the second part of the paper, where we shall apply 7.3 to deduce Wagner's conjecture from a theorem about hypergraphs drawn on a fixed surface. In this paper, by a *surface* we mean a compact connected 2-manifold with (possibly null) boundary. If Σ is a surface, its boundary is denoted by $bd(\Sigma)$, and each component of $bd(\Sigma)$ is a *cuff* of Σ. An *O-arc* in Σ is a subset of Σ homeomorphic to a circle; every cuff is thus an *O-arc*. A *line* is a subset homeomorphic to the closed interval [0,1]. If $X \subseteq \Sigma$ the closure of X is denoted by $\bar{X}$ and $\bar{X} \setminus X$ by $\tilde{X}$.

A *painting* Γ in a surface Σ is a triple (U, N, γ), where $U \subseteq \Sigma$ is closed, $N \subseteq U$ is finite, and

- $bd(\Sigma) \subseteq U$, and $U \setminus N$ has only finitely many arc-wise connected components, called *cells*

- for each cell c, $\bar{c}$ is a closed disc and $|\tilde{c}| = 2$ or 3 and $\bar{c} \cap N = \tilde{c} \subseteq bd(\bar{c})$

- for each cell c, if $c \cap bd(\Sigma) \neq \emptyset$ then $|\tilde{c}| = 2$, and $\bar{c} \cap bd(\Sigma)$ is a line and its ends are the members of $\tilde{c}$
- for each cell c, $\gamma(c)$ is a march μ with $\bar{\mu} = \tilde{c}$.

We write $U(\Gamma) = U, N(\Gamma) = N, \gamma_\Gamma = \gamma$, and denote the set of cells of Γ by $\mathcal{C}(\Gamma)$. The members of $N(\Gamma)$ are called *nodes*. If $c \in \mathcal{C}(\Gamma)$ and $1 \leq i \leq |\tilde{c}|$, we call the ith term of $\gamma(c)$ the ith *node of* c; and in particular, the first node of c is its *tail*. A cell c is a *border cell* if $c \cap bd(\Sigma) \neq \emptyset$, and otherwise is *internal*. Nodes in $bd(\Sigma)$ are *border nodes* and the others are *internal*. If Θ is a cuff, we say a cell c or node n *borders* Θ if $c \cap \Theta \neq \emptyset$ or $n \in \Theta$. The *size* of a cell c is $|\tilde{c}|$. The components of $\Sigma \setminus U(\Gamma)$ are the *regions* of Γ. A subset $X \subseteq \Sigma$ is Γ-*normal* if $X \cap U(\Gamma) \subseteq N(\Gamma)$. A painting Γ is *3-connected* if

- for every Γ-normal O-arc F in Σ with $|F \cap N(\Gamma)| \leq 2$ there is a closed disc $\Delta \subseteq \Sigma$ with $bd(\Delta) = F$ which includes at most one cell of Γ and with $\Delta \cap N(\Gamma) \subseteq F$
- for every Γ-normal line F in Σ with $|F \cap N(\Gamma)| \leq 2$ and with both ends in $bd(\Sigma)$ and with no other point in $bd(\Sigma)$, there is a closed disc $\Delta \subseteq \Sigma$ with $F \subseteq bd(\Delta) \subseteq F \cup bd(\Sigma)$ which includes at most one cell of Γ and with $\Delta \cap N(\Gamma) \subseteq F$.

Let Γ be a painting in Σ. We define its *skeleton* $sk(\Gamma)$ to be the subgraph of $K_{N(\Gamma)}$ with vertex set $N(\Gamma)$ in which for distinct $n_1, n_2 \in N(\Gamma), n_1$ and n_2 are adjacent in $sk(\Gamma)$ if and only if there is a cell $c \in \mathcal{C}(\Gamma)$ with $n_1, n_2 \in \tilde{c}$.

Let Γ, Γ' be paintings in Σ. Let ζ be a function with domain $\mathcal{C}(\Gamma) \cup N(\Gamma)$ and with the following properties:

- $\zeta(c) \in \mathcal{C}(\Gamma')$ for each $c \in \mathcal{C}(\Gamma)$, and $\zeta(c)$ has the same size as c, and for each cuff Θ, c borders Θ if and only if $\zeta(c)$ does (and hence c is internal if and only if $\zeta(c)$ is)
- $\zeta(c_1) \neq \zeta(c_2)$ for all distinct $c_1, c_2 \in \mathcal{C}(\Gamma)$
- for each cuff Θ, if $c \in \mathcal{C}(\Gamma)$ borders Θ and we orient Θ so that the tail of c immediately precedes $c \cap \Theta$, then the tail of $\zeta(c)$ immediately precedes $\zeta(c) \cap \Theta$ under the same orientation of Θ.
- for each $n \in N(\Gamma)$, $\zeta(n)$ is a non-null induced connected subgraph of $sk(\Gamma')$
- $\zeta(n_1)$ and $\zeta(n_2)$ are disjoint for distinct $n_1, n_2 \in N(\Gamma)$
- for all $n \in N(\Gamma)$ and $c \in \mathcal{C}(\Gamma)$ and $1 \leq i \leq |\tilde{c}|$, n is the ith node of c if and only if $\zeta(n)$ contains the ith node of $\zeta(c)$
- for every border cell $c' \in \mathcal{C}(\Gamma')$, if $c' \notin \zeta(\mathcal{C}(\Gamma))$ then the nodes of c' are adjacent in $\zeta(n)$ for some $n \in N(\Gamma)$.

We call such a function ζ a *linear inflation* of Γ in Γ'. (There are no "nonlinear" inflations in this paper, but there were in [5].) Theorem 2.1 of [5] implies the following. (Note that there is a minor discrepancy between the meanings of "painting" in these two papers; in this paper, if $|\tilde{c}| = 2$ then the closure of c is a disc, while in [5], the closure of c is a line. But it is easy to convert from one version to the other; make the discs narrow and the lines thick.)

8.1 *Let Σ be a surface and let Ω be a well-quasi-order. For each $i \geq 1$ let Γ_i be a 3-connected painting in Σ and let $\phi_i : \mathcal{C}(\Gamma_i) \to E(\Omega)$ be a function. Then there exist $j > i \geq 1$ and a linear inflation ζ of Γ_i in Γ_j such that $\phi_i(c) \leq \phi_j(\zeta(c))$ for each $c \in \mathcal{C}(\Gamma_i)$.*

The objective of the next two sections is to deduce Wagner's conjecture from 8.1 and the main theorem of [3].

9 Patchworks from a surface

We wish now to discuss certain patchworks associated with paintings in a surface. Let Σ be a surface, and for each cuff Θ let $\rho(\Theta) \geq 0$ be an integer. We call (Σ, ρ) a *graded surface*. Let Γ be a 3-connected painting in Σ, and let G be a hypergraph with $N(\Gamma) \subseteq V(G)$ and $(\mathcal{C}(\Gamma) = E(G)$, such that for each $n \in N(\Gamma)$ and $c \in \mathcal{C}(\Gamma)$, $n \in \tilde{c}$ if and only if n is incident with c in G. For each border node $n \in N(\Gamma)$, let $\beta(n) \subseteq V(G)$, such that

- for each $n \in N(\Gamma) \cap bd(\Sigma)$, $\beta(n) \cap N(\Gamma) = \emptyset$ and $|\beta(n)| = \rho(\Theta)$, where Θ is the cuff bordered by n; for nodes n_1, n_2 bordering distinct cuffs, $\beta(n_1) \cap \beta(n_2) = \emptyset$; and
$$V(G) = N(\Gamma) \cup \bigcup(\beta(n) : n \in N(\Gamma) \cap bd(\Sigma))$$
- for each internal cell c, the set of ends of c in G is $\tilde{c}$; and for each border cell c with n_1, n_2 the set of ends of c in G is $\beta(n_1) \cup \beta(n_2) \cup \{n_1, n_2\}$
- if n_1, n_2, n_3, $n_4 \in N(\Gamma)$ border the same cuff in order, then $\beta(n_1) \cap \beta(n_3) \subseteq \beta(n_2) \cup \beta(n_4)$.

In these circumstances, (Γ, β) is said to be a (Σ, ρ)-*hull* for G. Now let $P = (G, \mu, \Delta)$ be a patchwork. We say that P is (Σ, ρ)-*hulled* if there is a (Σ, ρ)-hull (Γ, β) for G^- such that

- for each internal cell $c \in \mathcal{C}(\Gamma)$, $\Delta(c)$ is free
- for each border cell $c \in \mathcal{C}(\Gamma)$ with $\tilde{c} = \{n_1, n_2\}$, there is a pairing M_c with $V(M_c) = \beta(n_1) \cup \beta(n_2) \cup \{n_1, n_2\}$, such that n_1, n_2 are adjacent in M_c, and M_c has $|\beta(n_1)| + 1 = |\beta(n_2)| + 1$ components, each containing one vertex of $\beta(n_1) \cup \{n_1\}$ and one of $\beta(n_2) \cup \{n_2\}$ (possibly the same), and either
  - $M_c \in \Delta(c)$ or
  - $M_c \setminus n_1n_2 \in \Delta(c)$ (where n_1n_2 denotes the edge of M_c joining n_1, n_2) and there is an internal cell c' of Γ with $n_1, n_2 \in \tilde{c'}$.
- $\pi(G) = 0$ and $dom(\mu) = E(G)$; and for each internal cell c, and for $1 \leq i \leq |\tilde{c}|$ the ith term of $\mu(c)$ is the ith node of c.

The main result of this section is the following.

9.1 *For every graded surface (Σ, ρ), the set of all (Σ, ρ)-hulled patchworks is well-behaved.*

Proof. Let Ω be a well-quasi-order. Let $r = \max \rho(\Theta)$, taken over all cuffs Θ, and $r = 0$ if $bd(\Sigma) = \emptyset$. Let Ω_0 be the well-quasi-order with $E(\Omega_0)$ the set of all 7-tuples $(\mu, \pi_0, \pi_1, \pi_2, \Delta, \omega, t)$ where

- μ is a march with $\leq 2r+3$ terms
- π_0, π_1, π_2 are marches in $\bar{\mu}$
- Δ is a patch with $V(\Delta) = \bar{\mu}$
- $\omega \in E(\Omega)$
- $t = 0$ or 1

where we say that $(\mu, \pi_0, \pi_1, \pi_2, \Delta, \omega, t) \leq (\mu', \pi'_0, \pi'_1, \pi'_2, \Delta', \omega', t')$ if $t = t'$, $\omega \leq \omega'$, μ and μ' have the same length k say, and the bijection from $\bar{\mu}$ to $\bar{\mu}'$ mapping μ to μ' also maps π_i to π'_i $(i = 0, 1, 2)$ and maps Δ to Δ'. It is easy to see that Ω_0 is indeed a well-quasi-order. We may assume that $E(\Omega_0) \cap E(\Omega) = \emptyset$; let $\Omega_1 = \Omega \cup \Omega_0$.

Now let $P = (G, \mu, \Delta, \phi)$ be an Ω-completion of a (Σ, ρ)-hulled patchwork. Let (Γ, β) be a (Σ, ρ)-hull for P. For each cuff Θ let c_Θ be a cell of Γ bordering Θ. For each node n bordering Θ let us choose a march $\pi(n)$ with $\bar{\pi}(n) = \beta(n)$, such that for each cell $c \neq c_\Theta$ bordering Θ with nodes n_1, n_2 and for $1 \leq i \leq \rho(\Theta)$, the ith term of $\pi(n_1)$ and the ith term of $\pi(n_2)$ belong to the same component of M_c (where M_c is as in the second part of the definition of (Σ, ρ)-hulled patchwork).

For each $c \in \mathcal{C}(\Gamma)$ we define $\psi(c)$ as follows. If c is internal we let $\psi(c) = \phi(c)$, and so we assume that c borders a cuff Θ, with nodes n_1, n_2, where n_1 is the first node of c. We define

$$\psi(c) = (\mu(c), (n_1, n_2), \pi(n_1), \pi(n_2), \Delta(c), \phi(c), t)$$

where $t = 0$ if $c \neq c_\Theta$ and $t = 1$ if $c = c_\Theta$.

In view of 8.1, to complete the proof it suffices (cf. 4.3) to show that if $P = (G, \mu, \Delta, \phi)$ and (Γ, β) is a (Σ, ρ)-hull for P with groupings denoted by M_c as before, and ψ is defined as above, and also $P' = (G', \mu', \Delta', \phi')$, (Γ', β'), $M'_{c'}$, ψ' are related similarly (with the same graded surface and same well-quasi-orders Ω, Ω_1) and ζ is a linear inflation of Γ in Γ' such that $\psi(c) \leq \psi'(\zeta(c))$ for each $c \in \mathcal{C}(\Gamma)$, then P is simulated in P'. Let $\pi(n)$ (for each border node n) be defined as before, and let $\pi'(n')$ be defined analogously for each border node n' of Γ'.

(1) *For each cuff* Θ, $\zeta(c_\Theta) = c'_\Theta$.

Subproof. Let $\zeta(c_\Theta) = c'$. Then c' borders Θ (since ζ is a linear inflation) and $\psi(c_\Theta) \leq \psi'(c')$, and so the seventh term of $\psi'(c')$ is 1. This proves (1).

For $v \in V(G) \setminus N(\Gamma)$ we define $\eta(v)$ to be the set of all vertices $v' \in V(G')$ such that there exist a cuff Θ and $n \in N(\Gamma) \cap \Theta$ and $n' \in V(\zeta(n)) \cap \Theta$ and an integer $i > 0$ such that v is the ith term of $\pi(n)$ and v' is the ith term of $\pi'(n')$. For $n \in N(\Gamma)$ we define $\eta(n) = V(\zeta(n))$. For $c \in \mathcal{C}(\Gamma)$ we define $\eta(c) = \zeta(c)$. Our next objective is to show that η is an expansion of P in P'.

(2) *For each* $v \in V(G)$, $\eta(v) \neq \emptyset$.

Subproof. If $v \in N(\Gamma)$ then $\zeta(v)$ is not null and so $\eta(v) \neq \emptyset$. If $v \in \beta(n)$ for some $n \in N(\Gamma) \cap \Theta$ where Θ is a cuff, let v be the ith term of $\pi(n)$, let $n' \in V(\zeta(n)) \cap \Theta$, and let v' be the ith term of $\pi'(n')$. Then $v' \in \eta(v)$ and so $\eta(v) \neq \emptyset$. This proves (2).

(3) *Let* $v \in V(G) \setminus N(\Gamma)$ *and let* $v' \in \eta(v)$*. For each* $n \in N(\Gamma) \cap bd(\Sigma)$ *and* $n' \in V(\zeta(n)) \cap bd(\Sigma)$*, if* v' *is the* i*th term of* $\pi'(n')$ *then* v *is the* i*th term of* $\pi(n)$*.*

Subproof. By the third condition in the definition of a (Σ, ρ)-hull, there is a line $F \subseteq \Theta$ for some cuff Θ, such that for each $n' \in N(\Gamma) \cap bd(\Sigma)$, $v' \in \beta'(n')$ if and only if $n' \in F$. Let us say that $n' \in N(\Gamma') \cap F$ is *good* if for some $i > 0$, v' is the ith term of $\pi'(n')$ and v is the ith term of $\pi(n)$ where $n' \in V(\zeta(n))$. Certainly some node in $N(\Gamma') \cap F$ is good since $v' \in \eta(v)$; and we wish to prove that all are good. It suffices therefore to show that if n'_1, $n'_2 \in N(\Gamma') \cap F$ are consecutive and n'_1 is good then so is n'_2. Let v' be the ith term of $\pi'(n'_1)$ and the jth term of $\pi'(n'_2)$; and let $n'_1 \in V(\zeta(n_1))$, $n'_2 \in V(\zeta(n_2))$. Then v is the ith term of $\pi(n_1)$, and we must show that it is the jth term of $\pi(n_2)$. Let $c' \in \mathcal{C}(\Gamma)$ border Θ with nodes n'_1, n'_2. If $n_1 = n_2$ then $c' \notin \zeta(\mathcal{C}(\Gamma))$ and so $c' \neq c'_\Theta$ by (1); hence $i = j$ because the ith term of $\pi'(n'_1)$ and the jth term of $\pi'(n'_2)$ are equal and hence belong to the same component of $M'_{c'}$ and the claim is trivial. We assume then that $n_1 \neq n_2$. Hence $c' = \zeta(c)$ for some $c \in \mathcal{C}(\Gamma)$ (because otherwise n'_1, n'_2 would be adjacent in and hence both belong to some $\zeta(n)$ for $n \in N(\Gamma)$, contrary to $n_1 \neq n_2$). Since $\psi(c) \leq \psi'(c')$ and the ith term of $\pi'(n'_1)$ is the jth term of $\pi'(n'_2)$ it follows that the ith term of $\pi(n_1)$ is the jth term of $\pi(n_2)$, that is, v is the jth term of n_2. This proves (3).

(4) *For distinct* v_1, $v_2 \in V(G)$, $\eta(v_1) \cap \eta(v_2) = \emptyset$*.*

Subproof. Let $v' \in \eta(v_1) \cap \eta(v_2)$. If $v' \in N(\Gamma')$ then v_1, $v_2 \in N(\Gamma)$ and hence $V(\zeta(v_1)) \cap V(\zeta(v_2)) \neq \emptyset$ and so $v_1 = v_2$. If $v' \notin N(\Gamma')$ then v_1, $v_2 \notin N(\Gamma)$ and there exist $n_1 \in N(\Gamma) \cap bd(\Sigma)$ and $n'_1 \in V(\zeta(n_1)) \cap bd(\Sigma)$ and $i > 0$ such that v_1 is the ith term of $\pi'(n'_1)$ and v' is the ith term of $\pi'(n'_1)$. Since $v' \in \eta(v_2)$ it follows from (3) that v_2 is the ith term of $\pi(n_1)$ and hence $v_1 = v_2$. This proves (4).

(5) *For each* $c \in \mathcal{C}(\Gamma)$, $\mu(c)$ *and* $\mu'(\eta(c))$ *have the same length* k *say, and for* $1 \leq i \leq k$, $\eta(v)$ *contains the* i*th term of* $\mu'(\eta(c))$ *where* v *is the* i*th term of* $\mu(c)$*.*

Subproof. Let $c' = \eta(c)$. Since $\psi(c) \leq \psi'(c')$ and $|\tilde{c}| = |\tilde{c}'|$ it follows that $\mu(c)$ and $\mu'(c')$ have the same length k say. Let $1 \leq i \leq k$, let v be the ith term of $\mu(c)$, and let v' be the ith term of $\mu'(c')$. We must show that $v' \in \eta(v)$. If c is internal then so is c', and v is the ith node of c and hence $\eta(v) = V(\zeta(v))$ contains the ith node of c', that is, v' as required. (We are using here the third condition in the definition of (Σ, ρ)-hulled.) We assume then that c and hence c' are border cells. If $v \in N(\Gamma)$ then $v \in \tilde{c}$; let v be the jth node of c. Then since $\psi(c) \leq \psi'(c')$, v' is the jth node of c', and hence belongs to $\eta(v) = V(\zeta(v))$ since ζ is a linear inflation. We assume then that $v \notin N(\Gamma)$. Choose $n \in \tilde{c}$ with $v \in \beta(n)$, and let v be the jth term of $\pi(n)$. Let n' be the corresponding node of c' (that is, the first node of c' if and only if n is the first node of c). Since $\psi(c) \leq \psi'(c')$, v' is the jth term of $\pi'(n')$ and so $v' \in \eta(v)$. This proves (5).

(6) *For each* $c \in \mathcal{C}(\Gamma)$, $\phi(c) \leq \phi'(\eta(c))$ *and the bijection from* $\bar{\pi}(c)$ *to* $\bar{\pi}'(\eta(c))$ *mapping* $\mu(c)$ *to* $\mu'(\eta(c))$ *also maps* $\Delta(c)$ *to* $\Delta'(\eta(c))$*.*

Subproof. If c is internal then $\phi(c) = \psi(c) \leq \psi'(\eta(c)) = \phi'(\eta(c))$ and $\Delta(c)$, $\Delta'(\eta(c))$ are both free. If c is a border cell the claim follows since $\psi(c) \le \psi'(\eta(c))$. This proves (6).

From (2), (3), (4), (5), (6) we deduce

(7) *η is an expansion of P in P'.*

For each $c' \in \mathcal{C}(\Gamma') \setminus \zeta(\mathcal{C}(\Gamma))$ we choose $\delta_{c'} \in \Delta(c')$ as follows. If c' is a border cell and $M'_{c'} \in \Delta(c')$, let $\delta_{c'} = M'_{c'}$. If c' is a border cell and $M'_{c'} \notin \Delta(c')$, let $\delta_{c'} = M'_{c'} \setminus e$, where e is the edge of $M'_{c'}$ joining the two nodes of c'. If c' is internal let $\delta_{c'}$ be the grouping K with $V(K) = \tilde{c'}$ in which distinct n_1, $n_2 \in \tilde{c'}$ are adjacent in K if and only if there exists $n \in N(\Gamma)$ with n_1, $n_2 \in V(\zeta(n))$. Then $\delta_{c'} \in \Delta(c')$ since $\Delta(c')$ is free. Let

$$H = N_{V(G')} \cup \bigcup(\delta_{c'} : c' \in \mathcal{C}(\Gamma') \setminus \zeta(\mathcal{C}(\Gamma))).$$

Then H is a realization of $P' \backslash \eta(E(G))$. We shall show that it realizes η.

(8) *For each $n \in N(\Gamma)$ there is a component J of H with $V(J) = V(\zeta(n))$; and for every component J of H not of this form with $E(J) \neq \emptyset$ there is a cuff Θ such that $V(J) \subseteq \bigcup(\beta(n) : n \in N(\Gamma) \bigcap \Theta)$.*

Subproof. Every edge of H either joins two nodes in $N(\Gamma')$ or joins two vertices both in $\bigcup(\beta(n) : n \in N(\Gamma) \cap \Theta)$ for some cuff Θ. Let $n'_1, n'_2 \in N(\Gamma')$; we claim that they are connected in H if and only if they both belong to $V(\zeta(n))$ for some $n \in N(\Gamma)$. First we prove the "only if" portion. If n'_1, n'_2 are connected in H then they are joined by a path of H, all the vertices of which belong to $N(\Gamma')$, and so it suffices to prove the claim when n'_1, n'_2 are adjacent in H. Choose $c' \in \mathcal{C}(\Gamma') \setminus \zeta(\mathcal{C}(\Gamma))$ such that the edge of H joining n'_1, n'_2 belongs to $\delta_{c'}$. If c' is internal, then it follows from the definition of $\delta_{c'}$ that there exists $n \in N(\Gamma)$ with n_1, $n_2 \in V(\zeta(n))$ as required. If c' is a border cell then from the seventh condition in the definition of "linear inflation", it follows that n'_1, n'_2 are adjacent in $V(\zeta(n))$ for some n, and again the claim holds. This proves "only if". Now for the "if" portion, assume that $n'_1, n'_2 \in V(\zeta(n))$. Since $\zeta(n)$ is a connected subgraph of $sk(\Gamma')$, we may assume that n'_1, n'_2 are adjacent in $sk(\Gamma')$ and hence in $V(\zeta(n))$. Let c' be a cell of Γ' such that $n'_1, n'_2 \in \tilde{c'}$. Since $\zeta(n)$ contains two different nodes of $\tilde{c'}$, it follows (from the sixth condition in the definition of "linear inflation") that $c' \notin \zeta(\mathcal{C}(\Gamma))$. If c' is internal, it follows that n'_1, n'_2 are adjacent in H from the definition of $\delta_{c'}$, so we may assume that c' is a border cell, and there is no internal cell $c'' \in \mathcal{C}(\Gamma') \setminus \zeta(\mathcal{C}(\Gamma))$ with $n'_1, n'_2 \in \tilde{c''}$. But then again it follows that n'_1, n'_2 are adjacent in H from the definition of $\delta_{c'}$. This proves the "if" assertion, and thereby proves (8).

(9) *Let $n \in N(\Gamma) \cap \Theta$, for some cuff Θ. Let n'_1, $n'_2 \in V(\zeta(n)) \cap \Theta$ and let $1 \le i \le \rho(\Theta)$. Then the ith terms of $\pi'(n'_1)$ and $\pi'(n'_2)$ are connected in H.*

Subproof. Since there is a line $F \subseteq \Theta$ such that for $n' \in N(\Gamma') \cap \Theta$, $n' \in V(\zeta(n))$ if and only if $n' \in F$, we may assume that n'_1, n'_2 are both nodes of some cell $c' \in \mathcal{C}(\Gamma)$ bordering Θ. Since n'_1, $n'_2 \in V(\zeta(n))$ it follows that $c' \notin \eta(\mathcal{C}(\Gamma))$ and so $c' \neq c'_\Theta$ by (1). Hence v'_1, v'_2 are connected in $M'_{c'}$ from the defining property of π', and hence they are connected in H. This proves (9).

(10) *Let n_1, $n_2 \in N(\Gamma) \cap \Theta$ for some cuff Θ, let $i > 0$, and let the ith term of $\pi(n_1)$ be the ith*

term of $\pi(n_2)$*. Let* $n_1' \in V(\zeta(n_1)) \cap \Theta$ *and* $n_2' \in V(\zeta(n_2)) \cap \Theta$*. Then the* i*th term of* $\pi'(n_1')$ *and the* i*th term of* $\pi'(n_2')$ *are connected in* H*.*

Subproof. By (9) the result holds if $n_1 = n_2$. Let v be the ith term of $\pi(n_1)$. Since there is a line $F \subseteq \Theta$ such that for $n \in N(\Gamma) \cap \Theta$, $v \in \beta(n)$ if and only if $n \in F$, we may assume (by the argument used in the proof of (3)) that n_1, n_2 are both nodes of some cell c bordering Θ. By (9) we may replace n_1' by any other element of $V(\zeta(n_1)) \cap \Theta$, for the result holds for the old element if and only if it holds for the new; and hence we may assume that n_1' and similarly n_2' are nodes of $c' = \eta(c)$. Since $\psi(c) \leq \psi'(c')$ and the ith term of $\pi(n_1)$ is the jth term of $\pi(n_2)$ we deduce that the ith term of $\pi'(n_1')$ is the jth term of $\pi'(n_2')$. This proves (10).

(11) *For each* $v \in V(G)$ *every two members of* $\eta(v)$ *are connected in* H*.*

Subproof. If $v \in N(\Gamma)$ then this follows from (8). If $v \notin N(\Gamma)$ it follows from (10).

(12) *If* v_1', v_2' *are adjacent in* H *then there exists* $v \in V(G)$ *with* $v_1', v_2' \in \eta(v)$*.*

Subproof. Let $e \in E(H)$ have ends v_1', v_2'. From (8) we may assume that $v_1' \in \beta'(n_1')$, $v_2' \in \beta'(n_2')$ where n_1', n_2' are the nodes of some border cell $c' \in \mathcal{C}(\Gamma')$ with $e \in E(M'_{c'})$ and $c' \notin \eta(\mathcal{C}(\Gamma))$. Let v_1' be the ith term of $\pi'(n_1')$; then since $c' \neq c'_\Theta$ by (1) it follows from the property of π' that v_2' is the ith term of $\pi'(n_2')$. Since $c' \notin \eta(\mathcal{C}(\Gamma))$ there exists $n \in N(\Gamma)$ with $n_1', n_2' \in V(\zeta(n))$; let v be the ith term of $\pi(n)$. Then $v_1', v_2' \in \eta(v)$. This proves (12).

From (11) and (12) it follows that H realizes η. This completes the proof of 9.1. ■

10 Excluding a minor

If G is a hypergraph, its *1-skeleton* $sk(G)$ is the subgraph of $K_{V(G)}$ with vertex set $V(G)$ in which distinct $v_1, v_2 \in V(G)$ are adjacent if there is an edge of G incident with both v_1 and v_2.

10.1 *Let* $P = (G, \mu, \Delta)$ *be a free patchwork, and let* C *be a subgraph of* $sk(G)$*. Then there is a realization* H *of* P *such that for all* $x, y \in V(C)$*,* x *and* y *are connected in* C *if and only if they are connected in* H*.*

Proof. For each $e \in E(G)$, choose $\delta_e \in \Delta(e)$ such that for distinct $x, y \in V(\delta_e)$, x and y are adjacent in δ_e if and only if they belong to $V(C)$ and are connected in C. (This is possible since P is free.) Let

$$H = N_{V(G)} \cup \bigcup(\delta_e : e \in E(G))).$$

Clearly if $x, y \in V(C)$ are connected in H then they are connected in C. On the other hand, C is a subgraph of H; for if $x, y \in V(C)$ are adjacent in C, choose $e \in E(G)$ such that x, y are ends of e; then x, y are adjacent in H. The result follows. ■

Let (Σ, ρ) be a graded surface, let $P = (G, \mu, \Delta)$ be a free rootless patchwork, and let $\mathcal{L}$ be a rooted location in G. We say that $(P, \mathcal{L})$ is (Σ, ρ)-*shelled* if $\mathcal{L}$ is fine and there is a heart P' of $(P, \mathcal{L})$

where $P' = (G', \mu', \Delta')$ and $E(G') = \{e(A) : A \in \mathcal{L}\}$, and there is a (Σ, ρ)-hull (Γ, β) for G'^- such that

- if $c \in \mathcal{C}(\Gamma)$ is internal and $c = e(A)$ where $A \in \mathcal{L}$, then the ith node of c is the ith term of $\pi(A)$, for $1 \le i \le |\tilde{c}|$, and for every grouping K with $V(K) = \tilde{c}$ there is a subgraph C of $sk(A^-)$ such that for distinct x, $y \in \tilde{c}$, x and y are connected in C if and only if they are adjacent in K
- if $c \in \mathcal{C}(\Gamma)$ borders a cuff Θ, with nodes n_1, n_2, and $\rho(\Theta) = r$, and $c = e(A)$ where $A \in \mathcal{L}$, then there are r mutually disjoint paths $P_1, \ldots, P_r$ of $sk(A^-) \setminus \{n_1, n_2\}$ from $\beta(n_1)$ to $\beta(n_2)$, and either there is another path P_0 of $sk(A^-)$ from n_1, n_2 disjoint from $P_1 \cup \cdots \cup P_r$, or there is an internal cell c' of Γ with $n_1, n_2 \in \tilde{c'}$.

10.2 *Let $(P, \mathcal{L})$ be (Σ, ρ)-shelled. Then it has a heart which is (Σ, ρ)-hulled.*

The proof is immediate from 10.1.

Let $\mathcal{T}$ be a tangle in a hypergraph G, and let H be a graph. We say that $\mathcal{T}$ *controls an H-minor of $sk(G)$* if there is a function α with domain $V(H) \cup E(H)$, such that

- for each $v \in V(H)$, $\alpha(v)$ is a non-null connected subgraph of $sk(G)$, and $\alpha(u)$ and $\alpha(v)$ are disjoint for all distinct $u, v \in V(H)$
- $\alpha(e) \in E(sk(G))$ for each $e \in E(H)$, and $\alpha(e) \neq \alpha(f)$ for all distinct e, $f \in E(H)$
- for each $e \in E(H)$ with distinct ends u, v, $\alpha(e) \in E(sk(G))$ with one end in $V(\alpha(u))$ and the other in $V(\alpha)v))$
- for each loop $e \in E(H)$ with end v, $V(\alpha(v))$ contains both ends of $\alpha(e)$ and $e \notin E(\alpha(v))$
- there do not exist $(A, B) \in \mathcal{T}$ of order $< |V(H)|$ and $v \in V(H)$ such that $V(\alpha(v)) \subseteq V(A)$.

Next we convert a theorem of [3] into the language of this paper.

10.3 *For every graph H there exist $\theta \ge 1$ and a set $\mathcal{S}$ of graded surfaces, finite up to homeomorphism, with the following property. Let $P = (G, \mu, \Delta)$ be a rootless free patchwork, and let $\mathcal{T}$ be a tangle in G of order $\ge \theta$ controlling no H-minor of $sk(G)$. Then there exist $W \subseteq V(G)$ with $|W| < \theta$ and a fine rooted location $\mathcal{L}$ in G/W, such that*

- *$(P/W,\ \mathcal{L})$ is (Σ, ρ)-shelled for some $(\Sigma, \rho) \in \mathcal{S}$, and*
- *$\mathcal{L}$ is linked to $\mathcal{T}/W$.*

Proof. By theorem 14.2 of [3], there are integers $p, q, z \ge 0$ and $\theta > z$ with the property that, for every hypergraph G and tangle $\mathcal{T}$ in G of order $\ge \theta$, if $\mathcal{T}$ controls no H-minor of $sk(G)$, then there exists $W \subseteq V(G)$ with $|W| \le z$ and a $\mathcal{T}/W$-central portrayal $\pi = (\Sigma, \Gamma, \alpha, \beta, \nu)$ of G/W with warp $\le p$, such that Σ has at most q cuffs and H cannot be drawn in Σ, and π is true and $(2p+7)$-redundant. (We omit the definitions of these terms; see [3]. Note in particular that "paintings" in [3] are defined slightly differently, in that they are not equipped with the march function γ_Γ as in this paper.) Let $\mathcal{S}$ be the set of all graded surfaces (Σ, ρ) such that Σ has the property just mentioned (that is, Σ has at most q cuffs and H cannot be drawn in Σ), and $\rho(\Theta) \leq p$ for each cuff Θ of Σ. Thus $\mathcal{S}$ is finite up to homeomorphism. We claim that θ and $\mathcal{S}$ satisfy the theorem.

For let $P = (G, \mu, \Delta)$ be a rootless free patchwork, and let $\mathcal{T}$ be a tangle in G of order $\geq \theta$ controlling no H-minor of $sk(G)$. By the theorem just quoted, applied to G, we deduce that there exist W and $\pi = (\Sigma, \Gamma, \alpha, \beta, \nu)$ as above. Thus $|W| \leq z < \theta$. Now Γ is a painting in the sense of [3], but not yet a painting in the sense of this paper, because it lacks a function γ_Γ; choose such a function, arbitrarily, and therefore we may regard Γ as a painting in our sense. By theorems 8.3 and 8.5 of [3], it follows that Γ is 3-connected. By replacing Σ with a homeomorphic surface, we may assume that $\nu(n) = n$ for every $n \in N(\Gamma)$ (this is just to simplify notation a little). Let G' be the hypergraph with

$$V(G') = N(\Gamma) \cup \bigcup(\beta(n) : n \in N(\Gamma) \cap bd(\Sigma))$$

and $E(G') = \mathcal{C}(\Gamma)$, in which $c \in \mathcal{C}(\Gamma)$ is incident with $v \in V(G')$ if and only if either $v \in \tilde{c}$, or c is a border cell and $v \in \beta(n)$ for some $n \in \tilde{c}$. It follows that (Γ, β) is a (Σ, ρ)-hull for G', for some $(\Sigma, \rho) \in \mathcal{S}$.

For each cell c of Γ, let A_c be a rooted hypergraph with $A^-_C = \alpha(c)$, and with $\pi(A_c)$ as follows. If c is internal, let $\pi(A_c) = \gamma_\Gamma(c)$, and if c is a border cell with nodes n_1, n_2 say, let $\pi(A_c)$ be some march with $\overline{\pi(A_c)} = \{\nu(n_1), \nu(n_2)\} \cup \beta(n_1) \cup \beta(n_2)$. Let $\mathcal{L}$ be the set $\{A_c : c \in \mathcal{C}(\Gamma)\}$. Then $\mathcal{L}$ is a fine rooted location in G/W, and G' is a heart of $(G/W, \mathcal{L})$. It follows from theorems 9.1 and 9.8 of [3] (and from the definition of "warp") that $(P/W, \mathcal{L})$ is (Σ, ρ)-shelled.

It remains to check that $\mathcal{L}$ is linked to $\mathcal{T}/W$. Let $c \in \mathcal{C}(\Gamma)$, and suppose that $(A, B) \in \mathcal{T}/W$ with $A^-_c \subseteq A$. By theorem 11.7 of [3], (A, B) has order at least $|\overline{\pi(A_c)}|$; and so $\mathcal{L}$ is linked to $\mathcal{T}/W$. This proves 10.3. ■

We deduce

10.4 *Let Ω be a well-quasi-order and let $p \geq 0$. Let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \dots)$ be a countable sequence of free rootless Ω-patchworks such that for all $i \geq 1$, $sk(G^-_i)$ has no K_p minor. Then there exist $j > i \geq 1$ such that P_i is simulated in P_j.*

Proof. Take θ and $\mathcal{S}$ such that 10.3 holds (with $H = K_p$). Let $\mathcal{F}$ be the set of all patchworks which are (Σ, ρ)-hulled for some $(\Sigma, \rho) \in \mathcal{S}$. Since $\mathcal{S}$ is finite, $\mathcal{F}$ is well-behaved by 9.1. For all $i \geq 1$, if $\mathcal{T}$ is a tangle of order $\geq \theta$ in G_i, then $\mathcal{T}$ controls no K_p-minor of $sk(G^-_i)$, because there is no K_p-minor of $sk(G^-_i)$. By 10.3, there exists W and $\mathcal{L}$ as in 10.3. By 10.2, $((G_i, \mu_i, \Delta_i)/W, \mathcal{L})$ has a heart in $\mathcal{F}$. The result follows from 7.3. ■

As a corollary, we deduce the following form of Wagner's conjecture for directed graphs (which immediately implies the standard form of the conjecture for undirected graphs). A directed graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges.

10.5 *Let G_i $(i = 1, 2, \dots)$ be a countable sequence of directed graphs. Then there exist $j > i \geq 1$ such that G_i is isomorphic to a minor of G_j.*

Proof. Let $p = 2|E(G_1)| + |V(G_1)|$; then every tournament with p vertices has a minor isomorphic to G_1. We may therefore assume for each $i \geq 2$ that the (undirected) graph G'_i underlying G_i has no minor isomorphic to K_p, for otherwise G_i has a minor isomorphic to G_1. Take $\theta = 1$, and let Ω be the well-quasi-order with $E(\Omega) = \{0\}$. For each $i \geq 2$ let H_i be the rooted hypergraph $(G'_i, 0)$.

Let $P_i = (H_i, \mu, \Delta, \phi)$ where for $e \in E(G_i)$, $\mu(e)$ is the one- or two-vertex sequence enumerating the ends of e in G_i (tail first), $\Delta(e)$ is $\{N_X, K_X\}$ where X is the set of ends of e, and $\phi(e) = 0$. Then P_i is a free Ω-patchwork. The hypotheses of 10.4 are satisfied by the sequence P_i $(i = 2, 3, \dots)$ because no $sk(G'_i)$ has a minor isomorphic to K_p. Thus there exist $j > i \geq 2$ such that P_i is simulated in P_j. By the discussion in section 7 of [2], it follows that G_i is isomorphic to a minor of G_j, as required. ■

11 A refinement

The reader will see that we threw away a great deal in the proof of 10.4 and 10.5. If we repeat essentially the same argument a little more conservatively, we can obtain a stronger result which will be of use in the proof of Nash-Williams' "immersions" conjecture. That is our next objective.

11.1 *For every $p \geq 0$, there exist $\theta > 0$ and a well-behaved set of patchworks $\mathcal{F}$ with the following property. Let $P = (G, \mu, \Delta)$ be a rootless free patchwork, and let $\mathcal{T}$ be a tangle in G of order $\geq \theta$, controlling no K_p-minor of $sk(G^-)$. Then there is a fine rooted location $\mathcal{L}$ in G such that*

- *$(P, \mathcal{L})$ has a heart in $\mathcal{F}$, and*
- *$\mathcal{L}$ θ-isolates $\mathcal{T}$ for every edge-based tie-breaker of G.*

Proof. Take θ_1 and $\mathcal{S}$ such that 10.3 holds (with $H = K_p$ and θ replaced by θ_1). Let $\mathcal{F}_1$ be the set of all patchworks which are (Σ, ρ)-hulled for some $(\Sigma, \rho) \in \mathcal{S}$. Since $\mathcal{S}$ is finite, $\mathcal{F}$ is well-behaved by 9.1. Let $\mathcal{F}_2$ be related to $\mathcal{F}_1$ as $\mathcal{F}'$ is related to $\mathcal{F}$ in 7.2 (with θ replaced by θ_1). Let $\mathcal{F}$ be related to $\mathcal{F}_2$ as $\mathcal{F}'$ is related to $\mathcal{F}$ in 5.6, with θ replaced by $\theta_1 + 1$. Let $\theta = (\theta_1 + 1)^2$.

We claim that θ, $\mathcal{F}$ satisfy the theorem. For let $P = (G, \mu, \Delta)$ be a rootless free patchwork, and let $\mathcal{T}$ be a tangle in G of order $\geq \theta$, controlling no K_p-minor of $sk(G^-)$. From 10.3 applied to the set $\mathcal{T}_1$ of all $(A, B) \in \mathcal{T}$ of order $< \theta_1$, and 10.2, we deduce that there exists $W \subseteq V(G)$ with $|W| < \theta_1$ and a fine rooted location $\mathcal{L}_1$ in G/W such that $(P/W,\ \mathcal{L}_1)$ has a heart in $\mathcal{F}_1$ and $\mathcal{L}_1$ is linked to $\mathcal{T}_1/W$.

By 7.2 it follows that there is a fine rooted location $\mathcal{L}_2$ in G such that $W \subseteq \bar{\pi}(A)$ for all $A \in \mathcal{L}_2$, $\mathcal{L}_2/W$ is linked to $\mathcal{T}_1/W$ and $(P,\ \mathcal{L}_2)$ has a heart in $\mathcal{F}_2$. In particular, $\mathcal{L}_2$ has order $< \theta_1$, and $\mathcal{L}_2/W$ is linked to $\mathcal{T}/W$.

Choose $f \in E(G)$ and let λ be a tie-breaker defined by f. It follows that $\mathcal{L}_2$ θ_1-isolates (and hence $(\theta_1 + 1)$-isolates) $\mathcal{T}$ modulo $W \cup \{f\}$, by 6.2. By 5.6, there is a fine rooted location $\mathcal{L}_3$ in G such that $\mathcal{L}_3$ $(\theta_1 + 1)^2$-isolates $\mathcal{T}$ and $(P, \mathcal{L}_3)$ has a heart in $\mathcal{F}$, as required. ■

11.2 *Let Ω be a well-quasi-order, let $\mathcal{F}$ be a well-behaved set of partial Ω-patchworks, and let $\theta \geq 1$ and $p \geq 0$. Let $P_i = (G_i, \mu_i, \Delta_i, \phi_i)$ $(i = 1, 2, \dots)$ be a countable sequence of free rootless Ω-patchworks. For each $i \geq 1$, let λ_i be an edge-based tie-breaker in G_i. Suppose that for each $i \geq 1$ and each tangle $\mathcal{T}$ in G_i of order $\geq \theta$ which controls a K_p-minor of $sk(G_i^-)$, there is a rooted location $\mathcal{L}$ in G_i which θ-isolates $\mathcal{T}$ such that $(P_i,\ \mathcal{L})$ has a heart in $\mathcal{F}$. Then there exist $j > i \geq 1$ such that P_i is simulated in P_j.*

Proof. Choose θ_1 and $\mathcal{F}_1$ such that 11.1 holds (with θ, $\mathcal{F}$ replaced by θ_1, $\mathcal{F}_1$). Let $\mathcal{F}_2$ be the set of partial Ω-patchworks (G, μ, Δ, ϕ) with $dom(\phi) = \emptyset$ and $(G, \mu, \Delta) \in \mathcal{F}_1$. Then $\mathcal{F}_2$ is well-behaved.

Let $\mathcal{F}_3 = \mathcal{F} \cup \mathcal{F}_2$; then $\mathcal{F}_3$ is well-behaved. Moreover, for each $i \geq 1$ and each tangle $\mathcal{T}$ in G_i of order $\geq \theta_2 = \max(\theta, \theta_1)$, there is a rooted location $\mathcal{L}$ in G_i such that $\mathcal{L}$ θ_2-isolates $\mathcal{T}$ and $(P_i, \mathcal{L})$ has a heart in $\mathcal{F}_3$; for if $\mathcal{T}_2$ controls a K_p-minor of $sk(G_i^-)$, this is true by hypothesis, and if not then this is true by 11.1. The result follows from 3.1. ■

12 Acknowledgements

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SYMBOLS

Greek $\beta, \gamma, \delta, \zeta, \eta, \theta, \lambda, \mu, \pi, \rho, \tau, \phi, \psi, \omega, \Gamma, \Delta, \Theta, c_\Theta, \Sigma, \Omega$

Script $\mathcal{C}, \mathcal{F}, \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{T}$

Bold $\mathbf{H}$

Mathematical $\cup, \cap, \bigcup, \bigcap$ (cup, cap, bigcup, bigcap), $\sum$ (summation), $\lceil\rceil, \lfloor\rfloor$ (rounding), $\emptyset$ (null set), $*, \tilde{X}, A^-, P|A, G\backslash F, G/F$.