Erdős-Hajnal for graphs with no 5-hole

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Abstract

The Erdős-Hajnal conjecture says that for every graph H there exists $\tau > 0$ such that every graph G not containing H as an induced subgraph has a clique or stable set of cardinality at least $|G|^\tau$. We prove that this is true when H is a cycle of length five.

We also prove several further results: for instance, that if C is a cycle and H is the complement of a forest, there exists $\tau > 0$ such that every graph G containing neither of C, H as an induced subgraph has a clique or stable set of cardinality at least $|G|^\tau$.

1 Introduction

A cornerstone of Ramsey theory is the theorem of Erdős and Szekeres [14] from the 1930s, that every graph on n vertices has a clique or stable set of size $\Omega(\log n)$. This order of magnitude cannot be improved, as Erdős [11] showed that there are infinitely many graphs G whose cliques and stable sets all have size $O(\log(|G|))$. Indeed, in the class of all graphs, a typical graph only has cliques and stable sets of at most logarithmic size.

The celebrated Erdős-Hajnal conjecture asserts that for every other hereditary class of graphs, the picture is dramatically different: in every such class, all the graphs have cliques or stable sets of *polynomial* size. We say that a graph G *contains* a graph H if some induced subgraph of G is isomorphic to H, and G is H*-free* otherwise. Let $\alpha(G)$ and $\omega(G)$ denote the cardinalities of (respectively) the largest stable sets and cliques in G. The Erdős-Hajnal conjecture [12, 13] asserts the following:

1.1 Conjecture: *For every graph H, there exists $\tau > 0$ such that every H-free graph G satisfies*

$$\max(\alpha(G), \omega(G)) \geq |G|^{\tau}.$$

The Erdős-Hajnal conjecture is only known for a small family of graphs. It is trivially true for $H = K_2$; it is true for $H = P_4$, the four-vertex path (the P_4-free graphs form the well-known class of cographs); and Chudnovsky and Safra [7] showed that it is true when H is the bull (P_4 with an additional vertex adjacent to the two central vertices). It is easy to see that if the conjecture holds for H then it also holds for the complement $\overline{H}$. An important result of Alon, Pach and Solymosi [2] shows that if the conjecture holds for H and H' then it also holds for the graph obtained by substituting H' into a vertex of H. The Erdős-Hajnal conjecture therefore holds for every graph H in the closure of $\{K_2, P_4, \text{bull}\}$ under complements and substitution, but these are all the graphs (with at least two vertices) for which the conjecture was previously known. In particular, the conjecture holds for all graphs H with at most four vertices, but until now, has been open for three graphs on five vertices: C_5, P_5 and $\overline{P_5}$.

The five-vertex cycle C_5 has been a particularly frustrating open case, and has attracted a good deal of unsuccessful attention over the last 30 years (for example, it was highlighted by Erdős and Hajnal [13] and also by Gyárfás [16]). So we are happy to report some progress at last: in this paper, we will prove the conjecture for C_5, and present a number of other results.

It is known that for any H, excluding H makes a difference. Erdős and Hajnal [13] showed the following (and this is still the best known bound for a general graph H):

1.2 *For every graph H, there exists $c > 0$ such that*

$$\max(\alpha(G), \omega(G)) \geq 2^{c\sqrt{\log |G|}}$$

for every H-free graph G with $|G| \geq 1$.

It was also known that we can do better than 1.2 when $H = C_5$. In an earlier paper [6], with Jacob Fox, we showed:

1.3 *There exists $c > 0$ such that*

$$\max(\alpha(G), \omega(G)) \geq 2^{c\sqrt{\log |G| \log\log |G|}}$$

for every C_5-free graph G with $|G| \geq 2$.

Our first result in this paper is the following, proving the Erdős-Hajnal conjecture for C_5:

1.4 *There exists $\tau > 0$ such that every C_5-free graph G satisfies*

$$\max(\alpha(G), \omega(G)) \geq |G|^{\tau}.$$

The proof of 1.4 is novel, but the same proof method, with some extra twists, yields some other results about the Erdős-Hajnal conjecture. It does not seem to show that P_5 has the Erdős-Hajnal property, which, with its complement, is the other open case of 1.1 with $|H| = 5$; but it does give other nice things. In particular, it gives results when certain pairs or small families of induced subgraphs are excluded.

If $\mathcal{H}$ is a set of graphs, G is *$\mathcal{H}$-free* if it is H-free for each $H \in \mathcal{H}$. Every hereditary class of graphs is defined by its excluded subgraphs (that is, the graphs not in the class). Let $\mathcal{H}$ be a set of graphs (or a single graph); we say that *$\mathcal{H}$ has the Erdős-Hajnal property*[1] if there exists $\tau > 0$ such that $\max(\alpha(G), \omega(G)) \geq |G|^{\tau}$ for all $\mathcal{H}$-free graphs (if $\mathcal{H} = \{H\}$ we simply say that *H has the Erdős-Hajnal property*). Thus 1.1 says that every graph has the Erdős-Hajnal property, and 1.4 says that C_5 has the Erdős-Hajnal property. Note that if $\mathcal{H}$ has the Erdős-Hajnal property then so does the set $\{\overline{H} : H \in \mathcal{H}\}$ of complements of members of $\mathcal{H}$.

There has been some recent progress on small sets of graphs with the Erdős-Hajnal property. After partial results by a number of authors (see [4, 5, 17]), the following result was shown in [8]:

1.5 *If F and H are forests then $\{F, \overline{H}\}$ has the Erdős-Hajnal property.*

In this paper, we will show that one of the forests in 1.5 can be replaced by a cycle:

1.6 *If C is a cycle and H is a forest then $\{C, \overline{H}\}$ has the Erdős-Hajnal property.*

We will also show:

1.7 *If C is a cycle and ℓ is an integer, the set consisting of C and the complements of all cycles of length at least ℓ has the Erdős-Hajnal property.*

This strengthens the result of Bonamy, Bousquet and Thomassé [3] that the set consisting of all cycles of length at least ℓ and their complements has the Erdős-Hajnal property (see [9] for a substantial strengthening of this result).

In addition, we will show that a number of other sets of graphs have the Erdős-Hajnal property. For instance, let $\widehat{C_5}$ be the graph obtained from a cycle C of length five by adding a new vertex with neighbours two adjacent vertices of C. We will show:

1.8 $\{\widehat{C_5}, \overline{\widehat{C_5}}\}$ *has the Erdős-Hajnal property.*

[1] Some papers say "the class of $\mathcal{H}$-free graphs has the Erdős-Hajnal property" in this situation, but here the definition we give is more convenient.

Since $\overline{\widehat{C_5}}$ contains $\overline{P_5}$, this implies that $\{\widehat{C_5}, \overline{P_5}\}$ has the Erdős-Hajnal property, strengthening the theorem of [10] that the set of all "cap-free" graphs has the Erdős-Hajnal property. It also implies the result of Chudnovsky and Safra [7] that the bull has the Erdős-Hajnal property, because both $\widehat{C_5}, \overline{\widehat{C_5}}$ contain the bull.

We will further show

1.9 *$\{C_6, \overline{C_6}\}$ has the Erdős-Hajnal property.*

and

1.10 *$\{C_7, \overline{C_7}\}$ has the Erdős-Hajnal property.*

It would be nice to know if the same is true for $\{C_8, \overline{C_8}\}$, but this remains open. We are hopeful that the methods in this paper will lead to further results.

An important ingredient in the paper is a lemma about bipartite graphs that we will prove in the next section. This originated in a powerful lemma that was proved by Tomon [21], and developed further by Pach and Tomon [19]. We prove a significant strengthening of Tomon's result, and use it to prove a key lemma that will be used in the proofs of all our main theorems.

We note that there are a number of different ways to phrase the Erdős-Hajnal conjecture. Let us define $\kappa(G) = \alpha(G)\omega(G)$. For a set $\mathcal{H}$ of graphs, the following are equivalent:

- there exists $\tau > 0$ such that every $\mathcal{H}$-free graph G satisfies $\max(\alpha(G), \omega(G)) \geq |G|^\tau$;
- there exists $\tau > 0$ such that every $\mathcal{H}$-free graph G contains as an induced subgraph a cograph with at least $|G|^\tau$ vertices (this was implicitly used by Erdős and Hajnal [13]);
- there exists $\tau > 0$ such that every $\mathcal{H}$-free graph G contains as an induced subgraph a perfect graph with at least $|G|^\tau$ vertices (this is discussed in [16]);
- there exists $\tau > 0$ such that every $\mathcal{H}$-free graph G satisfies $\kappa(G) \geq |G|^\tau$.

The version using κ is sometimes easier to work with, and we will frequently use it below.

We use standard notation throughout. All graphs in this paper are finite and have no loops or parallel edges. We denote by $|G|$ the number of vertices of a graph G. If $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced on X. We write C_k for the cycle of length k, and P_k for the path with k vertices. Logarithms are to base two.

The paper is organized as follows. First we prove the strengthening of Tomon's theorem that we need, and then apply it to prove our key lemma; then we prove 1.4; then we extend this approach to see what else we can obtain, in particular proving the other theorems mentioned above.

2 A lemma about bipartite graphs

Let G be a graph. We say that two sets of vertices $A, B \subseteq V(G)$ are *complete* if they are disjoint and every element of A is adjacent to every element of B, and *anticomplete* if they are disjoint and no element of A is adjacent to an element of B. We say that a set $\mathcal{H}$ of graphs *has the strong Erdős-Hajnal property* if there exists $c > 0$ such that for every $\mathcal{H}$-free graph G with at least two vertices there are sets $A, B \subseteq V(G)$ with $|A|, |B| \geq c|G|$ such that the pair A, B is either complete or anticomplete. It is easy to prove that if $\mathcal{H}$ has the strong Erdős-Hajnal property then it has the Erdős-Hajnal property (see [1, 15]). This approach has been used in a number of papers to prove the Erdős-Hajnal property for various sets $\mathcal{H}$ (see, for example, [3, 4, 5, 8, 9, 17]).

If a finite set $\mathcal{H}$ of graphs has the strong Erdős-Hajnal property then, by considering sparse random graphs, it is easy to see that it is necessary for $\mathcal{H}$ to contain a forest; and similarly it is necessary for $\mathcal{H}$ to contain the complement of a forest (see [8]). It follows from 1.5 that these conditions are also sufficient, and so 1.5 characterizes finite sets $\mathcal{H}$ that have the strong Erdős-Hajnal property (infinite sets are a different matter: for example the set of all cycles has the strong Erdős-Hajnal property, but does not contain a forest).

Tomon [21] made the nice observation that there is a similar but weaker property that can also be used to prove the Erdős-Hajnal property. Suppose that $\mathcal{H}$ is a set of graphs and there are $c, k > 0$ such that, for every $\mathcal{H}$-free graph G with $|G| \geq 2$, there is some $t = t(G) \geq 2$ such that $V(G)$ includes t sets of size at least $c|G|/t^k$ that are pairwise complete or pairwise anticomplete (note that the strong Erdős-Hajnal property is the special case where we can always choose $t = 2$). We recall that $\kappa(G) = \alpha(G)\omega(G)$; let us write $\kappa(n)$ for the minimum of $\kappa(G)$ over $\mathcal{H}$-free graphs G with n vertices. It follows that $\kappa(G) \geq t\kappa(c|G|/t^k)$, and it is easily checked that this implies that $\kappa(n) \geq n^\tau$ for all n, provided $\tau > 0$ is sufficiently small, and so $\mathcal{H}$ has the Erdős-Hajnal property.

In order to find the required disjoint sets of vertices, Tomon [21] proved a powerful lemma about bipartite graphs, which was developed further by Pach and Tomon [19]. We will make use of the same idea, but will need to prove a significantly stronger form of the lemma.

Let G be a graph, and let $t, k \geq 0$ where t is an integer. We say $((a_i, B_i) : 1 \leq i \leq t)$ is a (t, k)-*comb* in G if:

- $a_1, \dots, a_t \in V(G)$ are distinct, and $B_1, \dots, B_t$ are pairwise disjoint subsets of $V(G) \setminus \{a_1, \dots, a_t\}$;
- for $1 \leq i \leq t$, a_i is adjacent to every vertex in B_i;
- for $i, j \in \{1, \dots, t\}$ with $i \neq j$, a_i has no neighbour in B_j; and
- $B_1, \dots, B_t$ all have cardinality at least k.

If $A, B \subseteq V(G)$ are disjoint and $a_1, \dots, a_t \in A$, and $B_1, \dots, B_t \subseteq B$, we call this a (t, k)-*comb in* (A, B). Our strengthening of Tomon's lemma [21] is as follows:

2.1 *Let G be a graph with a bipartition (A, B), such that every vertex in B has a neighbour in A; and let $\Gamma, \Delta, d > 0$ with $d < 1$, such that every vertex in A has at most Δ neighbours in B. Then either:*

- *for some integer $t \geq 1$, there is a $(t, \Gamma t^{-1/d})$-comb in (A, B); or*
- $|B| \leq \frac{3^{d+1}}{3/2-(3/2)^d}\Gamma^d\Delta^{1-d}$.

Proof. We define a partition of B, formed by pairwise disjoint subsets $C_1, C_2, \dots$ of B, defined inductively as follows. Let $s \geq 1$, and suppose that $C_1, \dots, C_{s-1}$ are defined, and every vertex in A has at most $(2/3)^{s-1}\Delta$ neighbours in D, where $D = B \setminus (C_1 \cup \dots \cup C_{s-1})$. Choose $a_1, a_2, \dots, a_k \in A$ with k maximum such that for $1 \leq i \leq k$, there are at least $(2/3)^s\Delta$ vertices in D that are adjacent to a_i and to none of $a_1, \dots, a_{i-1}$. Let C_s be the set of vertices in D adjacent to one of $a_1, \dots, a_k$;

then from the maximality of k, every vertex in A has at most $(2/3)^s\Delta$ neighbours in $D \setminus C_s$. This completes the inductive definition of $C_1, C_2, \ldots$. Since every vertex in B has a neighbour in A, it follows that every vertex in B belongs to some C_s.

(1) *For all $s \geq 1$, we may assume that $|C_s| \leq 2^{d+1}(2/3)^{s-sd-1}\Gamma^d\Delta^{1-d}$.*

Let $s \geq 1$, and let $a_1, \ldots, a_k$ be as above (that is, chosen with k maximum such that for $1 \leq i \leq k$, there are at least $(2/3)^s\Delta$ vertices in $D = B \setminus (C_1 \cup \cdots \cup C_{s-1})$ that are adjacent to a_i and to none of $a_1, \ldots, a_{i-1}$.) For $1 \leq i \leq k$ let P_i be the set of vertices in D that are adjacent to a_i and to none of $a_1, \ldots, a_{i-1}$; thus each $|P_i| \geq (2/3)^s\Delta$. For $1 \leq i \leq k$, let Q_i be the set of vertices in $D \setminus P_i$ adjacent to a_i; thus every vertex in Q_i is adjacent to one of $a_1, \ldots, a_{i-1}$, and

$$|Q_i| \leq (2/3)^{s-1}\Delta - (2/3)^s\Delta = (2/3)^s\Delta/2$$

since a_i has at most $(2/3)^{s-1}\Delta$ neighbours in D. Inductively, for $i = k, k-1, \ldots, 1$ in turn, we say that a_i is *good* if at most $|P_i|/2$ vertices in P_i are adjacent to a good vertex in $\{a_{i+1}, \ldots, a_k\}$. (Thus a_k is good, if $k > 0$.) Let $\{a_i \ : i \in I\}$ be the set of all good vertices; we claim that $|I| \geq k/2$. Let Q be the union of the sets Q_i $(i \in I)$; then Q has cardinality at most $|I|(2/3)^s\Delta/2$. If $i \in \{1, \ldots, k\} \setminus I$, then at least $|P_i|/2 \geq (2/3)^s\Delta/2$ vertices in P_i belong to Q; and so

$$(k - |I|)(2/3)^s\Delta/2 \leq |Q| \leq |I|(2/3)^s\Delta/2.$$

Consequently $|I| \geq k/2$. For each $i \in I$, let B_i be the set of vertices in P_i that are not in Q; then $|B_i| \geq (2/3)^s\Delta/2$, and $((a_i, B_i) : i \in I)$ is an $(|I|, (2/3)^s\Delta/2)$-comb in (A, B). Let $t = |I|$; so we may assume that either $t = 0$, or $(2/3)^s\Delta/2 < \Gamma t^{-1/d}$ (since otherwise the theorem holds); and in either case, $t < (2\Gamma(3/2)^s/\Delta)^d$. Hence $k \leq 2(2\Gamma(3/2)^s/\Delta)^d$, and

$$|C_s| \leq 2(2\Gamma(3/2)^s/\Delta)^d\Delta(2/3)^{s-1}.$$

This proves (1).

Now since $d < 1$, the sum of $(2/3)^{s-sd-1}$ over all integers $s \geq 1$ equals

$$\frac{(3/2)^d}{1 - (2/3)^{1-d}},$$

and so

$$|B| = |C_1| + |C_2| + \cdots \leq \frac{2^{d+1}(3/2)^d}{1 - (2/3)^{1-d}}\Gamma^d\Delta^{1-d} = \frac{3^{d+1}}{3/2 - (3/2)^d}\Gamma^d\Delta^{1-d}.$$

This proves 2.1. ∎

3 Applying the bipartite lemma

In this section we use 2.1 to prove our key lemma. Given a vertex $x \in V(G)$, we will apply 2.1 to the bipartite graph of edges between $A = N(x)$ and $B = V(G) \setminus (A \cup \{x\})$. By 2.1, this will either give us a large comb $((a_i, B_i)\ : 1 \le i \le t)$, or it will show that A has poor expansion. In the first case, we try to use the comb either to find H or to find many large sets of vertices that are pairwise complete or pairwise anticomplete; in the second, as there are no edges between $G[A]$ and $G[B \setminus N(A)]$ we can handle them separately. In both cases, it will be helpful if the set $\{a_1, \dots, a_t\}$ is a stable set: it turns out that we can build this into the key lemma.

Let $\tau > 0$. We say that a graph G is *τ-critical* if $\kappa(G) < |G|^\tau$, and $\kappa(G') \ge |G'|^\tau$ for every induced subgraph G' of G with $G' \neq G$. The next result is the key lemma that unlocked all the main results in this paper:

3.1 *For all $\delta, \varepsilon > 0$ with $\varepsilon < 1/20$, there exists $\tau > 0$ with the following property. Let G be a τ-critical graph, and let $X \subseteq V(G)$ with $|X| \ge \delta|G|$, such that $G[X]$ has maximum degree at most $\varepsilon\delta|G|$. Then there is a $(t, \delta|G|/(400\varepsilon t^2))$-comb $((a_i, B_i) : 1 \le i \le t)$ of $G[X]$ such that $t \ge 1/(400\varepsilon)$ and $\{a_1, \dots, a_t\}$ is stable, and there is a vertex $v \in X$ adjacent to $a_1, \dots, a_t$ and with no neighbours in $B_1 \cup \dots \cup B_t$.*

Proof. Choose τ with $0 < \tau < 1$ so small that

$$\frac{2^{1-1/\tau}}{\delta} + \left(\varepsilon + \frac{19}{20}\right)(\varepsilon\delta)^{-\tau} < 1.$$

(This is possible since $\varepsilon < 1/20$.) We claim that τ satisfies the theorem.

Let G, X be as in the theorem. We may assume that $\kappa(G) \ge 2$. It follows that $2 < |G|^\tau$, and so $|G| > 2^{1/\tau}$.

Let $X_0 = X$. Inductively, given a set $X_{i-1} \subseteq X$ with $X_{i-1} \neq \emptyset$, we make the following definitions:

- Let $v_i \in X_{i-1}$ have maximum degree in $G[X_{i-1}]$.
- Let A_i be the set of neighbours of v_i in $G[X_{i-1}]$ (possibly $A_i = \emptyset$).
- Let $C_i \subseteq A_i$ be a stable set with $|C_i| \ge |A_i|^\tau/\omega(G)$. (This exists, since G is τ-critical. Possibly $C_i = \emptyset$, but only if $A_i = \emptyset$.)
- Let X_i be the set of vertices in X_{i-1} with no neighbour in $\{v_i\} \cup C_i$.

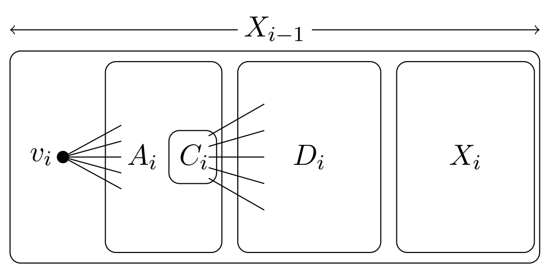

Figure 1: Figure for 3.1

The inductive definition stops when $|X_i| = \emptyset$; let this occur when $i = s$ say. Thus we define a nested sequence of subsets

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_s = \emptyset;$$

and also vertices $v_i \in X_{i-1} \setminus X_i$ and subsets $A_i, C_i \subseteq X_{i-1} \setminus X_i$ for $1 \le i \le s$. Note that there are no edges between $\{v_i\} \cup C_i$ and X_j for $i < j$.

For $1 \le i \le s$, let D_i be the set of vertices in X_{i-1} not in $A_i \cup \{v_i\}$, and with a neighbour in C_i. Let $\gamma = \delta/(400\varepsilon)$.

(1) *We may assume that* $|D_i| \le 19\,(\gamma|A_i| \cdot |X|/\delta)^{1/2}$ *for* $1 \le i < s$.

From the choice of v_i, every vertex in C_i has at most $|A_i|$ neighbours in D_i. By 2.1 applied to the bipartite graph between C_i and D_i, replacing Γ, Δ, d by $\gamma|X|/\delta, |A_i|, 1/2$, we deduce that either for some integer $t \ge 1$, there is a $(t, \gamma|X|/(\delta t^2))$-comb in (C_i, D_i), or

$$|D_i| \le \frac{3^{3/2}}{3/2 - (3/2)^{1/2}} (\gamma|X| \cdot |A_i|/\delta)^{1/2}.$$

Suppose the first holds, and let the comb be $((a_j, B_j) : 1 \le j \le t)$. The sets $B_1, \dots, B_t$ are pairwise disjoint subsets of X, and so $t\gamma|X|/(\delta t^2) \le |X|$, that is, $t \ge \gamma/\delta = 1/(400\varepsilon)$. Since $|X| \ge \delta|G|$, it follows that

$$\gamma|X|/(\delta t^2) \ge \gamma|G|/t^2 = \delta|G|/(400\epsilon t^2);$$

and therefore in this case the conclusion of the theorem is true. So we may assume that the second bullet holds. Since $3^{3/2}/(3/2 - (3/2)^{1/2}) \le 19$, this proves (1).

For $1 \le i \le s$, let $x_i = |A_i|/|X|$. Since $C_1 \cup \dots \cup C_s$ is stable, and hence has cardinality at most $\alpha(G)$, and

$$|C_i| \ge \frac{|A_i|^\tau}{\omega(G)} = \frac{(x_i|X|)^\tau}{\omega(G)} \ge \frac{(x_i\delta|G|)^\tau}{\omega(G)} \ge (x_i\delta)^\tau \alpha(G)$$

for each i, it follows that $\sum_{1 \le i \le s} x_i^\tau < \delta^{-\tau}$.

Now X is partitioned into the sets $\{v_i\}$ $(1 \le i \le s)$, A_i $(1 \le i \le s)$ and D_i $(1 \le i \le s)$, and so

$$\sum_{1 \le i \le s} (1 + |A_i| + |D_i|) = |X|,$$

that is,

$$\frac{s}{|X|} + \sum_{1 \le i \le s} \frac{|A_i|}{|X|} + \sum_{1 \le i \le s} \frac{|D_i|}{|X|} = 1.$$

We will bound these three terms separately.

First, since $\{v_1, \dots, v_s\}$ is stable, it follows that

$$\frac{s}{|X|} \le \frac{\alpha(G)}{|X|} \le \frac{|G|^\tau}{|X|} \le \frac{|G|^{\tau-1}}{\delta},$$

and since $|G|^{\tau-1} \le 2^{1-1/\tau}$ (because $|G| \ge 2^{1/\tau}$), it follows that $s/|X| < 2^{1-1/\tau}/\delta$.

Second,

$$\sum_{1 \le i \le s} \frac{|A_i|}{|X|} = \sum_{1 \le i \le s} x_i = \sum_{1 \le i \le s} x_i^\tau x_i^{1-\tau} \le \sum_{1 \le i \le s} x_i^\tau \varepsilon^{1-\tau} \le \varepsilon(\varepsilon\delta)^{-\tau}$$

since $x_i = |A_i|/|X| \le \varepsilon\delta|G|/|X| \le \varepsilon$.

Third,

$$\sum_{1\le i\le s} \frac{|D_i|}{|X|} \le 19\left(\frac{\gamma}{\delta}\right)^{1/2} \sum_{1\le i\le s} x_i^{1/2} = \frac{19}{20}\varepsilon^{-1/2} \sum_{1\le i\le s} x_i^{1/2}$$

by (1) and the definition of γ; and

$$\sum_{1\le i\le s} x_i^{1/2} = \sum_{1\le i\le s} x_i^{\tau} x_i^{1/2-\tau} \le \sum_{1\le i\le s} x_i^{\tau}\varepsilon^{1/2-\tau} \le \varepsilon^{1/2}(\varepsilon\delta)^{-\tau}.$$

Consequently

$$\sum_{1\le i\le s} \frac{|D_i|}{|X|} \le \frac{19}{20}(\varepsilon\delta)^{-\tau}.$$

Summing, we deduce that

$$\frac{2^{1-1/\tau}}{\delta} + \left(\varepsilon + \frac{19}{20}\right)(\varepsilon\delta)^{-\tau} \ge 1,$$

contrary to the choice of τ. This proves 3.1. ∎

This gives us a $(t, \delta|G|/(400\varepsilon t^2))$-comb. The t^2 in the denominator comes from applying 2.1 with $d = 1/2$; as was observed by Pach and Tomon [19], we could apply 2.1 with $d = 1/k$, for any real $k > 1$, and produced a comb with t^k in the denominator, but there is no gain for us in the applications.

4 The simplest application: excluding C_5

In this section we prove 1.4. This is implied by each of several stronger results later in the paper, but since the C_5 result is of great interest, and the argument for C_5 is easier than the material to come later (which will require additional ideas), we give a separate proof. We will need Rödl's theorem [20]:

4.1 *For every graph H and all $\varepsilon > 0$, there exists $\delta > 0$ such that for every H-free graph G, there exists $X \subseteq V(G)$ with $|X| \ge \delta|G|$, such that one of $G[X], \overline{G}[X]$ has at most $\varepsilon|X|(|X|-1)$ edges.*

We also need the following:

4.2 *Let G be a graph with at most $\varepsilon|G|(|G|-1)$ edges; then for every integer $m \ge 0$ with $m \le (|G|+1)/2$, there exists $X \subseteq V(G)$ with $|X| = m$ such that $G[X]$ has maximum degree less than $4\varepsilon(m-1)$.*

Proof. By averaging over all subsets Y of $V(G)$ with cardinality $2m-1$, it follows that there exists such a set Y where $G[Y]$ has at most $\varepsilon(2m-1)(2m-2) < 4\varepsilon m(m-1)$ edges. Choose $X \subseteq Y$ with cardinality m, such that $|E(G[X])|$ is as small as possible. Suppose that some $u \in X$ has at least $4\varepsilon(m-1)$ neighbours in X. It follows from the choice of X that for each $v \in Y \setminus X$, $|E(G[X'])| \ge |E(G[X])|$, where $X' = (X \setminus \{u\}) \cup \{v\}$, and so v has at least $4\varepsilon(m-1)$ neighbours in $X \setminus \{u\}$. Since $(Y \setminus X) \cup \{u\}$ has cardinality m, the total number of edges between $X \setminus \{u\}$ and $(Y \setminus X) \cup \{u\}$ is at least $4\varepsilon m(m-1) > |E(G[Y])|$, a contradiction. This proves 4.2. ∎

We deduce a slight but convenient strengthening of 4.1, the following (this is well-known, but we include the proof for completeness):

4.3 *For every graph H and all $\varepsilon > 0$, there exists $\delta > 0$ such that for every H-free graph G, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$, such that one of $G[X], \overline{G}[X]$ has maximum degree at most $\varepsilon\delta|G|$.*

Proof. Let $\varepsilon' = \varepsilon/4$. By 4.1 there exists $\delta' > 0$ such that for every H-free graph G, there exists $Z \subseteq V(G)$ with $|Z| \geq \delta'|G|$, such that one of $G[Z], \overline{G}[Z]$ has at most $\varepsilon'|Z|(|Z|-1)$ edges. Let $\delta = \delta'/2$; we claim that δ satisfies the theorem. Thus, let G be H-free. By the choice of δ', there exists $Z \subseteq V(G)$ with $|Z| \geq \delta'|G|$, such that one of $G[Z], \overline{G}[Z]$ has at most $\varepsilon'|Z|(|Z|-1)$ edges. By replacing G by its complement if necessary, we may assume the first. Let $m = \lceil \delta|G| \rceil$; then

$$|Z| \geq \lceil \delta'|G| \rceil \geq 2m - 1.$$

By 4.2 applied to $G[Z]$, there exists $X \subseteq Z$ with $|X| = m$ such that $G[X]$ has maximum degree less than $4\varepsilon'(m-1) \leq \varepsilon\delta|G|$. This proves 4.3. ∎

If $X \subseteq V(G)$, we sometimes write $\alpha(X)$ for $\alpha(G[X])$ and so on. Now we can prove the main result of this section, which we restate:

4.4 *C_5 has the Erdős-Hajnal property.*

Proof. Choose ε with $0 < \varepsilon < 1/400$, and choose δ satisfying 4.3 with $H = C_5$. Let $\tau > 0$ satisfy 3.1. Every positive number smaller than τ also satisfies 3.1, and since $400\varepsilon < 1$, by reducing τ we may assume that $(400\varepsilon)^{2-1/\tau} > 400\varepsilon/\delta$. We will show that $\kappa(G) \geq |G|^\tau$ for every C_5-free graph G. By the remarks in the introduction, this is equivalent to showing that C_5 has the Erdős-Hajnal property.

Suppose that there is a C_5-free graph G with $\kappa(G) < |G|^\tau$, and choose G minimal; then G is τ-critical. By 4.3 there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$, such that one of $G[X], \overline{G}[X]$ has maximum degree at most $\varepsilon\delta|G|$. By replacing G with its complement if necessary (this is legitimate since $\overline{G}$ is also C_5-free and τ-critical) we may assume that $G[X]$ has maximum degree at most $\varepsilon\delta|G|$. By 3.1 and the choice of τ, there is a $(t, \delta|G|/(400\varepsilon t^2))$-comb $((a_i, B_i) : 1 \leq i \leq t)$ of $G[X]$ such that $t \geq 1/(400\varepsilon)$ and $\{a_1, \dots, a_t\}$ is stable, and there is a vertex $v \in X$ adjacent to $a_1, \dots, a_t$ and with no neighbours in $B_1 \cup \dots \cup B_t$.

If there exist i, j with $1 \leq i < j \leq t$ such that some vertex $b_i \in B_i$ has a neighbour $b_j \in B_j$, then the subgraph induced on $\{b_1, b_2, a_1, a_2, v\}$ is isomorphic to C_5, a contradiction. So the sets $B_1, \dots, B_t$ are pairwise anticomplete. Since G is τ-critical, it follows that $\kappa(B_i) \geq |B_i|^\tau$ for each i, and since $\kappa(B_i) \leq \alpha(B_i)\omega(G)$, we have

$$\alpha(B_i) \geq |B_i|^\tau/\omega(G) \geq (\delta|G|/(400\varepsilon t^2))^\tau/\omega(G).$$

Since $B_1, \dots, B_t$ are pairwise anticomplete, it follows that

$$\alpha(G) \geq \sum_{1 \leq i \leq t} \alpha(B_i) \geq t(\delta|G|/(400\varepsilon t^2))^\tau/\omega(G),$$

and so $\kappa(G) \geq t(\delta|G|/(400\varepsilon t^2))^\tau$. Since $\kappa(G) < |G|^\tau$, it follows that $400\varepsilon/\delta \geq t^{1/\tau-2}$. But $t \geq 1/(400\varepsilon)$, and $\tau < 1/2$, and so $400\varepsilon/\delta \geq (400\varepsilon)^{2-1/\tau}$, contrary to the choice of τ. This proves 4.4. ∎

5 Blockades

Next we will add some refinements to the proof of 1.4, but first let us set up some more terminology. Let G be a graph. A *pure pair* in G is a pair A, B of disjoint subsets of $V(G)$ such that A is either complete or anticomplete to B. A *blockade* $\mathcal{B}$ in G is a sequence $(B_1, \dots, B_t)$ of pairwise disjoint subsets of $V(G)$ called *blocks*. (In this paper the order of the blocks $B_1, \dots, B_t$ in the sequence will not matter.) We denote $B_1 \cup \dots \cup B_t$ by $V(\mathcal{B})$. The *length* of a blockade is the number of blocks, and its *width* is the minimum cardinality of a block.

A blockade $\mathcal{B} = (B_1, \dots, B_t)$ in G is *pure* if (B_i, B_j) is a pure pair for all i, j with $1 \leq i < j \leq t$. Let P be the graph with vertex set $\{1, \dots, t\}$, in which i, j are adjacent if B_i is complete to B_j. We say P is the *pattern* of the pure blockade $\mathcal{B}$. A *cograph* is a P_4-free graph. Every cograph P with more than one vertex admits a pure pair (A, B) with $A, B \neq \emptyset$ and with $A \cup B = V(P)$.

We need:

5.1 *Let $\mathcal{B} = (B_1, \dots, B_t)$ be a pure blockade with a cograph pattern. Then*

$$\kappa(B_1 \cup \dots \cup B_t) \geq \sum_{1 \leq i \leq t} \kappa(B_i).$$

Proof. We proceed by induction on t. If $t = 1$ the claim is true, so we assume $t > 1$. Hence there is a partition (I, J) of $\{1, \dots, t\}$, with $I, J \neq \emptyset$, such that either B_i is complete to B_j for all $i \in I$ and $j \in J$, or B_i is anticomplete to B_j for all $i \in I$ and $j \in J$. We may assume the second by replacing G by its complement if necessary. Let $V = V(\mathcal{B})$, and $U = \bigcup_{i \in I} B_i$, and $W = \bigcup_{j \in J} B_j$. Thus U is anticomplete to W. From the inductive hypothesis, $\kappa(U) \geq \sum_{i \in I} \kappa(B_i)$ and $\kappa(W) \geq \sum_{j \in J} \kappa(B_j)$. But

$$\kappa(V) = \alpha(V)\omega(V) = (\alpha(U) + \alpha(W))\omega(V) \geq \alpha(U)\omega(U) + \alpha(W)\omega(W) = \kappa(U) + \kappa(W),$$

and the result follows. This proves 5.1. ∎

The following is a slight extension of an idea of Pach and Tomon [19] (which they called the "quasi-Erdős-Hajnal property"):

5.2 *Let $\tau > 0$, and suppose that G is τ-critical. Then for every integer $t > 0$, there is no pure blockade in G with a cograph pattern, of length t and width at least $|G|t^{-1/\tau}$, such that $B_i \neq V(G)$ for each i.*

Proof. Suppose that $\mathcal{B} = (B_1, \dots, B_t)$ is such a blockade. Since G is τ-critical, $\kappa(B_i) \geq |B_i|^\tau \geq |G|^\tau / t$ for each i, and so by 5.1,

$$\kappa(G) \geq \kappa(B_1 \cup \dots \cup B_t) \geq \sum_{1 \leq i \leq t} \kappa(B_i) \geq |G|^\tau,$$

a contradiction. This proves 5.2. ∎

6 Forests and their complements

The proof of 4.4 can be developed to give more. We have two ways to do so, and in this section we explain the first.

If H is a graph, we wish to augment it in two ways. Let the vertices of H be $\{b_1, \dots, b_k\}$, and add $k+1$ new vertices $a_1, \dots, a_k, v$ to $V(H)$, where a_i is adjacent to b_i for $1 \le i \le k$, and v is adjacent to $a_1, \dots, a_k$, and there are no other edges. Let the graph we obtain be H'. We call H' a *star-expansion* of H.

In this section we will prove:

6.1 *Let H be a forest. Let H_1 be the star-expansion of H, and let H_2 be the star-expansion of $\overline{H}$. Then*

$$\{H_1, H_2, \overline{H_1}, \overline{H_2}\}$$

has the Erdős-Hajnal property.

6.1 is particularly nice when $H = P_4$, since P_4 is isomorphic to its complement, and so we only have to exclude two graphs instead of four. We obtain:

6.2 *Let H be the graph of figure 2; then $\{H, \overline{H}\}$ has the Erdős-Hajnal property.*

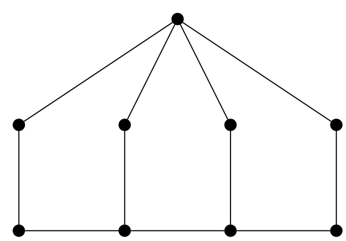

Figure 2: The star-expansion of P_4.

This contains 1.4, 1.9 and 1.10, because the graph of figure 2 contain C_5, C_6 and C_7. (The approach via 6.1 does not work for $C_8, \overline{C_8}$, because there is no forest H such that the star-expansion of $\overline{H}$ contains one of $C_8, \overline{C_8}$.)

If $\mathcal{B} = (B_1, \dots, B_t)$ is a blockade in G, we say an induced subgraph H of G is *$\mathcal{B}$-rainbow* if $V(H) \subseteq V(\mathcal{B})$ and $|B_i \cap V(H)| \le 1$ for $1 \le i \le t$. To prove 6.1 we need the following theorem of [8]:

6.3 *For every forest H, there exist $d > 0$ and an integer K with the following property. Let G be a graph with a blockade $\mathcal{B}$ of length at least K, and let W be the width of $\mathcal{B}$. If every vertex of G has degree less than W/d, and there is no anticomplete pair $A, B \subseteq V(G)$ with $|A|, |B| \ge W/d$, then there is a $\mathcal{B}$-rainbow copy of H in G.*

We used this in [8] to deduce that for every forest H, the set $\{H, \overline{H}\}$ has the Erdős-Hajnal property. We see that 6.1 (applied to the forest H) will be an extension of that result, since the four graphs of 6.1 all contain one of $H, \overline{H}$.

To prove 6.1, we need to bootstrap 6.3 into something stronger, and we do so in several stages. We will use a strengthening of 4.1, due to Nikiforov [18], the following:

6.4 *For all $\varepsilon > 0$ and every graph H on h vertices, there exist $\gamma, \delta > 0$ such that if G is a graph containing fewer than $\gamma|G|^h$ induced labelled copies of H, then there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that one of $G[X], \overline{G}[X]$ has at most $\varepsilon|X|(|X|-1)$ edges.*

Applying 4.2 as before yields:

6.5 *For all $\varepsilon > 0$ and every graph H on h vertices, there exist $\gamma, \delta > 0$ such that if G is a graph containing fewer than $\gamma|G|^h$ induced labelled copies of H, then there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that one of $G[X], \overline{G}[X]$ has maximum degree at most $\varepsilon\delta|G|$.*

Let us prove a version of 6.3 without the sparsity hypothesis:

6.6 *For every forest H, there exist $d > 0$ and an integer K, such that, for every graph G with a blockade $\mathcal{B}$ of length at least K, if there is no pure pair $A, B \subseteq V(G)$ with $|A|, |B| \geq W/d$, where W is the width of $\mathcal{B}$, then there is a $\mathcal{B}$-rainbow copy of one of $H, \overline{H}$ in G.*

Proof. Choose d', K' to satisfy 6.3 (with d, K replaced by d', K'). Let $\varepsilon \leq 1/(2d'K')$ with $\varepsilon > 0$, and choose $\gamma, \delta > 0$ to satisfy 6.5. Choose $K \geq 2K'/\delta$, and such that $(1 - h/K)^h > 1 - \gamma$. Choose d such that $d \geq d'K'/(\delta K - K')$. We claim that K, d satisfy the theorem.

(1) $\delta K/K' - 1 \geq \max(\varepsilon\delta d'K, d'/d)$.

To see that $\delta K/K' - 1 \geq \varepsilon\delta d'K$, observe that $\delta K/(2K') \geq 1$, and $\delta K/(2K') \geq \varepsilon\delta d'K$. The second part, that $\delta K/K' - 1 \geq d'/d$, is true from the choice of d. This proves (1).

Let G be a graph with a blockade $\mathcal{B} = (B_1, \ldots, B_K)$ and of width W. We may assume that $|B_i| = W$ for each i, and so $|V| = KW$, where $V = B_1 \cup \cdots \cup B_K$. We assume that there is no $\mathcal{B}$-rainbow copy of H. But the number of sequences $(v_1, \ldots, v_h)$ with $v_1, \ldots, v_h \in V$, such that $v_1, \ldots, v_h$ all belong to different blocks of the blockade, is

$$|V|(|V| - W)(|V| - 2W) \cdots (|V| - (h-1)W) > (1 - h/K)^h|V|^h \geq (1 - \gamma)|V|^h,$$

and since none of them induce a $\mathcal{B}$-rainbow copy of H, it follows that the number of induced labelled copies of H in $G[V]$ is less than $|V|^h - (1-\gamma)|V|^h = \gamma|V|^h$. By 6.5 applied to $G[V]$, there exists $X \subseteq V$ with $|X| \geq \delta KW$, such that one of $G[X], \overline{G}[X]$ has maximum degree less than $\varepsilon\delta KW$; and by replacing G by $\overline{G}$ if necessary, we may assume that $G[X]$ has maximum degree less than $\varepsilon\delta KW$. By (1), there exists a real number W' such that

$$\frac{\delta K}{K'} - 1 \geq \frac{W'}{W} \geq \max\left(\varepsilon\delta d'K, \frac{d'}{d}\right).$$

The sets $B_1 \cap X, \ldots, B_K \cap X$ each have cardinality at most W, but their union has cardinality at least δKW. Let us choose pairwise disjoint subsets $I_1, \ldots, I_t$ of $\{1, \ldots, K\}$, with t maximum such that $|B'_h| \geq W'$ for $1 \leq h \leq t$, where $B'_h = \bigcup_{i \in I_h} B_i \cap X$. We may assume that $I_1, \ldots, I_t$ are minimal with this property, and so $|B'_h| \leq W' + W$ for $1 \leq h \leq t$. From the maximality of t,

$$\sum(|B_i \cap X| : i \in \{1, \ldots, t\} \setminus I_1 \cup \cdots \cup I_t) < W';$$

and so $\delta KW \leq |X| \leq t(W'+W)+W'$. Since $\delta K/K'-1 \geq W'/W$ it follows that $t \geq K'$.

Let $\mathcal{B}'$ be the blockade $(B'_1,\dots,B'_{K'})$; it has width at least W'.

(2) *There is a $\mathcal{B}'$-rainbow copy of H.*

Suppose not. Let $G' = G[B'_1 \cup \dots \cup B'_{K'}]$. By 6.3 applied to G', either

- some vertex of G' has degree at least W'/d'; or
- there is an anticomplete pair $A, B \subseteq V(G')$ with $|A|,|B| \geq W'/d'$.

But the first does not hold, since G' has maximum degree less than $\varepsilon\delta KW \leq W'/d'$; and the second does not hold, since $W'/d' \geq W/d$ and there is no pure pair (A,B) in G with $|A|,|B| \geq W/d$. This proves (2).

The $\mathcal{B}'$-rainbow copy of H in (2) is also $\mathcal{B}$-rainbow. This proves 6.6. ∎

6.7 *For every forest H, there exist an integer $d > 0$, such that, for every integer $s \geq 1$ and every graph G, the following holds. Let $D = 2^{s-1}d^{2s-1}$, and let $\mathcal{B}$ be a blockade in G of length D. Then either*

- *G admits a pure blockade $\mathcal{A}$ with a cograph pattern, of length 2^s and width at least W/D, where W is the width of $\mathcal{B}$; or*
- *there is a $\mathcal{B}$-rainbow copy of one of $H, \overline{H}$ in G.*

Proof. Choose K, d to satisfy 6.6. Then any pair of numbers K', d' with $K' \geq K$ and $d' \geq d$ also satisfy 6.6, so by increasing K or d if necessary, we may assume that $K = d$. We claim that d satisfies 6.7. This is true if $s = 1$, from the choice of d, and so we assume it is true for some $s \geq 1$ and prove it for $s+1$.

Let $D = 2^s d^{2s+1}$, and let G be a graph with a blockade $\mathcal{B} = (B_1,\dots,B_D)$ of width W. Partition $\{1,\dots,D\}$ into d sets of cardinality D/d, say $I_1,\dots,I_d$. Let $B'_h = \bigcup_{i \in I_h} B_i$ for $1 \leq i \leq d$; then $\mathcal{B}' = (B'_1,\dots,B'_d)$ is a blockade of length d and width WD/d. Let $G' = G[B_1 \cup \dots \cup B_D]$. We may assume there is no $\mathcal{B}'$-rainbow copy of H or of $\overline{H}$ in G'; so from the choice of d, there is a pure pair (A,B) of G' with $|A|,|B| \geq WD/d^2$.

Let $W' = W/(2d^2)$, and $D' = 2^{s-1}d^{2s-1}$. Let p be the number of $i \in \{1,\dots,D\}$ such that $|A \cap B_i| \geq W'$. Then $pW + DW' \geq |A| \geq WD/d^2$, and so $p \geq D/(2d^2) = D'$. Let $\mathcal{C}$ be the blockade formed by the D' largest sets of the form $A \cap B_i$; then $\mathcal{C}$ has width at least W', and we may assume that there is no $\mathcal{C}$-rainbow copy of H or of $\overline{H}$. Thus the inductive hypothesis, applied to the blockade $\mathcal{C}$ of $G[A]$ implies that $G[A]$ admits a pure blockade with a cograph pattern, of width at least $W'/D' = W/D$ and length 2^s; and similarly so does $G[B]$. But then combining these gives a pure blockade in G with a cograph pattern, of width at least W/D and length 2^{s+1}. This proves 6.7. ∎

Now finally we can prove 6.1, which we restate:

6.8 *Let H be a forest. Let H_1 be the star-expansion of H, let H_2 be the star-expansion of $\overline{H}$, and let*

$$\mathcal{H} = \{H_1, H_2, \overline{H_1}, \overline{H_2}\}.$$

Then $\mathcal{H}$ has the Erdős-Hajnal property.

Proof. Much of the proof is the same as for 4.4. Let d satisfy 6.7. Choose $\varepsilon > 0$ with $\varepsilon < 1/(400d)$, choose δ to satisfy 4.3 with $H = H_1$, and let $\gamma = \delta/(400\varepsilon)$. Choose $\tau > 0$ satisfying 3.1, such that $1/\tau > 3 + 6\log_2(d)$, and such that $2^q d^{2q+1} < 1/(400\varepsilon)$ where

$$q = \frac{\log_2(d) - \log_2(\gamma)}{1/\tau - 3 - 6\log_2(d)}.$$

(We can satisfy the last condition since by making τ sufficiently small we can make q arbitrarily close to 0, and hence make $2^q d^{2q+1}$ arbitrarily close to $d < 1/(400\varepsilon)$.)

As in the proof of 4.4, we may assume that there is a τ-critical $\mathcal{H}$-free graph G, and there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$, such that $G[X]$ has maximum degree at most $\varepsilon\delta|G|$.

By 3.1 and the choice of τ, there is a $(t, \gamma|G|/t^2)$-comb $((a_i, B_i) : 1 \leq i \leq t)$ of $G[X]$ such that $t \geq 1/(400\varepsilon)$ and $\{a_1, \ldots, a_t\}$ is stable, and there is a vertex $v \in X$ adjacent to $a_1, \ldots, a_t$ and with no neighbours in $B_1 \cup \cdots \cup B_t$. Let $\mathcal{B} = (B_1, \ldots, B_t)$.

(1) *There is a $\mathcal{B}$-rainbow copy of H or of $\overline{H}$.*

Suppose not. Choose an integer s maximum such that $D_s \leq t$, where $D_s = 2^{s-1}d^{2s-1}$. Thus $D_{s+1} > t$. Since $D_{q+1} \leq 1/(400\varepsilon) \leq t$, it follows that $s > q$.

By 6.7, G admits a pure blockade $\mathcal{A}$ with a cograph pattern, of width at least $\gamma|G|/(t^2 D_s)$ and length 2^s. By 5.2, $\gamma|G|/(t^2 D_s) < |G|(2^s)^{-1/\tau}$, that is, $\gamma < t^2 D_s 2^{-s/\tau}$. The maximality of s implies that $2^s d^{2s+1} \geq t$, and so, substituting for t and for D_s, we obtain

$$\gamma < 2^{2s} d^{4s+2} 2^{s-1} d^{2s-1} 2^{-s/\tau}.$$

It follows that $\log_2(\gamma) + s/\tau - 3s + 1 < (6s+1)\log_2(d)$, and so

$$\left(\frac{1}{\tau} - 3 - 6\log_2(d)\right) s < \log_2(d) - \log_2(\gamma).$$

Hence $s < q$, a contradiction. This proves (1).

But now the result follows as in 4.4. This proves 6.1. ∎

7 Excluding a forest complement

If H is a forest, then since two of the four graphs of 6.1 contain $\overline{H}$, it follows that the set consisting of $\overline{H}$ and the remaining two graphs in 6.1 has the Erdős-Hajnal property. But we can do better than this: it is sufficient just to exclude one of the remaining two, as we show in this section. This is proved by a slight variation in the proof of 6.1.

We will need the following theorem of [8] (it is a consequence of 6.3):

7.1 *For every forest H, there exists $\varepsilon > 0$ such that if a graph G with $|G| > 1$ has maximum degree less than $\varepsilon|G|$, and has no anticomplete pair of sets $A, B \subseteq V(G)$ with $|A|, |B| \geq \varepsilon|G|$, then G contains H.*

We use this to prove:

7.2 *Let H be a forest, and let H' be the star-expansion of H. Then $\mathcal{H} = \{\overline{H}, H'\}$ has the Erdős-Hajnal property.*

Proof. We define $d, \varepsilon, \delta, \tau$ and the rest, exactly as in the proof of 6.1, except we choose ε satisfying 7.1 as well as the other conditions, and choose τ such that $\varepsilon\delta > 2^{-1/\tau}$ as well as the other conditions.

As before, we may assume that there is a τ-critical $\mathcal{H}$-free graph G, and there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$, such that one of $\overline{G}[X], G[X]$ has maximum degree at most $\varepsilon\delta|G|$. (We are not free to replace G by its complement, since the class of $\mathcal{H}$-free graphs is not closed under taking complements.)

Suppose that $\overline{G}[X]$ has maximum degree at most $\varepsilon\delta|G|$. By 7.1 applied to $\overline{G}$, there exist disjoint $A, B \subseteq X$, with A complete to B, and with $|A|, |B| \geq \varepsilon\delta|G|$. By 5.2, $\varepsilon\delta|G| < |G|2^{-1/\tau}$, and so $\varepsilon\delta < 2^{-1/\tau}$, contrary to the choice of τ.

Thus $G[X]$ has maximum degree at most $\varepsilon\delta|G|$. Exactly as in the proof of 6.1, we obtain the blockade $\mathcal{B}$, and prove there is a $\mathcal{B}$-rainbow copy of H or of $\overline{H}$. The second is impossible since G is $\overline{H}$-free; and so G contains the star-expansion of H. This proves 7.2. ∎

We see that 1.6 follows from 7.2, by applying 7.2 to a forest H' containing H and containing a path of length at least $|E(C)| - 4$ (because then the star-expansion of H' contains C.) The following is a theorem of Bonamy, Bousquet and Thomassé [3]:

7.3 *For every integer $\ell > 0$, there exists $\varepsilon > 0$ such that if G has maximum degree less than $\varepsilon|G|$, and G has no anticomplete pair (A, B) with $|A|, |B| \geq \varepsilon|G|$, then G has a hole of length at least ℓ.*

The proof of 7.2 can be modified to show the following, by using 7.3 in place of 7.1 (we omit the details):

7.4 *Let H be the star-expansion of a forest; then for every integer $\ell \geq 3$,*

$$\{H, \overline{C_\ell}, \overline{C_{\ell+1}}, \overline{C_{\ell+2}}, \ldots\}$$

has the Erdős-Hajnal property.

This implies 1.7, by letting H be the star-expansion of a path of length $|E(C)| - 4$. (We may assume that C has length at least five, because it is known that C_3, C_4 both have the Erdős-Hajnal property.)

8 C_5 with a hat

There is still one result mentioned in the introduction that is not contained in any of the results we proved so far, namely 1.8, and now we will prove that.

8.1 $\mathcal{H} = \{\widehat{C_5}, \overline{\widehat{C_5}}\}$ *has the Erdős-Hajnal property.*

Proof. We proceed as usual: as in all these proofs, we choose a suitable $\varepsilon \le 1/20$, choose δ satisfying 4.3, and then choose $\tau > 0$ satisfying 3.1, and we can also make τ less than any positive function of the other parameters we choose. Let us see what we need.

We may assume (for a contradiction) that there is a τ-critical $\mathcal{H}$-free graph G; and there exists $X \subseteq V(G)$ with $|X| \ge \delta|G|$, such that $G[X]$ has maximum degree at most $\varepsilon\delta|G|$. (We can pass to the complement if necessary.) Let $\gamma = \delta/(400\varepsilon)$. By 3.1, there is a $(t, \gamma|G|/t^2)$-comb $((a_i, B_i) : 1 \le i \le t)$ of $G[X]$ such that $t \ge 1/(400\varepsilon)$ and $\{a_1, \dots, a_t\}$ is stable, and there is a vertex $v \in X$ adjacent to $a_1, \dots, a_t$ and with no neighbours in $B_1 \cup \dots \cup B_t$.

(1) *For $1 \le i \le t$, there is a component $G[D_i]$ of $G[B_i]$ with $|D_i| \ge \gamma|G|/t^3$.*

Suppose not, say for $i = 1$. Choose s maximum such that there are s subgraphs $F_1, \dots, F_s$ of $G[B_1]$, pairwise disjoint, each a union of components of $G[B_1]$, and each with at least $\gamma|G|/t^3$ vertices. We may assume that each F_j is minimal, and so has at most $2\gamma|G|/t^3$ vertices, since each component of $G[B_1]$ has at most $\gamma|G|/t^3$ vertices. Thus $F_1 \cup \dots \cup F_s$ has at most $2s\gamma|G|/t^3$ vertices, and so there are at least $\gamma|G|/t^2 - 2s\gamma|G|/t^3$ vertices of B_1 not in any of $F_1, \dots, F_s$. From the maximality of s, $\gamma|G|/t^2 - 2s\gamma|G|/t^3 < \gamma|G|/t^3$, and so $t - 2s < 1$. Hence $s \ge t/2$. But this contradicts 5.2, since we will arrange that $\gamma|G|/t^3 \ge |G|(t/2)^{-1/\tau}$. To ensure this last, arrange at the start of the proof that $t^{1-3\tau} \ge 4$, by choosing $1/(400\varepsilon) \ge 16$ and $\tau \le 1/6$, and arrange that $\gamma^\tau \ge 1/2$, by choosing τ very small. This proves (1).

(2) *$\mathcal{D} = (D_1, \dots, D_t)$ is a pure blockade.*

Suppose not; then there exist distinct $i, j \in \{1, \dots, t\}$, such that some vertex $u \in D_j$ has both a neighbour and a non-neighbour in D_i. Since $G[D_i]$ is connected, there is an edge xy of $G[D_i]$ such that u is adjacent to x and not to y; and then the subgraph induced on $\{v, a_i, a_j, x, y, u\}$ is isomorphic to $\widehat{C_5}$, a contradiction. This proves (2).

(3) *There is no $\mathcal{D}$-rainbow triangle.*

Suppose there is, and so G contains the star-expansion of K_3; but the star-expansion of K_3 contains $\widehat{C_5}$, a contradiction. This proves (3).

Let P be the pattern of the pure blockade $\mathcal{D}$. Since P is triangle-free by (3), and $|P| = t$, it follows that there is a stable set I of P with cardinality at least $t^{1/2}/2$. Hence the sets D_i $(i \in I)$ are pairwise anticomplete, but we will arrange that $\gamma|G|/t^3 \ge |G|(t^{1/2}/2)^{-1/\tau}$, a contradiction to 5.2. To ensure this last, we arrange at the start of the proof that $1/(400\varepsilon) \ge 256$ and $\tau \le 1/12$, implying that $t \ge 256$ and $t^{1/2-3\tau} \ge 4$; and arrange that $\gamma^\tau \ge 1/2$ by choosing τ sufficiently small. This proves 1.8. ∎

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