Asymptotic structure. VII. The coarse Menger conjecture in series-parallel graphs

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Abstract

The (false) "coarse Menger conjecture" asserts that for all integers $k, c \geq 1$ there exists ℓ such that if G is a graph and $S, T \subseteq V(G)$, either there are k paths of G between S, T, pairwise at distance more than c, or there is a set $X \subseteq V(G)$ with $|X| \leq k-1$, such that every path between S, T has distance at most ℓ from X. This is known to be false for general graphs G, and indeed for graphs G with tree-width at most six. Here we prove that it is true for graphs with tree-width at most two.

1 Introduction

Coarse graph theory is a new and rapidly developing area of research, concerned with extending theorems about disjoint subgraphs of graphs to "far apart" subgraphs of graphs. (See, for instance, [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16]). In particular, Albrechtsen, Huynh, Jacobs, Knappe, and Wollan (in [1]) and Georgakopoulos and Papasoglu (in [6]) independently conjectured the following, called the "coarse Menger conjecture", which gave rise to a great deal of research.

1.1 False conjecture: *For all integers $k, c \geq 1$ there exists ℓ such that if G is a graph and $S, T \subseteq V(G)$, either there are k paths of G between S, T, pairwise at distance more than c, or there is a set $X \subseteq V(G)$ with $|X| \leq k-1$, such that every path between S, T has distance at most ℓ from X.*

This was shown to be false in [10], which gave a counterexample with $k = 3$ and $c = 2$ for each value of ℓ. But it is known to be true or still open in some interesting special cases:

- It is true when $k \leq 2$ ([1, 6] and see also [10]); the counterexamples in [10] have $k = 3$.
- It may be true when $c = 1$ (see for instance [5, 7]); the counterexamples in [10] have $c = 2$.
- It may be true if G is planar, or has some fixed genus; the counterexamples in [10] have unbounded genus. It is true if G is planar and every vertex in $S \cup T$ is incident with the infinite region [16].
- It may be true when G has tree-width at most five; the counterexamples in [10] have tree-width six.
- It is true if G has bounded path-width [15].

In this paper we take a step towards the third and fourth bullets above. We will prove that the coarse Menger conjecture is true for series-parallel graphs. In other words:

1.2 *For all integers $k, c \geq 1$ there exists ℓ such that if G is a series-parallel graph and $S, T \subseteq V(G)$, then either:*

- *there are k paths of G between S, T, pairwise at distance more than c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k-1$, such that every path between S, T has distance at most ℓ from X.*

Series-parallel graphs are the graphs not containing K_4 as a minor, or equivalently, the graphs with tree-width at most two. It is very easy to prove the conjecture for graphs of tree-width one, so one would think that tree-width two cannot be much more difficult. But in fact we found it challenging, and worked on it for several months before we found a solution.

The proof breaks into four parts:

- We show that for any path P in a series-parallel graph, there is a path Q with the same ends as P that is "locally" a geodesic, such that every vertex of Q is close to P (this is not true in general graphs, but it is true in series-parallel graphs, and something like it is true in graphs of bounded path-width).

- We use this and ideas of [15] to reduce the problem to the case when $|S| = |T| = k$. (This reduction does not use that G is series-parallel.)
- Then it follows relatively easily that any minimal counterexample G has a tree-decomposition of width two such that the tree indexing the tree-decomposition has bounded path-width, and so G also has bounded path-width.
- Finally, we apply the theorem of [15] to deduce the result.

The most non-trivial part of the proof is the second, reducing the question to the case when $|S| = |T| = k$.

The final step feels like overkill, because we are applying a hard theorem to a very simple graph (one can show without much difficulty that at this stage G is a subdivision of a graph with a bounded number of vertices), and we tried to find a direct proof without applying the powerful machinery of [15]. But that led us into such a mess of cases that we gave up and fell back on [15].

2 Near-geodesic paths

All graphs in this paper are finite. (It would be straightforward to extend the theorem to infinite graphs, but we have not done so, preferring to keep the proof as simple as possible.) Also, all graphs have no loops or parallel edges, although at one stage we need "multigraphs", which do have loops or parallel edges. If X is a vertex of G, or a subset of the vertex set of G, or a subgraph of G, and the same for Y, then $\text{dist}_G(X, Y)$ denotes the distance in G between X, Y, that is, the number of edges in the shortest path of G with one end in X and the other in Y. (If no path exists we set $\text{dist}_G(X, Y) = \infty$.) If $Z \subseteq V(G)$, we write $\text{dist}_Z(u, v)$ instead of $\text{dist}_{G[Z]}(u, v)$ for convenience when the meaning is clear. The *interior* of a path P is the set of $v \in V(P)$ with degree two in P. A *geodesic* in G between X, Y (where again X, Y are vertices, or sets of vertices, or subgraphs) is a path of G between X, Y of length $\text{dist}_G(X, Y)$.

To prove 1.2, we will obtain (by induction on k) $k - 1$ paths between S, T that are pairwise far apart; and it would be helpful if these paths all were geodesics. We cannot arrange that, but we can arrange either of two other properties both meaning that the paths are "nearly" geodesics in a sense. We say a path P is:

- a *c-geodesic* if $c > 0$ and $\text{dist}_P(u, v) = \text{dist}_G(u, v)$ for all $u, v \in V(P)$ with $\text{dist}_G(u, v) \leq c$;
- a *(c, d)-near-geodesic* if $c, d > 0$ and $\text{dist}_P(u, v) \leq d$ for all $u, v \in V(P)$ with $\text{dist}_G(u, v) \leq c$.

Every c-geodesic is a (c, d)-near-geodesic for all $d \geq c$, and we have some elbow room here: we can arrange that the paths we care about are c-geodesics, and really we only need that they are (c, d)-near-geodesics. (We use (c, d)-near-geodesics instead of c-geodesics when we can, in the hope of future applications to graphs in which we can obtain (c, d)-near-geodesics but not c-geodesics, such as graphs of bounded path-width.)

2.1 *Let $c \geq 1$ be an integer, and let R be a path of a series-parallel graph G. Then there is a c-geodesic R' in G with the same ends as R, such that $\text{dist}_G(v, R) \leq 2c$ for every $v \in V(R')$.*

Proof. Let R have ends s, t. If $Z \subseteq V(G)$, with $V(R) \subseteq Z$, we say a *Z-hop* is a geodesic of length at most c, with distinct ends both in Z and with interior disjoint from Z, with length strictly less than the distance between its ends in $G[Z]$. We define $n \geq 0$, and $Z_0, \ldots, Z_n \subseteq V(G)$ and paths $P_1, \ldots, P_n$ inductively, as follows. Let $Z_0 = V(R)$. Inductively, suppose that $i \geq 0$ and P_i, Z_i have been defined. Let S be a shortest path in $G[Z_i]$ between s, t. If S is a c-geodesic, let $n = i$; the inductive definition is complete. Now we assume that S is not a c-geodesic, and so there exist $u, v \in V(S)$ such that $\text{dist}_G(u, v) \leq c$ and $\text{dist}_G(u, v) < \text{dist}_S(u, v) = \text{dist}_{Z_i}(u, v)$. Let Q be a geodesic between u, v, chosen with $Z_i \cup V(Q)$ minimal.

Since $|E(Q)| < \text{dist}_{Z_i}(u, v)$, some vertex of Q is not in Z_i, and so there is a subpath P_{i+1} of Q with both ends in Z_i, with length at least two, and with no internal vertex in Z_i. Moreover, $|E(P_{i+1})|$ is strictly less than the distance between the ends of P_{i+1} in $G[Z_i]$, because otherwise we could reroute Q by replacing P_{i+1} with a path of $G[Z_i]$, contrary to the minimality of $Z_i \cup V(Q)$. Consequently P_{i+1} is a Z_i-hop. Let $Z_{i+1} = Z_i \cup V(P_{i+1})$. This completes the inductive step of the definition. Since the graph is finite, this process terminates, and so n exists.

This defines $Z_0, \ldots, Z_n$ and $P_1, \ldots, P_n$. Moreover, we have the following properties:

- $V(R) = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_n$;
- for $1 \leq i \leq n$, P_i is a Z_{i-1}-hop, and $Z_i = Z_{i-1} \cup V(P_i)$;
- some shortest path in $G[Z_n]$ between s, t is a c-geodesic.

If $u, v \in Z_n$, we say u is *earlier* than v if for some i, $u \in Z_i$ and $v \notin Z_i$. We observe first that for $1 \leq i \leq n$, there is a path of $G[Z_i]$ including P_i with both ends in $V(R)$ and with no internal vertex in $V(R)$, and we call it a *link* for P_i.

For $1 \leq i \leq n$, we say P_i has *height* 1 if some end of P_i is in $V(R)$, and for $h \geq 2$ we say P_i has *height* h if h is minimum such that some end of P_i belongs to the interior of some P_j of height less than h. Thus, the height of each P_i is defined. We will show that each P_i has height one or two.

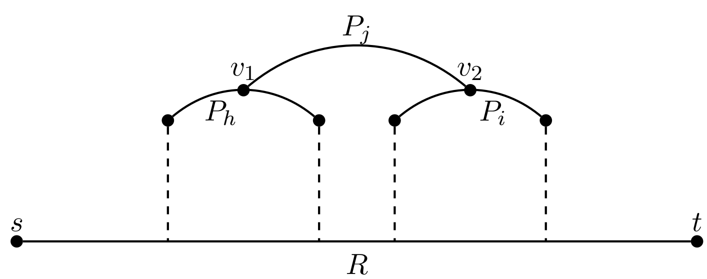

Figure 1: When P_j has height 2. The solid and dashed lines both represent paths, and the four vertical paths are vertex-disjoint except that the second and third might have a common end in R.

Suppose that P_j has height two; let us examine its structure. Let P_j have ends v_1, v_2, and let $h, i \in \{1, \ldots, n\}$ with $h \leq i < j$ such that v_1, v_2 belong to the interiors of P_h, P_i respectively. It is impossible that $h = i$, because then the subpath of P_i between v_1, v_2 would be a geodesic, and yet strictly longer than P_j (since P_j is a Z_{j-1}-hop). Thus $h < i < j$. Let L_h, L_i be links for P_h, P_i respectively. It follows that every vertex in $L_h \cap L_i$ is an end of both, and the subpaths of R joining the ends of L_h and joining the ends of L_i are edge-disjoint, because in any other case there would be a K_4-minor (we leave the reader to check this). See Figure 1. Let us call the union of P_j, L_h, L_i, R a *support* of P_j.

Suppose that some P_k has height 3. Then one of its ends is in the interior of some P_j of height 2 (again, see Figure 1). Choose a support C for P_j. Thus, every vertex in C is earlier than every vertex of the interior of P_k, and so P_k intersects C only in one end of P_k. The other end of P_k, w say, belongs to the interior of some $P_{j'}$ where $j' < k$, and as before $j' \neq j$; let L be a link for $P_{j'}$. Thus, $L \cap P_k$ contains only the vertex w. The two subpaths of L between w and $V(P)$, together with P_k, each include a path between w and C, and these three subpaths are pairwise vertex-disjoint except for w; and in every case this yields a K_4 minor, a contradiction (again, we leave checking this to the reader).

This proves that no P_i has height three, and so they all have height one or two. Consequently, every vertex of Z_n has distance at most $2c$ from $V(R)$. Since there is a c-geodesic path in $G[Z_n]$ between s, t, this proves 2.1. ▮

3 Leaps

Let $S, T \subseteq V(G)$, and let $P_1, \dots, P_k$ be S-T paths, pairwise at distance at least $2r+2$, where $r \geq 1$ is an integer we will specify later. For $1 \leq i \leq k$, let P_i have ends $s_i \in S$ and $t_i \in T$, where $V(P_i) \cap S = \{s_i\}$ and $V(P_i) \cap T = \{t_i\}$. (Possibly $s = t_i \in S \cap T$.) Let $\mathcal{P} = \{P_1, \dots, P_k\}$, and we denote $V(P_1) \cup \dots \cup V(P_k)$ by $V\mathcal{P}$. For $1 \leq i \leq k$, let A be the set of all vertices with distance at most r from $V\mathcal{P}$. Let $B = V(G) \setminus A$. Let $\mathrm{bd}(A)$ be the set of vertices in A with a neighbour in B. A *rib* is a geodesic of G between $V\mathcal{P}$ and $\mathrm{bd}\, A$ (necessarily of length exactly r).

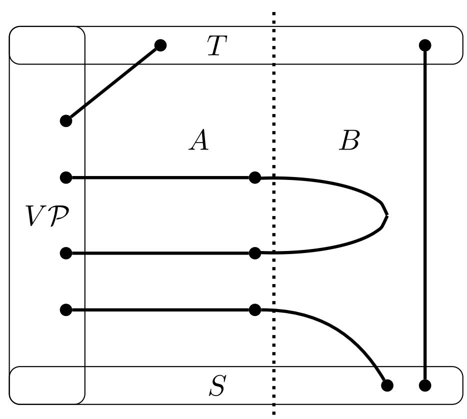

Figure 2: The four types of leaps. (The thick lines represent paths.)

Let L be a path of G, with ends u, v. We say L is a *leap* if (see Figure 2) $u, v \in S \cup T \cup V\mathcal{P}$, and either:

- $u, v \in V\mathcal{P}$ are distinct, and $|V(L) \cap \mathrm{bd}(A)| = 2$, say $V(L) \cap \mathrm{bd}(A) = \{u', v'\}$, where u, u', v', v are in order in L, and the subpaths of L from u to u', from u' to v', and from v' to v are respectively a rib, a path with interior in B (of length at least two), and a rib; or

- u, v are distinct, and exactly one of $u, v \in V\mathcal{P}$, say $u \in V\mathcal{P}$; and $v \in (S \cup T) \cap B$, and $|V(L) \cap \mathrm{bd}(A)| = 1$, say $V(L) \cap \mathrm{bd}(A) = \{u'\}$, and the subpaths of L from u to u', and from u' to v are respectively a rib and a path of length at least one with no vertex in A except u'; or
- u, v are distinct, and exactly one of $u, v \in V\mathcal{P}$, say $u \in V\mathcal{P}$; and $v \in (S \cup T) \cap A$, and L is a geodesic from $V\mathcal{P}$ to v (thus, in this case L is a path of $G[A]$ and has bounded length); or
- L is a path of $G[B]$ between S, T (possibly $u = v$).

Let F_0 be the set of all ordered pairs (u, v) of vertices of G such that there is a leap with ends u, v (thus, if $u = v$ then $u \in (S \cup T) \cap B$). An *ℓ-barrier* is a k-tuple $(Q_1, \dots, Q_k)$, where Q_i is a subpath of P_i of length at most ℓ for $1 \le i \le k$. We say that $(u, v) \in F_0$ *jumps* an ℓ-barrier $(Q_1, \dots, Q_k)$ if

- either $u \in S$, or u, s_i belong to the same component of $P_i \setminus V(Q_i)$ for some $i \in \{1, \dots, k\}$; and
- either $v \in T$, or v, t_i belong to the same component of $P_i \setminus V(Q_i)$ for some $i \in \{1, \dots, k\}$.

We say that $F \subseteq F_0$ is *ℓ-jumping* (with respect to $S, T, P_1, \dots, P_k$) if for every ℓ-barrier $(Q_1, \dots, Q_k)$, some member of F jumps $(Q_1, \dots, Q_k)$.

Let $X \subseteq V(G)$. We say that X is *s-obstructing* (with respect to S, T) if every path in G between S, T has distance at most s from X. Suppose that $P_1, \dots, P_k$ are (c, d)-near-geodesics. Our goal in this section is to show that if F_0 is not ℓ-jumping, then there is an ℓ'-obstructing set of size k, where ℓ' is not much bigger than ℓ; and conversely, if there is an ℓ'-obstructing set of size k, then F_0 is not ℓ-jumping, where ℓ is not much bigger than ℓ'. We begin with:

3.1 *Let $c, d, \ell, \ell', r \ge 1$ be integers, with $2r + 1 \le c$, and let $P_1, \dots, P_k$ be as above, pairwise at distance at least $2r + 2$. Suppose that $P_1, \dots, P_k$ are (c, d)-near-geodesics, and that F_0 is not ℓ-jumping with respect to $S, T, P_1, \dots, P_k$. If $\ell' \ge r + (d + 1)/2$ and $\ell' \ge r + (\ell + 1)/2$, then there is an ℓ'-obstructing set (with respect to S, T) of size k.*

Proof. Since F_0 is not ℓ-jumping, there is an ℓ-barrier $(Q_1, \dots, Q_k)$ such that no member of F_0 jumps $(Q_1, \dots, Q_k)$. For $1 \le i \le k$, choose $x_i \in V(Q_i)$ such that every vertex of Q_i has distance in Q_i at most $(\ell + 1)/2$ from x_i, and let S_i, T_i be respectively the subpaths of P_i between s_i, x_i, and between x_i, t_i. Let S', T' be respectively the sets of all vertices v such that $\mathrm{dist}_G(v, S_1 \cup \dots \cup S_k) \le r$, and $\mathrm{dist}_G(v, T_1 \cup \dots \cup T_k) \le r$.

(1) *If $u \in S'$ belongs to or has a neighbour in T', then* $\mathrm{dist}_G(u, X) \le \ell'$.

Choose $u \in S'$ and $v \in T'$, equal or adjacent. Choose $i, j \in \{1, \dots, k\}$ and $s \in V(S_i)$ and $t \in V(T_j)$ such that $\mathrm{dist}_G(u, s), \mathrm{dist}_G(v, t) \le r$. Hence $\mathrm{dist}_G(P_i, P_j) \le \mathrm{dist}_G(s, t) \le 2r + 1$. Since $P_1, \dots, P_k$ pairwise are at distance at least $2r + 2$, it follows that $i = j$. But $\mathrm{dist}_G(s, t) \le 2r + 1 \le c$, and P_i is (c, d)-geodesic, and so $\mathrm{dist}_{P_i}(s, t) \le d$. Moreover, x_i belongs to the subpath of P_i between s, t, and so $\mathrm{dist}_{P_i}(x_i, s) + \mathrm{dist}_{P_i}(x_i, t) \le d$. Therefore, $\mathrm{dist}_G(u, x_i) + \mathrm{dist}(v, x_i) \le 2r + d$, and since $\mathrm{dist}_G(u, x_i) \le \mathrm{dist}(v, x_i) + 1$, it follows that $\mathrm{dist}_G(u, x_i) \le (2r + d + 1)/2 \le \ell'$. This proves (1).

We claim that $\{x_1, \dots, x_k\}$ is ℓ'-obstructing. Suppose not, and choose a path M from S to T such that $\mathrm{dist}_G(M, X) > \ell'$.

(2) *$V(M) \cap S \cap T' = \emptyset$, and similarly $V(M) \cap S' \cap T = \emptyset$.*

Suppose that there exists $s \in V(M) \cap S \cap T'$. Choose $i \in \{1, \ldots, k\}$ such that $\text{dist}_G(s, T_i) \leq r$, let R be a geodesic between $s, V\mathcal{P}$ (necessarily between s, T_i, by (1)), and let u be the end of R in $V(T_i)$. If $u \in V(Q_i)$, then $\text{dist}_G(x_i, u) \leq (\ell+1)/2$, and so $\text{dist}_G(s, x_i) \leq (\ell+1)/2 + r \leq \ell'$, a contradiction. So $u \notin V(T_i) \setminus V(Q_i)$, and so R is a leap jumping the barrier $(Q_1, \ldots, Q_k)$, a contradiction. This proves that $V(M) \cap S \cap T' = \emptyset$, and similarly $V(M) \cap S' \cap T = \emptyset$, and so proves (2).

Since one end of M is in S and hence in $(S \cap B) \cup S'$ by (2), and similarly the other end is in $(T \cap B) \cup T'$, there is a minimal subpath N of M that intersects both $(S \cap B) \cup S', (T \cap B) \cup T'$. Since $A = S' \cup T'$, all internal vertices of N belong to B; so if N is between $S \cap B$ and $T \cap B$, then N is a leap, and its ends form a member of F_0 jumping the barrier, a contradiction. So, from the symmetry, we may assume that one end u' of N is in S'. Choose $i \in \{1, \ldots, k\}$ such that $\text{dist}_G(u', S_i) \leq r$. By (2), $u' \notin T'$. Let R be a geodesic between $u', V\mathcal{P}$ (necessarily between u', S_i, since $u' \notin T'$), and let u be the end of R in $V(S_i)$. If $u \in V(Q_i)$, then $\text{dist}_G(x_i, u) \leq (\ell+1)/2$, and so $\text{dist}_G(u', x_i) \leq (\ell+1)/2 + r \leq \ell'$, a contradiction. So $u \notin V(Q_i)$. Let N be between u', v'. If $v' \in T \cap B$, then $u' \in \text{bd}(A)$, and R is a rib, and N is therefore a leap jumping the barrier, a contradiction. So $v' \in T'$, and hence by (1), u', v' are not equal or adjacent. Hence N has length at least two, and its internal vertices are in B, so $u', v' \in \text{bd}(A)$, and R is a rib. Choose a rib R' between $v', V(T_i)$ similarly, and then $R \cup N \cup R'$ is a leap jumping the barrier, a contradiction. This proves 3.1. ∎

The proof just given is closely related to step (4) of the proof of theorem 5.1 in [15].

3.2 *Let $c, d, \ell, \ell', r \geq 1$ be integers, with $2r + 1 \leq c$, and let $P_1, \ldots, P_k$ be as above, pairwise at distance at least $2r + 2$. Suppose that $P_1, \ldots, P_k$ are (c, d)-near-geodesics, and that there is an ℓ'-obstructing set (with respect to S, T) of size k. If $\ell \geq 2d$ and $\ell' \leq r/2$, then F_0 is not ℓ-jumping with respect to $S, T, P_1, \ldots, P_k$.*

Proof. Let $Y \subseteq V(G)$ be an ℓ'-obstructing set with $|Y| = k$. For each $y \in Y$, there is at most one $i \in I$ such that $\text{dist}_G(y, P_i) \leq \ell'$, since $\text{dist}_G(P_i, P_j) \geq 2r + 2 > 2\ell'$. On the other hand, for each $i \in I$ there exists $y_i \in Y$ such that $\text{dist}_G(y_i, P_i) \leq \ell'$, since Y is ℓ'-obstructing, and therefore $\{y_1, \ldots, y_k\} = Y$. Choose $x_i \in V(P_i)$ with $\text{dist}_G(y_i, x_i) \leq \ell'$ for $1 \leq i \leq k$. Since every vertex in Y has distance at most ℓ' from $\{x_1, \ldots, x_k\}$, it follows that X is $2\ell'$-obstructing, where $X = \{x_1, \ldots, x_k\}$.

For $1 \leq i \leq k$, let Q_i be the subpath of P_i consisting of all vertices v of P_i such that $\text{dist}_{P_i}(v, x_i) \leq \ell/2$. Thus, Q_i has length at most ℓ, and $(Q_1, \ldots, Q_k)$ is an ℓ-barrier. We claim that no member of F_0 jumps this barrier. Suppose this is false, and $(u, v) \in F_0$ jumps the barrier. Thus, either $u \in S$, or $u \in V(P_i)$ for some $i \in \{1, \ldots, k\}$, and in the second case $s_i \notin V(Q_i)$ and s_i, u belong to the same component (S_i say) of $P_i \setminus V(Q_i)$. Similarly, either $v \in T$, or there exists $j \in \{1, \ldots, k\}$ such that t_j, v belong to the same component (T_j say) of $P_j \setminus V(Q_j)$.

(1) *If i exists then $\text{dist}_G(S_i, X) > 2\ell'$, and similarly if j exists then $\text{dist}_G(T_j, X) > 2\ell'$.*

Suppose that i exists (and hence $u \in V(S_i)$), and $\text{dist}_G(S_i, X) \leq 2\ell'$, and hence $\text{dist}_G(S_i, x_i) \leq 2\ell'$ (since $\text{dist}_G(S_i, P_{i'}) \geq 2r + 2 > 2\ell'$ if $i' \neq i$). Choose $w \in V(S_i)$ with $\text{dist}_G(w, x_i) \leq 2\ell'$. Since

$2\ell' \le c$, and P_i is a (c,d)-near-geodesic, it follows that $\text{dist}_{P_i}(w,x_i) \le d \le \ell/2$, and so $w \in V(Q_i)$, a contradiction. This proves (1).

Let L be a leap with ends u,v. Since X is $2\ell'$-obstructing, and the union of L, S_i (if i exists) and T_j (if j exists) includes an S-T path, some vertex of this path has distance at most $2\ell'$ from X, and by (1) every such vertex is in L. But every such vertex is in A, since $r \ge 2\ell'$, and so some component R of $L[A]$ satisfies $\text{dist}_G(R,X) \le 2\ell'$. Choose $x_i \in X$ with $\text{dist}_G(R,x_i) \le 2\ell'$. From the definition of a leap, R is either a rib, or a geodesic between $V\mathcal{P}$ and $(S \cup T) \cap A$, and in particular, one end of R is in $V(P_i)$ and is an end of L. So we may assume that $u \in V(R) \cap V(P_i)$. Since R has length at most r, and $\text{dist}_G(R,x_i) \le 2\ell'$, it follows that $\text{dist}_G(u,x_i) \le r + 2\ell' \le c$. Since P_i is a (c,d)-near-geodesic, we deduce that $\text{dist}_{P_i}(x_i,u) \le d \le \ell/2$, and therefore $u \in V(Q_i)$, a contradiction. This proves 3.2. ▮

We need the following, an immediate consequence of theorem 4.2 of [15]:

3.3 *With notation as before, if F_0 is ℓ-jumping, there exists $F \subseteq F_0$ that is also ℓ-jumping, such that exactly one member of F has first term in $S \setminus \{s_1,\dots,s_k\}$, and exactly one member of $\mathcal{F}$ has second term in $T \setminus \{t_1,\dots,t_k\}$. Consequently, there exist $S' \subseteq S$ and $T' \subseteq T$ with $|S'| = |T'| = k+1$ such that $P_1,\dots,P_k$ are S'-T' paths, and there exist $F \subseteq F_0$ such that F is ℓ-jumping with respect to $S', T', P_1, \dots, P_k$.*

We combine these pieces to prove the following:

3.4 *Let $\ell, d \ge 1$. Let $S,T \subseteq V(G)$, and suppose that $P_1,\dots,P_k$ are S-T paths in G, pairwise at distance at least $4\ell+2$, and each a $(4\ell+1,d)$-near-geodesic. Suppose that there is no set with size $\le k$ that is $(2\ell+d+1)$-obstructing with respect to S,T. Then there exist $S' \subseteq S$ and $T' \subseteq T$ with $|S'| = |T'| = k+1$ such that there is no set with size $\le k$ that is ℓ-obstructing with respect to S',T'.*

Proof. By replacing $P_1,\dots,P_k$ by subpaths if necessary, we may assume that for $1 \le i \le k$, P_i has ends $s_i \in S$ and $t_i \in T$, and $V(P_i) \cap S = \{s_i\}$, and $V(P_i) \cap T = \{t_i\}$. By hypothesis, there is no set of size at most k that is $(2\ell+d+1)$-obstructing with respect to S,T. Let $\ell_1 = 2d$ and $r = 2\ell$ and $c = 4\ell+1$. By 3.1, since $2\ell+d+1 \ge r + (\ell_1+1)/2$, it follows that F_0 is ℓ_1-jumping with respect to $S,T,P_1,\dots,P_k$.

By 3.3, there exist $S' \subseteq S$ and $T' \subseteq T$ with $|S'| = |T'| = k+1$ and $F \subseteq F_0$ such that F is ℓ_1-jumping with respect to $S',T',P_1,\dots,P_k$. From 3.2, since $2r+1 \le c$, and $\ell_1 \ge 2d$ and $\ell \le r/2$, there is no ℓ-obstructing set (with respect to S',T') of size k. This proves 3.4. ▮

4 The case when S,T have bounded size

The result 3.4 of the last section serves to reduce proving 1.2 to proving the same statement when $|S|,|T|$ are bounded, and so next we handle that case. Suppose that G is a series-parallel graph, and $S,T \subseteq V(G)$; and for some choice of k,c,ℓ,r, $|S|+|T| \le r$, and neither of the conclusions of 1.2 hold (with k replaced by $k+1$), that is:

- there do not exist $k+1$ paths of G between S,T, pairwise at distance more than c; and

- there is no set $X \subseteq V(G)$ with $|X| \le k$, such that every path between S, T has distance at most ℓ from X.

Let us call such a triple (G, S, T) a (k, c, ℓ, r)-*counterexample*, and in this section we look at the possible quadruples (k, c, ℓ, r) such that there is a (k, c, ℓ, r)-counterexample. A (k, c, ℓ, r)-counterexample (G, S, T) is *minimal* if there is no (k, c, ℓ, r)-counterexample with smaller size, where the *size* of G means $|V(G)| + |E(G)|$.

A *tree-decomposition* $(H, (W_h : h \in V(H)))$ of a graph or multigraph G consists of a tree H, and a subset $W_h \subseteq V(G)$ for each $h \in V(H)$, satisfying:

- $G = \bigcup_{h \in V(H)} G[W_h]$; and
- $W_h \cap W_{h''} \subseteq W_{h'}$ for all $h, h', h'' \in V(H)$ such that h' lies on the path of H between h, h''.

Its *width* is the maximum of $|W_h| - 1$ over all $h \in V(H)$, and the *tree-width* of G is the minimum width of its tree-decompositions.

Similarly, a *path-decomposition* of G is a sequence $(U_1, \dots, U_n)$ of subsets of $V(G)$ with $G = \bigcup_{1 \le i \le n} G[U_i]$, such that $U_i \cap U_k \subseteq U_j$ for all i, j, k with $1 \le i \le j \le k \le n$. Its *width* is the maximum of $|U_i| - 1$ for $1 \le i \le n$, and the *path-width* of G is the minimum width of all path-decompositions of G.

4.1 *Let $k, c, \ell, r \ge 1$, and suppose that (G, S, T) is a minimal (k, c, ℓ, r)-counterexample. Then G has path-width at most $3r + 1$.*

Proof. By repeatedly contracting an edge incident with some vertex of degree two until the process stops, we deduce that G can be obtained from a multigraph G' with no vertex of degree two by edge-subdivision. Since G is series-parallel, so is G'. Take a tree-decomposition $(H, (W_h : h \in V(H)))$ of width at most two, with $|V(H)|$ as small as possible. Thus, each W_h has size at most three. If $|V(H)| = 1$, then G has path-width at most four, as is easly seen, so we assume that $|V(H)| \ge 2$, and therefore H has minimum degree one. A vertex of degree one is a *leaf*.

Let g be a leaf of the tree H. From the minimality of $|V(H)|$, there is a vertex $v \in W_g$ such that $v \notin \bigcup_{h \in V(H) \setminus \{g\}} W_h$. We call v a *private* vertex of W_g. Since $v \in V(G')$, its degree in G' is either at most one or at least three. Since every edge of G' incident with v has both ends in W_h, it follows that either:

- v is incident in G' with a loop of G'; or
- v has degree at most one in G'; or
- there are two (non-loop) parallel edges of G' incident with v.

Now G is obtained by subdividing G'; so each non-loop edge e of G' is subdivided to become a path P_e of G joining the ends of e, and each loop e of G' is subdivided to become a cycle P_e through the vertex incident with e. In both cases, the vertices of P_e not incident with e in G' are called *internal* vertices of P_e; they all have degree two in G.

We call the vertices in $S \cup T$ the *terminals.* Let us say a terminal t *is attached* to a leaf g of H if either t is a private vertex of W_g, or t is an internal vertex of P_e for some $e \in E(G')$ incident in G' with a private vertex of W_g.

(1) *For every leaf g of H, some terminal is attached to g.*

Let v be a private vertex for W_g. If v is incident in G' with a loop e, then some vertex of P_e different from v is a terminal (and therefore attached to g), since otherwise $(G \setminus (V(P_e) \setminus \{v\}), S, T)$ is a (k, c, ℓ, r)-counterexample, contradicting the minimality of (G, S, T). If v has degree at most one in G', then it has degree at most one in G, and so v is a terminal (attached to g), since otherwise $(G \setminus v, S, T)$ is a (k, c, ℓ, r)-counterexample. Thus we may assume that there are two parallel edges e, f of G' both incident with v. Let P_e have length at most that of P_f, and choose x in the interior of P_f (since G has no parallel edges, P_f has length at least two). If no internal vertex of P_f is a terminal (and therefore attached to g), then $(G \setminus x, S, T)$ is a (k, c, ℓ, r)-counterexample, as is easily seen. This proves (1).

Since each terminal is attached to at most one leaf, it follows that H has at most r leaves, and so has at most $r-2$ vertices with degree at least three; and by deleting all such vertices, we obtain a forest with path-width at most one. Hence H has path-width at most $r-1$. (This is wasteful – one can prove that H has path-width at most $O(\log r)$, but for our purposes any bound will do.)

Let $(U_1, \dots, U_n)$ be a path-decomposition of H such that $|U_i| \le r$ for each i. For $1 \le i \le n$, let $V_i = \bigcup_{h \in U_i} W_h$.

(2) *$(V_1, \dots, V_n)$ is a path-decomposition of G' with width at most $3r-1$.*

To see this, let $v \in V(G')$; it suffices to show that for $1 \le i < i' < i'' \le n$, if $v \in V_i \cap V_{i''}$ then $v \in V_{i'}$. Let M be the set of $h \in V(H)$ such that $v \in W_h$; then M is the vertex set of a subtree of H, since $(H, (W_h : h \in V(H)))$ is a tree-decomposition of G'. Since $v \in V_i$, it follows that $M \cap U_i \neq \emptyset$, and similarly $M \cap U_{i''} \neq \emptyset$. Since M is the vertex set of a connected subgraph of H and $i <' < i''$, it follows that $M \cap U_{i'} \neq \emptyset$, and so $v \in W_{i'}$. Thus, $(V_1, \dots, V_n)$ is a path-decomposition of G'. Each V_i has size at most $3r$, and this proves (2).

Now G is obtained from the multigraph G' by subdividing edges, and repeated edge-subdivision increases path-width by at most two, as is easily seen. From (2), G has path-width at most $3r+1$, so this proves 4.1. ∎

It is proved in [15] that:

4.2 *Let $k, c, d \ge 1$ be integers. Then there exists $\ell > 0$ such that for every graph G with path-width at most d, and all $S, T \subseteq V(G)$, either:*

- *there are $k+1$ paths between S, T, pairwise at distance more than c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \le k$ such that every path between S, T contains a vertex with distance at most ℓ from some member of X.*

We deduce:

4.3 *Let $k, r, c \ge 1$; then there exists $\ell \ge 1$ such that there is no (k, c, ℓ, r)-counterexample.*

Proof. Choose $\ell \geq 1$ to satisfy 4.2, taking $d = 3r+1$. Suppose there is a (k, c, ℓ, r)-counterexample, and hence there is a minimal (k, c, ℓ, r)-counterexample (G, S, T) say. By 4.1, G has path-width at most $3r + 1$; but this contradicts 4.2. This proves 4.3. ∎

5 Combining the parts

Now we are ready to prove 1.2, which we restate:

5.1 *For all integers $k, c \geq 1$ there exists $\ell(k, c) \geq 1$ such that if G is a series-parallel graph and $S, T \subseteq V(G)$, either there are k paths of G between S, T, pairwise at distance more than c, or there is a set $X \subseteq V(G)$ with $|X| \leq k - 1$, such that every path between S, T has distance at most $\ell(k, c)$ from X.*

Proof. The result is true if $k = 1$, and we proceed by induction on k; thus, we assume that $\ell(k, c')$ exists for all $c' \geq 1$, and we will show that $\ell(k + 1, c)$ also exists for all $c \geq 1$. Let $c \geq 1$. By 4.3 we may choose ℓ_1 such that there is no $(k, c, \ell_1, 2k + 2)$-counterexample. Define

$$\ell(k + 1, c) = \max(\ell(k, 20\ell_1 + 6), 6\ell_1 + 2).$$

Now let G be a series-parallel graph, and let $S, T \subseteq V(G)$. We will show that either there are $k + 1$ S-T paths of G, pairwise at distance more than c, or there is a set $X \subseteq V(G)$ with $|X| \leq k$, such that every S-T path has distance at most $\ell(k + 1, c)$ from X.

From the inductive hypothesis, we may assume that there are k S-T paths $P_1, \dots, P_k$, pairwise at distance more than $20\ell_1+6$, since otherwise there is a set $X \subseteq V(G)$ with $|X| \leq k-1$, such that every path between S, T has distance from X at most $\ell(k, 20\ell_1+6) \leq \ell(k+1, c)$. By 2.1, for $1 \leq i \leq k$ there is a $(4\ell_1 + 1)$-geodesic path P_i' with the same ends as P_i, such that $\text{dist}_G(v, P_i) \leq 2(4\ell_1 + 1)$ for every $v \in V(P_i')$. Consequently $P_1', \dots, P_k'$ are pairwise at distance more than $(20\ell_1+6)-4(4\ell_1+1) = 4\ell_1+2$.

We may assume that there is no set with size $\leq k$ that is $(6\ell_1 + 2)$-obstructing with respect to S, T, since $6\ell_1 + 2 \leq \ell(k + 1, c)$. By 3.4, since $P_1', \dots, P_k'$ are $(4\ell_1 + 1, 4\ell_1 + 1)$-near-geodesics, there exist $S' \subseteq S$ and $T' \subseteq T$ with $|S'| = |T'| = k + 1$ such that there is no set with size $\leq k$ that is ℓ_1-obstructing with respect to S', T'. But (G, S', T') is not a $(k, c, \ell_1, 2k+2)$-counterexample, by the choice of ℓ_1, and hence there are $k+1$ S-T paths in G, pairwise at distance more than c. This proves 5.1. ∎

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