Asymptotic structure. VI. Distant paths across across a disc.

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Abstract

The (false) "coarse Menger" conjecture proposed that for all k, c there exists ℓ, such that for every graph G and subsets $S, T \subseteq V(G)$, either there are $k+1$ $S-T$ paths pairwise with distance more than c, or there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every $S-T$ path has distance at most ℓ from X. This is known to be false, but may be true if G is planar. Here we show that it is true if G is planar and all vertices in $S \cup T$ are on the same region.

1 Introduction

The "disjoint paths problem" asks when there is a set of $k+1$ vertex-disjoint paths between sets S, T of vertices of a graph G; and it is answered by a theorem of K. Menger from 1927 [8], that such paths exist if and only if there is no subset $X \subseteq V(G)$ of size $\leq k$ such that every $S-T$ path has a vertex in X. But what if we want the paths to be at least a certain distance from one another?[1] This question is motivated both by the developing area of "coarse graph theory", which is concerned with the large-scale geometric structure of graphs (see Georgakopoulos and Papasoglu [5]), and by the algorithmic question of deciding whether such paths exist (see Bienstock [3], Kawarabayashi and Kobayashi [7], and Baligács and MacManus [2]). We say X, Y are *c-distant* if their distance is at least $c+1$.

A coarse analogue of Menger's theorem was conjectured by Albrechtsen, Huynh, Jacobs, Knappe and Wollan [1], and independently by Georgakopoulos and Papasoglu [5]:

1.1 False conjecture: *For all integers $k, c \geq 0$ there exists $\ell > 0$ with the following property. Let G be a graph and let $S, T \subseteq V(G)$. Then either*

- *there are $k+1$ paths between S, T, pairwise c-distant; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex with distance at most ℓ from some member of X.*

Both sets of authors proved the result for $k = 1$, but we showed in [9] that 1.1 is false for all $k \geq 2$, even if $c = 2$ and G has maximum degree three.

On the other hand, our counterexample is highly non-planar (it has unbounded genus), and one might hope that the conjecture 1.1 is true for planar graphs, or perhaps even for graphs drawn in any fixed surface. In this paper we show that it is true if G is planar and can be drawn in the plane such that all vertices in $S \cup T$ are incident with the infinite region (briefly, "with S, T on the outside"). More exactly:

1.2 *Let $c, k \geq 0$, and let G be drawn in the plane, with S, T on the outside. Then either:*

- *there are $k+1$ paths between S, T, pairwise c-distant; or*
- *there is a set of at most k connected subgraphs of G, such that every path between S, T intersects one of these subgraphs, and the sum of the diameters of the subgraphs is at most $200k^3c$.*

(All graphs in this paper are finite, and have no loops or parallel edges.) To prove this, we first obtain a necessary and sufficient condition for the existence of k paths $P_1, \ldots, P_k$ pairwise at distance at least c, where P_i joins two given vertices s_i, t_i, and $s_1, \ldots, s_k, t_1, \ldots, t_k$ all belong to the infinite region.

Finally, we obtain a linear-time algorithm to test whether there is a set of $k+1$ $S-T$ paths that are pairwise c-distant (for fixed c, k, and still assuming that S, T are on the outside of the drawing).

[1] If X and Y are vertices, or sets of vertices, or subgraphs, of a graph G, then $\text{dist}_G(X, Y)$ denotes the distance between X, Y, that is, the number of edges in the shortest path of G with one end in X and the other in Y.

2 The linkage problem

The linkage problem is different from the question answered by Menger's theorem: now we are given k pairs (s_i, t_i) $(1 \le i \le k)$ of vertices of a graph G, and a *linkage* for this set of pairs means a set of k vertex-disjoint paths $P_1, \dots, P_k$, where P_i joins s_i, t_i for $1 \le i \le k$. Testing whether a linkage exists is NP-complete if k is part of the input [6], and solvable in polynomial time if k is fixed [11]. But it is much simpler if G is drawn in a closed disc Σ and $s_1, \dots, s_k, t_1, \dots, t_k$ belong to $\mathrm{bd}(\Sigma)$: it is shown in [10] that in that case, a linkage for the pairing (s_i, t_i) $(1 \le i \le k)$ exists if and only if:

- no two of the pairs (s_i, t_i) "cross" (that is, for $1 \le i < j \le k$, there is a line segment of $\mathrm{bd}(\Sigma)$ that contains both of s_i, t_i and neither of s_j, t_j); and
- for every two points $a, b \in \mathrm{bd}(\Sigma)$ and every simple curve L between a, b in Σ that meets the drawing only in vertices, $|L \cap V(G)|$ is at least the number of values of $i \in \{1, \dots, k\}$ such that both line segments of $\mathrm{bd}(\Sigma)$ between a, b contain one of s_i, t_i.

Here we are asking for a linkage of c-distant paths, but it turns out that, in the same disc case, a similar theorem holds; and this is a key lemma for the proof of 1.2. In this section we state and prove it.

If G is drawn in Σ, each vertex of G is a point of Σ and each edge of G is a line segment, that is, a subset of Σ homeomorphic to the closed interval $[0, 1]$. We denote the union of the set of vertices and the set of edges of G by $U(G)$. A *region* of G in Σ means an arc-wise connected component of $\Sigma \setminus U(G)$.

Now us fix some number $c \ge 0$ (throughout the paper); and again, G be drawn in a closed disc Σ. A *log* is a path of G of length at most c. A *boom*[2] of *length* t is a sequence $Q_1, \dots, Q_t$ of logs, such that for $1 \le i < t$, there is a region of G incident with a vertex of Q_i and with a vertex of Q_{i+1} (this is true, for instance, if $Q_i \cap Q_{i+1}$ is non-null). If $b \in \mathrm{bd}(\Sigma)$, a boom $Q_1, \dots, Q_t$ *attaches to* b if either:

- $b \in V(Q_1 \cup \dots \cup Q_t)$, or
- b belongs to a region of G that is incident with a vertex of $Q_1 \cup \dots \cup Q_t$.

(Note that the regions are disjoint from $U(G)$, so if b satisfies the first bullet it is a vertex, and if it satisfies the second then it belongs to $\mathrm{bd}(\Sigma) \setminus U(G)$.) If $a, b \in \mathrm{bd}(\Sigma)$, a boom $Q_1, \dots, Q_t$ *joins* a, b if either it attaches to both a, b, or $t = 0$ and a, b belong to a common region. We will prove the following (see Figure 1):

2.1 *Let Σ be a closed disc, let G be a graph drawn in Σ, and let $c \ge 0$ be an integer. Let $s_1, t_1, \dots, s_k, t_k \in V(G) \cap \mathrm{bd}(\Sigma)$, such that $(s_i, t_i), (s_j, t_j)$ do not cross for $1 \le i < j \le k$. Then exactly one of the following holds:*

- *There is a linkage for the pairs $(s_1, t_1), \dots, (s_k, t_k)$, of paths that are pairwise c-distant;*
- *there exist distinct $a, b \in \mathrm{bd}(\Sigma)$, and a boom joining a, b of length less than the number of $i \in \{1, \dots, k\}$ such that (s_i, t_i) crosses (a, b).*

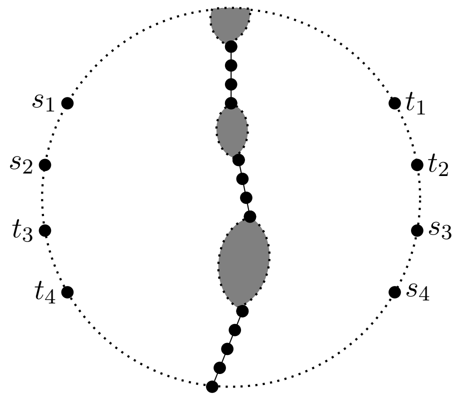

Figure 1: A boom of length three, obstructing a linkage for (s_i, t_i) $(1 \leq i \leq 4)$ of pairwise 4-distant paths. The gray areas are regions.

There is a problem that is slightly more general than the linkage problem across a disc described above, and it is useful for our application to prove this more complicated version. Let C be a circle. A subset of C that is nonempty, closed and arcwise connected is called an *interval* of C; so intervals are either line segments, or single points or the whole of C. Two intervals are *internally-disjoint* if no internal point of either of them belongs to the other. If I_1, I_2, I_3, I_4 are pairwise internally-disjoint intervals of C, we say that (I_1, I_3) *crosses* (I_2, I_4) if there is a rotation around C that traverses the sets I_1, I_2, I_3, I_4 in this order. (This extends the definition for crossing of pairs of points given earlier.) For intervals I with $|I| = 1$ and $|I| = \{x\}$ say, we often write x for I.

Now let Σ be a closed disc, and let us assign a direction of rotation "clockwise" to $\mathrm{bd}(\Sigma)$. If A, B are distinct, internally-disjoint intervals of $\mathrm{bd}(\Sigma)$, $[A \to B]$ denotes the interval L of $\mathrm{bd}(\Sigma)$ that is minimal subject to $L \cap A, L \cap B \neq \emptyset$, and such that A, L, B are in clockwise order. Let $\mathcal{I} = \{I_1, \ldots, I_k\}$ be a set of pairwise internally-disjoint intervals of $\mathrm{bd}(\Sigma)$, numbered in clockwise order. We call $\mathcal{I} = \{I_1, \ldots, I_k\}$ an *interval system* for $\mathrm{bd}(\Sigma)$. For $1 \leq i < j \leq k$, let $d_{i,j} \geq 0$ be an integer, such that if $1 \leq p, q, r, s \leq k$ then one of $d_{p,r}, d_{q,s} = 0$. We call the function d a *demand function* for $\mathcal{I}$. If G is drawn in Σ, a *linkage in G for $d, \mathcal{I}$* is a family $\mathcal{L} = (\mathcal{L}_{i,j} : 1 \leq i < j \leq k)$ of pairwise disjoint sets of paths, where $\mathcal{L}_{i,j}$ contains $d_{i,j}$ paths, all between I_i, I_j, and the paths in $U(\mathcal{L})$ are pairwise vertex-disjoint where $U(\mathcal{L})$ denotes the set of paths $\bigcup_{1 \leq i < j \leq k} \mathcal{L}_{ij}$. (The condition "one of $d_{p,r}, d_{q,s} = 0$" is assumed because there could not be a corresponding linkage if this condition was not satisfied.) A linkage $\mathcal{L}$ is *c-distant* if every two members of $U(\mathcal{L})$ are c-distant.

The next result implies 2.1 by taking each I_i to be a singleton. We will use this, rather than 2.1, in the proof of 1.2.

2.2 *Let Σ be a closed disc, let $\mathcal{I} = \{I_1, \ldots, I_k\}$ be an interval system, and let d be a demand function for $\mathcal{I}$. Let G be a graph drawn in Σ, and let $c \geq 0$ be an integer. Then exactly one of the following holds:*

- *there is a c-distant linkage in G for $d, \mathcal{I}$;*

[2] Historically, a boom was an obstruction, made by logs chained together and strung across a harbour mouth, so it seems an appropriate name.

- *there exist $a, b \in \mathrm{bd}(\Sigma)$, and a boom joining a, b, of length less than the sum of $d_{i,j}$ over all $1 \le i < j \le k$ such that (I_i, I_j) crosses (a, b).*

Proof. It is easy to see that not both hold and we omit the argument. Now we prove that one of the statements holds. We suppose for a contradiction that neither holds. By slightly enlarging Σ, we may assume that no edge of G intersects Σ, and the only vertices of G in Σ are the vertices in $I_1 \cup \cdots \cup I_k$. (This is not quite trivial: the enlargement does not change whether the first outcome of the theorem holds, but in the second outcome one of a, b, say x, might belong to $V(G) \setminus (S \cup T)$ and to the boundary of the old disc, and needs to be replaced with some point in a region with x of the boundary of the new disc.) We may also assume that $k \ge 2$, and so none of the intervals in $\mathcal{I}$ is the whole of $\mathrm{bd}(\Sigma)$. Moreover, we can assume that for each of the intervals I_i, both its ends (or its point, if it is a singleton) belong to $V(G)$.

If P, P' are paths of G, with ends s, t and s', t' respectively, and $s, s', t, t' \in \mathrm{bd}(\Sigma)$, we say that P' is *inside* P if P, P' are vertex-disjoint and either

- $s \ne t$ and $s', t' \in [s \to t]$; or
- $s = t$ and $[I_k \to I_1] = V(P)$.

Now let $0 \le d'_{ij} \le d_{ij}$ for $1 \le i < j \le k$, such that for $1 \le i \le j \le k$, if $d'_{ij} > 0$ then $d'_{i'j'} = d_{i'j'}$ for all i', j' with $i \le i' < j' \le j$ and $(i', j') \ne (i, j)$. We call d' a *partial d-demand*. Let $\mathcal{L} = (\mathcal{L}_{ij} : 1 \le i < j \le k)$ be a c-distant linkage for the demand function d' for $\mathcal{I}$. For $1 \le i < j \le k$ and $P \in L_{ij}$, we say that P is *pushed* if for each vertex $v \in V(P)$, either:

- $v \in [I_i \to I_j]$; or
- v is incident with a region that contains a point of $[I_i \to I_j]$; or
- there is a region incident with v and some vertex u, and a log Q of length $\le c$ with ends u, w, where w belongs to some path $P' \ne P$ in $U(\mathcal{L})$ that is inside P.

We say that $\mathcal{L}$ is *pushed* if for $1 \le i < j \le k$, each $P \in \mathcal{L}_{i,j}$ is pushed.

Let us choose a partial d-demand d', with $\sum d'_{ij}$ maximum such that there is a pushed c-distant linkage $\mathcal{L} = (\mathcal{L}_{ij} : 1 \le i < j \le k)$ for d'. We may further assume that for $1 \le i < j \le k$ and each $P \in \mathcal{L}_{ij}$, $V(P) \cap I_i$ contains only one vertex, an end of P, and the same for $V(P) \cap I_j$. For every two internally-disjoint intervals A, B of $\mathrm{bd}(\Sigma)$, let $f(A, B)$ be the number of paths in $U(\mathcal{L})$ such that its pairs of ends crosses (A, B).

(1) *For $1 \le i < j \le k$, and each $P \in \mathcal{L}_{i,j}$, and for each $v \in V(P)$, there exists b in $[I_i \to I_j]$ and not in the interior of any member of $\mathcal{I}$, and a boom $Q_1, \ldots, Q_t$ attaching to b and with $V(Q_1) = \{v\}$, of length $t = f(V(P) \cap I_i, b)$.*

Let P have ends $s \in I_i$ and $t \in I_j$. Define $\phi(P)$ by:

- if $s \ne t$, let $\phi(P)$ be the number of paths in $U(\mathcal{L})$ with both ends in $[s \to t]$ (including P itself);
- if $s = t$ and $V(P) = [I_i \to I_j]$, let $\phi(P) = 1$; and
- if $s = t$ and $V(P) = [I_j \to I_i]$, let $\phi(P) = |U(\mathcal{L})|$.

We proceed by induction on $\phi(P)$. Let $v \in V(P)$. Since P is pushed, either:

- $v \in [I_i \to I_j]$; or
- v is incident with a region that contains a point of $[I_i \to I_j]$; or
- there is a region incident with v and some vertex u, and a log Q with ends u, w, where w belongs to some path P' of $\mathcal{L}$ inside P.

In the first two cases, the boom we want has length one, with a one-vertex log. In the third case, let u, Q, w, P' be as in that case. Since P' is inside P and hence $\phi(P') < \phi(P)$, we can apply the inductive hypothesis to P', w. The result follows by concatenating Q with the boom for P', w. This proves (1).

We suppose (for a contradiction) that $d'_{ij} \neq d_{ij}$ for some i, j with $1 \leq i < j \leq k$. Choose such a pair (i, j) with $[I_i \to I_j]$ minimal. So we know:

- $d_{i',j'} = 0$ for all i', j' with $1 \leq i' < i$ and $i < j' < j$, and $d_{i',j'} = 0$ for all i', j' with $i < i' < j$ and $j < j' \leq k$, since d is a demand function and $d_{ij} \neq 0$;
- $d'_{i',j'} = d_{i',j'}$ for all i', j' with $i \leq i' < j' \leq j$ and $(i', j') \neq (i, j)$, from the minimality of $[I_i \to I_j]$; and
- $d'_{i',j'} = 0$ for all i', j' with $1 \leq i' \leq i$ and $j \leq j' \leq k$ and $(i', j') \neq (i, j)$, since d' is a partial d-demand and $d'_{i,j} < d_{i,j}$.

It follows that for $1 \leq i' < j' \leq k$, if $\mathcal{L}_{i'j'} \neq \emptyset$, then either

- $i \leq i' < j' \leq j$; or
- $1 \leq i' < j' \leq i$, or $j \leq i' < j' \leq k$.

Let $\mathcal{P}_1, \mathcal{P}_2$ be the union of all the sets $\mathcal{L}_{i'j'}$ satisfying the first bullet and second bullet respectively. Thus, $\mathcal{P}_1, \mathcal{P}_2$ are disjoint and have union $U(\mathcal{L})$. For $i = 1, 2$ let Z_i be the set of all vertices with distance at most c from some member of $\mathcal{P}_i$.

(2) *There is a path from I_i to I_j in $G \setminus (Z_1 \cup Z_2)$.*

Suppose that there is no such path. Consequently there is a region of $G \setminus (Z_1 \cup Z_2)$ containing points of $[I_i \to I_j], [I_j \to I_i]$. (Note that this is true even if one of $I_i \cap V(G), I_j \cap V(G)$ is a subset of $Z_1 \cup Z_2$.) Hence either:

- there is a region of G containing points $a_1 \in [I_i \to I_j]$ and $a_2 \in [I_j \to I_i]$; or
- there is a region of G containing a point $a_2 \in [I_j \to I_i]$ and incident with a vertex $z_1 \in Z_1$; or
- there is a region of G containing a point $a_1 \in [I_i \to I_j]$ and incident with a vertex $z_2 \in Z_2$; or
- there is a region of G incident with a vertex $z_1 \in Z_1$ and with a vertex $z_2 \in Z_2$.

(Note that the points a_1, a_2 might lie in the interior of members of $\mathcal{I}$.) In the first case, there is a boom of length zero joining a_1, a_2, and yet (I_i, I_j) crosses (a_1, a_2) and $d_{i,j} \geq 1$, a contradiction.

In the second case, since $z_1 \in Z_1$, there is a log Q between z_1 and some $u \in V(P)$, for some $P \in \mathcal{P}_1$. Let $P \in \mathcal{L}_{i'j'}$. By (1), there exists $b \in [I_{i'} \to I_{j'}]$, not in the interior of any member of $\mathcal{I}$, such that there is a boom $Q_1, \ldots, Q_t$ with $u \in V(Q_1)$, attaching to b and with $V(Q_1) = \{u\}$, of length $t = f(V(P) \cap I_{i'}, b)$. Consequently $Q, Q_2, \ldots, Q_t$ is a boom of the same length t, joining a_2 and b. Now there are two subcases. If $(i', j') \neq (i, j)$, then $f(V(P) \cap I_{i'}, b)$ equals the sum of $d_{i''j''}$ over all choices of $i'' < j''$ such that $I_{i''} \subseteq [I'_i \to b]$ and $I_{j''} \subseteq [b \to I_{j'}]$. This sum is less than the sum of $d_{i''j''}$ over all choices of i'', j'' with $i'' < j''$ such that $(I_{i''}, I_{j''})$ crosses (a_2, b), since this latter sum also contains $d_{ij} > 0$. Thus t is less than the sum of $d_{i'',j''}$ over all i'', j'' such that $(I_{i''}, I_{j''})$ crosses (a_2, b), a contradiction. In the second subcase, when $(i', j') = (i, j)$, then $f(V(P) \cap I_{i'}, b)$ equals $d'_{i,j}$ plus the sum of $d_{i''j''}$ over all choices of $i'' < j''$ such that $I_{i''} \subseteq [I_i \to b]$ and $I_{j''} \subseteq [b \to I_j]$ and $(i'', j'') \neq (i, j)$. Again, this sum is less than the sum of $d_{i''j''}$ over all choices of $i'' < j''$ such that $(I_{i''}, I_{j''})$ crosses (a_2, b), since this latter sum contains $d_{ij} > d'_{ij}$, and again we obtain a contradiction.

The third and fourth cases are similar, using $[I_j \to I_i]$ in place of $[I_i \to I_j]$ in the third case, and combining both arguments for the fourth case. This proves (2).

Since there is a path P from I_i to I_j in $G \setminus (Z_1 \cup Z_2)$, we can choose such a path P to minimize the set of regions that belong to the side of P in Σ (in the natural sense) that contains $[I_i \to I_j]$. It follows that if we define $\mathcal{L}'_{ij} = \mathcal{L}_{ij} \cup \{P\}$, and $\mathcal{L}'_{i'j'} = \mathcal{L}_{i'j'}$ for all $(i', j') \neq (i, j)$, then $(\mathcal{L}_{i'j'} : 1 \leq i' < j' \leq k)$ is pushed, contrary to the maximality of the choice of d'. This contradiction shows that $d' = d$, and so proves 2.2. ∎

As we said, this result is a strengthening of 2.1. There are other strengthenings possible, that can be proved in the same way. First, instead of specifying pairs of vertices and asking for paths joining them with pairwise distance more than c, one could specify disjoint subsets of vertices, and ask for connected subgraphs, each including one of the subsets, again with pairwise distance more than c; and again such subgraphs exist if and only if there is no obstructing boom. (When $c = 0$, this was done in [10].) Second, instead of asking for the paths pairwise to have distance more than c, one could specify a minimum distance for each pair of paths that lie on a common region of the union of the paths, and a similar result holds (varying the lengths of logs in a boom appropriately). Third, we could make the graph a digraph, and ask for each path to be a directed path from a specified end to its other end. We omit the details.

3 The coarse Menger conjecture with a bounded number of alternations

Now we turn to the main topic, the proof of 1.2. The proof uses edge-contraction, which is easy enough for general graphs but difficult to be precise about when graphs are drawn in a surface. In all cases, edge-contraction is to be applied to the drawing, not just to the graph, and we beg the reader's indulgence for any imprecision. We are avoiding loops and parallel edges in this paper, but they might appear when we contract edges, so we need some way to get rid of them. Let us just say that, when we do a contraction, any loops that appear are deleted, and if we make a set of parallel edges, then all but one of the edges in the set are deleted, choosing the one survivor arbitrarily.

We are given S, T, subsets of vertices of a graph G drawn in the plane, with S, T on the outside. With such a drawing, we can choose a simple closed curve in the plane that passes through each vertex in $S \cup T$, bounding an open disc in which all the remainder of the graph is drawn. Let us call such a curve a *bounding curve*. (We stress that the only vertices drawn in the bounding curve itself are those in $S\cup$; all the others are strictly inside.) If G has cut-vertices (or even worse, if G is not connected), the order in which a bounding curve passes through the vertices of $S \cup T$ may not be unique, but that will not matter.

Given G, S, T as above, if we contract some set F of edges of G, making a graph G' say, let H be the subgraph of G with vertex set $V(G)$ and edge set F. Each vertex v of G' is made by identifying the vertices of some component $\eta(v)$ of H under contraction, and we call $\eta(v)$ the *pre-image of v under contracting F*. Let S' be the set of $v \in V(G')$ such that $S \cap V(\eta(v)) \neq \emptyset$, and define T' similarly. Then G' is also drawn in the plane and S', T' are on the outside. We say that "G', S', T' are obtained from G, S, T by contracting F".

Let C be a bounding curve, and let us assign it a direction of rotation called "clockwise". We may assume that $S, T \neq \emptyset$. If I is an interval of C with $|I| = \{x\}$, we say x is the *end* of I and the *interior* of I is null. We can choose a set $\mathcal{I} = \{I_1, I_2, \ldots, I_{2n}\}$ of pairwise internally-disjoint intervals of C with the following properties:

- $I_1, I_2, \ldots, I_{2n}$ are numbered in clockwise order on $\mathrm{bd}(\Sigma)$;
- for $1 \leq i \leq 2n$ with i odd, the end(s) of I_i belong to S, and for i even, the end(s) of I_i belong to T;
- $I_1, I_3, I_5, \ldots, I_{2n-1}$ are pairwise disjoint, and $I_2, I_4, \ldots, I_{2n}$ are pairwise disjoint;
- $S \subseteq I_1 \cup I_3 \cup I_5 \cup \cdots \cup I_{2n-1}$, and $T \subseteq I_2 \cup I_4 \cup \cdots \cup I_{2n}$;

Thus, $\mathcal{I}$ is an interval system for $\mathrm{bd}(\Sigma)$. We call such a set $\mathcal{I}$ an *interval covering* for S, T; and we care about the size of $\mathcal{I}$. The next result shows that, if $\mathcal{I}$ is bounded, we can find the k connected subgraphs as in 1.2 not only with bounded diameter, but with a bounded number of edges in total.

3.1 *Let G be drawn in the plane, and let $S, T \subseteq V(G)$ be on the outside. Let C be a bounding curve, and let $\mathcal{I}$ be an interval covering in C of S, T with size $2n$. Suppose that there do not exist $k+1$ paths between S and T, pairwise with distance more than c. Then there exist k connected subgraphs $B_1, \ldots, B_k$, such that every $S-T$ path in G contains a vertex of $B_1 \cup \cdots \cup B_k$, and $B_1 \cup \cdots \cup B_k$ has at most $8kn^2c$ edges.*

Proof. Let $\mathcal{I} = \{I_1, I_2, \ldots, I_{2n}\}$, numbered as in the definition. Let I_i have ends a_i, b_i, where b_i is the clockwise end of I_i. Moreover there is an interval L_i of C with ends b_i, a_{i+1} (where a_{2n+1} means a_1) such that a_{i+1} is the clockwise end of L_i. Thus $I_1, \ldots, I_{2n}, L_1, \ldots, L_{2n}$ are pairwise internally-disjoint intervals of C with union C. For all choices of $J_1, J_2 \in \{I_1, \ldots, I_{2n}, L_1, \ldots, L_{2n}\}$, if there is a boom of length at most k joining some point of J_1 and some point of J_2, let $F(J_1, J_2) = F(J_2, J_1)$ be the union of the edge-sets of the logs in some shortest such boom. If there is no such boom, let $F(J_1, J_2) = \emptyset$. Thus each $F(J_1, J_2)$ has size at most kc. Let F be the union of the sets $F(J_1, J_2)$ over all choices of $\{J_1, J_2\}$. Since there are at most $8n^2$ choices of the unordered pair $\{J_1, J_2\}$, F has size at most $8kn^2c$.

Let G', S', T' be obtained from G, S, T by contracting F. For each $v \in V(G')$, let $\eta(v)$ its pre-image under contracting F.

(1) *There do not exist $k+1$ vertex-disjoint paths in G' from S' to T'.*

Suppose that there are such paths $P_1, \ldots, P_{k+1}$. For $1 \le h \le k+1$, let P_h have ends $s_h \in S$ and $t_h \in T$. (The choice of s_h, t_h is not always uniquely determined by P_h, since it might have both ends in $S \cap T$, but let us choose some such labeling of its ends.) For $1 \le i < j \le 2n$:

- if i, j have the same parity define $d_{ij} = 0$,
- if i is odd and j is even let d_{ij} be the number of $h \in \{1, \ldots, k+1\}$ with $s_h \in I_i$ and $t_h \in I_j$; and
- if i is even and j is odd, let d_{ij} be the number of $h \in \{1, \ldots, k+1\}$ with $t_h \in I_i$ and $s_h \in I_j$.

(Thus each of $P_1, \ldots, P_{k+1}$ is counted exactly once.) Thus d is a demand function for $\mathcal{I}$. By hypothesis, there is no c-distant linkage in G for $d, \mathcal{I}$. Hence by 2.2, there exist $a, b \in C$, and a boom of length less than t joining a, b, where t is the sum of d_{ij} over $1 \le i < j \le n$ such that (I_i, I_j) crosses (a, b) in C. Choose $X, Y \in \{I_1, \ldots, I_{2n}, L_1, \ldots, L_{2n}\}$ such that $a \in X$ and $b \in Y$. Since $t \le k+1$, there is a boom of length at most k joining X, Y, such that F contains the edges of all its logs. But all these edges are contracted in making G'; and this contradicts that $P_1, \ldots, P_{k+1}$ exist. This proves (1).

From (1) and Menger's theorem, there is a set $X \subseteq V(G')$ with $|X| \le k$ such that every $S' - T'$ path in G' contains a vertex of X. Consequently every $S - T$ path in G contains a vertex of one of the subgraphs $\eta(x)$ ($x \in X$). But each $\eta(x)$ has all its edges in F. This proves 3.1. ∎

4 The coarse Menger conjecture in the general disc case

To complete the proof of 1.2, we need to show how to reduce the general case to the case covered in the previous section. If G is connected and non-null and drawn in the plane, let r be its infinite region. There is a closed walk $v_0, e_1, v_1, \ldots, e_n, v_n = v_1$ that traces its boundary in the natural sense: that is, every edge incident with r appears once or twice in this walk, and for $1 \le i \le n$, the edges e_i, e_{i+1} and their common end v_i make an "angle" of the boundary of r. We call this a *boundary walk* of the drawing.

If W is a closed walk $v_0, e_1, \ldots, e_n, v_n = v_0$, a *subwalk* of W is a walk of the form $v_i, e_{i+1}, v_{i+1}, \ldots, e_j, v_j$ or of the form $v_j, e_{j+1}, \ldots, v_n = v_0, e_1, v_1, \ldots, v_i$, where $0 \le i \le j \le k$. In particular, if $1 \le i < j \le k$, we call the first the (i, j)-*subwalk* and the second the (j, i)-*subwalk*. We need the following lemma.

4.1 *Let G be a connected graph, drawn in the plane, and let $v_0, e_1, v_1, \ldots, e_n, v_n = v_0$ be a boundary walk. Let $X \subseteq V(G)$, such that for $1 \le i < j \le n$, if $v_i = v_j$ then one of $v_{i+1}, \ldots, v_{j-1} \in X$, and one of $v_{j+1}, \ldots, v_n, v_1, \ldots, v_{i-1}$ is in X. Then the number of $i \in \{1, \ldots, n\}$ with $v_i \in X$ is at most $2|X|$.*

Proof. There is a hypothesis that for $1 \le i < j \le n$, if $v_i = v_j$ then one of $v_{i+1}, \dots, v_{j-1} \in X$, and one of $v_{j+1}, \dots, v_n, v_1, \dots, v_{i-1}$ is in X; let us call this the *betweenness hypothesis*. For any walk W with terms $w_0, f_1, w_1, \dots, f_m, w_m = w_0$ in G, let $\phi(W)$ be the number of $i \in \{1, \dots, m\}$ such that $w_i \in X$.

Now let W be the given boundary walk of G. We proceed by induction on the length of W. If for each $x \in X$ there is at most one $i \in \{1, \dots, n\}$ such that $v_i = x$, then $\phi(W) \le |X|$ and the claim is true; so we assume that there exist $1 \le i < j \le n$ such that $v_i = v_j \in X$. Choose such i, j with $j - i$ minimum, and let W_1, W_2 be the (i,j)-subwalk and the (j,i)-subwalk respectively. Thus there are two subdrawings G_1, G_2 of the drawing of G, such that W_i is a boundary walk of G_i for $i = 1, 2$, and $V(G_1 \cap G_2) = \{v_i\}$. Let $X_i = X \cap V(G_i)$ for $i = 1, 2$. From the minimality of $j - i$, it follows that $\phi(W_1) = |X_1|$; and from the hypothesis, there exists h with $i < h < j$ such that $v_h \in X$, and so $|X_1| \ge 2$. Moreover, W_2 satisfies the betweenness hypothesis, so we can apply the inductive hypothesis to it, and therefore $\phi(W_2) \le 2|X_2|$. But $\phi(W) = \phi(W_1) + \phi(W_2)$, and $|X_1| + |X_2| = |X| + 1$, and so

$$\phi(W) \le |X_1| + 2|X_2| = 2(|X_1| + |X_2|) - |X_1| = 2|X| + 2 - |X_1| \le 2|X|$$

since $|X_1| \ge 2$. This proves 4.1. ∎

We also need the following.

4.2 *Let G be a connected graph drawn in the plane, let C be a bounding curve, let Σ be the closed disc bounded by C, and let $\mathcal{I} = \{I_1, \dots, I_{2n}\}$ be an interval system in C. For $1 \le i \le 2n$, let P_i be an $S - T$ path with one end in I_i and the other in I_{i+1}, such that if P_i has length > 0, then $[I_i \to I_{i+1}]$ has nonnull interior, and all its vertices are incident with the region of G in Σ that contains the interior of $[I_i \to I_{i+1}]$ (where I_{2n+1} means I_1). Let $k \ge 1$. Then either*

- *there are $k+1$ of the paths $P_1, \dots, P_{2n}$ pairwise c-distant; or*
- *there are connected subgraphs $B_1, \dots, B_t$ with $t \le 5k/2 - 1$, such that each B_i has diameter at most $3c$, and each P_i has a vertex in some B_j.*

Proof. We proceed by induction on n. (I_{2n+1} means I_1 throughout.) Suppose that $\text{dist}_G(P_i, P_j) \le c$ for some non-conseutive i, j with $1 \le i \le j \le 2n$ (that is, with $j \ge i + 2$ and $i + 2n \ge j + 2$). We may assume that $i = 1$. Let Q be a log between P_1, P_j, and let B be the maximal subgraph of G such that each of its vertices has distance at most c from Q. Thus B has diameter at most $3c$. Let k_1 be the maximum number of $P_2, \dots, P_{j-1}$ that have distance more than c from Q and more than c from each other. These k_1 paths also have distance at least $c+1$ from each of $P_{j+1}, \dots, P_{2n}$, since if some $P_{j'}$ with $j + 1 \le j \le 2n$ has positive length then all of its vertices are incident with the region of G in Σ that contains the interior of $[I_{j'} \to I_{j'+1}]$ (and so are on the side of Q not containing P_2). In particular, they have distance $> c$ from P_{2n}, and consequently we may assume that $k_1 \le k - 1$.

Similarly, let k_2 be the maximum number of $P_{j+1}, \dots, P_{2n}$ that have distance more than c from Q and more than c from each other; and then $k_2 \le k - 1$. From the inductive hypothesis applied to $P_2, \dots, P_{j-1}$, if $k_1 \ge 1$, there is a set of at most $5k_1/2 - 1$ connected subgraphs of G, each of diameter at most $3c$, such that each of $P_2, \dots, P_{j-1}$ intersects one of them, or intersects B; and similarly if $k_2 \ge 1$, there is a set of at most $5k_2/2 - 1$ connected subgraphs of G, each of diameter at most $3c$, such that each of $P_{j+1}, \dots, P_{2n}$ intersects one of them, or intersects B.

Assume first that $k_1, k_2 \geq 1$. Consequently, we have in total at most $5k_1/2 - 1 + 5k_2/2 - 1 + 1$ connected subgraphs, each of diameter at most $3c$, such that each of $P_1, \ldots, P_{2n}$ intersects one of them. If $k_1 + k_2 > k$ then the first outcome holds, and otherwise the second holds.

So we may assume that $k_1 = 0$ say. But then all of $P_1, \ldots, P_j$ meet B, and so if $k_2 > 0$, we have $(5k_2/2 - 1) + 1 \leq 5k/2 - 1$ connected subgraphs that work, and if also $k_2 = 0$ then B suffices by itself. In either case, the result holds.

So we can assume that there is no such log Q. Hence $P_1, P_3, \ldots, P_{2n-1}$ are pairwise c-distant, and so $n \leq k$. But we may choose $2n$ singleton sets such that each P_i contains one of them, and so we may assume that $2n > 5k/2 - 1$, and so $n = k = 1$. If $\text{dist}_G(P_1, P_2) > c$ then the first outcome holds, and otherwise there is a log joining P_1, P_2 and the second holds. This proves 4.2. ∎

We remark that $5/2$ might not be the best possible constant in 4.2, although it is easy to show that no constant less than $3/2$ works in general.

We also need:

4.3 *Let G be a connected graph drawn in the plane, with S, T on the outside, and let W be its boundary walk. Suppose that $X \subseteq V(G)$ has the property that every subwalk of W that contains a vertex in S and a vertex in T also contains a vertex in X; and suppose that there do not exist $k+1$ $S-T$ paths pairwise with distance more than c. Then there exist k connected subgraphs $B_1, \ldots, B_k$, such that every $S-T$ path in G contains a vertex of $B_1 \cup \cdots \cup B_k$, and $B_1 \cup \cdots \cup B_k$ has at most $32k|X|^2c$ edges.*

Proof. We proceed by induction on the length of W. Let W be $v_0, e_1, v_1, \ldots, e_n, v_n$, and suppose first that there exist $1 \leq i < j \leq n$ such that $v_i = v_j$ and none of $v_{i+1}, \ldots, v_{j-1}$ belong to X. It follows that at least one of S, T is disjoint from $\{v_{i+1}, \ldots, v_{j-1}\}$. Let W_1, W_2 be the (i,j)- and (j,i)-subwalks respectively. There are subdrawings G_1, G_2 such that $G_1 \cup G_2 = G$ and $V(G_1 \cap G_2) = \{v_i\}$, such that W_i is a boundary walk of G_i for $i = 1, 2$. If both S, T are disjoint from $\{v_{i+1}, \ldots, v_{j-1}\}$, then deleting $V(G_1) \setminus \{v_i\}$ makes no difference, and the result follows by induction applied to W_2, G_2. We assume then that $S \cap \{v_{i+1}, \ldots, v_{j-1}\} \neq \emptyset$, and so $T = T \cap V(G_2)$. Let $S_2 = (S \cap V(G_2)) \cup \{v_i\}$. Then every subwalk of W_2 between S_2, T contains a vertex in X, because either it is a subwalk of W, or it passes through v_i, and then it can be extended to a subwalk of W by adding a portion of W_1. But then the result follows from the inductive hypothesis applied to $G_2, S_2, T, X \cap V(G_2)$.

We may therefore assume that there is no such pair i, j. Consequently the hypotheses of 4.1 are satisfied, and so the number of $i \in \{1, \ldots, n\}$ with $v_i \in X$ is at most $2|X|$. Hence there is a partition of $\{1, \ldots, n\}$ into at most $4|X|$ intervals (where we count a set of the form $\{j, \ldots, n, 1, \ldots, i\}$ as an interval if $i < j$), such that for each of these intervals I say, one of S, T is disjoint from $\{v_i : i \in I\}$. (To see this, take $\{i\}$ for each $v_i \in X$ as a singleton interval, and also take all the gaps between them.) Let C be a bounding curve. It follows (by intersecting each of our intervals with $V(C) \cap V(G)$) that there is an interval system $\mathcal{I}$ in C of size at most $4|X|$, such that $S \cup T \subseteq \bigcup_{I \in \mathcal{I}} I$, and such that for each $I \in \mathcal{I}$, one of S, T is disjoint from I. By replacing by their union any two consecutive intervals in $\mathcal{I}$ that are both disjoint from S or both disjoint from T, we may assume that $\mathcal{I}$ is an interval covering of S, T of size at most $4|X|$. Hence the result follows from 3.1. This proves 4.3. ∎

Now we deduce our main theorem, which we restate:

4.4 *Let $c, k \geq 0$, let G be drawn in the plane, with S, T on the outside. Then either:*

- *there are $k+1$ paths between S, T, pairwise at distance more than c; or*
- *there is a set of at most k connected subgraphs of G, such that every path between S, T intersects one of these subgraphs, and the sum of the diameters of the subgraphs is at most $200k^3c$.*

Proof. We may assume that G is connected. Let C be a bounding curve, let Σ be the closed disc bounded by C, and let $\mathcal{I} = \{I_1, \dots, I_{2n}\}$ be an interval covering of S, T. For $1 \leq i \leq 2n$, let P_i be an $S-T$ path with one end in I_i and the other in I_{i+1}, such that for each $v \in V(P_i)$, v is incident with a region of G in Σ that contains a point of $[I_i \to I_{i+1}]$. By 4.2, we may assume that there is a set of at most $5k/2-1$ connected subgraphs of G, each of diameter at most $3c$, such that each of $P_1, \dots, P_{2n}$ intersects one of them. These subgraphs need not be vertex-disjoint, but their union has at most $5k/2$ components, and the sum of the diameters of these components is at most $15kc/2$. Let F be the union of the edge sets of these components, and let G', S', T' be obtained from G, S, T by contracting F. Each of the components with union F contracts to a vertex in G'; let X be the set of such vertices. Thus $|X| \leq 5k/2-1$. For each $v \in V(G')$, let $\eta(v)$ be the pre-image of v under contracting F. Thus the sum of the diameters of $\eta(v)$, over all $v \in V(G')$, is at most $15kc/2$. (Note that, typically, $\eta(v)$ consists just of the vertex v and so has diameter zero: the only terms that contribute to the sum are when $v \in X$.)

Now G' is connected; let W be a boundary walk, where W is $v_0, e_1, \dots, e_t, v_t = v_0$ say. Every subwalk of W that contains a vertex of S' and a vertex of T', also contains a vertex of X, because of the way we constructed X. We may assume that there do not exist $k+1$ paths of G' between S', T', pairwise at distance more than c, since otherwise the first outcome of the theorem holds. Thus, by 4.3, there exist k connected subgraphs $B'_1, \dots, B'_k$ of G', such that every $S'-T'$ path in G' contains a vertex of $B'_1 \cup \dots \cup B'_k$, and $B'_1 \cup \dots \cup B'_k$ has at most $32k(5k/2-1)^2c$ edges. For $1 \leq i \leq k$, let B_i be the subgraph of G formed by the union of the edges of B'_i and the subgraphs $\eta(v)$ ($v \in V(B'_i)$). Thus $B_1, \dots, B_k$ are connected, and every $S-T$ path in G meets one of them. Moreover, the sum of the diameters of $B_1, \dots, B_k$ is at most $32k(5k/2-1)^2c + 15kc/2 \leq 200k^3c$, since $B'_1 \cup \dots \cup B'_k$ has at most $32k(5k/2-1)^2c$ edges and the sum of the diameters of $\eta(v)$ for $v \in V(G')$ is at most $15kc/2$. This proves 4.4. ∎

5 An algorithm

Suppose we are given a graph drawn in the plane, with S, T on the outside. Can we check in polynomial time whether there exist k $S-T$ paths pairwise c-distant? If c, k are constants, the answer is yes, as follows. Let us say the *depth* of a vertex v is the minimum n such that there is a sequence

$$v = v_0, r_1, v_1, r_2, \dots, v_n$$

of alternating vertices and regions, where v_n is incident with the infinite region, and r_i is incident with v_{i-1}, v_i for $1 \leq i \leq n$. The $S-T$ paths we want exist if and only if there exist distinct $s_1, \dots, s_k$ in S and distinct $t_1, \dots, t_k \in T$, such that there are k paths joining s_i, t_i for $1 \leq i \leq k$, pairwise c-distant. And for a given choice of $s_1, \dots, s_k, t_1, \dots, t_k$, whether such paths exist is determined by the existence of certain booms of length at most $k-1$, attaching to two points of some bounding curve, as in 2.1. The vertices in every such boom have depth at most $c(k-1)/2$, so we can delete all vertices with depth more than $c(k-1)/2+1$ without changing whether the booms exist, and therefore without changing whether the paths we want exist. (The "+1" is to avoid making new regions incident with vertices at depth $c(k-1)/2$.) But after this, the graph has bounded tree-width, and the problem can be solved in linear time, by Courcelle's theorem [4], since the question can be expressed by a monadic second-order formula.

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