Asymptotic structure. V. The coarse Menger conjecture in bounded path-width

WORKING DRAFT

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Abstract

Menger's theorem tells us that if S, T are sets of vertices in a graph G, then (for $k \geq 0$) either there are $k+1$ vertex-disjoint paths between S and T, or there is a set of k vertices separating S and T. But what if we want the paths to be far apart, say at distance at least c? One might hope that we can find either $k+1$ paths pairwise far apart, or k sets of bounded radius that separate S and T, where the bound on the radius is some ℓ that depends only on k, c (the "coarse Menger conjecture"). We showed in an earlier paper that this is false for all $k \geq 2$ and $c \geq 3$. To do so we gave a sequence of finite graphs, counterexamples for larger and larger values of ℓ with $k = 2$, $c = 3$. Our counterexamples contained subdivisions of uniform binary trees with arbitrarily large depth as subgraphs, and so had unbounded path-width.

Here we show that, if H is a graph that can be drawn in the plane such that each region shares a vertex with the infinite region, then the coarse Menger conjecture is true for all graphs not containing H as a minor. Consequently, the conjecture is true for all graphs with bounded path-width (by taking H to be a sufficiently large tree), and it is true for series-parallel graphs (by taking $H = K_4$). The first is somewhat surprising, since the conjecture is false for bounded tree-width.

1 Introduction

Let S, T be sets of vertices of a graph G. (In this paper, all graphs are finite and have no loops or multiple edges.) Menger's theorem [7] tells us that either there are $k+1$ pairwise vertex-disjoint paths between S and T, or there is a set X of at most k vertices such that every S-T path in G meets X. But what if we want the paths to be pairwise far apart? In this case, the question is much harder. Bienstock [3] showed that it is NP-hard to decide whether, given four vertices s_1, s_2, t_1, t_2 of a graph G, there are two paths between $\{s_1, s_2\}$ and $\{t_1, t_2\}$ that have distance ≥ 2, that is, they are vertex-disjoint and there is no edge joining them. This was recently extended by Baligács and MacManus [2], who showed the same thing for distance $\geq c$, for each $c \geq 3$.

Since the problem is NP-complete, one would not expect to find a necessary and sufficient condition for the existence of $k+1$ S-T paths at distance at least c; but still one could hope for some sort of obstruction that is necessary for excluding $k+1$ S-T paths at distance at least c, and sufficient for excluding $k+1$ S-T paths at distance more than some larger number depending on k, c. Two groups of researchers, Albrechtsen, Huynh, Jacobs, Knappe and Wollan [1], and independently Georgakopoulos and Papasoglu [5], proposed such a statement:

1.1 Coarse Menger Conjecture: *For all integers $k \geq 0$ and $c \geq 1$ there exists $\ell \geq 0$ with the following property. Let G be a graph and let $S, T \subseteq V(G)$. Then either*

- *there are $k+1$ paths between S, T, pairwise at distance at least c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex at distance at most ℓ from some member of X.*

Both groups showed that this is true for $k = 1$, and Gartland, Korhonen and Lokshtanov [4] and Hendrey, Norin, Steiner, and Turcotte [6] proved it for bounded degree graphs when $c = 2$. However, three of us showed in [8] that the coarse Menger conjecture is false for all $k \geq 2$, for any fixed $c \geq 3$, and with all degrees at most three. Indeed, it remains false even if we weaken the bound $|X| \leq k$ in the second bullet to $|X| \leq m$, where m is any constant depending on k, c [12].

Thus, we need to lower our sights a little, and one way to do so is to work in restricted classes of graphs. The counterexamples of [8] have unbounded genus, and have unbounded "path-width" (defined in the next section). It might be true that the coarse Menger conjecture holds for graphs of bounded genus, but this is open; see [13] for some progress in this direction. Our result implies that the coarse Menger conjecture is true for graphs of bounded path-width.

More exactly, it implies:

1.2 *Let $k, d \geq 0$ and $c \geq 1$ be integers. Then there exists $\ell \geq 0$, such that for every graph G with path-width at most d, and all $S, T \subseteq V(G)$, either:*

- *there are $k+1$ paths between S, T, pairwise at distance at least c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex at distance at most ℓ from some member of X.*

(For fixed k, d, the number ℓ depends linearly on c, as we will see.) We will deduce in the conclusion that the coarse Menger conjecture also holds for graphs of "bounded coarse path-width". Curiously, the coarse Menger conjecture is *not* true for graphs of bounded tree-width, since the counterexamples of [8] have tree-width six.

Our main result is more general than this. Let us say a graph H is *near-outerplanar* if it is planar, and has a planar drawing such that every region is incident with a vertex of the infinite region (see Figure 1). Thus, outerplanar graphs are near-outerplanar, but so are many other graphs, such as wheels.

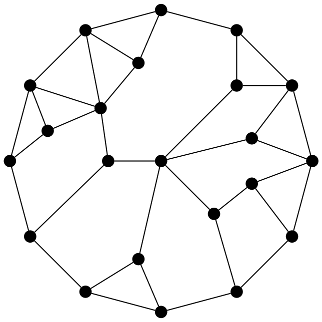

Figure 1: A near-outerplanar graph.

We will prove that:

1.3 *For every near-outerplanar graph H, and all integers $k \geq 0$ and $c \geq 1$, there exists $\ell \geq 0$, such that for every graph G not containing H as a minor, and all $S, T \subseteq V(G)$, either:*

- *there are $k+1$ paths between S, T, pairwise at distance at least c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex at distance at most ℓ from some member of X.*

(We could assume that $k \geq 2$ if we wanted, because the coarse Menger conjecture is known to be true for $k = 1$ [1, 5, 8], but there is no need.) In particular, every tree H is near-outerplanar, and so the coarse Menger conjecture is true for graphs that do not contain H as a minor; and this implies 1.2, as we discuss in the next section. Moreover, the graph K_4 is near-outerplanar, and so the coarse Menger conjecture is true for series-parallel graphs, since they are the graphs with no K_4 minor. (This was an open question.)

2 Subdivisions and path-width

In this paper, for $d \geq 1$, the "uniform binary tree of depth $d+1$" is the tree H such that for some $r \in V(H)$ (the "root"), r has degree two, all other vertices have degree one or three, and every vertex of degree one has distance exactly d from r. Thus, H has $2^{d+1} - 1$ vertices. We denote this tree by H_d.

If H is a graph, a *subdivision* of H is a graph obtained from H by replacing each of its edges by a path of length at least one joining the same pair of vertices, where these paths are pairwise vertex-disjoint except for their ends. For $\ell \geq 1$, let us say an $(< \ell)$*-subdivision* of H is a subdivision obtained by replacing each edge by a path of length $\leq \ell$ (and at least one). (This is to be consistent with the standard term "1-subdivision", which means replacing each edge with a path of length two.)

Let us define path-width. A graph G has *path-width* at most d if and only if there is a sequence $W_1, \ldots, W_n$ of subsets of its vertex set, satisfying:

- $|W_i| \leq d+1$ for $1 \leq i \leq n$;
- $G[W_1] \cup \cdots \cup G[W_n] = G$; and
- $W_i \cap W_k \subseteq W_j$ for $1 \leq i \leq j \leq k \leq n$.

We do not really need this definition. The only thing about bounded path-width that concerns us is a theorem of Robertson and Seymour [14]:

2.1 *For every integer $\delta \geq 1$, there exists k, such that every graph that contains no subdivision of H_δ as a subgraph has path-width at most k; and conversely, every graph that contains a subdivision of H_δ as a subgraph has path-width at least $\delta/2$.*

Thus, knowing that there is a bound on path-width is the same as knowing that for some δ, no subgraph is a subdivision of H_δ. Indeed, in this paper it is more natural to work with the "excluded tree subdivision" version directly, rather than working with path-width. So we could reformulate 1.2 as follows:

2.2 *For all integers $k, c \geq 0$ and $\delta \geq 1$, there exists $\ell \geq 0$, with the following property. Let G be a graph that contains no subdivision of H_δ as a subgraph, and let $S, T \subseteq V(G)$. Then either*

- *there are $k+1$ paths between S, T, pairwise at distance greater than c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex at distance at most ℓ from some member of X.*

Indeed, one can easily modify the proof we will give (although we will not do so), to show that the same conclusion holds even if we just assume that G contains no $(< \ell)$-subdivision of H_δ.

3 Near-outerplanar graphs and the bordered binary tree.

But, as we said, our main result is more general than 2.2. For $\delta \geq 1$, the graph H_δ is a uniform binary tree of depth $\delta + 1$. Let us add a path running through the leaves in this order, as in Figure 2. We call the graph that results H_δ^+. The vertex of degree two at the top of the figure is its *root*.

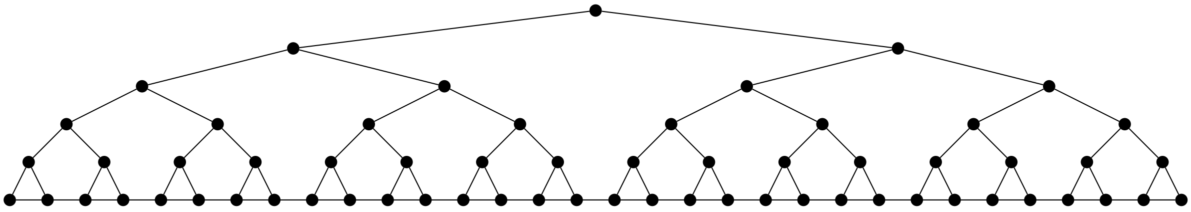

Figure 2: The graph H_5^+.

The added path in the graph of the figure runs through all 32 leaves of H_5; let us number them $v_1, \ldots, v_{32}$ in order. There are many other orders that would give an isomorphic graph: for instance a path running though the leaves in the order $v_2, v_1, v_3, v_4, \ldots, v_{32}$ would give a graph isomorphic to H_5^+, as would the order $v_3, v_4, v_1, v_2, v_5, \ldots, v_{32}$. But not every order works; let us say a linear order of the leaves of H_δ is *normal* if adding a path through the leaves in that order gives a copy of H_δ^+.

Evidently, H_δ^+ is near-outerplanar, and it is an easy exercise to show that every near-outerplanar graph is isomorphic to a minor of H_δ^+ for sufficiently large δ. To see this, let H be near-outerplanar, drawn such that every region is incident with a vertex on the infinite region.

- By adding an edge between two components if possible, and repeating, we may assume that H is connected.
- By adding an edge between ac if a-b-c is a path of the infinite boundary and b is a cut-vertex separating a, c, and repeating, we may assume the boundary of the infinite region is a cycle C.
- If some vertex of C has degree at least four, we may uncontract an edge (making C longer), "splitting" this vertex into two adjacent vertices of smaller degree. By repeating, we may assume that every vertex of C has degree at most three.
- Hence, $H \setminus E(C)$ is a forest, and by subdividing and adding edges we may assume it is a tree T, and all leaves of T are in C.
- We may assume this tree has maximum degree at most three, by splitting vertices into two adjacent vertices of smaller degree as before.
- And then the result is clear.

We will show that:

3.1 *For all integers $\delta \geq 5$, $k \geq 0$, and $c \geq 0$, let G be a graph with no H_δ^+ minor, and let $S, T \subseteq V(G)$. Then either*

- *there are $k+1$ paths between S, T, pairwise at distance greater than c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex at distance at most $(k\delta)^{4k\delta}c$ from some member of X.*

This strengthens 2.2, and will be our main result. We remark that here we are asking for paths at distance $> c$ rather than $\geq c$ as in 1.2; we find this form slightly more convenient. We are assuming $\delta \geq 5$ for numerical reasons.

The main purpose of this section is to prove that if we have a sufficiently large uniform binary tree with a path running through many of its leaves in any order, then G contains H_δ^+ as a minor.

3.2 *Let $\delta, h \geq 1$; let T be a subgraph of G isomorphic to a subdivision of H_h, with root r, and let P be a path of G that contains more than $(2h)^{\delta-1}$ leaves of T, and contains no other vertices of T. Then there is a subtree J of T, such that:*

- *J, rooted at s, is isomorphic to a subdivision of the rooted tree H_δ, where s is the vertex of J closest in T to r;*
- *all leaves of J are leaves of T; and*

- *P contains all the leaves of J in normal order, and contains no other vertices of J.*

Consequently G contains a subdivision of H_δ^+ as a subgraph.

Proof. We may assume that T is isomorphic to H_h, and we proceed by induction on h. If $\delta = 1$, then P contains at least two leaves of H_h and the claim holds, so we assume that $\delta \geq 2$. If $h = 1$, then T has only two leaves, and yet by hypothesis P contains more than $2^{\delta-1}$ of them, which is impossible. Thus we may assume that $h \geq 2$ and the result holds for $h - 1$. Let r_1, r_2 be the neighbours of r in T, and let T_1, T_2 be the components of $T \setminus r$, where $r_i \in V(T_i)$ for $i = 1, 2$. Suppose first that $|V(P \cap T_2)| \leq 2(2h)^{\delta-2}$. Then

$$|V(P \cap T_1)| > (2h)^{\delta-1} - 2(2h)^{\delta-2} = (2h-2)(2h)^{\delta-2} \geq (2h-2)^{\delta-1},$$

and the result follows from the inductive hypothesis applied to T_1 and P. So we may assume that $|V(P \cap T_2)| > 2(2h)^{\delta-2}$, and similarly $|V(P \cap T_1)| > 2(2h)^{\delta-2}$.

Choose an edge e of P such that some component P_1 of $P \setminus e$ contains exactly $2(2h)^{\delta-2} + 1$ vertices of T (this is possible since $2(2h)^{\delta-2} + 1 \leq (2h)^{\delta-1}$), and let P_2 be the other component of $P \setminus e$. The vertices of P in T all belong to T_1 or to T_2, so by exchanging T_1, T_2 if necessary, we may assume that $|V(P_1 \cap T_1)| > (2h)^{\delta-2}$. Hence $|V(P_1 \cap T_2)| \leq (2h)^{\delta-2}$, and so

$$|V(P_2 \cap T_2)| \geq |V(P \cap T_2)| - |V(P_1 \cap T_2)| > (2h)^{\delta-2}.$$

From the inductive hypothesis, for $i = 1, 2$, there is a subtree J_i of T_i, satisfying the three bullets above, with T, r, δ, J replaced by $T_i, r_i, \delta - 1, J_i$. Let J be the union of J_1, J_2 and the path of T between r_1, r_2 (which passes through r). Then J, rooted at r, is isomorphic to a subdivision of H_δ. Moreover, all leaves of J are leaves of one of J_1, J_2 and so are leaves of T; and P contains all the leaves of J, and no other vertices of J. But all vertices of J in J_i belong to P_i for $i = 1, 2$, and since P passes through the leaves of J_1 and of J_2 in normal order, it also passes through the leaves of J in normal order. This proves 3.2. ∎

4 A key lemma

If we contract an edge of a graph, then distances do not change by much, but if we delete an edge or a vertex, they might change considerably. In this section, we prove a lemma that allows us to bypass this problem to some extent, in graphs excluding subdivisions of some H_δ^+. The proofs in this section are the only places in the paper where we use the hypothesis about subdivisions of H_δ^+.

If X is a vertex of a graph G, or a subset of the vertex set of G, or a subgraph of G, and the same for Y, then $\text{dist}_G(X, Y)$ denotes the distance in G between X, Y, that is, the number of edges in the shortest path of G with one end in X and the other in Y. (If no path exists we set $\text{dist}_G(X, Y) = \infty$.)

For convenience, when $Y \subseteq V(G)$, we sometimes write $\text{dist}_Y(u, v)$ for $\text{dist}_{G[Y]}(u, v)$. If G is a graph, a *tie-breaker weighting* is a map $\lambda : E(G) \to \mathbb{R}$ with the following properties, where $\lambda(X)$ means $\sum_{e \in X} \lambda(e)$:

- if $X, Y \subseteq E(G)$ and $\lambda(X) \leq \lambda(Y)$ then $|X| \leq |Y|$;
- if $X, Y \subseteq E(G)$ are distinct, then $\lambda(X) \neq \lambda(Y)$.

To obtain such a map, one could define $\lambda(e) = 1 + \varepsilon(e)$, where the numbers $\varepsilon(e)$ are independent and very small. If λ is a tie-breaker weighting, we say a path P is a *λ-geodesic* if there is no path Q with the same ends as P and with $\lambda(E(Q)) < \lambda(E(P))$. For all u, v in the same component, there is a unique λ-geodesic between u, v.

Let $Z \subseteq V(G)$, and let $H \subseteq G$ be a tree with $|V(H)| \geq 2$, with all its leaves in Z and no other vertices in Z. We say that H is *Z-leaved.*

4.1 *Let $\delta, \ell \geq 1$, let G be a graph, and let $Z \subseteq V(G)$ such that no subgraph is a Z-leaved $(< 2^{\delta-1}\ell)$-subdivision of H_δ. Then there exists $Y \supseteq Z$ with the following properties:*

- *every vertex in Y has distance at most $2^{\delta-1}\ell$ from Z;*
- *for all $u, v \in Y$, if $\operatorname{dist}_{G[Y]}(u, v) > \delta 2^\delta \ell$, then $\operatorname{dist}_G(u, v) > \ell$.*

Proof. For $1 \leq t \leq \delta$, define $f_t = 2^{\delta-t}\ell$, and $g_t = 2tf_1$. Choose a tie-breaker weighting λ in G. Let $Z_0 = Z$. We will define a sequence of subsets $Z = Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_\delta$ inductively, as follows. For $1 \leq t \leq \delta$, suppose that $Z_0, \ldots, Z_{t-1}$ have been defined. A *t-bite* is a path P such that:

- P is a λ-geodesic of length at least two;
- the ends u_1, u_2 of P belong to Z_{t-1};
- P has length at most f_t;
- if $t = 1$ then no internal vertex of P is in Z; and
- if $t \geq 2$ then $\operatorname{dist}_{Z_{t-1}}(u_1, u_2) > g_t$.

Let Z_t be the union of Z_{t-1} and the vertex sets of all t-bites. This completes the inductive definition of $Z_0, \ldots, Z_\delta$. We observe:

(1) *For $1 \leq t \leq \delta$, if $u, v \in Z_{t-1}$, and $\operatorname{dist}_G(u, v) \leq f_t$, and either $t = 1$ or $\operatorname{dist}_{Z_{t-1}}(u, v) > g_t$, then $\operatorname{dist}_{Z_t}(u, v) = \operatorname{dist}_G(u, v) \leq f_t$.*

To see this, we may assume that u, v are distinct and nonadjacent. Let P be the λ-geodesic between u, v. If $t = 1$, then every vertex of P not in Z belongs to a 1-bite and hence to Z_1, and so $V(P) \subseteq Z_t$. If $t \geq 2$, then P is a t-bite, so again $V(P) \subseteq Z_t$. Hence, in either case,

$$\operatorname{dist}_{Z_t}(u, v) = |E(P)| = \operatorname{dist}_G(u, v) \leq f_t.$$

This proves (1).

For $1 \leq t \leq \delta$ and for each $v \in Z_t \setminus Z_{t-1}$, v belongs to the interior of a t-bite (perhaps of several t-bites). Choose one such t-bite, arbitrarily, and denote it by S_v. Thus, S_v is defined for each $v \in Z_\delta \setminus Z$.

For each $v \in Z$, let L_v be the one-vertex subgraph with vertex v. Inductively, for $1 \leq t \leq \delta$, if $v \in Z_t \setminus Z_{t-1}$, we define $L_v = S_v \cup L_{u_1} \cup L_{u_2}$, where u_1, u_2 are the ends of S_v. (These graphs L_v will turn out to be Z-leaved subdivisions of the binary tree H_t; so that will show that t cannot be large.)

(2) *For* $1 \le t \le \delta$*, if* $v \in Z_t$*, then every vertex* w *of* L_v *is joined to* v *by a path of* L_v *with length at most* $f_t + f_{t-1} + \cdots + f_1 = 2f_1 - f_t$*.*

We proceed by induction on t. Let $w \in V(L_v)$. If $w \in V(S_v)$, the statement is clear, since S_v has length at most f_t, so we may assume that $w \in V(L_u)$ for some end u of S_v. Since $u \in Z_{t-1}$, it follows inductively that w is joined to u by a path of L_u of length at most $f_{t-1} + f_{t-2} + \cdots + f_1$. Since L_u is a subgraph of L_v, the claim follows by adjoining to this path the subpath of S_v between v, u. This proves (2).

(3) *For* $1 \le t \le \delta$ *and every* t*-bite* P *with ends* u_1, u_2*,* L_{u_1} *is vertex-disjoint from* L_{u_2}*.*

We may assume that $t \ge 2$. Suppose that there exists $w \in V(L_{u_1} \cap L_{u_2})$. By (2),

$$\mathrm{dist}_{Z_{t-1}}(w, u_i) \le \mathrm{dist}_{L_{u_i}}(w, u_i) \le 2f_1$$

for $i = 1, 2$, and so $\mathrm{dist}_{Z_{t-1}}(u_1, u_2) \le 4f_1$. Since $\mathrm{dist}_{Z_{t-1}}(u_1, u_2) > g_t \ge 4f_1$ (because $t \ge 2$) there is no such vertex w. This proves (3).

(4) *For* $2 \le t \le \delta$*, if* P *is a* t*-bite with ends* u_1, u_2*, then* $u_1, u_2 \in Z_{t-1} \setminus Z_{t-2}$*.*

Suppose first that $u_1, u_2 \in Z_{t-2}$. Then

$$\mathrm{dist}_G(u_1, u_2) = |E(P)| \le f_t \le f_{t-1}, \text{ and}$$
$$\mathrm{dist}_{Z_{t-2}}(u_1, u_2) \ge \mathrm{dist}_{Z_{t-1}}(u_1, u_2) > g_t \ge g_{t-1},$$

so $\mathrm{dist}_{Z_{t-1}}(u_1, u_2) \le f_{t-1} \le g_t$ by (1), a contradiction. Thus, at least one of u_1, u_2, say u_1, belongs to $Z_{t-1} \setminus Z_{t-2}$. Suppose that $u_2 \in Z_{t-2}$. Since S_{u_1} has length at most f_{t-1}, it includes a subpath of length at most $f_{t-1}/2$ between u_1 and one of the ends w_1 of S_{u_1}. It follows that $\mathrm{dist}_G(w_1, u_2) \le f_t + f_{t-1}/2 = f_{t-1}$. Moreover,

$$\mathrm{dist}_{Z_{t-1}}(u_1, u_2) \le \mathrm{dist}_{Z_{t-1}}(w_1, u_1) + \mathrm{dist}_{Z_{t-1}}(w_1, u_2),$$

and so

$$\mathrm{dist}_{Z_{t-2}}(w_1, u_2) \ge \mathrm{dist}_{Z_{t-1}}(w_1, u_2) \ge \mathrm{dist}_{Z_{t-1}}(u_1, u_2) - \mathrm{dist}_{Z_{t-1}}(w_1, u_1) > g_t - f_{t-1}/2 \ge g_{t-1}.$$

By (1), $\mathrm{dist}_{Z_{t-1}}(w_1, u_2) \le f_{t-1}$. It follows that $\mathrm{dist}_{Z_{t-1}}(u_1, u_2) \le f_{t-1} + f_{t-1}/2 \le g_t$, contradicting that $\mathrm{dist}_{Z_{t-1}}(u_1, u_2) > g_t$. Thus $u_2 \notin Z_{t-2}$, and so $u_2 \in Z_{t-1} \setminus Z_{t-2}$. This proves (4).

(5) *For* $2 \le t \le \delta$*, if* P *is a* t*-bite with ends* u_1, u_2*, then for* $i = 1, 2$*, every vertex of* $P \cap L_{u_i}$ *belongs to* $Z_{t-1} \setminus Z_{t-2}$ *and hence belongs to the interior of* S_{u_i}*.*

Suppose that $w \in V(P \cap L_{u_2})$, and $w \in Z_{t-2}$. By (2),

$$\mathrm{dist}_{Z_{t-1}}(u_2, w) \le 2f_1 - f_{t-1},$$

and so

$$\mathrm{dist}_{Z_{t-1}}(u_1, w) > g_t - (2f_1 - f_{t-1})\,.$$

As in the proof of (4), there is a subpath P_1 of S_{u_1}, between u_1 and some $w_1 \in Z_{t-2}$, of length at most $f_{t-1}/2$. Hence

$$\begin{aligned}\mathrm{dist}_{Z_{t-2}}(w_1, w) \geq \mathrm{dist}_{Z_{t-1}}(w_1, w) > g_t - (2f_1 - f_{t-1}) - f_{t-1}/2 \geq g_{t-1}, \text{ and}\\ \mathrm{dist}_G(w_1, w) \leq |E(P)| + |E(P_1)| \leq f_t + f_{t-1}/2 = f_{t-1}.\end{aligned}$$

By (1), $\mathrm{dist}_{Z_{t-1}}(w_1, w) \leq f_{t-1}$, and so

$$\begin{aligned}\mathrm{dist}_{Z_{t-1}}(u_1, u_2) &\leq \mathrm{dist}_{Z_{t-1}}(u_1, w_1) + \mathrm{dist}_{Z_{t-1}}(w_1, w) + \mathrm{dist}_{Z_{t-1}}(w, u_2)\\ &\leq f_{t-1}/2 + f_{t-1} + (2f_1 - f_{t-1}) \leq g_t,\end{aligned}$$

a contradiction. This proves (5).

(6) *For $1 \leq t \leq \delta$, if $v \in Z_t \setminus Z_{t-1}$, then L_v (rooted at v) is a ($< f_1$)-subdivision of the binary tree H_t, and all its leaves are in Z.*

We proceed by induction on t. If $t = 1$ the statement is clear, so we assume that $t \geq 2$. Let S_v have ends $u_1, u_2 \in Z_{t-1}$. By (4), $u_1, u_2 \in Z_{t-1} \setminus Z_{t-2}$. From the inductive hypothesis, for $i = 1, 2$, L_{u_i} (rooted at u_i) is a ($< f_1$)-subdivision of the binary tree H_{t-1}, and all its leaves are in Z. By (3), L_{u_1}, L_{u_2} are vertex-disjoint. By (5), for $i = 1, 2$, every vertex of S_v in L_{u_i} belongs to $Z_{t-1} \setminus Z_{t-2}$ and hence belongs to the interior of S_{u_i}. But since S_{u_i} and S_v are both λ-geodesics, their intersection is also a λ-geodesic A_i say. Thus, for $i = 1, 2$, A_i is a subpath of S_v containing the end u_i of S_v, possibly of length zero; and A_1 is vertex-disjoint from A_2 since $v \in V(S_v)$ and $v \notin V(S_{u_i})$ for $i = 1, 2$. Let A_i have ends u_i, a_i for $i = 1, 2$, and let A_0 be the subpath of S_v between a_1, a_2. Thus, S_v is the concatenation of the paths A_1, A_0, A_2. Moreover, for $i = 1, 2$, every vertex of A_i belongs to $Z_{t-1} \setminus Z_{t-2}$ by (5); so A_i contains neither end of S_{u_i}. The interior of the path A_0 is disjoint from $L_{u_1} \cup L_{u_2}$ and contains v, and so this proves (6).

We want to show that each L_v is Z-leaved, and to do so it only remains to prove that L_v has no vertices in Z except its leaves. For that we use the following.

(7) *For $1 \leq t \leq \delta$, if $v \in Z_t \setminus Z$ and $v' \in V(S_v)$, then* $\mathrm{dist}_{L_v}(v', Z) \leq (f_t + f_{t-1} + \cdots + f_1)/2 = f_1 - f_t/2$.

We proceed by induction on t. Since S_v has length at most f_t, v' is joined to an end u of S_v by a subpath of S_v with length at most $f_t/2$. If $t = 1$ then $u \in Z$ and the claim holds, so we assume that $t \geq 2$. Inductively, $\mathrm{dist}_{L_u}(u, Z) \leq (f_{t-1} + \cdots + f_1)/2$, and the claim follows. This proves (7).

(8) *For $1 \leq t \leq \delta$, if $v \in Z_t \setminus Z_{t-1}$, then the tree L_v is Z-leaved.*

We proceed by induction on t. The result is true if $t = 1$ from the definition of a 1-bite, so we assume that $t \geq 2$. Let u_1, u_2 be the ends of S_v. Inductively, the trees L_{u_1}, L_{u_2} are Z-leaved, so we just need to show that no internal vertices of S_v belong to Z.

Suppose that $z \in Z$ is an internal vertex of S_v. As in (6), S_v is the concatenation of a subpath A_1 of S_{u_1} between u_1 and some a_1, a path A_0 say between a_1, a_2, and a subpath A_2 of S_{u_2} between a_2, u_2, where:

- a_i is an internal vertex of S_{u_i} for $i = 1, 2$; and
- no internal vertex of A_0 belongs to $V(L_{u_1}) \cup V(L_{u_2})$.

It follows that z, v are both internal vertices of A_0. Let B_i be the subpath of A_0 between z, a_i for $i = 1, 2$. By (7), for $i = 1, 2$ there exists $z_i \in V(L_{u_i}) \cap Z$ and a path Q_i of L_{u_i} between a_i, z_i of length at most $f_1 - f_{t-1}/2$. Hence $\text{dist}_G(z, z_i) \leq (f_1 - f_{t-1}/2) + f_t = f_1$ since B_i has length at most f_t, and consequently the λ-geodesic between z and z_1 has length at most f_1. Each vertex of this λ-geodesic either belongs to Z or to the interior of a 1-bite, and therefore belongs to Z_1; and so $\text{dist}_{Z_1}(z_i, z) \leq f_1$ for $i = 1, 2$. Consequently,

$$\text{dist}_{Z_{t-1}}(z_1, z_2) \leq \text{dist}_{Z_1}(z_1, z_2) \leq 2f_1.$$

Now, for $i = 1, 2$,

$$\text{dist}_{Z_{t-1}}(z_i, u_i) \leq |E(Q_i)| + |E(A_i)| \leq (f_1 - f_{t-1}/2) + f_t = f_1,$$

since A_i is a subpath of S_v and so has length at most f_t. Hence $\text{dist}_{Z_{t-1}}(u_1, u_2) \leq 4f_1 \leq g_t$ (since $t \geq 2$), a contradiction. This proves (8).

Since, by hypothesis, there is no Z-leaved $(< f_1)$-subdivision of H_δ in G, it follows from (8) that if $Z_t \setminus Z_{t-1} \neq \emptyset$ then $t \leq \delta - 1$. In particular, $Z_\delta = Z_{\delta-1}$, and it follows that every δ-bite is contained in $G[Z_{\delta-1}]$. We deduce that if $u, v \in Z_{\delta-1}$ are distinct, and $\text{dist}_G(u, v) \leq f_\delta$, then $\text{dist}_{Z_{\delta-1}}(u, v) \leq g_\delta$. Moreover, by (7), every vertex $v \in Z_{\delta-1}$ has distance at most f_1 from Z, since L_v has a vertex in Z by (6). Thus, setting $Y = Z_{\delta-1}$ satisfies the theorem. This proves 4.1. ▮

In 4.1 we assumed that no subdivision of H_δ has all its leaves in Z and no other vertices in Z, but now we need to refine that.

4.2 *Let $d \geq 1$ be an integer, let G be a graph, let Γ be a subgraph of G, such that no subgraph is a $V(\Gamma)$-leaved subdivision of H_d whose leaves all belong to different components of Γ. Suppose that $M_1, M_2, \ldots, M_t$ are paths of G, each of length at most ℓ. Let $\Gamma_i = \Gamma \cup (M_1 \cup \cdots \cup M_i)$ for $0 \leq i \leq t$; and suppose in addition that for each $i \geq 1$, the ends of M_i lie in different components of Γ_{i-1}, and none of its internal vertices lie in $V(\Gamma_{i-1})$. Then for each $v \in V(\Gamma_t) \setminus V(\Gamma)$, either v lies in the interior of some M_i with both ends in $V(\Gamma)$, or there are at least three components C of Γ such that v is joined to C by a path in Γ_t of length at most $(d+1)(\ell-1)$.*

Proof. For $h \geq 1$, we say that a $V(\Gamma)$-leaved subdivision of H_h in G is *Γ-pruned* if its leaves are all in distinct components of Γ. We say the *height* of each vertex in $V(\Gamma)$ is zero; and inductively, for $1 \leq i \leq t$, let us say that for each vertex in the interior of M_i, its *height* is one more than the minimum of the heights of u_1, u_2, where u_1, u_2 are the ends of M_i. Then:

(1) *For each $i \geq 0$ and each $v \in V(\Gamma_i)$ with height at least $h \geq 1$, there is a subgraph of Γ_i that is a*

Γ-pruned subdivision of H_h rooted at v.

We use induction on h. The statement is clear if $h = 1$, so we assume $h \geq 2$. We may assume that i is minimum such that $v \in V(\Gamma_i)$, and consequently v belongs to the interior of M_i. Let u_1, u_2 be the ends of M_i, joining components C_1, C_2 of Γ_{i-1}. Thus, u_1, u_2 have height at least $h-1 \geq 1$. From the inductive hypothesis there is a subgraph L_j of C_j rooted at u_j that is a Γ-pruned subdivision of H_{h-1}. But L_1, L_2 are disjoint, and every component of Γ containing a leaf of L_1 is different from every component of Γ containing a leaf of L_2, since they belong to different components of Γ_{i-1}. Moreover, L_1, L_2 are both disjoint from the interior of M_i, since the latter is disjoint from $V(\Gamma_{i-1})$. Consequently, $L_1 \cup L_2 \cup M_i$ (rooted at v) is the desired subdivision of H_h. This proves (1).

From (1) and the hypothesis, it follows that every vertex has height at most $d-1$.

(2) *For each $i \geq 0$ and each $v \in V(\Gamma_i)$ with height $h \geq 0$, v is joined to $V(\Gamma)$ by a path in Γ_i of length at most $h(\ell - 1)$.*

We prove this by induction on $h \geq 0$. If $h = 0$, the statement is clear, so we assume that $h \geq 1$. Choose i minimum with $v \in V(\Gamma_i)$. Then v is joined to a vertex u of height $h-1$ by a path of Γ_i of length at most $\ell - 1$ (a subpath of M_i); and from the inductive hypothesis, u is joined to $V(\Gamma)$ by a path in Γ_{i-1} (and hence of Γ_i) of length at most $(h-1)(\ell-1)$. Consequently v is joined to $V(\Gamma)$ by a path in Γ_i of length at most $h(\ell-1)$. This proves (2).

(3) *For each $i \geq 1$ and each $v \in V(\Gamma_i) \setminus V(\Gamma)$, there are at least two components C of Γ such that v is joined to C by a path in Γ_i of length at most $d(\ell-1)$.*

Choose i minimum with $v \in V(\Gamma_i)$. Thus, v belongs to the interior of M_i; let M_i have ends u_1, u_2. Each of u_1, u_2 belongs to $V(\Gamma_{i-1})$, and since u_1, u_2 have height at most $d-1$ by (1), it follows from (2) that each of u_1, u_2 is joined to a component of Γ by a path in Γ_{i-1} of length at most $(d-1)(\ell-1)$. These components of Γ are different since u_1, u_2 are in different components of Γ_{i-1}, and v is joined to each of them by a path in Γ_i of length at most $d(\ell-1)$. This proves (3).

In particular, for each $v \in V(M_1 \cup \cdots \cup M_t) \setminus V(\Gamma)$, v has height at least one; choose i minimum with $v \in V(\Gamma_i)$. Thus, v belongs to the interior of M_i; let M_i have ends u_1, u_2. If u_1, u_2 both have height zero then M_i has both ends in $V(\Gamma)$ and the theorem holds; so we assume that $u_1 \notin V(\Gamma)$. By (3), there are at least two components C_1, C_2 of Γ that are joined to u_1 by a path in Γ_{i-1} of length at most $d(\ell-1)$; and by (2), there is a component C_3 of Γ such that u_2 is joined to C_3 by a path in Γ_{i-1} of length at most $(d-1)(\ell-1)$. Moreover, $C_3 \neq C_1, C_2$ since u_1, u_2 belong to different components of Γ_{i-1}. Consequently there are at least three components C of Γ such that v is joined to C by a path in Γ_t of length at most $(d+1)(\ell-1)$. This proves 4.2. ∎

5 Augmenting paths

Let us extend the definition of $\operatorname{dist}_G(u,v)$ a little, to accommodate vertices $u, v \notin V(G)$: if one of $u, v \notin V(G)$ then $\operatorname{dist}_G(u,v) = \infty$.

Some more notation: if P is a path and $u, v \in V(P)$, $P[u, v]$ denotes the subpath between u, v. If $\mathcal{P}$ is a set of vertex-disjoint paths of a graph G, we denote $P_1 \cup \cdots \cup P_k$ by $U\mathcal{P}$, and its vertex set by $V\mathcal{P}$. Let G be a graph, let $S, T \subseteq V(G)$ be disjoint, and let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a set of k vertex-disjoint S-T paths, with

$$V\mathcal{P} \cup S \cup T = V(G),$$

such that for $1 \leq h \leq k$, no proper subpath of P_h is an S-T path. Let P_h have ends $s_h \in S$ and $t_h \in T$. If $u, v \in V(P_h)$ are distinct, and v belongs to $P_h[u, t_h]$, we say that v is *later than* u *in* P_h, and u *is earlier than* v *in* P_h.

It is an elementary theorem (a special case of the theory of augmenting paths) that:

5.1 *Given G, S, T and $\mathcal{P} = \{P_1, \ldots, P_k\}$ as above, the following are equivalent:*

1. *For every choice of $v_i \in V(P_i)$ for $1 \leq i \leq k$, there is an edge ab of G with $a, b \notin \{v_1, \ldots, v_k\}$, such that*
   - *either $a \in S \setminus V\mathcal{P}$ or for some $h \in \{1, \ldots, k\}$, $a \in V(P_h)$, and a is earlier than v_h in P_h, and*
   - *either $b \in T \setminus V\mathcal{P}$ or for some $h \in \{1, \ldots, k\}$, $b \in V(P_h)$ and b is later than v_h in P_h.*
2. *There is a sequence $a_1b_1, a_2b_2, \ldots, a_nb_n$ of oriented edges of G, not in $E(P_1 \cup \cdots \cup P_k)$, such that*
   - *$a_1 \in S \setminus V\mathcal{P}$, and $b_n \in T \setminus V\mathcal{P}$;*
   - *for $1 \leq i < n$, b_i, a_{i+1} belong to the same path P_h say (where $1 \leq h \leq k$), and a_{i+1} is earlier than b_i in P_h.*
3. *There is a sequence $a_1b_1, a_2b_2, \ldots, a_nb_n$ as above, satisfying in addition that for $1 \leq h \leq k$, and $1 \leq i < j \leq n$, if $u \in \{a_i, b_i\} \cap V(P_h)$ and $v \in \{a_j, b_j\} \cap V(P_h)$, then either*
   - *u is earlier than v in P_h, or*
   - *$b_i = u = v = a_j$; or*
   - *$b_i = u$ and $a_j = v$ and $j = i + 1$.*
4. *There are $k + 1$ vertex-disjoint S-T paths in G.*

We do not actually need this theorem, and we mention it just for comparison with the more complicated results that we will need.

Let S, T be sets, and let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a set of k vertex-disjoint S-T paths, each a minimal S-T path. (Possibly some P_i has length zero, and then its vertex is in $S \cap T$.) We call $(S, T, \mathcal{P})$ a *setting*. Let F_0 be the set of all ordered pairs of vertices ab with $a, b \in V\mathcal{P} \cup S \cup T$.

Let us fix some setting $(S, T, \mathcal{P})$ where $\mathcal{P} = \{P_1, \ldots, P_k\}$. Let $c \geq 0$ be an integer. A *c-barrier* (in the setting) is a k-tuple $Q_1, \ldots, Q_k$, where Q_h is a subpath of P_h of length at most c. We say $ab \in F_0$ *jumps* a c-barrier $Q_1, \ldots, Q_k$ (in the setting) if $a, b \notin V(Q_1 \cup \cdots \cup Q_k)$, and

- either $a \in S \setminus V\mathcal{P}$ or for some $h \in \{1, \ldots, k\}$, $a \in V(P_h)$, and a is earlier than each vertex of Q_h in P_h; and

- either $b \in T \setminus V\mathcal{P}$ or for some $h \in \{1, \ldots, k\}$, $b \in V(P_h)$ and b is later than each vertex of Q_h in P_h.

Let us say a set $F \subseteq F_0$ is *c-jumping* (in the setting $(S, T, \mathcal{P})$) if for every c-barrier, some member of F jumps the c-barrier.

A *partial c-augmenting sequence to b_n* is a sequence $a_1b_1, a_2b_2, \ldots, a_nb_n$ of elements of F_0, such that

- $a_1 \in S \setminus V\mathcal{P}$;
- for $1 \leq i < n$, there exists $h \in \{1, \ldots, k\}$ with $b_i, a_{i+1} \in V(P_h)$, and a_{i+1} is earlier than b_i in P_h, and $P_h[a_{i+1}, b_i]$ has length at least $c + 1$.

If in addition $b_n \in T \setminus V\mathcal{P}$, we call such a sequence a *c-augmenting sequence.* Thus, if $a_i \in S \setminus V\mathcal{P}$ and $b_i \in T \setminus V\mathcal{P}$ then $i = 1 = n$. For $F \subseteq F_0$, the sequence is *in F* if $a_ib_i \in F$ for $1 \leq i \leq n$.

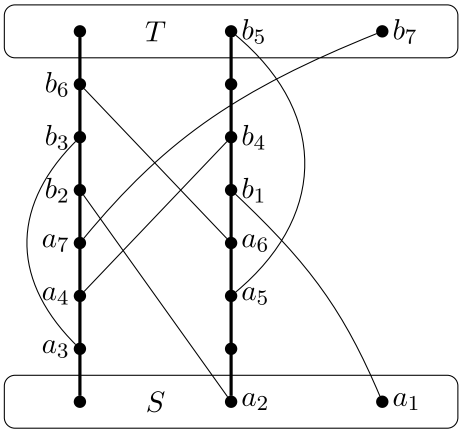

Figure 3: P_1, P_2 are the two paths of thick edges. With $k = 2$, the sequence $a_1b_1, \ldots, a_7b_7$ is minimal 2-augmenting, but it is not 1-separated (defined later). For every choice of three vertex-disjoint S-T paths, some edge of $P_1 \cup P_2$ joins two of them.

We begin with:

5.2 *Let $(S, T, \mathcal{P})$ be a setting, with $\mathcal{P} = \{P_1, \ldots, P_k\}$, and let $c \geq 0$ be an integer. With F_0 as before, let $F \subseteq F_0$. Then the following are equivalent:*

- *F is c-jumping;*
- *there is a c-augmenting sequence of elements of F.*

Proof. We show first that the second statement implies the first. To see this, assume that the sequence $a_1b_1, a_2b_2, \ldots, a_nb_n$ of pairs in F is c-augmenting, and let $Q_1, \ldots, Q_k$ be a c-barrier. Choose i maximum such that either $a_i \in S \setminus V\mathcal{P}$, or for some $h \in \{1, \ldots, k\}$, $a_i \in V(P_h) \setminus V(Q_h)$ and a_i is earlier in P_h than each vertex of Q_h. If $b_i \in T \setminus V\mathcal{P}$ then a_ib_i jumps the c-barrier, so we assume that $b_i \in V(P_j)$ for some $j \in \{1, \ldots, k\}$. Consequently $i < n$, and $a_{i+1} \in V(P_j)$, earlier than b_i in P_j. From the maximality of i, there exists $q \in V(Q_j)$ such that a_{i+1} is not earlier than q in P_j. Since

$P_j[a_{i+1}, b_i]$ has length at least $c+1$, it follows that b_i is later than q in P_j, and $P_j[q, b_i]$ has length at least $c+1$. Since Q_j has length at most c, it follows that b_i is later in P_j than every vertex of Q_j; and so a_ib_i jumps the c-barrier. This proves the first statement.

To show the converse, suppose that F is c-jumping, and for $1 \le h \le k$, choose $v_h \in V(P_h)$ with $P_h[s_h, v_h]$ maximal such that either $v_h = s_h$ or there is a partial c-augmenting sequence to v_h in F. For $1 \le h \le k$, let Q_h be the maximal subpath of $P_h[s_h, v_h]$ with length at most c, such that one of its ends is v_h. Thus, $Q_1, \dots, Q_k$ is a c-barrier, and so, since F is c-jumping, some $ab \in F$ jumps this c-barrier. Suppose first that $a \in S \setminus V\mathcal{P}$. If $b \in T \setminus V\mathcal{P}$, then ab is a c-augmenting sequence, so we assume that $b \in V(P_h)$ for some $h \in \{1, \dots, k\}$. Since ab jumps the c-barrier, it follows that b is later than v_h in P_h, contradicting the choice of v_h, since ab is a partial c-augmenting sequence to b. Thus, we may assume that for some $h \in \{1, \dots, k\}$, $a \in V(P_h)$, and a is earlier than each vertex of Q_h in P_h. Since $a \notin V(Q_h)$, it follows from the maximality of Q_h that Q_h has length exactly c, and therefore $P_h[a, v_h]$ has length at least $c+1$. Let $a_1b_1, \dots, a_sb_s$ be a partial c-augmenting sequence to v_h in F. Consequently $a_1b_1, \dots, a_sb_s, ab$ is a partial c-augmenting sequence to b in F. If $b \notin T \setminus V\mathcal{P}$, then, since ab jumps the c-barrier, there exists $h' \in \{1, \dots, k\}$ such that $b \in P_{h'}[v_{h'}, t_{h'}]$ and $b \neq v_{h'}$; but this contradicts the definition of $v_{h'}$. Thus, $b \in T \setminus V\mathcal{P}$, and so $a_1b_1, \dots, a_sb_s, ab$ is a c-augmenting sequence in F. This proves 5.2. ▌

This provides an analogue of the first two bullets of 5.1, and the next result gives an analogue of the third bullet.

5.3 *Let $(S, T, \mathcal{P})$ be a setting, with $\mathcal{P} = \{P_1, \dots, P_k\}$, and let $c \ge 0$ be an integer. Let $F \subseteq F_0$ be c-jumping, and choose a c-augmenting sequence $a_1b_1, \dots, a_nb_n$ of elements of F, with n minimum. For $1 \le h \le k$, and $1 \le i < j \le n$, if $u \in \{a_i, b_i\} \cap V(P_h)$ and $v \in \{a_j, b_j\} \cap V(P_h)$, then either*

- *u is earlier than v in P_h; or*
- *$b_i = u$ and $v = a_j$ and $P_h[u, v]$ has length at most c; or*
- *$b_i = u$ and $v = a_j$ and $j = i+1$.*

Proof. Suppose that $1 \le h \le k$, and $1 \le i < j \le n$, and $u \in \{a_i, b_i\} \cap V(P_h)$ and $v \in \{a_j, b_j\} \cap V(P_h)$, and u is not earlier than v in P_h. If $u = a_i$ and $v = a_j$, then $i \ge 2$ and

$$a_1b_1, \dots, a_{i-1}b_{i-1}, a_jb_j, \dots, a_nb_n$$

is a c-augmenting sequence in F, contrary to the minimality of n. Similarly, if $u = b_i$ and $v = b_j$, then

$$a_1b_1, \dots, a_ib_i, a_{j+1}b_{j+1}, \dots, a_nb_n$$

is a c-augmenting sequence, a contradiction; and if $u = a_i$ and $v = b_j$, then $i \ge 2$ and $j \le n-1$ and

$$a_1b_1, \dots, a_{i-1}b_{i-1}, a_{j+1}b_{j+1}, \dots, a_nb_n$$

is a c-augmenting sequence, a contradiction. Thus, we assume that $u = b_i$ and $v = a_j$. If $P_h[u, v]$ has length at least $c+1$, then

$$a_1b_1, \dots, a_ib_i, a_jb_j, \dots, a_nb_n$$

is a c-augmenting sequence, and so $j = i+1$; and otherwise $P_h[u, v]$ has length at most c. In either case the result holds. This proves 5.3. ▌

The results 5.2 and 5.3 do not quite provide an analogue of 5.1, because we have no counterpart to the fourth statement of 5.1, the existence of $k+1$ vertex-disjoint S-T paths. One might hope that

- *In the graph obtained from $U\mathcal{P}$ by adding the remainder of $S \cup T$ as extra vertices and the pairs in F as edges, there exist $k+1$ S-T paths, such that no two of them are joined by a path of $U\mathcal{P}$ of length at most c.*

could be added to the list of equivalent statements given by 5.2 and 5.3 to give an analogue of the fourth statement of 5.1, but that is wrong. This statement does imply the statements of 5.2, but the reverse implication does not hold. For instance, the graph of Figure 3 with $k = 2$ gives a 2-augmenting sequence, and yet for every three vertex-disjoint S-T paths, some two of them are joined by one of the edges of $P_1 \cup P_2$, which is more than we needed for a counterexample.

The property given by 5.3 implies that for each $h \in \{1, \dots, k\}$, the vertices a_i that lie in P_h are all distinct and in order in P_h, but it does not imply that they are far apart in P_h. For instance, if $k = 1$ and P_1 has vertices $s_1 = v_1\text{-}\cdots\text{-}v_n = t_1$, and $S = \{s_1, s_2\}$ and $T = \{t_1, t_2\}$, and F is the union of $\{s_2v_{c+2}, v_{n-c-1}t_2\}$ and the pairs v_iv_{i+c+2} for $1 \leq i \leq n-c-2$, then the only c-augmenting sequence in F uses all of F. Nevertheless, we can arrange that the a_i's are far apart, and the b_j's are far apart, by sacrificing some of the jumping power. We show this in two steps: first we arrange that the b_j's are far apart, in the following.

We recall that $\text{dist}_{U\mathcal{P}}(b, b') = \infty$ unless $b, b' \in V\mathcal{P}$ and b, b' belong to the same component of $U\mathcal{P}$.

5.4 *Let $p, q \geq 0$ be integers, and let $F \subseteq F_0$ be $(p+q)$-jumping. Then there exists $D \subseteq F$ that is p-jumping, such that if $ab, a'b' \in D$ are distinct then* $\text{dist}_{U\mathcal{P}}(b, b') > q$.

Proof. We will use a modified version of the second half of the proof of 5.2. We say a partial p-augmenting sequence $a_1b_1, \dots, a_sb_s$ is *end-separated* if $\text{dist}_{U\mathcal{P}}(b_i, b_j) > q$ for all distinct $i, j \in \{1, \dots, s\}$. By 5.2 it suffices to show that there is an end-separated p-augmenting sequence in F.

For $1 \leq h \leq k$, choose $v_h \in V(P_h)$ with $P_h[s_h, v_h]$ maximal such that either $v_h = s_h$ or there is an end-separated partial p-augmenting sequence to v_h in F. For $1 \leq h \leq k$, let Q_h be the maximal subpath of P_h containing v_h, such that $Q_h \cap P_h[s_h, v_h]$ has length at most p, and $Q_h \cap P_h[v_h, t_h]$ has length at most q. Thus, $Q_1, \dots, Q_k$ is a $(p+q)$-barrier, and so, since F is $(p+q)$-jumping, some $ab \in F$ jumps this $(p+q)$-barrier. Suppose first that $a \in S \setminus V\mathcal{P}$. If $b \in T \setminus V\mathcal{P}$, then ab is an end-separated p-augmenting sequence, so we assume that $b \in V(P_h)$ for some $h \in \{1, \dots, k\}$. Since ab jumps the $(p+q)$-barrier, it follows that b is later than v_h in P_h, contradicting the choice of v_h, since ab is an end-separated partial p-augmenting sequence to b in F.

Thus, we may assume that for some $h \in \{1, \dots, k\}$, $a \in V(P_h)$, and a is earlier than each vertex of Q_h in P_h. Let $a_1b_1, \dots, a_sb_s$ be an end-separated partial p-augmenting sequence to v_h in F. Since $a \notin V(Q_h)$, it follows that $Q_h \cap P_h[s_h, v_h]$ has length exactly p, and $P_h[a, v_h]$ has length at least $p+1$. Consequently $a_1b_1, \dots, a_sb_s, ab$ is a partial p-augmenting sequence to b in F. If $b \notin T \setminus V\mathcal{P}$, then, since ab jumps the $(p+q)$-barrier, there exists $h' \in \{1, \dots, k\}$ such that $b \in P_{h'}[v_{h'}, t_{h'}]$ and $b \notin V(Q_{h'})$; but then $Q_{h'} \cap P_{h'}[v_{h'}, t_{h'}]$ has length exactly q, and so $P_{h'}[v_{h'}, b]$ has length $> q$. Since each b_i in $V(P_{h'})$ belongs to $P_{h'}[s_{h'}, v_{h'}]$ from the definition of $v_{h'}$, it follows that $a_1b_1, \dots, a_sb_s, ab$ is an end-separated p-augmenting sequence to b in F, contrary to the definition of $v_{h'}$. Thus, $b \in T \setminus V\mathcal{P}$, and so $a_1b_1, \dots, a_sb_s, ab$ is an end-separated p-augmenting sequence in F. This proves 5.4. ∎

Let us say a subset $D \subseteq F_0$ is *ℓ-separated* if $\text{dist}_{U\mathcal{P}}(a, a') > \ell$ and $\text{dist}_{U\mathcal{P}}(b, b') > \ell$ for all distinct $ab, a'b' \in D$. We deduce:

5.5 *In the same notation, let $c \geq 0$ be an integer, and let $F \subseteq F_0$ be $5c$-jumping. Then there exists $D \subseteq F$ that is c-jumping and $2c$-separated.*

Proof. This follows from two applications of 5.4: first, to F with $(p,q) = (3c, 2c)$, giving some $3c$-jumping set F'; and then to F' with S,T exchanged and $(p,q) = (c, 2c)$. This proves 5.5. ▮

We remark that, even if F is c-jumping and $2c$-separated, and if $a_1b_1, \ldots, a_nb_n$ is a minimal c-augmenting sequence of members of F, it is possible that $a_i = b_j$ for some pairs i,j. Even so, now we can obtain something like an analogue of the fourth statement of 5.1:

5.6 *In the same notation, let $c \geq 0$ be an integer, and let $F \subseteq F_0$ be c-jumping and $2c$-separated. Let H be obtained from $U\mathcal{P}$ by adding the remainder of $S \cup T$ as vertices, and the pairs in F as edges. Then there exist $k+1$ vertex-disjoint S-T paths in H, such that no two of them are joined by a path of $U\mathcal{P}$ of length at most c.*

Proof. By 5.3, there is a c-augmenting sequence $a_1b_1, \ldots, a_nb_n$ in F such that:

(1) *For $1 \leq h \leq k$, and $1 \leq i < j \leq n$, if $u \in \{a_i, b_i\} \cap V(P_h)$ and $v \in \{a_j, b_j\} \cap V(P_h)$, then either*

- *u is earlier than v in P_h; or*
- *$b_i = u$ and $v = a_j$ and $P_h[a_j, b_i]$ has length at most c; or*
- *$b_i = u$ and $v = a_j$ and $j = i+1$.*

We deduce:

(2) *Let $1 \leq h \leq k$, and $1 \leq i \leq n$ with $b_i \in V(P_h)$ (and hence $a_{i+1} \in V(P_h)$); then for $1 \leq j \leq n$, if a_j belongs to $P_h[a_{i+1}, b_i]$ then either $j = i+1$, or $j > i+1$ and $P_h[a_j, b_i]$ has length at most c. Consequently there is at most one value of $j \neq i+1$ with $a_j \in V(P_h[a_{i+1}, b_i])$, and any such j satisfies $j \geq i+2$. Similarly there is at most one value of $j \neq i$ with $b_j \in V(P_h[a_{i+1}, b_i])$, and any such j satisfies $j \leq i-1$.*

By (1), $a_1, \ldots, a_n$ are all distinct, and $b_1, \ldots, b_n$ are all distinct. Suppose that a_j belongs to $P_h[a_{i+1}, b_i]$, and $j \neq i+1$. Thus, a_{i+1} is earlier than a_j in P_h. If $j \leq i$ then setting $u = a_j$ and $v = a_{i+1}$ in (1) yields a contradiction; so $i < j$, and hence $j \geq i+2$. By (1) with $u = b_i$ and $v = a_j$, it follows that $P_h[a_j, b_i]$ has length at most c. Consequently, if $j' \neq j$ also satisfies that $a_{j'}$ belongs to $P_h[a_{i+1}, b_i]$, and $j' \neq i+1$, then $P_h[a_j, a_{j'}]$ has length at most c, contradicting that F is $2c$-separated. This proves the first assertion of (2), and the second follows from the symmetry. This proves (2).

For $1 \leq i < n$, b_i and a_{i+1} both belong to the same member of $\mathcal{P}$, say P_h; let $R_i = P_h[a_{i+1}, b_i]$.

(3) *Every vertex in $V\mathcal{P}$ belongs to at most two of $R_1, \ldots, R_{n-1}$.*

Suppose that some vertex w of P_h belongs to $R_i, R_{i'}, R_{i''}$, where $i < i' < i''$. Thus,

$$a_{i+1}, a_{i'+1}, a_{i''+1}, b_i, b_{i'}, b_{i''}$$

are in order in P_h (and are all distinct except possibly $a_{i''+1} = b_i$), and $w \in P_h[a_{i''+1}, b_i]$. By (1) with $u = b_i, v = a_{i'+1}$, $P_h[a_{i'+1}, b_i]$ has length at most c. But it includes $P_h[a_{i'+1}, a_{i''+1}]$ as a subpath, and this has length at least $2c + 1$ since F is $2c$-separated, a contradiction. This proves (3).

For $i = 0, 1, 2$, let X_i be the set of edges of $U\mathcal{P}$ that belong to exactly i of $R_1, \ldots, R_{n-1}$. Let H' be the digraph obtained as follows:

- Direct the edges of P_h from s_h to t_h for $1 \leq h \leq k$, and let H'' be obtained from their union by adding the (directed) edges $a_i b_i$ for $1 \leq i \leq n$. (Thus the only vertices of S in $V(H'')$ are $s_1, \ldots, s_k$ and a_1.)
- Reverse the direction of all edges in X_2.
- Delete all edges in X_1.

We claim:

(4) *Every vertex of H' either has outdegree one and indegree one, or has outdegree zero and indegree zero, except for $a_1, s_1, \ldots, s_k$, which have outdegree one and indegree zero, and $t_1, \ldots, t_k, b_n$, which have indegree one and outdegree zero.*

Let $v \in V(H')$. The claim is true for v if $v \in \{a_1, b_n\}$, so we may assume that $v \in V\mathcal{P}$. Let $v \in V(P_h)$ where $1 \leq h \leq k$. Suppose that $v = s_h$, and let e be the edge of P_h incident with v. It follows that $v \neq b_1, \ldots, b_n$, and either $v \neq a_1, \ldots, a_n$ (and then $e \in X_0 \subseteq E(H')$) or $v = a_i$ for some $i < n$ (and then $e \in E(R_{i-1})$, and so $e \in X_1$ and $e \notin E(H')$). In either case v has outdegree one and indegree zero in H'. Thus we may assume that $v \neq s_h$ and similarly $v \neq t_h$, and so v is an internal vertex of P_h. Let e_1, e_2 be the two edges of P_h incident with v, where e_1 is in the subpath between s_h and v.

If $v \notin \{a_2, \ldots, a_n, b_1, \ldots, b_{n-1}\}$ then v is an internal vertex of the (at most two) paths R_i that contain v, and so satisfies the claim. Thus from the symmetry, we may assume that $v \in \{a_2, \ldots, a_n\}$; let $v = a_i$ say. Thus $i \geq 2$ and v is an end of R_{i-1}, and $e_2 \in E(R_{i-1})$.

Suppose next that $v \neq b_1, \ldots, b_{n-1}$. If $e_2 \in X_1$ then $e_2 \notin E(H')$, and $e_1 \in X_0 \subseteq E(H')$, and v therefore has indegree and outdegree one in H'; so we may assume that $e_2 \in X_2$. Let $e_2 \in E(R_j)$ say, where $1 \leq j \leq n-1$ and $j \neq i$. Since $a_j, b_j \neq v$, it follows that $e_1 \in E(R_j)$, and so $e_1 \in X_1$ by (3) applied to v; but then $e_2 \in E(H'), e_1 \notin E(H')$, and the claim is true for v (because v is the head of the directed edge e_2, since $e_2 \in X_2$).

So we may assume that $v \in \{b_1, \ldots, b_{n-1}\}$; let $v = b_j$. Hence $i \neq j$, and v is an end of both R_{i-1}, R_j, and both $e_1, e_2 \in X_1$ by (3). Hence $e_1, e_2 \notin E(H')$, and again the claim holds for v. This proves (4).

Let J be the undirected graph underlying H'. From (4), each component of J is either an S-T path or a cycle or a vertex of degree zero; and $a_1, s_1, \ldots, s_k$ all belong to different components. Since $a_1, s_1, \ldots, s_k$ all have outdegree one and indegree zero in H', and vice versa for $b_n, t_1, \ldots, t_k$, it follows that there are $k+1$ vertex-disjoint S-T paths $P'_1, \ldots, P'_{k+1}$ in J, each a component of J. It remains to show that no two of these paths are joined by a path of $U\mathcal{P}$ with length at most c. Suppose that Q is such a path; and we can assume that no internal vertex of Q belongs to any of

$P'_1, \dots, P'_{k+1}$. Consequently the first and last edges of Q are not edges of $H \setminus X_1$, and so they belong to X_1. Choose $h \in \{1, \dots, k\}$ such that Q is a subpath of P_h, with ends u, v say, where u is earlier than v in P_h. Consequently $u, v \in \{a_1, \dots, a_n, b_1, \dots, b_n\}$. Let $u \in \{a_i, b_i\}$ and $v \in \{a_j, b_j\}$. Since F is $2c$-separated, not both $u = a_i$ and $v = a_j$, and similarly not both $u = b_i$ and $v = b_j$. So either $u = a_i$ and $v = b_j$, or $u = b_i$ and $v = a_j$.

Suppose first that $u = a_i$ and $v = b_j$. Since $Q = P_h[a_i, b_j]$ has length at most c, and $R_j = P_h[a_{j+1}, b_j]$ has length at least $c+1$, and both a_i, a_{j+1} are earlier than b_j, it follows that R_j contains Q, and similarly R_{i-1} contains Q. In particular, both R_j, R_{i-1} contain the end-edges of Q, which are in X_1, and so $j = i - 1$. But then $Q = R_j$ and so has length more than c, a contradiction.

Finally, suppose that $u = b_i$ and $v = a_j$. Since $u \neq v$ and F is $2c$-separated, it follows that $v \notin \{b_1, \dots, b_n\}$, and so $a_j b_j$ is the only pair in F incident with v. Let e be the edge of $P_h[u, v]$ incident with v. Since $e \in X_1$, there exists $i' \in \{1, \dots, n\}$ such that $e \in E(R_{i'})$, and therefore $i' \neq j - 1$. Consequently $b_{i'}$ is in P_h and later than $v = a_j$ (and therefore later than b_i) in P_h, and so $i' > i$. Moreover, $a_{i'+1}$ is in P_h and earlier than v in P_h. Since F is $2c$-separated, it follows that $P_h[a_{i'+1}, a_j]$ has length more than $2c$, and so $a_{i'+1}$ is also earlier than b_i, and $P_h[a_{i'+1}, b_i]$ has length more than c since $Q = P_h[b_i, a_j]$ has length at most c. Since $i' \geq i + 1$, this contradicts (1) (taking u, v of (1) to be $b_i, a_{i'+1}$ respectively). This proves 5.6. ∎

Finally, here is another lemma we will need:

5.7 *In the same notation, let $a_1b_1, \dots, a_nb_n$ be a c-augmenting sequence, and let $\mathcal{J}$ be a partition of $\{1, \dots, n\}$. Then there is a c-augmenting sequence $a'_1b'_1, a'_2b'_2, \dots, a'_mb'_m$ such that*

- *for $1 \leq i' \leq m$, there exist $J \in \mathcal{J}$ and $i, j \in J$ such that $a'_{i'} = a_i$ and $b'_{i'} = b_j$;*
- *for each $J \in \mathcal{J}$ there is at most one $i \in J$ such that $a_i \in \{a'_1, \dots, a'_m\}$, and (therefore) at most one $j \in J$ such that $b_j \in \{b'_1, \dots, b'_m\}$.*

Proof. We proceed by induction on n. We may assume all members of $\mathcal{J}$ are nonempty. If they are all of size one, the result is true, so we may assume that $J_1 \in \mathcal{J}$ has size at least two. Choose $i, j \in J_1$ respectively minimum and maximum. If $a_i \neq b_j$ then

$$a_1b_1, \dots, a_{i-1}b_{i-1}, a_ib_j, a_{j+1}b_{j+1}, \dots, a_nb_n$$

is a c-augmenting sequence, and if $a_i = b_j$ then

$$a_1b_1, \dots, a_{i-1}b_{i-1}, a_{j+1}b_{j+1}, \dots, a_nb_n$$

is a c-augmenting sequence. Assume first that $a_i \neq b_j$. Let $m = n + i - j$, and define:

$$\begin{aligned}
a'_h &= a_h \text{ for } 1 \leq h \leq i \\
b'_h &= b_h \text{ for } 1 \leq h \leq i - 1 \\
a'_h &= a_{h+j-i} \text{ for } i + 1 \leq h \leq m \\
b'_h &= b_{h+j-i} \text{ for } i \leq h \leq m
\end{aligned}$$

Define $f(J_1) = \{i\}$, and for each $J \in \mathcal{J} \setminus \{J_1\}$, define

$$f(J) = \{h : 1 \leq h \leq i - 1 \text{ and } h \in J\} \cup \{h : i + 1 \leq h \leq m \text{ and } h + j - i \in J\}.$$

Then

$$\{f(J) : J \in \mathcal{J} \text{ and } f(J) \neq \emptyset\}$$

is a partition of $\{1, \ldots, m\}$, and the result follows from the inductive hypothesis applied to this partition and $a'_1b'_1, \ldots, a'_mb'_m$. This completes the proof when $a_i \neq b_j$. The argument when $a_i = b_j$ is similar, and we omit it. This proves 5.7. ▮

6 The main proof

If $v \in V(G)$ and A is a vertex or set of vertices or subgraph of G, a (v, A)-*geodesic* of G means a path between v and A of length $\operatorname{dist}_G(v, A)$. Now we prove 3.1, which we restate for convenience.

6.1 *Let $k, c \geq 0$ and $\delta \geq 5$ be integers, and let G be a graph with no subgraph that is a subdivision of H_δ^+. Let $S, T \subseteq V(G)$. Then either*

- *there are $k + 1$ paths between S, T, pairwise at distance greater than c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex at distance at most $(k\delta)^{4k\delta}c$ from some member of X.*

Proof. We proceed by induction on k; the result is trivial for $k = 0$, so we assume that $k \geq 1$. Define $d = \lceil 2\delta \log_2(k\delta) \rceil$. Then $2^{d-1}/k > (2d)^{\delta-1}$.

Let G be a graph with no subgraph that is a subdivision of H_δ^+, and let $S, T \subseteq V(G)$. We assume

(1) *There is no X with $|X| \leq k$, such that every path between S, T contains a vertex at distance at most $(k\delta)^{4k\delta}c$ from some member of X.*

We must therefore show that there are $k + 1$ paths between S, T, pairwise at distance more than c. Define $r = d(4d+14)2^dc$. An S-T path P is *near-geodesic* if for all $u, v \in V(P)$, either $\operatorname{dist}_P(u, v) \leq r$ or $\operatorname{dist}_G(u, v) > (4d + 14)c$. We claim that

(2) *There are k near-geodesic S-T paths in G, pairwise at distance more than $(4d + 14)c$.*

From the inductive hypothesis and (1), and since

$$(k\delta)^{4k\delta} \geq (4d + 14)(2^d + 1)((k - 1)\delta)^{4(k-1)\delta},$$

there are k S-T paths $Q_1, \ldots, Q_k$, pairwise at distance more than $(4d + 14)(2^d + 1)c$.

Let $Z = V(Q_1 \cup \cdots \cup Q_k)$. Since G contains no subdivision of H_δ^+, and $2^{d-1}/k > (2d)^{\delta-1}$, 3.2 implies that there is no subgraph of G that is a Z-leaved subdivision of H_d. By 4.1, taking $\ell = (4d + 14)c$, there exists $Y \supseteq Z$ such that

- every vertex in Y has distance at most $2^{d-1}(4d + 14)c$ from Z;
- for all $u, v \in Y$, if $\operatorname{dist}_G(u, v) \leq (4d + 14)c$, then $\operatorname{dist}_{G[Y]}(u, v) \leq d2^d(4d + 14)c = r$.

For $1 \le i \le k$, there is a path in $G[Y]$ between the ends of Q_i, since Q_i is such a path. Let P_i be a shortest such path. If $u, v \in V(P_i)$ with $\mathrm{dist}_{P_i}(u,v) > r$, then $\mathrm{dist}_{G[Y]}(u,v) > r$, and so $\mathrm{dist}_G(u,v) > (4d+14)c$, that is, P_i is near-geodesic, for $1 \le i \le k$.

For each $v \in V(P_i)$, since $v \in Y$, it follows that v has distance at most $2^{d-1}(4d+14)c$ from Z, that is, from some Q_j, say $Q(v)$. If $u, v \in V(P_i)$ are adjacent, then

$$\mathrm{dist}_G(Q(u), Q(v)) \le 2^d(4d+14)c + 1 \le (4d+14)(2^d+1)c$$

and so $Q(u) = Q(v)$ since $Q_1, \dots, Q_k$ pairwise have distance more than $(4d+14)(2^d+1)c$. Since P_i, Q_i have the same ends, and so $Q(v) = Q_i$ when v is an end of P_i, it follows that $Q(v) = Q_i$ for all $v \in V(P_i)$, that is, every vertex in P_i has distance at most $2^{d-1}(4d+14)c$ from Q_i. Consequently, $P_1, \dots, P_k$ pairwise have distance more than $(4d+14)(2^d+1)c - (4d+14)2^dc = (4d+14)c$. This proves (2).

Fix S-T paths $P_1, \dots, P_k$, each near-geodesic and pairwise at distance more than $(4d+14)c$, and we may choose them minimal with this property; so each has only one vertex in S and one in T, its ends. Let P_h have ends $s_h \in S$ and $t_h \in T$, for $1 \le h \le k$. Let $\mathcal{P} = \{P_1, \dots, P_k\}$.

For $p \ge 1$, let V_p be the set of vertices at distance more than p from $V\mathcal{P}$. Let L be a path of G with ends a, b. We say (see Figure 4):

- L is a *leap of type 1* if $a, b \in V\mathcal{P}$, and there exist $x, y \in V(L)$ with a, x, y, b in order, such that the subpaths $L[a,x], L[b,y]$ have length exactly $(d+3)c$, $L[x,y]$ has length at least two, and every internal vertex of $L[x,y]$ belongs to $V_{(d+3)c}$. (It follows that $L[a,x]$ is an $(x, V\mathcal{P})$-geodesic, and $L[b,y]$ is a $(y, V\mathcal{P})$-geodesic.)
- L is a *leap of type 2* if $a \in V\mathcal{P}$, $b \in (S \cup T) \cap V_{(d+3)c}$, and there exists $x \in V(L)$ such that $L[a,x]$ has length $(d+3)c$, and every internal vertex of $L[x,b]$ belongs to $V_{(d+3)c}$.
- L is a *leap of type 3* if $a \in V\mathcal{P}$, $b \in (S \cup T) \setminus V_{(d+3)c}$, and L is a $(b, V\mathcal{P})$-geodesic.
- L is a *leap of type 4* if $a \in S$ and $b \in T$ and $V(L) \subseteq V_{(d+3)c}$.

A *leap* is a leap of type 1, 2, 3 or 4.

Let F be the set of all ordered pairs uv such that some leap has ends u, v. (Thus, if $ab \in F$ then $ba \in F$.)

(3) *F is $5r$-jumping in the setting $(S, T, \mathcal{P} = \{P_1, \dots, P_k\})$.*

For $1 \le i \le k$ let Q_i be a subpath of P_i of length at most $5r$; thus, $Q_1, \dots, Q_k$ is a $5r$-barrier in the stated setting. We may assume (by extending Q_h) that for $1 \le h \le k$, either $Q_h = P_h$ or Q_h has length exactly $5r$. For $1 \le h \le k$, $P_h \setminus V(Q_h)$ has at most two components. If one of them contains s_h, call it A_h, and otherwise let A_h be the null graph; and if one contains t_h call it B_h, and otherwise B_h is null. Choose $q_h \in V(Q_h)$ for $1 \le h \le k$. Let X be the set of vertices v of G with $\mathrm{dist}_G(v, A_1 \cup \dots \cup A_k) \le (d+3)c$ and $\mathrm{dist}_G(v, \{q_1, \dots, q_k\}) > (k\delta)^{4k\delta}c$; and let Y be the set of v with $\mathrm{dist}_G(v, B_1 \cup \dots \cup B_k) \le (d+3)c$ and $\mathrm{dist}_G(v, \{q_1, \dots, q_k\}) > (k\delta)^{4k\delta}c$.

Suppose that $\mathrm{dist}_G(X, Y) \le 1$; then $\mathrm{dist}_G(A_1 \cup \dots \cup A_k, B_1 \cup \dots \cup B_k) \le 2(d+3)c + 1$. Choose $i, j \in \{1, \dots, k\}$ such that $\mathrm{dist}_G(A_i, B_j) \le 2(d+3)c+1$. Since $\mathrm{dist}_G(P_i, P_j) > (4d+14)c > 2(d+3)c+1$

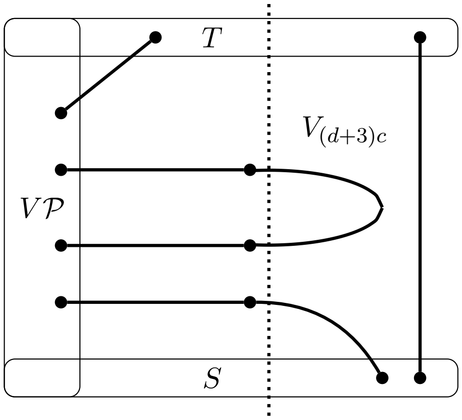

Figure 4: The four types of leaps. (The thick lines represent paths.)

for all distinct i,j, it follows that $i=j$. Since $\text{dist}_{P_i}(A_i,B_i)\ge 5r+2$ (because both intervals are disjoint from Q_i, which has length $5r$), there are vertices $u,v\in P_i$, such that $\text{dist}_{P_i}(u,v)\ge 5r+2$ and yet $\text{dist}_G(u,v)\le 2(d+3)c+1$, contradicting that P_i is near-geodesic, since $2(d+3)c+1\le(4d+14)c$ and $5r+2>r$. This proves that $\text{dist}_G(X,Y)\ge 2$, and in particular X,Y are disjoint.

We claim that for all $v\in V(G)\setminus V_{(d+3)c}$, if $\text{dist}_G(v,\{q_1,\dots,q_k\})>(k\delta)^{4k\delta}c$, then every $(v,V\mathcal{P})$-geodesic is either a $(v,A_1\cup\dots\cup A_k)$-geodesic or a $(v,B_1\cup\dots\cup B_k)$-geodesic. Let J be a $(v,V\mathcal{P})$-geodesic, and let b be its end in $V\mathcal{P}$. Then $b\notin V(Q_1\cup\dots\cup Q_k)$ since $\text{dist}_G(v,\{q_1,\dots,q_k\})>(k\delta)^{4k\delta}c\ge(4d+14)c+5r$ and J has length at most $(d+3)c$ and $Q_1,\dots,Q_k$ all have length at most $5r$. Thus, either $b\in V(A_1\cup\dots\cup A_k)$ or $b\in V(B_1\cup\dots\cup B_k)$, and so J is either $(v,A_1\cup\dots\cup A_k)$-geodesic or a $(v,B_1\cup\dots\cup B_k)$-geodesic as claimed. In particular, v belongs to one of X,Y.

If $S\cap Y\ne\emptyset$, let $s\in S\cap Y$ and let J be an $(s,V\mathcal{P})$-geodesic, and let $b\in B_1\cup\dots\cup B_k$ be the end of J in $V\mathcal{P}$. Then J is a leap of type 3, and so $sb\in F$ jumps the $5r$-barrier $Q_1,\dots,Q_k$. Thus, we may assume that $S\cap Y=\emptyset$, and similarly $T\cap X=\emptyset$.

From (1), applied to the set $\{q_1,\dots,q_k\}$, there is an S-T path P in G such that

$$\text{dist}_G(P,\{q_1,\dots,q_k\})>(k\delta)^{4k\delta}c.$$

Consequently, for each vertex $v\in V(P)\setminus(X\cup Y)$, $\text{dist}_G(v,Q_1\cup\dots\cup Q_k)>(k\delta)^{4k\delta}c-5r\ge(d+3)c$, and $\text{dist}_G(v,A_1\cup\dots\cup A_k)>(d+3)c$ (since $v\notin X$), and similarly $\text{dist}_G(v,B_1\cup\dots\cup B_k)>(d+3)c$; so $v\in V_{(d+3)c}$, and therefore $V(P)\subseteq X\cup Y\cup V_{(d+3)c}$. Since P has first vertex in $S\cup X$ and last vertex in $T\cup Y$, there is a minimal subpath Q of P between $S\cup X$ and $T\cup Y$, with ends $x\in S\cup X$ and $y\in T\cup Y$, say, and therefore with no internal vertex in $X\cup Y\cup S\cup T$.

If $x\in S\setminus X$, then $x\notin Y$ (since $S\cap Y=\emptyset$), and since $\text{dist}_G(x,Q_1\cup\dots\cup Q_k)>(k\delta)^{4k\delta}c-5r\ge(d+3)c$, it follows that $x\in V_{(d+3)c}$. So if both $x\in S\setminus X$ and $y\in T\setminus Y$, then Q is a leap of type 4 and $xy\in F$ jumps the $5r$-barrier; so from the symmetry we may assume that $x\in X$, and consequently $x\ne y$ since $X\cap(Y\cup T)=\emptyset$. Let J_x be an $(x,V\mathcal{P})$-geodesic, with ends x and $a\in V(A_1\cup\dots\cup A_k)$. Thus, J_x has length $(d+3)c$, since $x\in X$ and has a neighbour in $V_{(d+3)c}$. If $y\in T\setminus Y$, then $Q\cup J_x$ is a leap of type 2, and $ay\in F$ jumps the $5r$-barrier. Thus, we may assume that $y\in Y$; let J_y be a $(y,V\mathcal{P})$-geodesic, with ends y,b where $b\in V(B_1\cup\dots\cup B_k)$. Then Q has length at least two since $\text{dist}_G(X,Y)\ge 2$; and so $Q\cup J_x\cup J_y$ is a leap of type 1, and $ab\in F$ jumps the $5r$-barrier. This proves (3).

From 5.5, and 5.2, there is an r-augmenting, $2r$-separated sequence $v_1v_2, v_3v_4, \dots, v_{2n-1}v_{2n}$ in F. Let $W = \{v_2, v_3, \dots, v_{2n-1}\}$. Thus, $W \subseteq V\mathcal{P}$. For $2 \le i, j \le 2n-1$, let us say i, j are *mated* if $i \ne j$ and $\operatorname{dist}_{U\mathcal{P}}(v_i, v_j) \le r$. It follows that if i, j are mated, then one of i, j is odd and the other is even, because $v_1v_2, \dots, v_{2n-1}v_{2n}$ is $2r$-separated; and for the same reason for each $v_i \in W$, i is mated with j for at most one $j \in \{2, \dots, 2n-1\}$. Not all of $v_1, \dots, v_{2n}$ need be distinct, but if $i \ne j$ and $v_i = v_j$ then i, j are mated, because of the following.

(4) *If* $i, j \in \{2, \dots, 2n-1\}$ *are distinct and* $\operatorname{dist}_G(v_i, v_j) \le (4d+14)c$ *then* i, j *are mated.*

Suppose that $\operatorname{dist}_G(v_i, v_j) \le (4d+14)c$. Consequently $v_i, v_j \in V(P_h)$ for some $h \in \{1, \dots, k\}$, since $v_i, v_j \in V\mathcal{P}$. Since P_h is near-geodesic, $\operatorname{dist}_{P_h}(v_i, v_j) \le r$, and so i, j are mated. This proves (4).

Some notation: if P is a path of G and $X \subseteq V(G)$, we write $P[X]$ for $P[V(P) \cap X]$. For $1 \le i \le n$ choose a leap L_i with ends v_{2i-1}, v_{2i}. If some L_i has type 4 (and hence $i = n = 1$), then $P_1, \dots, P_k, L_i$ are S-T paths satisfying the theorem; so we may assume that each L_i has type 1, 2 or 3. Thus, L_1, L_n have types 2 or 3, and all the others have type 1.

Let K be the set of all sets $\{i, j\}$ with $i \ne j$ and $2 \le i, j \le 2n-1$ such that i, j are mated. For $2 \le i \le n$, let S_{2i-1} be the maximal subpath of L_i with one end v_{2i-1} and with length at most $(d+3)c$; and for $1 \le i \le n-1$, let S_{2i} be the maximal subpath of L_i with one end v_{2i} and with length at most $(d+3)c$. Thus, S_i is defined for $2 \le i \le 2n-1$, and S_i has length $(d+3)c$ for $3 \le i \le 2n-2$. (See Figure 5.) Let $S'_i = S_i[V_c]$ (thus, if S_i has length at most c then S'_i is the null graph). Let $S''_i = S_i \setminus V_c$, and let the ends of S''_i be v_i, v'_i. For $1 \le i \le n$, let $R_i = L_i[V_c]$. Thus, R_i is a path unless L_i is a leap of type 3 and has length at most c, and then R_i is null.

We need to be careful with L_1, L_n. There are three possibilities for L_n (and the same for L_1):

- L_n is a leap of type 2;
- L_n is a leap of type 3 and has length more than c;
- L_n is a leap of type 3 and has length at most c.

Note that, in the second case when L_n has length more than c, since L_n is a $(v_{2n}, V\mathcal{P})$-geodesic it follows that $V(L_n) \subseteq V(S_{2n-1}) \cup V_c$, and so R_n joins v_{2n} and a neighbour of v'_{2n-1}.

Let us digress a little, to see the route that lies ahead. We have an r-augmenting, $2r$-separated sequence $v_1v_2, v_3v_4, \dots, v_{2n-1}v_{2n}$, and we could apply 5.6 to them, and assemble them into $k+1$ paths between S, T in the graph H of 5.6. We could try to convert these to $k+1$ S-T paths in G by replacing each pair $v_{2i-1}v_{2i}$ by the corresponding leap L_i say. Certainly, each path of H is converted into a connected subgraph intersecting both S, T. It might not be a path; tracing it gives a walk from S to T that might intersect itself. This is not a problem, we can just shortcut this into a path in the natural way.

More seriously, while the $k+1$ paths in H are pairwise disjoint, when we convert them to paths of G, they might no longer be disjoint, since L_i, L_j might intersect for distinct i, j. This is a problem we can easily avoid, by applying 5.7 to arrange that for each component D of $G[V_{(d+3)c}]$ there is at most one pair $v_{2i-1}v_{2i}$ for which the corresponding leap passes through D. We still need to worry that, say, S_{2i-1} intersects S_{2j-1}; but then $2i-1, 2j-1$ would be mated, and hence be joined by a

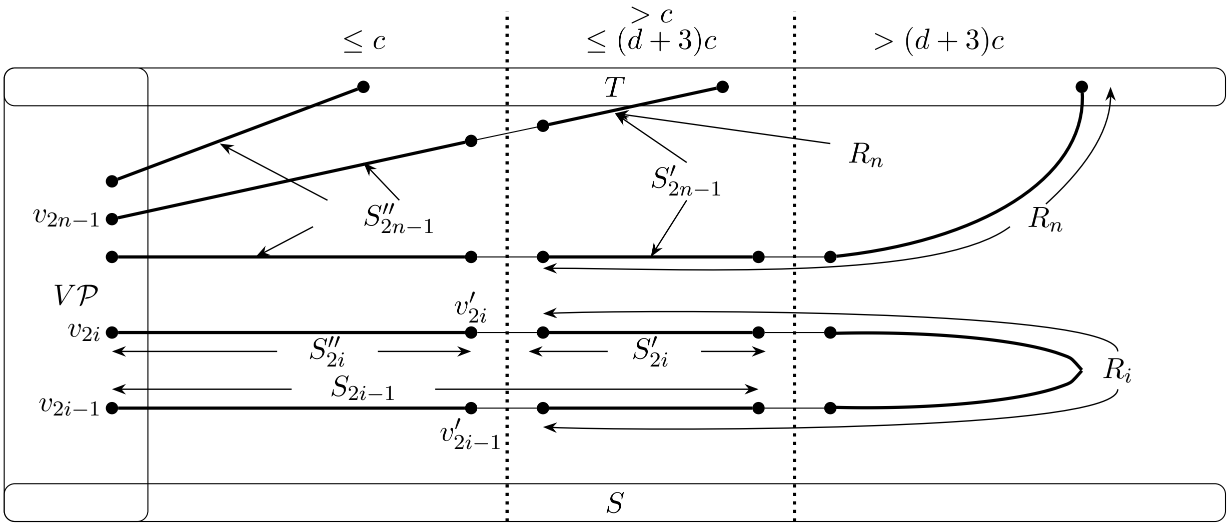

Figure 5: Definitions of R_i, S_i, S'_i, S''_i and v'_i.

subpath of some P_h of length at most $(4d+14)c$, and therefore would not belong to different paths in the output of 5.6, and that problem goes away.

So we obtain $k+1$ S-T paths of G, pairwise vertex-disjoint, and it would be nice if they pairwise had distance more than c. The parts of these paths that are close to $V\mathcal{P}$ are far apart, as we need. The difficulty is that the parts of these paths far away from $V\mathcal{P}$ might be too close to each other. More exactly, each L_i is composed of sections S''_{2i-1}, R_i and S''_{2i} (let us ignore the first and last leap for simplicity), and the distance between the S'' parts of distinct L_i, L_j is satisfactory. Indeed, the parts S_{2i-1}, S_{2i} of L_i are sufficiently far from the parts S_{2j-1}, S_{2j} of L_j. But the part of L_i within $V_{(d+3)c}$ (that is, the subpath joining S_{2i-1}, S_{2i}) might be uncomfortably close to the corresponding part of L_j, or to the subpaths S''_{2j-1}, S''_{2j}.

If so, we could choose a short path joining them, and reroute the paths via a process like that of 5.7. But we have to repeat this, and while initially these reroutings stay in or close to $V_{(d+3)c}$, when we iterate they can build on one another and become further from $V_{(d+3)c}$, and we lose control. The lemma 4.2 is designed to handle this.

That ends the digression; let us return to the proof proper. For each $\{i,j\} \in K$, if there is a path in G of length at most c between S'_i, S'_j, with all vertices in V_c, choose some such path and call it $N_{ij} = N_{ji}$. Let

$$\Gamma = \bigcup (R_i : 1 \le i \le n) \cup \bigcup (N_{ij} : \{i,j\} \in K).$$

Thus, $V(\Gamma) \subseteq V_c$.

(5) *There is no $V(\Gamma)$-leaved subdivision of H_d in $G[V_c]$ whose leaves are all in distinct components of Γ.*

Suppose that J is a $V(\Gamma)$-leaved subdivision of H_d in $G[V_c]$ whose leaves are all in distinct components of Γ. Each leaf u of J belongs to a different component D_u say of Γ, and u belongs either to some $V(R_i)(1 \le i \le n)$ or to some $V(N_{ij})$ ($\{i,j\} \in K$). In either case, there exists $i_u \in \{2, \dots, 2n-1\}$

with $S'_{i_u} \subseteq D_u$ and a path of D_u from u to S'_{i_u}; and hence there is a path Y_u from u to v_{i_u}, contained in $D_u \cup S_{i_u}$. Let U be the set of leaves of J. Each path Y_u is from u to $V\mathcal{P}$. They might not be vertex-disjoint, but if Y_u shares a vertex with $Y_{u'}$, this vertex belongs to S''_{i_u} and to $S''_{i_{u'}}$ (since $D_u \cap D_{u'}$ is null), and so $\text{dist}_G(v_{i_u}, v_{i_{u'}}) \leq 2c$, and hence $i_u, i_{u'}$ are mated by (4). Hence, for each $u \in U$, there is at most one $u' \in U$ different from u such that Y_u intersects $Y_{u'}$; and hence there is a subset $U' \subseteq U$ with $|U'| \geq 2^{d-1}$ such that $Y_u, Y_{u'}$ are vertex-disjoint for all distinct $u, u' \in U'$. By 3.2, since $2^{d-1}/k > (2d)^{\delta-1}$, it follows that G contains a subdivision of H^+_δ, a contradiction. This proves (5).

Choose t maximum such that there is a sequence of paths $M_1, \ldots, M_t$ of $G[V_c]$ satisfying, for each $i \geq 1$:

- M_i has length at most c;
- the ends of M_i lie in different components of $\Gamma \cup (M_1 \cup \cdots \cup M_{i-1})$ and none of its internal vertices lie in this graph; and
- at most one vertex of M_i belongs to $\bigcup_{2 \leq j \leq 2n-1} S'_j$.

Let $\Gamma' := \Gamma \cup (M_1 \cup \cdots \cup M_t)$. We claim:

(6) *Every vertex in $M_1 \cup \cdots \cup M_t$ has distance at most $(d+1)(c-1)+2$ from $V_{(d+3)c}$. Consequently, there is no path M of G (not necessarily of $G[V_c]$) satisfying*

- *M has length at most c;*
- *the ends of M lie in different components of Γ' and none of its internal vertices lie in this graph; and*
- *at most one vertex of M belongs to $\bigcup_{2 \leq j \leq 2n-1} V(S'_j)$.*

Let $x \in V(M_1 \cup \cdots \cup M_t)$, and suppose that $\text{dist}_G(x, V_{(d+3)c}) > (d+1)(c-1)+2$. If x is not equal or adjacent to any interior vertex of any M_i, then $x \in V(\Gamma)$ and there exists $i \in \{1, 2, \ldots, t\}$ such that M_i is an edge with one end x and the other end in $V(\Gamma)$. On the other hand, if x equals or is adjacent to a vertex x' in the interior of some M_i, then $x' \in V(M_1 \cup \cdots \cup M_t) \setminus V(\Gamma)$; and so $\text{dist}_G(x', V_{(d+3)c}) > (d+1)(c-1)+1$. By (5) and 4.2 (taking $\ell = c$ and replacing G by Γ'), either x' is in the interior of some M_i with both ends in $V(\Gamma)$, or there are three components of Γ such that x' has distance at most $(d+1)(c-1)$ from each of them in Γ' (and so in G).

In summary, one of the following two cases holds:

- there exists $i \in \{1, 2, \ldots, t\}$ such that M_i has both ends in $V(\Gamma)$ and contains a vertex of distance more than $(d+1)(c-1)+1$ to $V_{(d+3)c}$ in G; or
- there exists a vertex x' in the interior of some M_i, such that $\text{dist}_G(x', V_{(d+3)c}) > (d+1)(c-1)$, and x' has distance at most $(d+1)(c-1)$ from three components of Γ.

In the first case, let M_i have ends x_1, x_2. Since M_i has length at most c and contains a vertex of distance more than $(d+1)(c-1)+1 \geq c$ to $V_{(d+3)c}$ in G, it follows that $x_1, x_2 \notin V_{(d+3)c}$; but then, for $j = 1, 2$, x_j belongs either to S'_{i_j} for some $i_j \in \{2, \ldots, 2n-1\}$, or to $N_{i_j h}$ for some $\{i_j, h\} \in K$.

Thus, $\mathrm{dist}_G(v_{i_j}, x_j) \le (d+3)c + c$ for $j = 1, 2$. Moreover, S'_{i_1}, S'_{i_2} belong to different components of $\Gamma_{i-1} := \Gamma \cup (M_1 \cup \cdots \cup M_{i-1})$, and so $i_1 \neq i_2$. Since M_i has length at most c, it follows that

$$\mathrm{dist}_G(v_{i_1}, v_{i_2}) \le 2((d+3)c + c) + c \le (4d+14)c,$$

and so i_1, i_2 are mated by (4). We recall that x_1 belongs either to S'_{i_1} or to $N_{i_1 h}$ for some h. In the second case, $h = i_2$, since h, i_2 are both mated with i_1. But S'_{i_1}, S'_h belong to the same component of Γ (since $N_{i_1 h}$ exists), and yet S'_{i_1}, S'_{i_2} belong to different components of Γ_{i-1}, from the definition of M_i, a contradiction. Thus, x_1 belongs to S'_{i_1}, and similarly x_2 belongs to S'_{i_2}. This contradicts the hypothesis that at most one vertex of M_i is in $\bigcup_{2 \le j \le 2n-1} V(S'_j)$.

Thus, the second case holds; and so there are three components C_1, C_2, C_3 of Γ such that x' has distance at most $(d+1)(c-1)$ from each of them. For $j = 1, 2, 3$, choose $x_j \in C_j$ with distance at most $(d+1)(c-1)$ from x'. Since $\mathrm{dist}_G(x', V_{(d+3)c}) > (d+1)(c-1)$, it follows that $x_1, x_2, x_3 \notin V_{(d+3)c}$. Consequently, for $j = 1, 2, 3$ there exists $i_j \in \{2, \ldots, 2n-1\}$ such that v_{i_j} is at distance at most $(d+3)c + c$ from x_j, and $S'_{i_1}, S'_{i_2}, S'_{i_3}$ all belong to different components of Γ. In particular, i_1, i_2, i_3 are all different. Therefore, some pair of i_1, i_2, i_3 is not mated, say i_1, i_2; but

$$\mathrm{dist}_G(v_{i_1}, v_{i_2}) \le 2(d+1)c + 2((d+3)c + c) \le (4d+14)c,$$

contrary to (4). This proves the first statement of (6).

Suppose that there is a path M of G satisfying:

- M has length at most c;
- the ends of M lie in different components of Γ', and none of its internal vertices lie in this graph; and
- at most one vertex of M belongs to $\bigcup_{2 \le j \le 2n-1} V(S'_j)$.

If some vertex of M lies in $V(M_1 \cup \cdots \cup M_t) \cup V_{(d+3)c}$, then it has distance at most $(d+1)(c-1) + 2$ from $V_{(d+3)c}$; and therefore has distance at least $(d+2)c - ((d+1)(c-1) + 2) > c$ from $V(G) \setminus V_c$. Consequently M is a path of $G[V_c]$, contrary to the maximality of t. So $V(M)$ is disjoint from $V(M_1 \cup \cdots \cup M_t) \cup V_{(d+3)c}$, and hence both its ends lie in $V(\Gamma) \setminus V_{(d+3)c}$ and hence in

$$\bigcup \left(V(S'_j) : 2 \le j \le 2n-1\right) \cup \bigcup \left(V(N_{ij}) : \{i,j\} \in K\right).$$

At most one end lies in $\bigcup_{2 \le j \le 2n-1} V(S'_j)$; so we may assume that one end x_1 of M lies in $V(N_{i_1 h_1})$ for some pair $\{i_1, h_1\} \in K$, and its other end x_2 belongs either to $V(S'_{i_2})$ for some $i_2 \in \{2, \ldots, 2n-1\}$, or to $V(N_{i_2 h_2})$ for some $\{i_2, h_2\} \in K$. Thus, $\mathrm{dist}_G(x_j, v_{i_j}) \le c + (d+3)c$ for $j = 1, 2$, and so

$$\mathrm{dist}_G(v_{i_1}, v_{i_2}) \le 2(c + (d+3)c) + c \le (4d+14)c.$$

By (4) i_1, i_2 are equal or mated. They are not equal since x_1, x_2 belong to different components of Γ', so i_1, i_2 are mated. By the same argument with i_1, h_1 exchanged, h_1, i_2 are mated; and so every two of h_1, i_1, i_2 are mated, a contradiction. Thus, there is no such M. This proves (6).

Let $\mathcal{D}$ be the set of components of Γ'. Each $D \in \mathcal{D}$ includes at least one of the paths R_j ($1 \le j \le n$). For each $D \in \mathcal{D}$, let J_D be the set of $j \in \{1, \ldots, n\}$ such that R_j is a non-null subgraph of

D, and let I_D be the set of all $i \in \{2, \ldots, 2n-1\}$ such that either $i/2 \in J_D$ or $(i+1)/2 \in J_D$. Let D^+ be the union of D and all the paths S_i with $i \in I_D$, and for convenience we define $I_{D^+} = I_D$. (Incidentally, even if $D_1, D_2 \in \mathcal{D}$ are distinct and therefore disjoint, it is possible that D_1^+, D_2^+ might intersect, because there might exist $i_j \in I_{D_j}$ for $j = 1, 2$ such that S''_{i_1}, S''_{i_2} intersect. But then i_1, i_2 would be mated.) Let $\mathcal{D}^+$ be the set of all the graphs D^+ for $D \in \mathcal{D}$, together with either of L_1, L_n that has length at most c. If L_1 has length at most c, define $I_{L_1} = \{2\}$, and if L_n has length at most c, define $I_{L_n} = \{2n-1\}$. The sets I_X ($X \in \mathcal{D}^+$) are nonempty and pairwise disjoint, and their union equals $\{2, \ldots, 2n-1\}$.

For each $X \in \mathcal{D}^+$, we say $x \in V(X)$ is *innocuous in* X if there exists $i \in I_X$ such that $x \in V(S_i)$, and $S_i[x, v_i]$ has length at most $2c$, and for each vertex y of $S_i[x, v_i]$, all edges of X incident with y belong to S_i. (Thus, if L_1 has length at most c, then all vertices of L_1 are innocuous in L_1, and a similar statement holds if L_n has length at most c.) We claim:

(7) *If $X_1, X_2 \in \mathcal{D}^+$ are different, and M is a path of length at most c in G between X_1, X_2, then for $j = 1, 2$, the end of M in X_j is innocuous in X_j.*

Suppose first that $\mathrm{dist}_G(M, V\mathcal{P}) > c$, and let M' be a minimal subpath of M that has nonempty intersection with two of the graphs in $\mathcal{D}^+$ (and therefore with two members of $\mathcal{D}$). Let the ends of M' be $x'_1 \in D'_1$ and $x'_2 \in D'_2$, where $D'_1, D'_2 \in \mathcal{D}$. From the maximality of t in the definition of $M_1, \ldots, M_t$, it follows that at least two vertices of M' belong to $\bigcup_{2 \le j \le 2n-1} S'_j$, and since no internal vertex of M' belongs to this subgraph, we deduce that x'_1, x'_2 are both in $\bigcup_{2 \le j \le 2n-1} S'_j$. Therefore, for $j = 1, 2$, there exists $i_j \in I_{D'_j}$ such that x'_j belongs to S'_{i_j}, with $i_1 \neq i_2$. Thus, $\mathrm{dist}_G(v_{i_1}, v_{i_2}) \le 2(d+3)c + c \le (4d+14)c$, and so i_1, i_2 are mated by (4). Since $i_1 \neq i_2$, and $V(M) \subseteq V_c$, it follows that $N_{i_1 i_2}$ exists, and so S'_{i_1}, S'_{i_2} belong to the same component of Γ', contradicting that $D'_1 \neq D'_2$. Thus, $\mathrm{dist}_G(M, V\mathcal{P}) \le c$.

Let M have ends $x_j \in X_j$ for $j = 1, 2$. Since $\mathrm{dist}_G(M, V\mathcal{P}) \le c$, it follows that $\mathrm{dist}_G(x_j, V\mathcal{P}) \le 2c$, and therefore $\mathrm{dist}_G(x_j, V_{(d+3)c}) > (d+1)c$. Since $(d+1)c \ge (d+1)(c-1) + 2$, by (6), x_j is in none of $M_1, \ldots, M_t$. Choose $i_j \in I_{X_j}$ as follows. If X_j is one of L_1, L_n of length at most c, let i_j be 2 or $2n-1$ correspondingly. If $X_j = D_j^+$ for some $D_j \in \mathcal{D}$, then:

- if $x_j \in D_j^+ \setminus D_j$, choose $i_j \in I_{D_j}$ with $x_j \in V(S''_{i_j})$;
- otherwise, either there exists $i_j \in I_{D_j}$ with $x_j \in V(S'_{i_j})$, or
- there exist $i_j, h \in I_{D_j}$ such that i_j, h are mated and $x_j \in V(N_{i_j h})$.

Choose i_j as above, for $j = 1, 2$. Since $i_j \in I_{X_j}$ for $j = 1, 2$, and $I_{X_1} \cap I_{X_2} = \emptyset$ since $X_1 \neq X_2$, we deduce that $i_1 \neq i_2$. In each case, it follows that $\mathrm{dist}_G(x_j, v_{i_j}) \le (d+3)c + c$, and so $\mathrm{dist}_G(v_{i_1}, v_{i_2}) \le 3c + 2(d+3)c \le (4d+14)c$, and so i_1, i_2 are mated by (4). Thus, the third bullet above is impossible, as before, and so $x_1 \in V(S_{i_1})$ and $x_2 \in V(S_{i_2})$.

Since $\mathrm{dist}_G(x_1, V\mathcal{P}) \le 2c$, it follows that the subpath of S_{i_1} between x_1, v_{i_1} has length at most $2c$, because it is an $(x_1, V\mathcal{P})$-geodesic. Let y be a vertex of this subpath, and let e be an edge of X_1 incident with y. To show that x_1 is innocuous in X_1, it remains to show that e is an edge of S_{i_1}, for all such y, e. Since $\mathrm{dist}_G(y, v_{i_1}) \le 2c$, it follows that $\mathrm{dist}_G(y, V_{(d+3)c}) > (d+3)c - 2c \ge (d+1)(c-1) + 2$, and so $y \notin V_{(d+3)c}$ and y belongs to none of $M_1, \ldots, M_t$, by (6). Thus, $e \in E(X_1) \setminus E(M_1 \cup \cdots \cup M_t)$, and so either

- there exists $h_1 \in I_{X_1}$ with $e \in E(S_{h_1})$; or
- there is a mated pair h_1, h with $h_1, h \in I_{X_1}$ such that N_{h_1h} exists and e is an edge of N_{h_1h}.

In either case $h_1 \in I_{X_1}$, and $\text{dist}_G(v_{h_1}, v_{i_1}) \le 2c + (c + (d+3)c) \le (4d+14)c$. Hence h_1, i_1 are equal or mated, by (4). But i_1, i_2 are mated, and $h_1 \neq i_2$ since $h_1 \in I_{X_1}$; and so $h_1 = i_1$. Therefore the second bullet above is impossible, and so $e \in E(S_{i_1})$. This proves that x_1 is innocuous in X_1, and similarly x_2 is innocuous in X_2, and so proves (7).

We recall that for each $D \in \mathcal{D}$, J_D is the set of $j \in \{1, \dots, n\}$ such that R_j is a non-null subgraph of D. We would like to apply 5.7 to the set of sets $\{J_D : D \in \mathcal{D}\}$, but it might not be a partition of $\{1, \dots, n\}$. Certainly its union contains $\{2, \dots, n-1\}$, but we have to be careful about $1, n$. There is no $D \in \mathcal{D}$ with $1 \in J_D$ if and only if L_1 is a leap of type 3 of length at most c; and the same for n, L_n. Let $\mathcal{J}$ be the partition of $\{1, \dots, n\}$ formed by the sets $\{J_D : D \in \mathcal{D}\}$, together with $\{1\}$ if L_1 is a leap of type 3 of length at most c, and $\{n\}$ if L_n is a leap of type 3 of length at most c. The sequence $v_1v_2, \dots, v_{2n-1}v_{2n}$ is r-augmenting and $2r$-separated; and by applying 5.7 to the partition $\mathcal{J}$ and this sequence, we deduce that there is an r-augmenting, $2r$-separated sequence $w_1w_2, \dots, w_{2m-1}w_{2m}$ such that, writing $T_j = S_i$ and $T'_j = S'_i$ if $w_j = v_i$, we have:

- $w_1, w_3, \dots, w_{2m-1} \in \{v_1, v_3, \dots, v_{2n-1}\}$, and $w_2, w_4, \dots, w_{2m} \in \{v_2, v_4, \dots, v_{2n}\}$;
- for $1 \le i \le m$, either:
  - $T'_{2i-1} \cup T'_{2i}$ is non-null, and there exists $D_i \in \mathcal{D}$ such that $V(T'_{2i-1}), V(T'_{2i}) \subseteq D_i$; or
  - $i = 1$, and L_1 is a leap of type 3 with length at most c, and $(w_1, w_2) = (v_1, v_2)$, or
  - $i = m$, and L_n is a leap of type 3 with length at most c, and $(w_{2m-1}, w_{2m}) = (v_{2n-1}, v_{2n})$;

  and
- $D_2, \dots, D_{m-1}$ and (if they exist) D_1, D_m are all different.

To see this, observe that $w_1 \in S \setminus V\mathcal{P}$, and $w_1 \in \{v_1, v_3, \dots, v_{2n-1}\}$, and therefore $w_1 = v_1$, and so if $\{1\} \in \mathcal{J}$ then $(w_1, w_2) = (v_1, v_2)$; and similarly if $\{n\} \in \mathcal{J}$ then $(w_{2m-1}, w_{2m}) = (v_{2n-1}, v_{2n})$. For $2 \le j \le 2m-1$, define $T''_j = S''_i$ and $w'_j = v'_i$, where $v_i = w_j$. Thus, for $2 \le i \le 2m-1$, T''_i is the subpath of T_i between w_i and w'_i, of length c unless T_i has length less than c. Let $w'_1 = v_1 = w_1 \in S$, let T''_1 be the one-vertex graph with vertex w'_1, let $w'_{2m} = v_{2n} = w_{2m} \in T$, and let T''_{2m} be the one-vertex graph with vertex w'_{2m}.

If i, j are mated, and $w_{i'} = v_i$ and $w_{j'} = v_j$, we say i', j' are *checkmated.* For $1 \le i \le m$, if D_i exists (which it does unless $i \in \{1, m\}$), then both w'_{2i-1}, w'_{2i} belong to or have a neighbour in D_i; let Q_i be a path between w'_{2i-1}, w'_{2i} with interior in $V(D_i)$. If D_1 does not exist, then L_1 has length at most c and joins w_1 and w_2; in this case let Q_1 be the one-vertex graph with vertex w_1. Similarly if D_m does not exist let Q_m be the one-vertex graph with vertex w_{2m}. Thus, for $1 \le i \le m$, $T''_{2i-1} \cup Q_i \cup T''_{2i}$ is a path between w_{2i-1}, w_{2i}.

(8) *For all distinct $i, j \in \{1, \dots, m\}$, if the distance in G between $T''_{2i-1} \cup Q_i \cup T''_{2i}$ and $T''_{2j-1} \cup Q_j \cup T''_{2j}$ is at most c, then one of $2i-1, 2i$ is checkmated with one of $2j-1, 2j$.*

Let M be a path of length at most c with ends x_1, x_2, where $x_1 \in V(T''_{2i-1} \cup Q_i \cup T''_{2i})$ and $x_2 \in V(T''_{2j-1} \cup Q_j \cup T''_{2j})$. Suppose first that D_i exists. By (7), x_1 is innocuous in D_i^+. Choose $h_1 \in I_{D_i}$ with $x_1 \in V(T_{h_1})$, with $h_1 \in \{2i-1, 2i\}$ if possible. Since $T''_{2i-1} \cup Q_i \cup T''_{2i}$ is a path in D_i^+ containing x_1 with both ends in $W \cup \{w_1, w_{2m}\}$, and for each vertex y of $T_{h_1}[x_1, w_{h_1}]$, all edges of D_i^+ incident with y belong to T_{h_1}, it follows that $T_{h_1}[x_1, w_{h_1}]$ is a subpath of $T''_{2i-1} \cup Q_i \cup T''_{2i}$, and therefore one of $2i-1, 2i$ is also a valid choice for h_1; and therefore h_1 is one of $2i-1, 2i$. Moreover, $T_{h_1}[x_1, w_{h_1}]$ has length at most $2c$, and so $\operatorname{dist}_G(x_1, w_{h_1}) \le 2c$.

Now suppose that D_i does not exist. Then $i \in \{1, m\}$; suppose first that $i = 1$. Hence $L_1 = T''_2$ has length at most c, and contains x_1; let $h_1 = 2$. Then $\operatorname{dist}_G(x_1, w_{h_1}) \le c$. Similarly, if $i = m$, let $h_1 = 2m-1$ and it follows that $\operatorname{dist}_G(x_1, w_{h_1}) \le c$. Thus, whether D_i exists or not, there exists $h_1 \in \{2i-1, 2i\}$ such that $\operatorname{dist}_G(x_1, w_{h_1}) \le 2c$. Define h_2 similarly for x_2; then $h_1 \ne h_2$ (since $i \ne j$), and it follows that $\operatorname{dist}_G(w_{h_1}, w_{h_2}) \le 2(2c) + c \le (4d+14)c$, and so h_1, h_2 are checkmated by (4). This proves (8).

But now the result follows from 5.6 applied to $w_1w_2, \dots, w_{2m-1}w_{2m}$, replacing each pair $w_{2i-1}w_{2i}$ in the resulting paths by $T''_{2i-1} \cup Q_i \cup T''_{2i}$. Let us see this in more detail. Let

$$F = \{w_1w_2, \dots, w_{2m-1}w_{2m}\},$$

and let H be the graph obtained from $U\mathcal{P}$ by adding the remainder of $S \cup T$ as vertices, and the ordered pairs in F as (undirected) edges. Since F is r-jumping (by 5.2) and $2r$-separated, we deduce from 5.6 that there exist $k+1$ vertex-disjoint S-T paths $Z_1, \dots, Z_{k+1}$ in H, such that no two of them are joined by a subpath of $U\mathcal{P}$ of length at most r. Each Z_s is a concatenation of subpaths of $U\mathcal{P}$ and edges $w_{2i-1}w_{2i}$.

For $1 \le s \le k+1$, let F_s be the set of pairs in F that are edges of Z_s. Thus, $Z_s \setminus F_s$ is a subgraph of H, and each of its components is either a subpath of a member of $\mathcal{P}$ or a one-vertex subgraph with vertex w_1 or w_{2m}.

(9) *If $a, b \in \{1, \dots, k+1\}$ are distinct, and $x \in V(Z_a)$ and $y \in V(Z_b)$ with $x, y \notin \{w_1, w_{2m}\}$, then* $\operatorname{dist}_G(x, y) > (4d+14)c$.

Suppose not; then there exist $x \in V(Z_a)$ and $y \in V(Z_b)$, both in $V\mathcal{P}$, with $\operatorname{dist}_G(x, y) \le (4d+14)c$. Since $x, y \in V\mathcal{P}$, at distance at most $(4d+14)c$, both x, y belong to the same member of $\mathcal{P}$, say P_h. Since $\operatorname{dist}_G(x, y) \le (4d+14)c$ and P_h is near-geodesic, it follows that $\operatorname{dist}_{P_h}(x, y) \le r$. But Z_a, Z_b are not joined by a subpath of $U\mathcal{P}$ of length at most r, a contradiction. This proves (9).

For each $a \in \{1, \dots, k+1\}$, let Y_a be the union of $Z_a \setminus F_a$ and the path $T''_{2i-1} \cup Q_i \cup T''_{2i}$ for each pair $w_{2i-1}w_{2i} \in F_a$. Then Y_a is a connected subgraph of G, containing a vertex in S and a vertex in T.

(10) *$Y_1, \dots, Y_{k+1}$ pairwise are at distance more than c.*

Suppose that $a, b \in \{1, \dots, k+1\}$ are distinct, and there exist $x \in V(Y_a)$ and $y \in V(Y_b)$ such that $\operatorname{dist}_G(x, y) \le c$. By (9), it is not the case that $x \in V(Z_a \setminus F_a) \cap V\mathcal{P}$ and $y \in V(Z_b \setminus F_b) \cap V\mathcal{P}$, so we may assume that $y \notin V(Z_b \setminus F_b) \cap \mathcal{P}$. Choose $w_{2j-1}w_{2j} \in F_b$ such that $y \in V(T''_{2j-1} \cup Q_j \cup T''_{2j})$.

Suppose that $x \notin V(Z_a) \cap V\mathcal{P}$. Then $x \in V(T''_{2i-1} \cup Q_i \cup T''_{2i})$ for some $w_{2i-1}w_{2i} \in F_a$. From (8), one of $2i-1, 2i$ (say i') is checkmated with one of $2j-1, 2j$ (say j'). Hence $w_{i'}, w_{j'}$ belong to the same member of $\mathcal{P}$, say P_h, and $\text{dist}_{P_h}(w_{i'}, w_{j'}) \leq r$. Yet $w_{i'} \in V(Z_a)$ and $w_{j'} \in V(Z_b)$, contradicting that Z_a, Z_b are not joined by a subpath of $U\mathcal{P}$ of length at most r.

So $x \in V(Z_a) \cap V\mathcal{P}$. Since $x \in V\mathcal{P}$, $\text{dist}_G(x, Q_j) > c$ and so $y \notin V(Q_j)$; and so $y \in V(T''_h)$ for some $h \in \{2j-1, 2j\}$. Since T''_h is a $(y, V\mathcal{P})$-geodesic, and $x, w_h \in V\mathcal{P}$ and $w_h \in V(T''_h)$, it follows that

$$\text{dist}_G(y, w_h) = \text{dist}_G(y, V\mathcal{P}) \leq \text{dist}_G(y, x) \leq c,$$

and therefore $\text{dist}_G(x, w_h) \leq 2c \leq (4d+14)c$; but $x \in V(Z_a)$ and $w_h \in V(Z_b)$, contrary to (9). This proves (10).

From (10), this proves 6.1. ∎

7 Concluding remarks

What about infinite graphs? We assumed that all our graphs were finite at the start of the paper, but augmenting path arguments work fine in infinite graphs (provided we only want some finite number of paths), and the only place in the proof that we used finiteness was in the choice of $M_1, \ldots, M_t$ with t maximum just before step (6) of the main proof. An easy application of Zorn's lemma would do instead, so in fact our theorem works for infinite graphs. (And "path-width" is better replaced by "line-width" for infinite graphs: see [10] for example.)

And for free, we can get a strengthening to graphs with "bounded coarse line-width". A (p,q)-*line-decomposition* of G is a family $(B_t : t \in L)$ of subsets of $V(G)$, where L is a linearly ordered set, such that

- $\bigcup_{t \in L} G[B_t] = G$;
- for all $t_1, t_2, t_3 \in L$, if $t_1 \leq t_2 \leq t_3$ (where $\leq$ is the linear order on L) then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$; and
- for each $t \in L$, B_t is the union of at most p subsets each with diameter in G at most q.

A class of graphs has *bounded coarse line-width* if there are p, q such that every graph in the class has a (p,q)-line-decomposition (see [11] for a coarse structural characterization of graphs with bounded coarse line-width).

We showed in [9] that for all p, q, there exist ℓ, c such that every graph that admits a (p,q)-line-decomposition also admits an (ℓ, c)-quasi-isometry to a graph of line-width at most p. (See [9] for definitions.) So we can strengthen our theorem, since its conclusion is invariant under taking quasi-isometries, and obtain that the coarse Menger conjecture is true for graphs in any class with bounded coarse line-width:

7.1 *Let $k, c, p, q \geq 0$ be integers. Then there exists $\ell \geq 0$, such that for every graph G with a (p,q)-line-decomposition, and all $S, T \subseteq V(G)$, either:*

- *there are $k+1$ paths between S, T, pairwise at distance more than c; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq k$ such that every path between S, T contains a vertex at distance at most ℓ from some member of X.*

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