

# Asymptotic structure. IV. A counterexample to the weak coarse Menger conjecture

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## Abstract

Coarse graph theory concerns finding “coarse” analogues of graph theory theorems, replacing disjointness with being far apart. One of the most interesting open questions is to find a coarse analogue of Menger’s theorem, which characterizes when there are  $k$  vertex-disjoint paths between two given sets of vertices of a graph. We showed in an earlier paper that the most natural such analogue is false, but a weaker statement remained as a popular open question. Here we show that the weaker statement is also false.

More exactly, suppose that  $S, T$  are sets of vertices of a graph  $G$ , and there do not exist  $k$  paths between  $S, T$ , pairwise at distance at least  $c$ . To make an analogue of Menger’s theorem, one would like to prove that there must be a small set  $X \subseteq V(G)$  such that every  $S$ - $T$  path of  $G$  passes close to a member of  $X$ : but how small and how close? In view of Menger’s theorem, one would hope for  $|X| < k$  and “close” some function of  $k, c$  (and indeed, this was conjectured by Georgakopoulos and Papasoglu, and independently, by Albrechtsen, Huynh, Jacobs, Knappe and Wollan); but we showed that this is false, even if  $c = 3$  and  $k = 3$ .

Here we upgrade the counterexample: we show that, even if  $c = k = 3$ , no pair of constants (for “small” and “close”) works. For all  $\ell, m$ , there is a graph  $G$  and  $S, T \subseteq V(G)$ , such that there do not exist three  $S$ - $T$  paths pairwise at distance at least three, and yet there is no  $X$  with  $|X| \leq m$  such that every  $S$ - $T$  path passes within distance at most  $\ell$  of  $X$ .

# 1 Introduction

The “disjoint paths problem” asks when there is a set of  $k$  vertex-disjoint paths between sets  $S, T$  of vertices of a graph  $G$ ; and it is answered by a theorem of K. Menger from 1927 [11], that such paths exist if and only if there is no subset  $X \subseteq V(G)$  of size  $< k$  such that every  $S$ - $T$  path has a vertex in  $X$ . But what if we want the paths to be at least a certain distance from one another?<sup>1</sup> This question is motivated both by the developing area of “coarse graph theory”, which is concerned with the large-scale geometric structure of graphs (see Georgakopoulos and Papasoglu [7]), and by the algorithmic question of deciding whether such paths exist (see Bienstock [3], Kawarabayashi and Kobayashi [9], and Baligács and MacManus [2]).

A coarse analogue of Menger’s theorem was conjectured by Albrechtsen, Huynh, Jacobs, Knappe and Wollan [1], and independently by Georgakopoulos and Papasoglu [7]:

**1.1 False conjecture:** *For all integers  $k, c \geq 1$  there exists  $\ell > 0$  with the following property. Let  $G$  be a graph and let  $S, T \subseteq V(G)$ . Then either*

- *there are  $k$  paths between  $S, T$ , pairwise at distance at least  $c$ ; or*
- *there is a set  $X \subseteq V(G)$  with  $|X| \leq k - 1$  such that every path between  $S, T$  contains a vertex at distance at most  $\ell$  from some member of  $X$ .*

Thus, when  $c = 1$ , we are asking for paths that are pairwise vertex-disjoint, the usual version of Menger’s theorem; and when  $c = 2$ , we are essentially asking for the union of the paths to be induced. Both sets of authors proved the result for  $k = 2$ , but we showed in [12] that 1.1 is false for all  $k \geq 3$ , even if  $c = 3$  and  $G$  has maximum degree three. The case  $c = 3$  is of special interest, because it is easy to see that if the result is true when  $c = 3$  (for some value of  $k$ ) then it is true for all  $c \geq 3$  and the same value of  $k$  (apply the result when  $c = 3$  to the  $c$ th power of  $G$ ). And indeed, the conjecture remains open when  $c = 2$ , and we have nothing to say about that case in this paper.

Since the most natural extension of Menger’s theorem is false, we fall back onto what seems the next most natural, the following weaker statement:

**1.2 False conjecture:** *For all integers  $k, c \geq 1$  there exist  $m, \ell > 0$  with the following property. Let  $G$  be a graph and let  $S, T \subseteq V(G)$ . Then either*

- *there are  $k$  paths between  $S, T$ , pairwise at distance at least  $c$ ; or*
- *there is a set  $X \subseteq V(G)$  with  $|X| \leq m$  such that every path between  $S, T$  contains a vertex at distance at most  $\ell$  from some member of  $X$ .*

We proposed this in [12], but now we will show that this too is false, even if  $c = k = 3$ . More exactly, we will show:

**1.3** *For all integers  $\ell, m \geq 1$ , there is a graph  $G$  and subsets  $S, T \subseteq V(G)$  such that:*

- *there do not exist three paths between  $S, T$  that pairwise are at distance at least three; and*
- *for every set  $X \subseteq V(G)$  with  $|X| \leq m$ , there is a path  $P$  between  $S, T$  such that  $\text{dist}_G(P, X) > \ell$ .*

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<sup>1</sup>If  $X$  and  $Y$  are vertices, or sets of vertices, or subgraphs, of a graph  $G$ , then  $\text{dist}_G(X, Y)$  denotes the distance between  $X, Y$ , that is, the number of edges in a shortest path of  $G$  with one end in  $X$  and the other in  $Y$ .

Our counterexample has some vertices of large degree, but it can easily be modified into a counterexample with only one vertex of degree more than three, while keeping  $c = k = 3$ . We will explain this in Section 3.

## 2 The counterexample

For all integers  $\ell, m \geq 1$ , we will give a construction for a quadruple  $(G, r, S, T)$  called (in this paper) an  $(\ell, m)$ -block, where  $G$  is a graph,  $r \in V(G)$ , and  $S, T$  are disjoint subsets of  $V(G) \setminus \{r\}$ , both of size  $m$ . Later we will prove (by induction on  $m$ ) that it has the properties that:

- $\text{dist}_G(u, v) > 2\ell$  for every two vertices  $u, v \in \{r\} \cup S \cup T$ , and every two vertices in  $S$  are joined by a path of length exactly  $2\ell + 1$  in which all internal vertices have degree two (and the same for  $T$ );
- for every choice of  $X \subseteq V(G)$  with  $|X| < m$ , there is a path  $P$  between  $S, T$  such that  $\text{dist}_G(P, X \cup \{r\}) > \ell$ ; and
- for every two vertex-disjoint  $S$ - $T$  paths  $P, Q$ , either one of  $P, Q$  contains  $r$ , or  $\text{dist}_G(P, Q) \leq 2$ .

This will prove 1.3. (To see this, take an  $(\ell, m + 1)$ -block and replace  $S, T$  by  $S \cup \{r\}, T \cup \{r\}$ .) We call  $r$  the *root*.

The construction for an  $(\ell, 1)$ -block is easy: let  $G'$  be a path of length  $2\ell + 1$  with ends  $s, t$ , and let  $S = \{s\}$  and  $T = \{t\}$ ; let  $r$  be a new vertex, and let  $G$  be obtained from  $G'$  by adding  $r$  as a vertex of degree zero. Then  $(G, r, S, T)$  is an  $(\ell, 1)$ -block.

The construction for  $(\ell, 2)$ -blocks was given in [12], but here it is again (slightly modified for convenience), illustrated in Figure 1. Each dotted curve in the figure represents a path of length

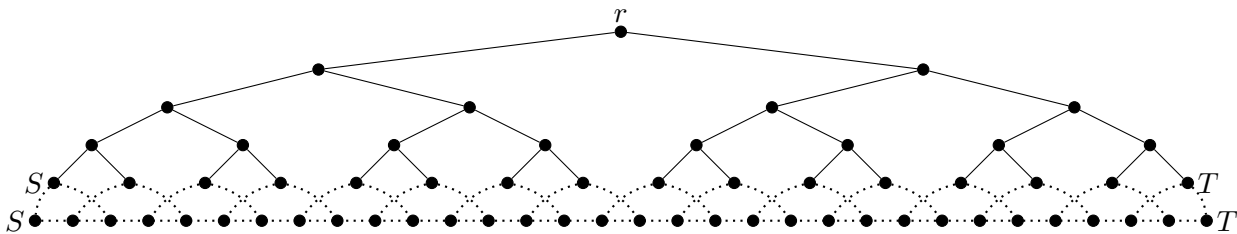


Figure 1: The dotted curves represent paths of length  $2\ell + 1$ .

$2\ell + 1$ , with interiors that are pairwise disjoint; let us call them “dotted paths”. If we delete the interiors of the “horizontal” dotted paths at the bottom of the figure, we obtain a subdivision of a uniform binary tree, rooted at  $r$ , of depth 6 in the figure<sup>2</sup>; and to make an  $(\ell, 2)$ -block, we need this tree to have depth at least  $2\ell + 3$ . The path formed by the horizontal edges in the figure is called its *base path*; it starts in  $S$  and ends in  $T$ , and has no other vertices in  $S \cup T$ . Let  $s_0 \in S$  and  $t_0 \in T$  be the vertices not in the base path. In the figure there are 32 vertices shown in the base path; they are the leaves of the binary tree. In general there are  $2^{2\ell+2}$  such vertices, if the binary tree has depth  $2\ell + 3$ . We call these the *anchors* of the base path. (The base path has many vertices that are not

<sup>2</sup>The “depth” of a uniform binary tree is the number of vertices in paths from root to leaf.

anchors, since it is a union of dotted paths). Let us number the anchors of the base path  $v_1 \cdots v_n$  in order, where  $v_1 \in S$  and  $v_n \in T$ .

An  $(\ell, m + 1)$ -block is defined to be a certain combination of  $(\ell, m)$ -blocks, explained below. Roughly, we replace each anchor of an  $(\ell, 2)$ -block by a set of  $m$  vertices, and replace each dotted path between anchors by an  $(\ell, m)$ -block, and then we identify all the roots.

Here then, more exactly, is the inductive construction. We assume that we have an  $(\ell, m)$ -block  $(H, r, S, T)$ , and we will assemble copies of it to make an  $(\ell, m + 1)$ -block. Take an  $(\ell, 2)$ -block  $(H_0, r, S_0, T_0)$  as in Figure 1, and number the anchors of its base path  $v_1, \dots, v_n$ , as described above. Let  $S_0 = \{v_1, s_0\}$  and  $T_0 = \{v_n, t_0\}$ . For  $1 \leq i \leq n$ , let  $V_i$  be a set of  $m$  new vertices; and for  $1 \leq i < n$ , let  $(H_i, r, V_i, V_{i+1})$  be a copy of  $(H, r, S, T)$ , where all vertices of  $H_i$  are new except those in  $V_i \cup V_{i+1} \cup \{r\}$ . (Thus, for  $2 \leq i \leq n - 1$ , the vertices in  $V_i$  belong to both  $V(H_i)$  and to  $V(H_{i-1})$ , and  $r$  belongs to all the  $H_i$ 's.) Let  $J$  be obtained from  $H_0$  by deleting all vertices of its base path and all internal vertices of all its dotted paths (so  $J$  is a binary tree of depth  $2\ell + 2$ ). Let  $G$  be the graph obtained from the union of  $J, H_1, \dots, H_{n-1}$  by adding a path of length  $2\ell + 1$  between  $u$  and each vertex of  $V_i$ , for each  $u \in V(J)$  and each  $v_i$  with  $1 \leq i \leq n$  such that  $u, v_i$  are joined by a dotted path of  $H_0$ . We call these last *spines* of  $G$ . Let  $S = V_1 \cup \{s_0\}$  and  $T = V_n \cup \{t_0\}$ . Then  $(G, r, S, T)$  is an  $(\ell, m + 1)$ -block. (We illustrate this when  $m = 2$  in Figure 2.)

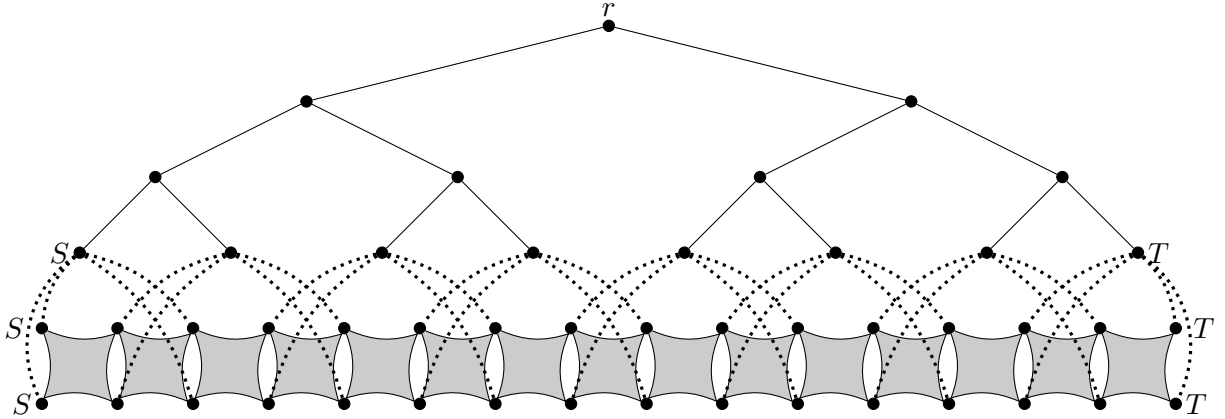


Figure 2: An  $(\ell, 3)$ -block (except that the binary tree should have depth  $2\ell + 3$ , and not just five as in the figure). The dotted curves represent paths of length  $2\ell + 1$ . Each gray area (together with  $r$ ) is an  $(\ell, 2)$ -block, and each contains neighbours of  $r$  (not shown in the figure.)

It is easy to check that if  $(G, r, S, T)$  is an  $(\ell, m)$ -block, then every two vertices in  $\{r\} \cup S \cup T$  are at distance at least  $2\ell + 1$ , and any two vertices in  $S$  are joined by a path of length  $2\ell + 1$  in which all internal vertices have degree two (and the same for  $T$ ). Now we will check the other properties mentioned in the bullets at the start of this section.

**2.1** *If  $(G, r, S, T)$  is an  $(\ell, m + 1)$ -block, and  $P, Q$  are paths between  $S, T$ , then either one of  $P, Q$  contains  $r$ , or  $\text{dist}_G(P, Q) \leq 2$ .*

**Proof.** We proceed by induction on  $m$ , and the result is true when  $m = 1$  by the result of [12], so we assume it is true for  $(\ell, m)$ -blocks. We use the notation given in the construction for an  $(\ell, m + 1)$ -block. Roughly,  $G$  was obtained from  $H_0$  by blowing up the vertices and edges of its base path; now

we want to shrink them back to  $H_0$ , and carry  $P, Q$  to some paths  $P', Q'$  of  $H_0$ . More exactly, let  $Z = V(J) \cup V_1 \cup \dots \cup V_n$ . The ends of  $P$  both belong to  $Z$ ; so  $P$  is the concatenation of a sequence of paths of  $G$ , each with distinct ends in  $Z$  and no internal vertex in  $Z$ , say  $P_1, \dots, P_g$ . Each  $P_i$  is either an edge of  $J$ , or a spine, or a path of some  $H_j$ . We may assume that no  $P_i$  contains  $r$ . For  $1 \leq i < n$ , let  $B_i$  be the dotted path of  $H_0$  with ends  $v_i, v_{i+1}$ . For  $1 \leq i \leq g$ , define a subgraph  $P'_i$  of  $H_0$  as follows.

- If  $P_i$  is an edge of  $J$  let  $P'_i = P_i$ .
- If  $P_i$  is a spine with ends  $u \in V(J)$  and some vertex in some  $V_j$ , let  $P'_i$  be the dotted path of  $H_0$  between  $u, v_j$ .
- If  $P_i$  is a path of some  $H_j$  with both ends in  $V_j$ , let  $P'_i$  be the one-vertex path with vertex  $v_j$ .
- If  $P_i$  is a path of some  $H_j$  with both ends in  $V_{j+1}$ , let  $P'_i$  be the one-vertex path with vertex  $v_{j+1}$ .
- If  $P_i$  is a path of some  $H_j$  with one end in  $V_j$  and the other in  $V_{j+1}$ , let  $P'_i = B_j$ .

It is not necessarily true that  $P'_1 \cup \dots \cup P'_g$  is a path of  $H_0$  from  $S_0$  to  $T_0$ , because it might pass through the same vertex more than once, and indeed  $P'_1, \dots, P'_g$  need not all be distinct. But concatenating  $P'_1, \dots, P'_g$  yields a walk from  $S_0$  to  $T_0$ , and therefore  $P'_1 \cup \dots \cup P'_g$  includes a path  $P'$  from  $S_0$  to  $T_0$ . Define  $Q'$  similarly, with  $P$  replaced by  $Q$ . We may assume that no internal vertex of  $P'$  or of  $Q'$  belongs to  $S_0 \cup T_0$ .

(1) *We may assume that there exists  $i \in \{1, \dots, n\}$  such that  $v_i \in V(P') \cap V(Q')$ .*

Since  $(H_0, r, S_0, T_0)$  is an  $(\ell, 2)$ -block, and neither of  $P', Q'$  contains  $r$ , it follows that  $\text{dist}_{H_0}(P', Q') \leq 2$ . Now there are two cases, depending on whether  $P', Q'$  are vertex-disjoint or not. The only vertices among  $V(J) \cup \{v_1, \dots, v_n\}$  that are at distance at most two in  $H_0$  are vertices of  $J$ , joined via paths in  $J$ , and such vertices are in  $P'$  (respectively,  $Q'$ ) if and only if they are in  $P$  (respectively,  $Q$ ). Thus, if  $P', Q'$  are vertex-disjoint, then since  $\text{dist}_{H_0}(P', Q') \leq 2$ , it follows that the corresponding path of  $H_0$  is a path of  $J$ ; and hence  $\text{dist}_G(P, Q) \leq 2$ , and the result holds. So we may assume that there is some vertex  $x \in V(P') \cap V(Q')$ . If  $x \in V(J)$ , then  $x \in V(P) \cap V(Q)$  and the result holds, so we may assume that  $x \notin V(J)$ . It remains to show that we may choose  $x$  to be an anchor of  $H_0$ . If not, then  $x$  belongs to the interior of some dotted path of  $H_0$ ; but then the whole of that path belongs to  $P' \cap Q'$ , and one end of the path is an anchor. This proves (1).

(2) *We may assume that there exists  $i \in \{1, \dots, n-1\}$  such that  $B_i \subseteq P' \cap Q'$ .*

By (1) we may assume that  $v_i \in V(P' \cap Q')$  for some  $i \in \{1, \dots, n\}$ . Suppose first that  $i = 1$ . Thus  $v_1$  is the first vertex of both  $P', Q'$ . Neither of  $P', Q'$  contains  $s_0$ , since no internal vertex of  $P'$  or of  $Q'$  belongs to  $S_0 \cup T_0$ ; so both  $P', Q'$  contain  $B_1$  and the claim holds. Similarly the claim holds if  $i = n$ , so we assume that  $2 \leq i \leq n-1$ . Now both  $P', Q'$  include two of the three dotted paths of  $H_0$  incident with  $v_i$ , and so include one in common. If that one is a spine, then its end in  $V(J)$  belongs to  $P' \cap Q'$  and we are done as before; and otherwise one of  $B_i, B_{i-1} \subseteq P' \cap Q'$ . This proves (2).

Consequently both  $P, Q$  have subpaths with one end in  $V_i$ , one end in  $V_{i+1}$ , and all internal vertices in  $H_i$ . Since  $(H_i, r, V_i, V_{i+1})$  is an  $(\ell, m)$ -block, and neither of  $P, Q$  contains  $r$ , it follows that  $\text{dist}_{H_i}(P, Q) \leq 2$ , and hence  $\text{dist}_G(P, Q) \leq 2$ . This proves 2.1.  $\blacksquare$

Next we need to show:

**2.2** *If  $(G, r, S, T)$  is an  $(\ell, m + 1)$ -block, and  $X \subseteq V(G)$  with  $|X| \leq m$ , then there is a path  $P$  of  $G$  between  $S, T$  with  $\text{dist}_G(P, X \cup \{r\}) > \ell$ .*

**Proof.** Again, we may assume the result holds for  $(\ell, m)$ -blocks, and use the notation of the construction. For each  $x \in X \cup \{r\}$ , let  $C_x$  be the set of vertices  $v$  of  $G$  with  $\text{dist}_G(v, x) \leq \ell$ , let  $C = \bigcup_{x \in X \cup \{r\}} C_x$ . We need to show that there is an  $S$ - $T$  path disjoint from  $C$ . In fact we will prove that there is such a path within the union of the  $H_i$ 's and the spines. In Figure 3 we show a section of the union of the  $H_i$ 's and spines. Let us label the vertices of  $J$  that are ends of spines  $u_1, u_3, u_5, \dots, u_{n-1}$  as in the figure. Thus,  $s_0 = u_1$ , and  $t_0 = u_{n-1}$ ; and for  $i$  odd with  $3 \leq i \leq n - 3$ , there is a spine between  $u_i$  and each vertex in  $V_{i-1} \cup V_{i+2}$ .

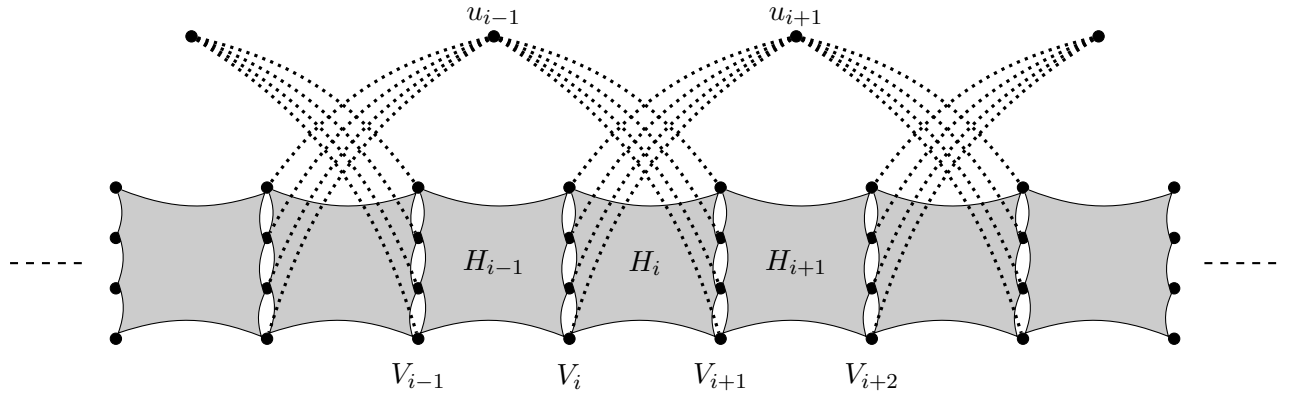


Figure 3: Part of the construction for an  $(\ell, m + 1)$ -block. The dotted curves represent paths of length  $2\ell + 1$ . Each gray area (together with  $r$ ) is an  $(\ell, m)$ -block, and really has  $m$  (not four as in the figure) vertices at either end. Each contains neighbours of  $r$  (not shown in the figure.)

We may assume that  $m \geq 2$ , by the result of [12], and so every vertex in  $S \cup T$  has degree at least three in  $G$ . We begin with:

(1) *We may assume that no internal vertex of  $R$  belongs to  $X$ , for every path  $R$  of  $G$  of length at least  $2\ell + 1$  in which all internal vertices have degree two in  $G$ .*

If  $x \in X$  is an internal vertex of such a path  $R$ , then by extending  $R$  as much as possible, we may assume that both ends of  $R$  have degree at least three. Let  $R$  have ends  $a, b$ . Since  $R$  has length at least  $2\ell + 1$ , not all of  $V(R)$  belongs to  $C_x$ , and we may assume that some vertex of  $R$  between  $x$  and  $b$  is not in  $C_x$ . Let  $X' = (X \setminus \{x\}) \cup \{a\}$ . Every path of  $G$  within distance  $\ell$  of  $x$ , and with no end in the interior of  $R$ , is within distance  $\ell$  of  $a$  (because  $C_x \setminus C_a$  only contains internal vertices of  $R$ ). So if the result is true for  $X'$  then it is true for  $X$ , so by repeating this at most  $|X|$  times, we

may assume there is no such  $x$ . This proves (1).

(2) We may assume that for some  $i \in \{1, \dots, n-1\}$ , every path of  $H_i$  between  $V_i, V_{i+1}$  intersects  $C$ .

Suppose not, and for  $1 \leq i \leq n-1$  let  $P_i$  be a path of  $H_i$  between  $V_i, V_{i+1}$  that is vertex-disjoint from  $C$ . For  $2 \leq i < n$ , the end of  $P_{i-1}$  in  $V_i$  and the end of  $P_i$  in  $V_i$  are joined by a path ( $Q_i$  say) of  $H_i$  of length  $2\ell + 1$  (or zero, if the two ends are equal) and all its internal vertices have degree two, so by (1), no vertex of  $Q_i$  belongs to  $X \cup \{r\}$ . Since its ends are not in  $C$ , no vertex of  $Q_i$  is in  $C$ , and it follows that

$$P_1 \cup Q_2 \cup P_2 \cup Q_3 \cup \dots \cup Q_{n-1} \cup P_{n-1}$$

is a path satisfying the theorem. This proves (2).

Let  $i$  be as in (2), choosing  $i$  to be odd if possible. For each  $x \in X$ ,  $C_x$  contains at most one vertex in  $V_i \cup V_{i+1} \cup \{r\}$ , since they pairwise are at distance at least  $2\ell + 1$ . Let  $X'$  be the set of all  $x \in X$  with  $C_x \cap V(H_i) \neq \emptyset$ ; and for each  $x \in X'$ , define  $f(x)$  as follows. If  $x \in V(H_i)$  let  $f(x) = x$ , and if  $x \notin V(H_i)$  let  $f(x)$  be the unique vertex of  $C_x$  in  $V_i \cup V_{i+1} \cup \{r\}$  (which exists, since  $C_x \cap V(H_i) \neq \emptyset$ ). Let  $f(r) = r$ . Then for each  $x \in X' \cup \{r\}$ ,  $C_x \cap V(H_i)$  is a subset of the set ( $C'_{f(x)}$ , say) of vertices of  $H_i$  that have distance at most  $\ell$  from  $f(x)$  in  $H_i$ . There is no path of  $H_i$  between  $V_i, V_{i+1}$  avoiding  $\bigcup_{x \in X' \cup \{r\}} C'_{f(x)}$ , so from the inductive hypothesis, it follows that  $|X'| = |X|$ , and the vertices  $f(x)$  ( $x \in X \cup \{r\}$ ) are all distinct, and therefore  $r \notin C_x$  and  $C_x \cap V(H_i) \neq \emptyset$  for each  $x \in X$ .

Hence  $X \subseteq V(H_{i-1}) \cup V(H_i) \cup V(H_{i+1})$  (defining  $H_0, H_n$  to be null), since no vertex of  $X$  is in a spine by (1). Moreover, since  $V_{i-1}$  is at distance  $2\ell + 1$  from  $V(H_i)$ , and  $C_x \cap V(H_i) \neq \emptyset$  for each  $x \in X$ , it follows that  $V_{i-1} \cap C = \emptyset$  (again, defining  $V_0, V_{n+1} = \emptyset$ ). If  $i = 1$ , then  $u_i = s_0 \in S$ , and the result holds since there is a path from  $u_i$  to  $T$  avoiding  $C$ , starting with one of the spines incident with  $u_i$ ; and similarly the result holds if  $i = n-1$ . If  $3 \leq i \leq n-3$  and  $i$  is odd, the vertex  $u_i$  exists and provides a way to avoid  $C$ , via two spines incident with  $u_i$ ; so we assume that  $i$  is even, and we cannot choose  $i$  to be odd. Thus  $H_{i-1}$  exists, and replacing  $i$  by  $i-1$  does not satisfy (2); and therefore there is a path of  $H_{i-1}$  between  $V_{i-1}, V_i$  that is disjoint from  $C$ . But then we can extend this path via two spines incident with  $u_{i+1}$  to provide a route that avoids  $C$ . This proves 2.2.  $\blacksquare$

### 3 What remains?

In view of this counterexample, is there anything even weaker that might be true and provide some sort of coarse extension of Menger's theorem? There are several possibilities to consider:

#### Bounded degree

Imposing an upper bound on the maximum degree looks promising at first, because Gartland, Korhonen and Lokshantov [6] and Hendrey, Norin, Steiner, and Turcotte [8] proved it with  $c = 2$ , that is, when we look for the union of the paths to be induced: they proved

**3.1** *For every integer  $\Delta \geq 1$  there exists  $C > 0$  with the following property. Let  $G$  be a graph with maximum degree at most  $\Delta$ , let  $k \geq 1$  be an integer, and let  $S, T \subseteq V(G)$ ; then either*

- there are  $k$  paths between  $S, T$ , pairwise at distance at least two; or
- there is a set  $X \subseteq V(G)$  with  $|X| < kC$  such that every path between  $S, T$  contains a vertex of  $X$ .

But when  $c \geq 3$ , we can nearly make a counterexample. In the definition of an  $(\ell, m + 1)$ -block, there are currently vertices of large degree: the vertices of  $J$  that are the ends of spines, and the root, and the vertices in the sets  $V_i$ . It is easy to modify the construction to keep the vertices in  $V_i$  of degree at most three: attach a leaf to each vertex in  $S \cup T$ , and call these new vertices  $S$  and  $T$  instead. (When we chain them together to make an  $(\ell, m + 1)$ -block, they become of degree three instead of degree one, but that is fine.) For a vertex  $u \in V(J)$  that is an end of spines, it is (in general) currently incident with  $2m$  spines, and one edge of  $J$ . But we can partition these spines into two groups in the natural way, and replace each group with a binary tree with root  $u$  and leaves the corresponding set  $V_i$ ; so that problem goes away as well. Thus, the only problem is the root vertex. In summary, we can make a counterexample to 1.2 in which only one vertex has degree more than three, but we don't see how to fix this last vertex.

## Bounded path-width

Our counterexamples contain arbitrarily large binary trees, and so have unbounded path-width. We will prove in a later paper [5] that not only 1.2 but also 1.1 is true in graphs of bounded path-width:

**3.2** *For all  $c, k, w \geq 1$ , there exists  $\ell$  such that if  $G$  has path-width at most  $w$ , and  $S, T \subseteq V(G)$ , then either:*

- there are  $k$  paths between  $S, T$ , pairwise at distance at least  $c$ ; or
- there is a set  $X \subseteq V(G)$  with  $|X| \leq k - 1$  such that every path between  $S, T$  contains a vertex at distance at most  $\ell$  from some member of  $X$ .

Our counterexample to 1.2 contains subdivisions of arbitrarily large complete graphs, and indeed contains them with arbitrarily large “fatness” (see [7] for definition). Is 1.2 true for every proper minor-closed class? (Since this paper was written, this has been proved by C.-H. Liu [10].) It is easy to show that 1.2 is true for graphs of bounded tree-width. Here is a proof of this “folklore” result, which has not previously been published as far as we know, but seems to be widely known. (And compare the proof of Theorem 2.6 in [4], which is essentially the same proof for a different result):

**3.3** *For all  $d, k, c \geq 1$ , if  $G$  has tree-width at most  $d$  and  $S, T \subseteq V(G)$ , then either there are  $k$   $S$ - $T$  paths in  $G$ , pairwise at distance at least  $c$ , or there is a set  $X \subseteq V(G)$  with  $|X| \leq (k - 1)(d + 1)$  such that every  $S$ - $T$  path in  $G$  is at distance at most  $c/2$  from  $X$ .*

**Proof.** (Sketch.) Let  $(W_r : r \in V(R))$  be a tree-decomposition of  $G$  of width at most  $d$ ; thus,

- $R$  is a tree, each  $W_r \subseteq V(G)$ , and  $G$  is the union of the subgraphs  $G[W_r]$ ;
- $W_r \cap W_{r''} \subseteq W_{r'}$  for all  $r, r', r'' \in V(R)$  such that  $r'$  is on the path of  $R$  between  $r, r''$ ; and
- $|W_r| \leq d + 1$  for each  $r \in V(R)$ .

For each  $S$ - $T$  path  $P$  of  $G$ , let  $A_P$  be the set of  $r \in V(R)$  such that some vertex of  $W_r$  has distance at most  $c/2$  from  $V(P)$ . Thus  $A_P$  is the vertex set of a subtree of  $R$ . From the Gallai packing/covering theorem for vertex sets of subtrees of a tree, either there are  $k$   $S$ - $T$  paths  $P_1, \dots, P_k$  such that the sets  $A_{P_1}, \dots, A_{P_k}$  are pairwise disjoint, or there is a set  $Q \subseteq V(R)$  with  $|Q| \leq k - 1$  such that  $Q \cap A_P \neq \emptyset$  for each  $S$ - $T$  path  $P$ . In the first case, the paths  $P_1, \dots, P_k$  are at distance at least  $c$ . In the second case, every  $S$ - $T$  path in  $G$  is at distance at most  $c/2$  from  $\bigcup_{q \in Q} W_q$ . This proves 3.3.  $\blacksquare$

## Planar graphs

The hope that 1.2 might be true for every proper minor-closed class suggests that we study whether 1.2 is true for planar graphs, but in that case, it seems possible that 1.1 itself is true. Indeed, 1.1 might be true for graphs of bounded genus: our counterexample to 1.1 in Figure 1 has unbounded genus, because it contains arbitrarily many vertex-disjoint nonplanar subgraphs.

We have begun to work on trying to prove 1.1 for planar graphs, and have been able to prove it in the case when the graph is planar and all the vertices in  $S \cup T$  are incident with the infinite region [13]. (Despite its appearance, that was difficult!) We think we can also prove it in the cylinder case, when the graph is planar and some two regions include  $S \cup T$ : and perhaps there is an approach to the general coarse Menger conjecture for planar graphs that is like that in [14, 15]. But this will need much more work.

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