Asymptotic structure. III. Excluding a fat tree
WORKING DRAFT

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Abstract

Robertson and Seymour proved that for every finite tree H, there exists k such that every finite graph G with no H minor has path-width at most k; and conversely, for every integer k, there is a finite tree H such that every finite graph G with an H minor has path-width more than k. If we (twice) replace "path-width" by "line-width", the same is true for infinite graphs G.

We prove an analogue in coarse graph theory, as follows. For every finite tree H, there exists k such that for every $c \geq 0$, there exist L, C such that every graph that does not contain H as a c-fat minor admits an (L, C)-quasi-isometry to a graph with line-width at most k; and conversely, for all k there is a finite tree H such that for all L, C there exists c such that no graph that contains H as a c-fat minor admits an (L, C)-quasi-isometry to a graph with line-width at most k.

1 Introduction

Graphs in this paper may be infinite, and have no loops or parallel edges. If G is a graph and $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced on X. If X is a vertex of G, or a subset of the vertex set of G, or a subgraph of G, and the same for Y, then $\operatorname{dist}_G(X, Y)$ denotes the distance in G between X, Y, that is, the number of edges in the shortest path of G with one end in X and the other in Y. (If no path exists we set $\operatorname{dist}_G(X, Y) = \infty$.)

Let G, H be graphs, and let $\phi : V(G) \to V(H)$ be a map. Let $L, C \geq 0$; we say that ϕ is an (L, C)-*quasi-isometry* if:

- for all u, v in $V(G)$, if $\operatorname{dist}_G(u, v)$ is finite then $\operatorname{dist}_H(\phi(u), \phi(v)) \leq L \operatorname{dist}_G(u, v) + C$;
- for all u, v in $V(G)$, if $\operatorname{dist}_H(\phi(u), \phi(v))$ is finite then $\operatorname{dist}_G(u, v) \leq L \operatorname{dist}_H(\phi(u), \phi(v)) + C$; and
- for every $y \in V(H)$ there exists $v \in V(G)$ such that $\operatorname{dist}_H(\phi(v), y) \leq C$.

If G is a graph, we write $U(G)$ for $V(G) \cup E(G)$. Let G, H be graphs, and let $c \geq 0$ be an integer. For each $x \in U(H)$, let $\eta(x)$ be a non-null connected subgraph of G, all pairwise vertex-disjoint, such that

- for each $uv \in E(H)$, there is an edge of G between $\eta(u)$ and $\eta(uv)$ $(= \eta(vu))$, and an edge between $\eta(v)$ and $\eta(uv)$;
- $\operatorname{dist}_G(\eta(x), \eta(y)) > c$ for all distinct $x, y \in U(H)$, except when one of x, y is in $V(H)$, the other is in $E(H)$, and the edge is incident in H with the vertex. (See Figure 1.)

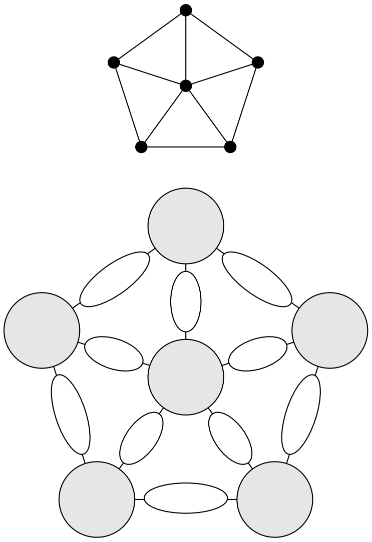

Figure 1: A fat minor.

In these circumstances, we say that G contains H as a *c-fat minor*, and η *exhibits* H as a c-fat minor of G. Sometimes, we are given a subset $X \subseteq V(G)$, and the function η satisfies that $\eta(x) \subseteq G[X]$

for all $x \in U(H)$. In this case, we say that X *includes a c-fat H-minor of G*. Note that this is not the same thing as saying that $G[X]$ contains H as a c-fat minor, because being c-fat depends on a distance function, and we are using the distance function of G rather than that of $G[X]$. If η exhibits H as a c-fat minor of G, $\eta(H)$ denotes the subgraph $\bigcup_{x\in U(H)} \eta(x)$.

It is easy to see that:

1.1 *Let $L, C, c \geq 0$, and suppose that G contains H as a c-fat minor, and there is an (L, C)-quasi-isometry from G to some graph G'. If $c \geq L(L + C) + C$ then G' contains H as a minor.*

A. Georgakopoulos and P. Papasoglu [8] conjectured that a converse also holds:

1.2 False conjecture: *For every graph H and all $c \geq 0$, there exists $L, C \geq 0$, such that if a graph G does not contain H as a c-fat minor, then G admits an (L, C)-quasi-isometry to a graph with no H minor.*

This is known to be true when:

- $H = K_3$, by Manning's theorem [13] (and see [8]);
- $H = K_{1,m}$, by Georgakopoulos and Papasoglu [8];
- $H = K_{2,3}$, by Chepoi, Dragan, Newman, Rabinovich, and Vaxès [3];
- $H = K_4^-$ (that is, K_4 with one edge deleted) by Fujiwara and Papasoglu [7];
- $H = K_{2,t}$, by Albrechtsen, Distel and Georgakopoulos [1].

The conjecture has recently been shown to be false in general, by Davies, Hickingbotham, Illingworth and McCarty [6]; and even more recently, Albrechtsen, Distel and Georgakopoulos [2] showed that it is false when H is the graph of the octahedron. There remains hope that 1.2 is true for some nontrivial classes of graphs H – for instance, it may be true for all trees H as far as we know.

Our result is not exactly of this type, but has the same flavour. A *path-decomposition* of a graph G is a pair $(T, (B_t : t \in V(T)))$, where T is a path (possibly infinite) and each B_t is a subset of $V(G)$ (called a *bag*), such that:

- $\bigcup_{t\in V(T)} B_t = V(G)$;
- for every edge $e = uv$ of G, there exists $t \in V(T)$ with $u, v \in B_t$; and
- for all $t_1, t_2, t_3 \in V(T)$, if t_2 lies between t_1, t_3 in T, then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$.

The *width* of a path-decomposition $(T, (B_t : t \in V(T)))$ is the maximum of the numbers $|B_t| - 1$ for $t \in V(T)$, or ∞ if there is no finite maximum; and the *path-width* of G is the minimum width of a path-decomposition of G.

Robertson and Seymour [16] proved:

1.3 *For every finite tree H, there exists k such that every finite graph G with no H minor has path-width at most k; and conversely, for every integer k, there is a finite tree H such that every finite graph with an H minor has path-width more than k.*

It is important here that G is finite: for instance, if G is the disjoint union of countably many infinite paths, then G does not contain the claw $K_{1,3}$ as a minor, and yet G does not have finite path-width. This can be fixed. Say a *line-decomposition* of G is a family $(B_t : t \in T)$ of subsets of $V(G)$, where T is a linearly-ordered set, satisfying the same three bullets above (with $V(T)$ replaced by T everywhere). The *width* of a line-decomposition $(B_t : t \in T)$ is the maximum of the numbers $|B_t| - 1$ for $t \in T$, or ∞ if there is no finite maximum; and the *line-width* of G is the minimum width of a line-decomposition of G. We proved in [5] that G has line-width at most k if and only if every finite subgraph has path-width at most k. Consequently, if we replace "path-width" by "line-width" in 1.3, then the statement of 1.3 is true even for infinite graphs G.

Our aim in this paper is to give a coarse graph theory analogue of this "line-width" version of 1.3. Our main result says:

1.4 *For every finite tree H there exists $k \geq 0$ such that for every $c \geq 0$, there exist $L, C \geq 0$ such that every graph that does not contain H as a c-fat minor admits an (L, C)-quasi-isometry to a graph with line-width at most k; and conversely, for all k there is a finite tree H such that for all L, C there exists c such that no graph that contains H as a c-fat minor admits an (L, C)-quasi-isometry to a graph with line-width at most k.*

The second half is easy: choose $c > L(L + C) + C$, and let H be a finite tree with line-width more than k. If G contains H as a c-fat minor, and there is an (L, C)-quasi-isometry from G to G', then by 1.1, G' contains H as a minor and hence has line-width more than k. The first half is much more difficult and will occupy the whole paper, except for the final section which concerns graph searching.

As a first step, let us eliminate the quasi-isometry. If $X \subseteq V(G)$, we say X has *quasi-size* at most (k, r) if there is a set $Y \subseteq V(G)$ with $|Y| \leq k$, such that $\text{dist}_G(x, Y) \leq r$ for each $x \in X$. If $X \subseteq V(G)$, let us say a line-decomposition $(B_t : t \in T)$ of $G[X]$ has *quasi-width* at most (k, r) in G if B_t has quasi-size at most (k, r) for each $t \in T$. Let us say X has *quasi-line-width* at most (k, r) if $G[X]$ admits a line-decomposition with quasi-width at most (k, r).

It is important that the distance function used in the definition of quasi-size and quasi-line-width is that defined by G, not that defined by $G[X]$. This will be the case throughout the paper. Even when speaking of subgraphs of G, we will never use their distance functions: the distance function in use will always be that of G. We sometimes write X for $G[X]$ when $X \subseteq V(G)$. This should cause no confusion since there is always only one graph G under consideration.

It was proved in [14], strengthening a result of [9], that:

1.5 *For all k, r, there exist $L, C \geq 1$ such that if G has quasi-line-width at most (k, r), then G admits an (L, C)-quasi-isometry to a graph with line-width at most k.*

Thus, to complete the proof of 1.4, it suffices to prove the following:

1.6 *For every finite tree H there exists $k \geq 0$ such that for every $c \geq 0$, there exists $r = O(c)$ such that every graph that does not contain H as a c-fat minor has quasi-line-width at most (k, r).*

H_0 denotes the tree with one vertex. For $\ell \geq 1$ an integer, let H_ℓ be the finite tree such that every vertex has degree one or three, and for some vertex r (called the *root*) every path from r to a vertex of degree one has length exactly ℓ. Every tree H is a minor of H_ℓ for some choice of ℓ, and then, if G does not contain H as a c-fat minor then it does not contain H_ℓ as a c-fat minor. Consequently it suffices to prove 1.6 when $H = H_\ell$, that is:

1.7 *For every $\ell \geq 1$, there exists $k \geq 0$ such that for every $c \geq 0$, there exists $r = O(c)$ such that every graph that does not contain H_ℓ as a c-fat minor has quasi-line-width at most (k, r).*

2 Buildings and tie-breakers

If $X \subseteq V(G)$, $\mathrm{bd}(X)$ denotes the set of vertices in X that have a neighbour in $V(G) \setminus X$, and is called the *boundary* of X. A key idea of the proof is that we work with subsets $X \subseteq V(G)$ such that, simultaneously, $G[X]$ has bounded quasi-line-width and $\mathrm{bd}(X)$ has bounded quasi-size, and it turns out that we can make the two bounds the same with little loss. Let us say $X \subseteq V(G)$ has *quasi-bound* at most (a, b) if $G[X]$ has quasi-line-width at most (a, b) and $\mathrm{bd}(X)$ has quasi-size at most (a, b).

A *tie-breaker* in G is a well-order Λ of the set of all edges of G (this exists, by the well-ordering theorem). If P, Q are distinct finite paths of G, we say P is Λ-*shorter* than Q if either

- $|E(P)| < |E(Q)|$; or
- $|E(P)| = |E(Q)|$, and the first element (under Λ) of $(E(P) \setminus E(Q)) \cup (E(Q) \setminus E(P))$ belongs to P.

This defines a total order on the set of all finite paths of G. A Λ-*geodesic* means a finite path P such that no other path joining its ends is Λ-shorter than P. Every Λ-geodesic of G is a geodesic of G, but the converse is false. (The point of the tie-breaker is that there is only one Λ-geodesic between any two vertices, while this is not true for geodesics.) It is easy to check that if P is a Λ-geodesic then so are all subpaths of P. Given a graph G, we will keep some tie-breaker Λ fixed, and usually suppress the dependence of other objects on the choice of Λ.

A *building* in a graph G is a nonempty subset $X \subseteq V(G)$ such that $G[X]$ is connected. If $\mathcal{T}$ is a set of pairwise vertex-disjoint buildings in a graph G, we define $V(\mathcal{T}) = \bigcup_{X \in \mathcal{T}} X$. Again, let $\mathcal{T}$ be a set of pairwise vertex-disjoint buildings, in a graph G, now with a tie-breaker Λ. We say that $X \in \mathcal{T}$ is Λ-*closest* to v if there is a path of G between v, X that is Λ-shorter than any other path between v and Y for $Y \in \mathcal{T}$. For each $X \in \mathcal{T}$, let $\Delta_{\mathcal{T}}(X)$ (or just $\Delta(X)$, when $\mathcal{T}$ is clear) be the set of all $v \in V(G)$ such that X is Λ-closest to v. We call $\Delta(X)$ the *Voronoi cell* of X, and the collection of subsets $\Delta(X)$ $(X \in \mathcal{T})$ is called the *Voronoi partition* defined by $\mathcal{T}$. (It is indeed a partition, provided that each component of G includes a member of $\mathcal{T}$; and this is true for us since we will always assume that G is connected and $\mathcal{T}$ is non-null.) The main purpose of the tie-breaker is to make the Voronoi partition defined by $\mathcal{T}$ well-defined. We see that:

- $X \subseteq \Delta(X)$ for each $X \in \mathcal{T}$;
- the sets $\Delta(X)$ $(X \in \mathcal{T})$ are pairwise disjoint and have union $V(G)$;
- for each $X \in \mathcal{T}$ and each $v \in \Delta(X)$, $\mathrm{dist}_G(v, X) \leq \mathrm{dist}_G(v, Y)$ for each $Y \in \mathcal{T}$, and there is a path of $G[\Delta(X)]$ between v and X of length $\mathrm{dist}_G(v, X)$.

If C is a set of buildings, we define $\Delta_{\mathcal{T}}(C) = \bigcup_{X \in C} \Delta_{\mathcal{T}}(X)$. If $\mathcal{T}$ is a set of pairwise vertex-disjoint buildings, and $X, Y \in \mathcal{T}$ are distinct, we say that X *adjoins* Y (in $\mathcal{T}$) or $\mathcal{T}$-*adjoins* Y if there is an edge between $\Delta_{\mathcal{T}}(X)$ and $\Delta_{\mathcal{T}}(Y)$. A set $C \subseteq \mathcal{T}$ is *adjoin-connected* (or $\mathcal{T}$-*adjoin-connected* in case of ambiguity) if for every partition of C into two nonempty sets A, B, some member of A adjoins some member of B.

Here is an easy lemma.

2.1 *Let $X \subseteq V(G)$ have quasi-line-width at most (a, b), and let $\mathrm{bd}(X)$ have quasi-size at most (a', b'). Let $Y \subseteq V(G)$ with $X \subseteq Y$, such that every vertex in $Y \setminus X$ has distance at most r from $\mathrm{bd}(X)$. Then $\mathrm{bd}(Y)$ has quasi-size at most $(a', b' + r)$, and Y has quasi-line-width at most $(a + a', \max(b, b' + r))$.*

Proof. Let $Z = Y \setminus X$. Every vertex in Z has distance at most r from $\mathrm{bd}(X)$, and the latter has quasi-size at most (a', b'); so $Z \cup \mathrm{bd}(X)$ has quasi-size at most $(a', b' + r)$. Since $\mathrm{bd}(Y) \subseteq Z \cup \mathrm{bd}(X)$ (since $X \subseteq Y$), it follows that $\mathrm{bd}(Y)$ has quasi-size at most $(a', b' + r)$. By adding Z to each bag of a line-decomposition of $G[X]$, we deduce that Y has quasi-line-width at most $(a + a', \max(b, b' + r))$. This proves 2.1. ▮

3 Societies

For inductive purposes, it is sometimes helpful to strengthen "fat" to "superfat". With notation as before, let $\ell \geq 1$, let r be the root of H_ℓ, let B_1, B_2, B_3 be the three components of $H_\ell \setminus \{r\}$, and for $i = 1, 2, 3$, let e_i be the edge of H_ℓ between r and $V(B_i)$. We say that η exhibits H_ℓ as a *c-superfat* minor of a graph G if η exhibits H_ℓ as a c-fat minor of G, and $\mathrm{dist}_G(x, y) > 3c$ for all i, j with $1 \leq i < j \leq 3$, and all $x \in V(\eta(B_i)) \cup V(\eta(e_i))$, and all $y \in V(\eta(B_j)) \cup V(\eta(e_j))$. When $\ell = 0$, we say that η exhibits H_0 as a c-superfat minor of a graph G if it exhibits H_0 as a c-fat minor of G.

We need some more definitions. Let $c, \ell, d_0 \geq 2$ be integers, fixed throughout the paper. We will be concerned with graphs that do not contain H_ℓ as a c-fat minor. (d_0 is a large number, much larger than c and ℓ, that we will specify later; in fact we will define $d_0 = 5 \cdot 4^{5\ell^2} c$, but its exact value will not matter until the end of the paper.) A *century* is an integer k with $0 \leq k \leq \ell$. Let k be a century, and let $\tau = (d, \alpha, \beta)$ be a triple of three positive integers; we call τ a *canon*. Let Λ be a tie-breaker in a connected graph G, and use it to define Voronoi cells as usual. A *kth-century τ-society* in G is a set $\mathcal{T}$ of pairwise vertex-disjoint buildings in G, where each member of $\mathcal{T}$ is assigned to be a "house" or "fort" of $\mathcal{T}$ (and not a house if $k = 0$), satisfying the following (where $\mathrm{rk}(X) = k - 1$ if X is a house and $\mathrm{rk}(X) = k$ if X is a fort):

- For all $v \in V(G)$, there exists $X \in \mathcal{T}$ such that $\mathrm{dist}_G(v, X) \leq d_0$;
- Every two distinct members X, Y of $\mathcal{T}$ have distance more than d.
- Every fort of $\mathcal{T}$ includes a c-superfat H_k-minor of G.
- Each fort of $\mathcal{T}$ has quasi-bound at most (α, β).
- $\Delta_{\mathcal{T}}(C)$ has quasi-bound at most (α, β), for every $\mathcal{T}$-village C. (A *$\mathcal{T}$-community* is an adjoin-connected set of houses, and a *$\mathcal{T}$-village* is a maximal $\mathcal{T}$-community.)

We will not need to work with arbitrary canons; we will mostly just be interested in canons $\tau = (d, \alpha, \beta)$ where d is large compared with c. Let us call them "suitable" for now. The strategy for the proof of the main theorem 1.7 is to prove that:

(*) *For every century k, and every suitable canon $\tau = (d, \alpha, \beta)$, if G has no c-fat H_ℓ minor, and admits a kth-century τ-society, then G has bounded quasi-line-width (where the bound depends on k, τ).*

If we could prove (*) when $k = 0$, then 1.7 follows, since every graph admits a 0th-century $(d, 1, 1)$-society (just pick a maximal set of vertices pairwise at distance more than d, and call them all forts). On the other hand, (*) is easy to prove when $k = \ell$. So we will work by induction on $\ell - k$: assume that (*) is true for $k + 1$, and try to prove that it is also true for k.

For the inductive step, we can assume that G admits a kth-century τ-society (for some $k < \ell$ and some suitable canon τ). We need to bound the quasi-line-width of G, and we do so by showing that G also contains some $(k + 1)$st-century τ'-society (for some suitable τ' depending on τ) and apply the inductive hypothesis. Getting from a kth-century society to a $(k + 1)$st-century one is thus the main part of the paper, and will be accomplished by first converting our society to a "realm", and optimizing the realm, and then extracting from that (via "governments" for the realm) the $(k+1)$st-century society we want. But in preparation for converting the given society to a realm, we first need to "civilize" it.

With τ as before, let $\mathcal{T}$ be a kth-century τ-society in a graph G. We say $\mathcal{T}$ is *civilized* if for every $\mathcal{T}$-village C, $V(C)$ has quasi-line-width at most (α, β) and $\bigcup_{X \in C} \mathrm{bd}(X)$ has quasi-size at most (α, β) (and hence the same holds for every $\mathcal{T}$-community C, since both these properties are inherited under taking subsets). This is a great strengthening of the final "society" axiom, and we will carry it out in the next two sections.

4 The grouping lemma

This section proves a key result that we call the "grouping lemma". Suppose we have some disjoint connected subgraphs of a graph G, and we would like to join some of them together with short paths, such that at the end, different components are not very close together, and each vertex in the short paths we used for joining is close to one of the original subgraphs. This is not always possible, for instance, if G is a uniform binary tree of large depth, and the subgraphs we begin with are the leaves of the tree G. But the grouping lemma says that it is possible, provided that G does not contain some tree as a fat minor. The argument is similar to that in section 4 of [15].

For each $i \geq 1$, let B_i be the rooted binary tree of depth i. Thus, B_i is the finite rooted tree such that every vertex has degree one or three except for the root r which has degree two, and every path from r to a vertex of degree one has length exactly i. (This differs from the tree H_i previously defined in that the root now has degree two, and in H_i the root has degree three.) Let B_0 be the tree with one vertex.

A $(\geq d)$-*subdivision* of a graph H is a graph H' obtained by subdividing each edge at least d times. Thus, $V(H) \subseteq V(H')$. Each edge e of H is replaced by a path joining the ends of e, and we call this path the *subdivided* e. For each subgraph X of H, the *subdivided* X means the subgraph of H' consisting of the vertices in $V(X)$ together with the union of the subdivided e over all edges e of X.

Suppose that G is a graph, with a subgraph J that is a $(\geq d)$-subdivision of B_ℓ, where $\ell \geq 1$. Suppose, moreover, that for each vertex $v \in B_\ell \subseteq V(J)$ with degree at least two in B_ℓ, if X, Y are the two components of $B_\ell \setminus v$ that do not contain the root of B_ℓ, and e, f are the edges of B_ℓ between v and X, Y respectively, then the union of the subdivided e and the subdivided f is a geodesic in G between the subdivided X and the subdivided Y. In this case we call J a *geodesic* $(\geq d)$*-subdivision* of B_ℓ.

In order to prove the grouping lemma, we first need the following:

4.1 *Let* $\ell \geq 1$ *and* $d \geq 3c \geq 6$ *be integers. Suppose that* G *is a graph, with a subgraph* J *that is a geodesic* $(\geq d)$*-subdivision of* B_ℓ*. Then* G *contains* B_ℓ *as a* c*-fat minor.*

Proof. From the definition of subdivision, $V(B_\ell) \subseteq V(G)$. For each edge or subgraph X of B_ℓ, let $\phi(X)$ be the subdivided X. Let r be the root of B_ℓ. For each $v \in V(B_\ell)$ of degree at least two, let X_v, Y_v be the two components of $B_\ell \setminus v$ that do not contain r, and let e_v, f_v be the edges of B_ℓ between v and X_v, Y_v respectively. For each $v \in V(B_\ell)$ of degree at least two, let $\eta(v)$ be the subpath of the geodesic $\phi(e_v) \cup \phi(f_v)$ with vertex set all vertices of $\phi(e_v) \cup \phi(f_v)$ with distance at most $2c$ from v. For each $v \in V(B_\ell)$ with degree one, let $\eta(v)$ be the one-vertex subgraph with vertex v. For each edge e of B_ℓ, with ends u, v say, where v is closer than u to r, let $\eta(e)$ be the subpath of $\phi(e)$ with vertex set all vertices not in $\eta(u) \cup \eta(v)$. We claim that η exhibits B_ℓ as a c-fat minor of G. To show this, it suffices to check that if x, y are both vertices or edges of B_ℓ, and it is not the case that one of x, y is in $V(B_\ell)$, the other is in $E(B_\ell)$, and the edge is incident in B_ℓ with the vertex, then $\operatorname{dist}_G(\eta(x), \eta(y)) > c$. Let P be the minimal path of B_ℓ that contains both x, y (thus, $V(P) \subseteq V(B_\ell) \subseteq V(G)$), and let v be the vertex of P that is closest to r in B_ℓ. (See Figure 2.)

Suppose first that $v = x$. Then we may assume that y belongs to Y_v, and hence $\eta(y)$ is a subgraph of $\phi(Y_v)$. Since $\phi(f_v)$ has length more than d, and is a geodesic between v and $\phi(Y_v)$, it follows that $\operatorname{dist}_G(v, \eta(y)) \geq d + 1$, and therefore $\operatorname{dist}_G(\eta(x), \eta(y)) \geq d + 1 - 2c > c$ as required.

Thus we may assume that $x, y \neq v$. Suppose next that $x = e_v$ and y belongs to X_v. Let the ends of $x = e_v$ in B_ℓ be u, v. Then $y \neq u$, and so we may assume that either $y = f_u$ or y belongs to Y_u. In either case, $\operatorname{dist}_G(u, \eta(y)) \geq 2c + 1$, since there is a geodesic between $u, \eta(y)$ that properly includes $\eta(u) \cap \phi(f_u)$, and the latter has length $2c$. Since $\phi(e_v)$ is a geodesic between $v, \phi(X_v)$, and includes $\eta(x)$, it follows that for every vertex $z \in \eta(x)$, $\operatorname{dist}_G(z, \eta(y)) \geq \operatorname{dist}_G(z, u)$. Therefore

$$2c + 1 \leq \operatorname{dist}_G(u, \eta(y)) \leq \operatorname{dist}_G(u, z) + \operatorname{dist}_G(z, \eta(y)) \leq 2 \operatorname{dist}_G(z, \eta(y)),$$

and consequently $\operatorname{dist}_G(z, \eta(y)) > c$ as required.

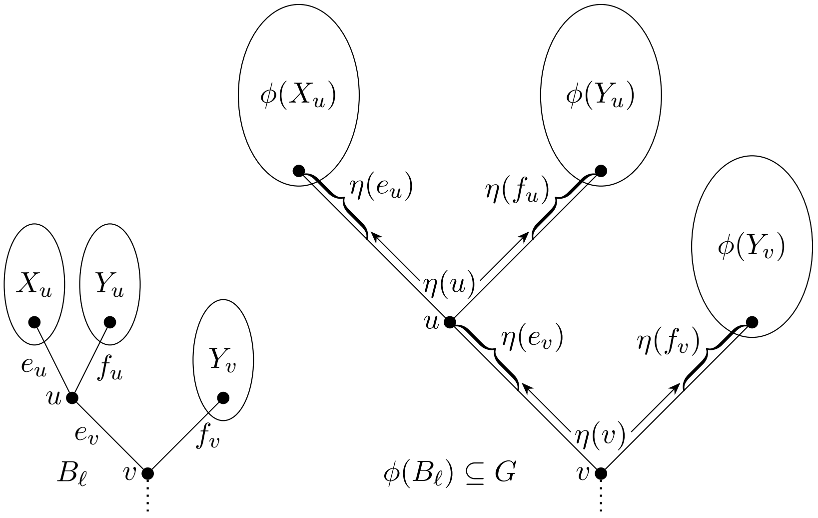

Figure 2: For the proof of 4.1.

Thus, we may assume that either x belongs to X_v or $x = e_v$, and either y belongs to Y_v or $y = f_v$. If $x = e_v$ and $y = f_v$, then since $\phi(e_v) \cup \phi(f_v)$ is a geodesic, and

$$\operatorname{dist}_G(\eta(e_v), v) = \operatorname{dist}_G(\eta(f_v), v) = 2c + 1,$$

it follows that $\operatorname{dist}_G(\eta(x), \eta(y)) \geq 4c + 2 > c$ as required. Thus, we may assume that x belongs to X_v, and hence $\eta(x)$ is a subgraph of $\phi(X_v)$. If $y = f_v$, then since $\phi(e_v) \cup \phi(f_v)$ is a geodesic between $\phi(X_v), \phi(Y_v)$, there is a geodesic between $\eta(y), \phi(X_v)$ that contains $\phi(e_v)$, and therefore has length at least $d+1$; and so $\operatorname{dist}_G(\eta(x), \eta(y)) \geq d + 1 > c$ as required. Finally, if y belongs to Y_v, then $\eta(y)$ is a subgraph of $\phi(Y_v)$, and so

$$\operatorname{dist}_G(\eta(x), \eta(y)) \geq \operatorname{dist}_G(\phi(X_v), \phi(Y_v)) \geq 2(d + 1) > c,$$

as required. Consequently, η exhibits B_ℓ as a c-fat minor. This proves 4.1. ▮

4.2 *Let $\ell \geq 1$ and $d \geq 3c \geq 6$, and suppose that G does not contain H_ℓ as a c-fat minor. Let $\mathcal{A}$ be a set of vertex-disjoint connected subgraphs of G. Then there is a set $\mathcal{P}$ of paths of G, with union $U\mathcal{P}$ say, such that:*

- *each member of $\mathcal{P}$ has length at most $4^{\ell+1}d$;*
- *for each $P \in \mathcal{P}$ and each $v \in V(P)$, there exists $A \in \mathcal{A}$ such that there is a path in $U\mathcal{P}$ of length $< 4^{\ell+1}d$ between v and A;*
- *let J be the subgraph consisting of the union of all members of $\mathcal{A}$ and all members of $\mathcal{P}$; then $\operatorname{dist}_G(u, v) > d$ for all vertices u, v in distinct components of J.*

Proof. Let J_0 be the union of all the subgraphs in $\mathcal{A}$. Inductively, we will define $\mathcal{P}_i$ and J_i for $1 \leq i \leq \ell + 2$ as follows. Suppose that $i \geq 1$ and J_{i-1} has been defined. Let $\mathcal{P}_i$ be the set of all paths P with the following properties:

- there exist distinct components X, Y of J_{i-1}, such that P is a shortest path of G between X, Y; and
- P has length at most $4^{\ell+2-i}d$.

Let J_i be the union of J_{i-1} and all the members of $\mathcal{P}_i$. This completes the inductive definition of J_i for $i \geq 1$.

(1) *For each $i \geq 1$, let $P \in \mathcal{P}_i$, joining components X, Y of J_{i-1}, and let the ends of P be $x \in V(X)$ and $y \in V(Y)$. Then there is a geodesic ($\geq d$)-subdivision of B_{i-1} in G that is a subgraph of X, with root x, and the same for Y, y.*

We proceed by induction on i. The result is clear for $i = 1$ (taking a "geodesic ($\geq d$)-subdivision of B_0" to mean a rooted tree with only one vertex), so we assume that $i \geq 2$ and the result holds for $i - 1$. Let $P \in \mathcal{P}_i$, joining components X, Y of J_{i-1}, and let the ends of P be $x \in V(X)$ and $y \in V(Y)$. (See Figure 3.) Thus, $x, y \in V(J_{i-1})$. Either $y \in V(J_{i-2})$, or y lies in a path in $\mathcal{P}_{i-1}$ included in Y with both ends in $V(J_{i-2})$, and in either case there is a path of Y between y, J_{i-2} of length at most $4^{\ell+2-i+1}d/2$. By combining this with P, we deduce that there is a path between x and a vertex in $J_{i-2} \cap Y$ of length at most $4^{\ell+2-i+1}d/2 + 4^{\ell+2-i}d \leq 3 \cdot 4^{\ell+2-i}d$.

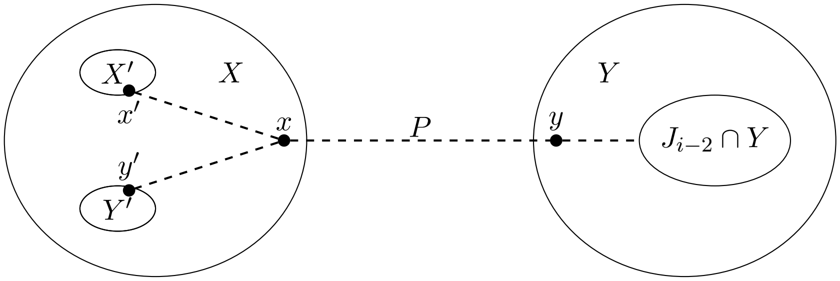

Figure 3: For the proof of 4.2.

But every path between vertices of J_{i-2} that belong to distinct components of J_{i-1} has length more than $4^{\ell+2-i+1}d$, so every path of X between x and J_{i-2} has length more than $4^{\ell+2-i}d$. In particular, $x \notin V(J_{i-2})$, and so there is a path $Q \in \mathcal{P}_{i-1}$ with x in its interior, and the two subpaths of Q between x and the ends of Q both have length more than $4^{\ell+2-i}d$. Let Q join components X', Y' of J_{i-2}, and let the corresponding ends of Q be x', y'. From the inductive hypothesis, there is a geodesic ($\geq d$)-subdivision of B_{i-2} in G that is a subgraph of X', with root x', and the same for Y', y'. But Q is a geodesic between X', Y', and hence the union of Q and these two geodesic ($\geq d$)-subdivisions of B_{i-2} makes a geodesic ($\geq d$)-subdivision of B_{i-1} with root x. This proves (1).

Since G does not contain H_ℓ as a c-fat minor, and therefore does not contain $B_{\ell+1}$ as a c-fat minor, it follows from 4.1 that no subgraph of G is a geodesic ($\geq d$)-subdivision of $B_{\ell+1}$. We deduce from (1) that $\mathcal{P}_{\ell+2} = \emptyset$. Consequently, every two components of $J_{\ell+1}$ have distance more than $4^{\ell+2-(\ell+2)}d = d$. Let $J = J_{\ell+1}$, and let $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_{\ell+1}$. Then every path in $\mathcal{P}$ has length at most $4^{\ell+2-1}d = 4^{\ell+1}d$. Moreover, for $1 \leq i \leq \ell + 1$, every vertex in J_i either belongs to J_{i-1} or to a path in $\mathcal{P}_i$, and so in either case has distance in $U\mathcal{P}$ at most $4^{\ell+2-i}d/2$ from some vertex in J_{i-1}; and so every vertex in J has distance from J_0 in $U\mathcal{P}$ at most

$$\sum_{1\le i\le \ell+1} 4^{\ell+2-i}d/2 < 4^{\ell+1}d.$$

This proves 4.2. ∎

5 Using the grouping lemma

We apply the grouping lemma to "civilize" a given society as follows.

5.1 *Let $k \ge 0$ be a century, and let $\tau = (d, \alpha, \beta)$ and $\tau' = (d', \alpha', \beta')$ be canons, where*

$$\begin{aligned} d' &\ge 3c \\ d &\le d_0 \\ d &\ge 4^{\ell+2}d' \\ \alpha' &= 2\alpha \\ \beta' &\ge \beta + 2d_0 + d/4. \end{aligned}$$

Let G be a graph that does not contain H_ℓ as a c-fat minor, and such that there is a kth-century τ-society in G. Then there is a civilized kth-century τ'-society in G.

Proof. Let $\mathcal{T}$ be a kth-century τ-society in G, and let W be the set of vertices $v \in V(G)$ such that $\mathrm{dist}_G(v, \mathrm{bd}(\Delta_{\mathcal{T}}(C))) \ge d_0$ for some $\mathcal{T}$-village C with $v \in \Delta_{\mathcal{T}}(C)$. Let Q be the union of W and all houses of $\mathcal{T}$, and let $\mathcal{A}$ be the set of all vertex sets of components of $G[Q]$. Let $\mathcal{T}''$ be the union of $\mathcal{A}$ and the set of all forts of $\mathcal{T}$. It follows that all members of $\mathcal{T}''$ are pairwise disjoint, and so $\mathcal{T}''$ defines Voronoi cells.

(1) *For each $A \in \mathcal{A}$ there is a $\mathcal{T}$-village C such that $\Delta_{\mathcal{T}''}(A) \subseteq \Delta_{\mathcal{T}}(C)$.*

For each $u \in A$ there is a unique $\mathcal{T}$-village C with $u \in \Delta_{\mathcal{T}}(C)$. Since there are no edges between $\Delta_{\mathcal{T}}(C)$ and $\Delta_{\mathcal{T}}(C')$ for distinct $\mathcal{T}$-villages C, C', and $G[A]$ is connected, it follows that there is a $\mathcal{T}$-village C such that $A \subseteq \Delta_{\mathcal{T}}(C)$. We claim that $\Delta_{\mathcal{T}''}(A) \subseteq \Delta_{\mathcal{T}}(C)$. Suppose not, and choose $w \in \Delta_{\mathcal{T}''}(A) \backslash \Delta_{\mathcal{T}}(C)$. Since $w \in \Delta_{\mathcal{T}''}(A)$, the Λ-shortest path P between $w, V(\mathcal{T}'')$ has an end $s \in A$, and $V(P) \subseteq \Delta_{\mathcal{T}''}(A)$. Since $w \notin \Delta_{\mathcal{T}}(C)$ and $s \in A \subseteq \Delta_{\mathcal{T}}(C)$, there is an edge uv of P such that $u \in \Delta_{\mathcal{T}}(C)$ and $v \notin \Delta_{\mathcal{T}}(C)$. The subpath of P between v, s is the Λ-shortest path between $v, V(\mathcal{T}'')$. Since $V(\mathcal{T}) \subseteq V(\mathcal{T}'')$, it follows that if $s \in V(C) \subseteq V(\mathcal{T})$ then this subpath is also the Λ-shortest path between $v, V(\mathcal{T})$, contradicting that $v \notin \Delta_{\mathcal{T}}(C)$. Thus $s \in W$, and hence $\mathrm{dist}_G(s, u) \ge d_0$, and so $\mathrm{dist}_G(s, v) > d_0$, which is impossible since $v \in V(P) \subseteq \Delta_{\mathcal{T}''}(A)$, and $\mathrm{dist}_G(s, v) = \mathrm{dist}_G(A, v)$. This proves that $\Delta_{\mathcal{T}''}(A) \subseteq \Delta_{\mathcal{T}}(C)$, and so proves (1).

(2) *If $A \in \mathcal{A}$ and Y is a fort of $\mathcal{T}$ then $\mathrm{dist}_G(A, Y) > d$.*

Let P be a shortest path in G between A, Y, with ends $x \in A$ and $y \in Y$. If $x \notin W$, then $x \in X$ for some house X of $\mathcal{T}$, and then $|E(P)| \geq \operatorname{dist}_G(X, Y) > d$ as required. If $x \in W$, choose a $\mathcal{T}$-village C such that $x \in \Delta_{\mathcal{T}}(C)$ and $\operatorname{dist}_G(x, \operatorname{bd}(\Delta_{\mathcal{T}}(C))) \geq d_0$. There is a vertex $u \in V(P) \setminus Y$ with $u \in \operatorname{bd}(\Delta_{\mathcal{T}}(C))$, and consequently

$$d \leq d_0 \leq \operatorname{dist}_G(x, u) < |E(P)| = \operatorname{dist}_G(A, Y)$$

as required. This proves (2).

We designate the members of $\mathcal{A}$ as houses of $\mathcal{T}''$, and the forts of $\mathcal{T}$ as forts of $\mathcal{T}''$. We do not claim that $\mathcal{T}''$ is a society (for any choice of canon) since its houses might be very close together. Nevertheless, let us use the same terminology: a *$\mathcal{T}''$-community* means a $\mathcal{T}''$-adjoin-connected set of its houses, and a *$\mathcal{T}''$-village* is a maximal $\mathcal{T}''$-community.

(3) *For every $\mathcal{T}''$-village C'', $\Delta_{\mathcal{T}''}(C'')$ has quasi-line-width at most (α, β) and $\bigcup_{A \in C''} \operatorname{bd}(A)$ has quasi-size at most $(\alpha, \beta + d_0)$.*

Let $A_1, A_2 \in C''$ such that A_1 $\mathcal{T}''$-adjoins A_2. By (1), there are $\mathcal{T}$-villages C_1, C_2 such that $\Delta_{\mathcal{T}''}(A_i) \subseteq \Delta_{\mathcal{T}}(C_i)$ for $i = 1, 2$. Since there is an edge between $\Delta_{\mathcal{T}''}(A_1), \Delta_{\mathcal{T}''}(A_2)$, and there is no edge between $\Delta_{\mathcal{T}}(C_1), \Delta_{\mathcal{T}}(C_2)$ if $C_1 \neq C_2$, it follows that $C_1 = C_2$; and since C'' is $\mathcal{T}''$-adjoin-connected, we deduce that there is a $\mathcal{T}$-village C such that $\Delta_{\mathcal{T}''}(C'') \subseteq \Delta_{\mathcal{T}}(C)$.

Now $\Delta_{\mathcal{T}}(C)$ has quasi-bound at most (α, β), since $\mathcal{T}$ is a τ-society. Since $\Delta_{\mathcal{T}''}(C'') \subseteq \Delta_{\mathcal{T}}(C)$, it follows that $\Delta_{\mathcal{T}''}(C'')$, and therefore $V(C'')$, has quasi-line-width at most (α, β). It remains to check that $\bigcup_{A \in C''} \operatorname{bd}(A)$ has quasi-size at most $(\alpha, \beta + d_0)$. Let $u \in \bigcup_{A \in C''} \operatorname{bd}(A)$; then $u \in \operatorname{bd}(A)$ for some $A \in C''$. We claim that $\operatorname{dist}_G(u, \operatorname{bd}(\Delta_{\mathcal{T}}(C))) \leq d_0$. To see this, choose $v \notin A$ adjacent to u (this exists since $u \in \operatorname{bd}(A)$). Hence $v \notin Q$, since A is the vertex set of a component of $G[Q]$. If $v \notin \Delta_{\mathcal{T}}(C)$, then $u \in \operatorname{bd}(\Delta_{\mathcal{T}}(C))$ and so $\operatorname{dist}_G(u, \operatorname{bd}(\Delta_{\mathcal{T}}(C))) = 0 \leq d_0$ as claimed, so we assume that $v \in \Delta_{\mathcal{T}}(C)$. Since $v \notin Q$, it follows that $\operatorname{dist}_G(v, \operatorname{bd}(\Delta_{\mathcal{T}}(C))) < d_0$. So $\operatorname{dist}_G(u, \operatorname{bd}(\Delta_{\mathcal{T}}(C))) \leq d_0$, as claimed. Since $\operatorname{bd}(\Delta_{\mathcal{T}}(C))$ has quasi-size at most (α, β), it follows that $\bigcup_{A \in C''} \operatorname{bd}(A)$ has quasi-size at most $(\alpha, \beta + d_0)$. This proves (3).

Let $\theta = 4^{\ell+1} d'$. Thus, $d \geq 4\theta$. By 4.2 applied to $\mathcal{A}$, there is a set $\mathcal{P}$ of paths of G, with union $U\mathcal{P}$ say, such that:

- for each $P \in \mathcal{P}$ and each $v \in V(P)$, there exists $A \in \mathcal{A}$ such that $\operatorname{dist}_{U\mathcal{P}}(v, A) < \theta$;
- let J be the subgraph of G consisting of the union of all members of $\mathcal{A}$ and all members of $\mathcal{P}$; then $\operatorname{dist}_G(u, v) > d'$ for all vertices u, v in distinct components of J.

Let $\mathcal{S}$ be the set of all vertex sets of components of J.

(4) *For each $S \in \mathcal{S}$, there is a $\mathcal{T}''$-village C'' such that every house of $\mathcal{T}''$ included in S belongs to C'', and consequently $S \subseteq \Delta_{\mathcal{T}''}(C'')$.*

The distance between distinct houses in $\mathcal{T}''$ may be as small as two; but if two houses A_1, A_2 of $\mathcal{T}''$ are at distance less than $d + 1$ then they belong to the same $\mathcal{T}''$-village. To see this, suppose that A_1, A_2 do not belong to the same $\mathcal{T}''$-village, and let P be a shortest path between A_1, A_2.

Since there are no edges between $\Delta_{\mathcal{T}''}(C_1)$ and $\Delta_{\mathcal{T}''}(C_2)$ for distinct $\mathcal{T}''$-villages C_1, C_2, some vertex w of P belongs to $\Delta_{\mathcal{T}''}(Y)$ for some fort Y of $\mathcal{T}''$. So both subpaths of P between w and its ends have subpaths between $\Delta_{\mathcal{T}''}(Y)$ and one of A_1, A_2, and so both have length at least $(d+1)/2$ since $\text{dist}_G(Y, A_i) \geq d+1$ for $i = 1, 2$. Hence P has length at least $d+1$, a contradiction.

Now let $S \in \mathcal{S}$. For each $v \in S$ there exists $A \in \mathcal{A}$ such that $\text{dist}_G(v, A) < \theta$; let $\mathcal{A}_v$ be the set of all such A. Since $\text{dist}_G(A, A') \leq 2(\theta - 1) \leq d$ for all distinct $A, A' \in \mathcal{A}_v$, it follows that they all belong to the same $\mathcal{T}''$-village C''_v, and if $v, v' \in S$ are adjacent, then any member of $\mathcal{A}_v$ has distance at most $2(\theta-1)+1$ from each member of $\mathcal{A}_{v'}$, and so $C''_v = C''_{v'}$. Consequently there is a $\mathcal{T}''$-village C'' containing A for all $A \in \mathcal{A}$ such that $\text{dist}_G(S, A) < \theta$.

It remains to show that $S \subseteq \Delta_{\mathcal{T}''}(C'')$. Let $v \in S$, choose $A \in \mathcal{A}$ with $\text{dist}_G(v, A) < \theta$, and choose $Y \in \mathcal{T}''$ with $v \in \Delta_{\mathcal{T}''}(Y)$. Thus, $\text{dist}_G(v, Y) \leq \text{dist}_G(v, A)$. Since $\text{dist}_G(Y, A) \leq 2(\theta-1) \leq d$, and every fort of $\mathcal{T}''$ has distance more than d from A (by (2)), it follows that Y is not a fort, and so $Y \in \mathcal{A}$. Since

$$\text{dist}_G(S, Y) \leq \text{dist}_G(v, Y) \leq \text{dist}_G(v, A) < \theta$$

it follows that $Y \in C''$ and hence $v \in \Delta_{\mathcal{T}''}(C'')$. This proves that $S \subseteq \Delta_{\mathcal{T}''}(C'')$, and so proves (4).

For each fort $Y \in \mathcal{T}''$, let $F(Y)$ be the set of all $v \in V(G)$ such that $\text{dist}_G(v, Y) \leq \theta$. Let $\mathcal{F}$ be the set of all sets $F(Y)$ for all forts Y of $\mathcal{T}''$. Let $\mathcal{T}' = \mathcal{S} \cup \mathcal{F}$.

(5) $\text{dist}_G(X', Y') > d'$ *for all distinct* $X', Y' \in \mathcal{T}'$.

We have seen this already if $X', Y' \in \mathcal{S}$, from the choice of $\mathcal{S}$. If $X' \in \mathcal{S}$ and $Y' \in \mathcal{F}$, let $v \in X'$, choose a house X of $\mathcal{T}''$ with $\text{dist}_G(v, X) < \theta$, and let $Y' = F(Y)$ for a fort Y of $\mathcal{T}''$. Then

$$\text{dist}_G(v, Y') \geq \text{dist}_G(X, Y) - 2\theta + 1 > d - 2\theta + 1 > d',$$

as required. Similarly, since every two forts of $\mathcal{T}''$ are at distance more than d, it follows that every two members of $\mathcal{F}$ are at distance more than $d - 2\theta \geq d'$. This proves (5).

In particular, the buildings in $\mathcal{T}'$ are pairwise disjoint, and we may speak of $\Delta_{\mathcal{T}'}(X)$ for $X \in \mathcal{T}'$. Assign the members of $\mathcal{S}$ to be houses of $\mathcal{T}'$, and the members of $\mathcal{F}$ to be forts of $\mathcal{T}'$. We will show that $\mathcal{T}'$ is a civilized kth-century τ'-society, and here are some steps to show that:

(6) *The following hold:*

- *For all* $v \in V(G)$, *there exists* $X \in \mathcal{T}'$ *such that* $\text{dist}_G(v, X) \leq d_0$.
- *Every fort of* $\mathcal{T}'$ *includes a* c*-superfat* H_k*-minor of* G.
- *Each fort of* $\mathcal{T}'$ *has quasi-bound at most* (α', β').

The first is true since it is true for $\mathcal{T}$ and each member of $\mathcal{T}$ is a subset of a member of $\mathcal{T}''$ and hence of $\mathcal{T}'$. The second is true because each fort of $\mathcal{T}'$ includes a fort of $\mathcal{T}''$ and hence one of $\mathcal{T}$.

To see the third, let Y be a fort of $\mathcal{T}''$, and hence of $\mathcal{T}$. Then Y has quasi-bound at most (α, β). By 2.1, $\text{bd}(F(Y))$ has quasi-size at most $(\alpha, \beta+\theta)$, and $F(Y)$ has quasi-line-width at most $(2\alpha, \beta+\theta)$. Since $2\alpha = \alpha'$ and $\beta + \theta \leq \beta'$, this proves the third statement, and hence proves (6).

For each $S \in \mathcal{S}$, let C_S be the $\mathcal{T}''$-village that contains each house of $\mathcal{T}''$ included in S (this exists, by (4)).

(7) *For each $S \in \mathcal{S}$, $\Delta_{\mathcal{T}'}(S) \subseteq \Delta_{\mathcal{T}''}(C_S)$.*

Suppose not; then since $S \subseteq \Delta_{\mathcal{T}''}(C_S)$ by (4), there are adjacent $u, v \in \Delta_{\mathcal{T}'}(S)$ such that $u \in \Delta_{\mathcal{T}''}(C_S)$ and $v \notin \Delta_{\mathcal{T}''}(C_S)$. Choose $Y \in \mathcal{T}''$ with $v \in \Delta_{\mathcal{T}''}(Y)$. Thus, $\text{dist}_G(v, C_S) \geq \text{dist}_G(v, Y)$. Since $v \notin \Delta_{\mathcal{T}''}(C_S)$, it follows that $Y \notin C_S$, and since no house of $\mathcal{T}''$ $\mathcal{T}''$-adjoins C_S (because C_S is a $\mathcal{T}''$-village), it follows that Y is a fort of $\mathcal{T}''$. (See Figure 4.)

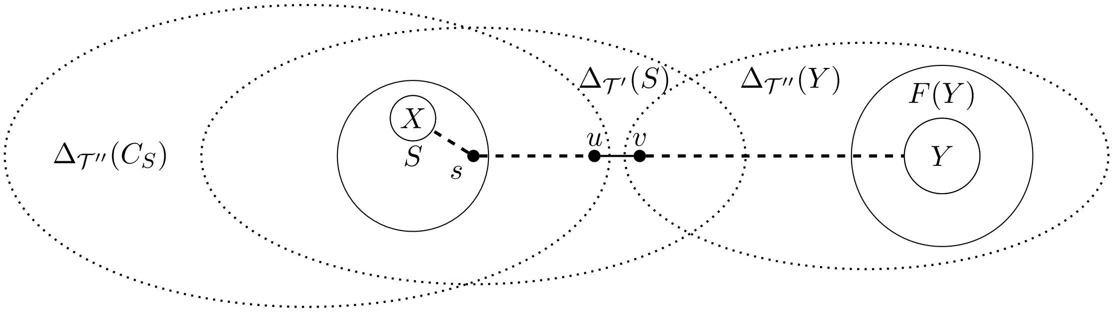

Figure 4: For the proof of (7).

Now $v \notin F(Y)$, since $v \in \Delta_{\mathcal{T}'}(S)$; and so $\text{dist}_G(v, F(Y)) \leq \text{dist}_G(v, Y) - \theta$. Since $v \in \Delta_{\mathcal{T}'}(S)$ and $F(Y) \in \mathcal{T}'$, it follows that

$$\text{dist}_G(v, S) \leq \text{dist}_G(v, F(Y)) \leq \text{dist}_G(v, Y) - \theta \leq \text{dist}_G(v, C_S) - \theta.$$

Choose $s \in S$ such that $\text{dist}_G(v, S) = \text{dist}_G(v, s)$; and choose a house X of $\mathcal{T}''$ with $\text{dist}_{G[S]}(s, X) < \theta$. Then $X \subseteq S$, and so $X \in C_S$ by (4). Consequently,

$$\text{dist}_G(v, C_S) \leq \text{dist}_G(v, X) < \text{dist}_G(v, s) + \theta,$$

a contradiction. This proves (7).

(8) *For every $\mathcal{T}'$-community C', $\Delta_{\mathcal{T}'}(C')$ has quasi-line-width at most (α, β); $\bigcup_{S \in C'} \text{bd}(S)$ has quasi-size at most $(\alpha, \beta + d_0 + \theta)$; and $\text{bd}(\Delta_{\mathcal{T}'}(C'))$ has quasi-size at most $(\alpha, \beta + 2d_0 + \theta)$.*

From (7), it follows that if $S_1, S_2 \in C'$ $\mathcal{T}'$-adjoin, then $C_{S_1} = C_{S_2}$, and since C' is $\mathcal{T}'$-adjoin-connected, all the $\mathcal{T}''$-villages C_S ($S \in C'$) are equal. Hence there is a $\mathcal{T}''$-village C'' such that $\Delta_{\mathcal{T}'}(C') \subseteq \Delta_{\mathcal{T}''}(C'')$. By (3), $\Delta_{\mathcal{T}''}(C'')$ has quasi-line-width at most (α, β), and since $\Delta_{\mathcal{T}'}(C') \subseteq \Delta_{\mathcal{T}''}(C'')$, it follows that $\Delta_{\mathcal{T}'}(C')$ has quasi-line-width at most (α, β). Now every vertex in $\bigcup_{S \in C'} \text{bd}(S)$ has distance at most θ from $\bigcup_{X \in C''} \text{bd}(X)$, and since the latter has quasi-size at most $(\alpha, \beta + d_0)$ by (3), it follows that $\bigcup_{S \in C'} \text{bd}(S)$ has quasi-size at most $(\alpha, \beta + d_0 + \theta)$. Since every vertex in $\Delta_{\mathcal{T}'}(C') \setminus V(C')$ has distance at most d_0 from $\bigcup_{S \in C'} \text{bd}(S)$, 2.1 implies that $\text{bd}(\Delta_{\mathcal{T}'}(C'))$ has quasi-size at most $(\alpha, \beta + 2d_0 + \theta)$. This proves (8).

From (6) and (8), $\mathcal{T}'$ is a civilized kth-century τ'-society. This proves 5.1. ∎

6 Realms

That concludes the first part of the proof. Now we come to the second part of the paper, and here our goal is to show that if G admits a civilized kth-century τ-society then it admits a $(k+1)$st-century τ'-society, for suitable τ, τ'. The proof uses "realms", which are basically civilized societies with castles added; and we need a number of further definitions.

For $\gamma \geq 0$ an integer, a *spacing with unit* γ is the function δ from $\{0, \ldots, 2\ell\}$ to $\mathbb{R}$ defined by $\delta(i) = \gamma 4^{-i}$ for $0 \leq i \leq 2\ell$. (We will arrange that δ is integer-valued, by choosing γ carefully.) Thus $\delta(i) = 4\delta(i+1)$ for each i with $0 \leq i < 2\ell$. A *spacing* is a spacing with unit γ for some choice of γ. A *standard* is a triple $\sigma = (\delta, \alpha, \beta)$, where $\alpha \geq 12$ and $\beta \geq 1$ are integers, and δ is a spacing with $\delta(2\ell) \geq 5c$ and $\delta(0) \leq d_0$. (The condition $\alpha \geq 12$ is just to make some arithmetic come out more smoothly.)

Let $0 \leq k < \ell$ be a century and let $\sigma = (\delta, \alpha, \beta)$ be as above. We say a *kth-century realm* with *standard* σ in a graph G is a set $\mathcal{T}$ of pairwise vertex-disjoint buildings of G, where each member of $\mathcal{T}$ is assigned to be a house, fort or castle of $\mathcal{T}$, with no houses if $k = 0$, satisfying the following (where $\text{rk}(X) = k-1, k$ or $k+1$ depending whether X is a house, fort or castle):

- For all $v \in V(G)$, there exists $X \in \mathcal{T}$ such that $\text{dist}_G(v, X) \leq d_0$;
- If $X, Y \in \mathcal{T}$ are distinct, then $\text{dist}_G(X, Y) > \delta(\text{rk}(X) + \text{rk}(Y))$;
- Every fort or castle X of $\mathcal{T}$ includes a c-superfat $H_{\text{rk}(X)}$-minor of G;
- Every fort of $\mathcal{T}$ has quasi-bound at most (α, β);
- Every castle of $\mathcal{T}$ has quasi-bound at most $(9\alpha, \beta + 2d_0)$;
- For every $\mathcal{T}$-community C, $V(C)$ has quasi-line-width at most (α, β) and $\bigcup_{X \in C} \text{bd}(X)$ has quasi-size at most (α, β). (Again, a *$\mathcal{T}$-community* of $\mathcal{T}$ is an adjoin-connected set of houses of $\mathcal{T}$, and a *$\mathcal{T}$-village* is a maximal $\mathcal{T}$-community.)

If X belongs to a realm $\mathcal{T}$, then it is a house, fort or castle of $\mathcal{T}$, and we call this its *class* under $\mathcal{T}$. It is easy to turn a civilized society into a realm for the same century (note, however, that we use the full strength of "civilized" – this was the point of converting a society into a civilized one):

6.1 *Let $0 \leq k < \ell$ be a century, and let $\sigma = (\delta, \alpha, \beta)$ be a standard. If $k = 0$, every civilized kth-century $(\delta(0), \alpha, \beta)$-society $\mathcal{T}$ in G is also a kth-century realm with standard σ. If $k > 0$, every civilized kth-century $(\delta(2k-2), \alpha, \beta)$-society $\mathcal{T}$ in G is also a kth-century realm with standard σ.*

Proof. If $k = 0$, and $X, Y \in \mathcal{T}$ are distinct, then $\text{dist}_G(X, Y) > \delta(0)$ since $\mathcal{T}$ is a 0th-century $(\delta(0), \alpha, \beta)$-society; and $\delta(0) \geq \delta(\text{rk}(X) + \text{rk}(Y))$ because 0th-century societies have no houses. If $k > 0$, then similarly $\text{dist}_G(X, Y) > \delta(2k-2) \geq \delta(\text{rk}(X) + \text{rk}(Y))$. So in either case, $\text{dist}_G(X, Y) > \delta(\text{rk}(X) + \text{rk}(Y))$. This proves 6.1. ∎

7 A sketch of the rest of the proof

Let us give an overall view of the remainder of the proof, before we embark on its details. To reach the goal expressed at the start of the previous section, it suffices (in view of 6.1) to be able to convert a kth-century realm (of a reasonably high standard) to a $(k+1)$st-century τ-society, where τ is some suitable canon. Suppose then that we have some kth-century realm $\mathcal{T}$.

7.1 Making it optimal

The first step is to "optimize" $\mathcal{T}$. This means, roughly, choosing $\mathcal{T}$ with the same standard and with as many castles as possible, combining houses and forts to make castles wherever we can. If some village adjoins three forts, then we could try to combine the Voronoi closures of these houses and forts into a castle; or if there are three villages joining a given fort to three other forts, again we could hope to combine their closures into a castle. (With some care: we need that the final approach to each of the three important forts is via a geodesic of length $c+1$, so we can apply 8.1). This is the content of 9.3. If we succeed, we make a new realm, still in the same century and with the same standard.

Suppose that, after a sequence of such promotions, we have converted $\mathcal{T}$ to a new realm $\mathcal{T}'$ (with the same century), and we want to show that $\mathcal{T}'$ has the same standard. Then in particular, we need that the boundary of the union of the Voronoi cells of each village of $\mathcal{T}'$ has bounded quasi-size. (Let us call this "fact F".) This is awkward to guarantee, because the Voronoi cells might change as we move to new realms, and therefore which sets are villages might also change. But for any building of $\mathcal{T}$ that remains in $\mathcal{T}'$, its Voronoi closure in $\mathcal{T}'$ is a subset of its closure in $\mathcal{T}$, so the villages of $\mathcal{T}'$ are communities (not necessarily maximal) of $\mathcal{T}$. We don't know in advance which communities of $\mathcal{T}$ might end up as villages in some $\mathcal{T}'$; so to guarantee that fact F will hold for $\mathcal{T}'$, we will arrange that fact F holds for $\mathcal{T}$ in a strengthened, hereditary form. This is the reason for the very strong realm axiom, that for every community X, the union of the boundaries of the members of X has bounded quasi-size. (The point is that the union of the boundaries may be very different from the boundary of the union.)

When $\mathcal{T}$ is optimal, the set of its forts and villages has a sort of linear structure; each village adjoins at most two forts, and for each fort, there are at most two other forts to which it is connected via a village. So each component of the graph of villages and forts defined by adjoining pairs has a "spine", a path or cycle that contains all forts in the component, together with some villages each attaching to one or two consecutive members of the spine. (This is 9.4.) By exploiting this linear structure, we prove (in 9.5 and 9.6) that the union of the Voronoi closures of all houses and forts has bounded quasi-line-width. This is a key step in our attempt to move to a new century; we hope to promote all the current houses and forts to be houses of a $(k+1)$st-century realm, and at least we have control of the quasi-line-width of the union of their Voronoi cells. The problem is to bound the quasi-size of their boundaries.

But we can get more than this from optimality of the realm. We might be able to combine some triple of forts into a castle even if they are *not* adjoin-connected via villages. If we can connect together three forts in a claw, moving between them either via villages or by paths ("passages") that are not very long and do not go very close to other forts or castles, then we want to do so. This step is critical later when we introduce "governments".

7.2 Governments and revolution

Castles have rank $k+1$. In the same way that we combined the Voronoi cells of three forts into a castle, we could try to combine the cells of three castles into some "super-castle" of rank $k+2$, and so on; these new higher-level objects are called "palaces", and the set of them, together with the members of $\mathcal{T}$ not included in any palace, is called a "government". But in a way this is simpler than combining forts into castles. There were several different scenarios for the latter; if some village adjoins three forts, or some fort can reach three others via villages, or if we can connect them via passages. For $j \geq k+1$, promoting three palaces of rank j to a palace of rank $j+1$ is simpler: we look for an adjoin-connected set of houses and forts that adjoins all three palaces, and if we find one with the appropriate properties (a "cabal") we promote it all to make a new palace with rank $j+1$, and thereby change the government (a "revolution"). (Except "adjoin" is the wrong term now, see below.) The more complex methods we used to make castles are still available at the higher levels, but seem not to help.

If a house or fort has been used to glue some three palaces into a higher-rank palace, then it is no longer available for glueing other things together. We call the houses and forts that have not yet been used "rebels". But we need to keep track of the buildings of the realm that have been combined into higher structures: the realm is not changing, but some of its buildings are *also* being used as parts of palaces.

There is an extra complication: we have a partition into Voronoi cells defined by the realm, and also a partition into Voronoi cells defined by the current government, and they are not the same. We use "$\mathcal{T}$-adjoin" and "$\mathcal{G}$-adjoin", to show which set of Voronoi cells we are referring to.

There is a delicate issue here, one that gave us a lot of trouble. Suppose we have found a set X of rebels, $\mathcal{G}$-adjoin-connected, and X $\mathcal{G}$-adjoins three palaces of rank j in the current government, and we want to promote all this to make a palace of rank $j+1$. We already saw that the union of the $\mathcal{T}$-Voronoi cells of X has bounded quasi-line-width, and therefore so does the union of the $\mathcal{G}$-cells (which are subsets); but we also need that the boundary of the latter has bounded quasi-size. That was not a problem when we built castles, because the boundaries of villages and forts have bounded quasi-size by definition, and we were only using at most three villages and at most four forts; but now it becomes a big problem.

We can get partway around it by making X minimal subject to its $\mathcal{G}$-adjoining three palaces of the same rank, and therefore the total number of palaces that X $\mathcal{G}$-adjoins is bounded (unless $|X| = 1$). So we would be happy if we could get a bound on the quasi-size of the interface between the union of the $\mathcal{G}$-cells of X and $\Delta_{\mathcal{G}}(S)$ of each palace S that X $\mathcal{G}$-adjoins. (We can get a bound on the quasi-size of the boundary of $\Delta_{\mathcal{G}}(S)$, from the definition of a palace; but we cannot use this to bound the interface, because there is a vicious circle among the constants.)

This is where we use "passages". Any term of X that $\mathcal{G}$-adjoins S is joined by a passage to one of the members of $\mathcal{T}$ within S (or it gives us no problem). But X is $\mathcal{G}$-adjoin-connected, and so $\mathcal{T}$-adjoin-connected, and this gives a linear structure as we saw earlier. Interior terms of this linear structure already $\mathcal{T}$-adjoin (possibly via villages) two forts in X, and they cannot reach three forts, even via a passage, from the optimality of $\mathcal{T}$. So, if an interior term $\mathcal{G}$-adjoins S, then its passage into S is very restricted. That gives us much more control of the interface between X and S, and eliminates the vicious circle. (This is 10.1 and 11.1.)

7.3 Societies and the next century

If some $\mathcal{G}$-adjoin-connected set of rebels has $\mathcal{G}$-Voronoi closure with a boundary of large quasi-size, then it must $\mathcal{G}$-adjoin many different palaces, and so three of the same rank, and we can have a revolution. So when we finally reach a stable government, each $\mathcal{G}$-adjoin-connected set of rebels has closure with boundary of small quasi-size. (This is 12.1.) Consequently, we can make a $(k+1)$st-century society $\mathcal{T}'$ by declaring that each rebel is a house of $\mathcal{T}'$, and each palace of the government is a fort of $\mathcal{T}'$; and that completes the proof.

8 Growing c-fat trees

We will try to grow a c-fat copy of H_{t+1} by taking three c-fat copies of H_t, sufficiently far apart, and connecting them together appropriately. This goes more smoothly if we work with "superfat" rather than "fat", as the next result shows.

8.1 *Let $c \geq 1$, let $t \geq 0$, and for $i = 1, 2, 3$, let η_i exhibit H_t as a c-superfat minor of a graph G, such that $\text{dist}_G(\eta_i(H_t), \eta_j(H_t)) > 5c + 2$ for all distinct $i, j \in \{1, 2, 3\}$. Let $W \subseteq V(G)$ such that $G[W]$ is connected, and $\text{dist}_G(\eta_i(H_t), W) = c + 1$ for $i = 1, 2, 3$; and for $i = 1, 2, 3$, let P_i be a geodesic from W to $\eta_i(H_t)$. Then there is a mapping η that exhibits H_{t+1} as a c-superfat minor of G, such that*

$$\eta(H_{t+1}) \subseteq G[W] \cup \eta_1(H_t) \cup \eta_2(H_t) \cup \eta_3(H_t) \cup P_1 \cup P_2 \cup P_3.$$

Proof. (See Figure 5.) For $1 \leq i < j \leq 3$, $\text{dist}_G(P_i, P_j) > (5c+2) - 2(c+1) = 3c$, since P_i, P_j both have length $c + 1$, and they have ends at distance $> 5c + 2$. If $t = 0$, let η map the root of H_{t+1} to W, its leaves to the three subgraphs $\eta_i(H_t)$ for $i = 1, 2, 3$, and each edge of H_{t+1} to the interior of the corresponding geodesic P_i; then η satisfies the theorem. Thus, we may assume that $t \geq 1$.

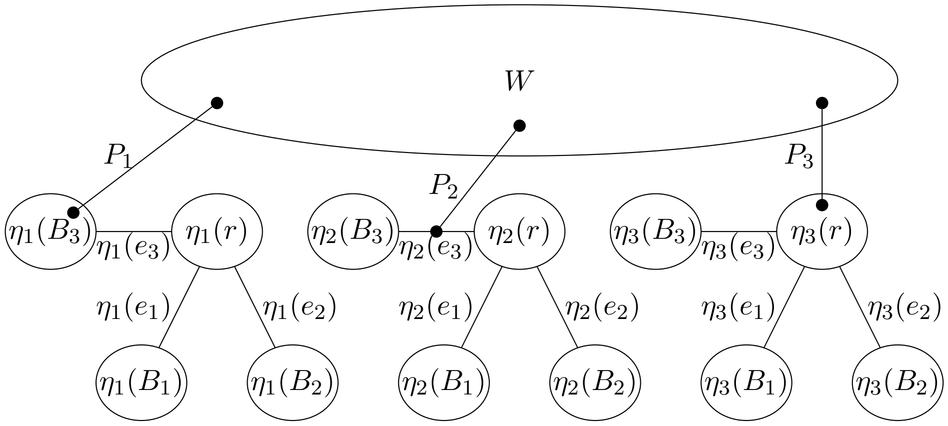

Figure 5: Growing a superfat minor. The figure shows the three ways that P_h might attach.

Let r be the root of H_t, let B_1, B_2, B_3 be the three components of $H_t \setminus \{r\}$, and for $h = 1, 2, 3$, let e_h be the edge of H_t between r and $V(B_h)$. For $h, i \in \{1, 2, 3\}$, let $\eta_h(B_i^+)$ denote the subgraph

$$\eta_h(e_i) \cup \bigcup_{x \in U(B_i)} \eta_h(x).$$

We know that for $h = 1, 2, 3$, and all distinct $i, j \in \{1, 2, 3\}$,

$$\operatorname{dist}_G(\eta_h(B_i^+), \eta_h(B_j^+)) \geq 3c + 1.$$

Consequently, there is at most one value of $i \in \{1, 2, 3\}$ such that $\operatorname{dist}_G(P_h, \eta_h(B_i^+)) \leq c$, since P_h has length $c + 1$ and η_h is c-superfat. By relabelling, we may assume that $\operatorname{dist}_G(P_h, \eta_h(B_i^+)) > c$ for $i = 1, 2$.

Since $\operatorname{dist}_G(\eta_i(H_t), \eta_j(H_t)) > 5c+2$ and P_i, P_j have length $c+1$, it follows that $\operatorname{dist}_G(P_i, \eta_j(H_t)) > 4c + 1$, and $\operatorname{dist}_G(P_i, P_j) > 3c$, for all distinct $i, j \in \{1, 2, 3\}$. Moreover, $\operatorname{dist}_G(P_h, \eta_h(B_i^+)) > c$ for $i = 1, 2$ and $h = 1, 2, 3$.

Let r' be the root of H_{t+1}, and let $B'_1, B'_2, B'_3, e'_1, e'_2, e'_3$ be defined as usual. Thus, for $h = 1, 2, 3$, B'_h is isomorphic to $H_t \setminus V(B_3)$; let ϕ_h be such an isomorphism. Define $\eta(r') = G[W]$, and for $h = 1, 2, 3$, let $\eta(x) = \eta_h(\phi_h(x))$ for each $x \in U(B'_h)$. For $h = 1, 2, 3$, we define $\eta(e'_h)$ as follows. If there is an end y_h of P_h in $\eta_h(r)$, let $\eta(e'_h) = (P_h \setminus W) \setminus \{y_h\}$ (since $c \geq 1$, this subgraph is non-null). If not, then there is an end of P_h in $\eta_h(B_3^+)$; let $\eta(e'_h) = (P_h \setminus W) \cup \eta_h(B_3^+)$. Then η exhibits H_{t+1} as a c-superfat minor of G. This proves 8.1. ▮

9 Optimizing within a century

We have a kth-century realm with some given standard, and we first want to choose an "optimal" one, by which we mean, roughly, one with the set of castles maximal; but since G and the realm might be infinite, we need to formulate this carefully. If $0 \leq k < \ell$, we say a kth-century realm $\mathcal{T}_2$ is an *extension* of a kth-century realm $\mathcal{T}_1$ if for each member $X_1 \in \mathcal{T}_1$ there exists $X_2 \in \mathcal{T}_2$ such that:

- $X_1 \subseteq X_2$; and
- either X_2 is a building of higher class than X_1 (that is, either X_1 is a house of $\mathcal{T}_1$ and X_2 is a fort or castle of $\mathcal{T}_2$, or X_1 is a fort of $\mathcal{T}_1$ and X_2 is a castle of $\mathcal{T}_2$), or $X_1 = X_2$ and X_1 has the same class under $\mathcal{T}_1$ and under $\mathcal{T}_2$.

Fix some standard σ. Let us say a kth-century realm $\mathcal{T}$ with standard σ is *optimal* (for σ) if no other kth-century realm with the same standard is an extension of $\mathcal{T}$. It follows from Zorn's lemma that if there is a kth-century realm with standard σ, then one of its extensions is optimal.

Next, an easy lemma. Let F be a subset of vertices of a graph H. Let P_1, P_2, P_3 be paths of H with a common end z that are otherwise pairwise vertex-disjoint. Let P_i have ends y_i, z for $i = 1, 2, 3$. We say the subgraph $P_1 \cup P_2 \cup P_3$ is an *F-claw* of H if for $i = 1, 2, 3$, $y_i \in F$, $y_i \neq z$, and no internal vertex of P_i is in F. The vertex z might belong to F. (See Figure 6.)

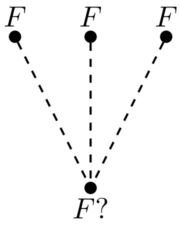

Figure 6: An F-claw. The dashed lines are paths.

9.1 *Let H be a connected graph and let $F \subseteq V(H)$, such that $H \setminus F$ has no edges. If no subgraph of H is an F-claw, then there is an induced path or cycle of H that contains all members of F, and every vertex of H not in F is adjacent only to one or two consecutive members of F. (See Figure 7.)*

We omit the proof, which is easy.

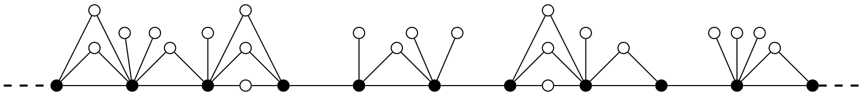

Figure 7: When there is no F-claw. The black vertices are in F. The horizontal path shown may be part of a cycle, or part of a finite or infinite path.

With H, F as before, suppose that no two vertices in $V(H) \setminus F$ are adjacent. Then each of P_1, P_2, P_3 has length one or two, and they all have length one unless their common end is in F. That gives us five cases (see Figure 8.)

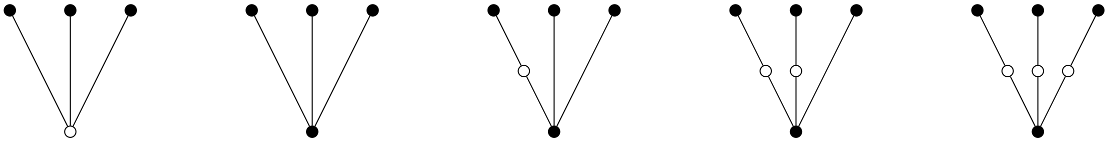

Figure 8: F-claws when F hits all edges. The black vertices are in F.

Suppose again that $F \subseteq V(H)$, but there is one special edge that may join two vertices in $V(H) \setminus F$ (we call it the "long edge"). If we enumerate the possible F-claws now, we still have the five cases of Figure 8 (and any of their edges might be the long edge), and in addition there are three more shown in Figure 9. (We omit the proof, which is easy case-checking.)

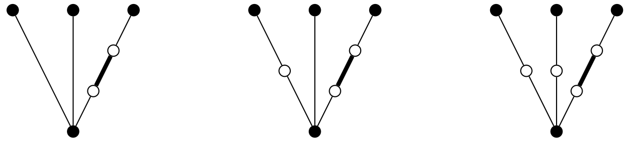

Figure 9: F-claws containing the long edge (drawn thick).

Let $\mathcal{T}$ be a kth-century realm in G, with standard σ, in the usual notation. A $\mathcal{T}$-community *adjoins* a fort X if one of its members adjoins X. A *passage* is a path P of G with the following properties:

- P has length at most $3d_0 + 1$;
- there exist distinct $X_1, X_2 \in \mathcal{T}$, each either a house or fort of $\mathcal{T}$, such that one end of P is in X_1 and the other in X_2;
- no internal vertex of P belongs to $X_1 \cup X_2$; and
- $\text{dist}_G(P, Y) > \delta(k + 1 + \text{rk}(Y))$ for each $Y \in \mathcal{T}$ with $Y \neq X_1, X_2$.

We say that P *joins* X_1, X_2, and P is *incident* with X_1, X_2. If X_1 is a house in a $\mathcal{T}$-community C, we also speak of P *joining* C, X_2.

9.2 *If X_1, X_2 are houses or forts of $\mathcal{T}$, and X_1 adjoins X_2, there is a passage of length at most $2d_0 + 1$ joining X_1, X_2.*

Proof. Choose $v_i \in \Delta_{\mathcal{T}}(X_i)$ for $i = 1, 2$ such that v_1, v_2 are adjacent. Let Q_i be the Λ-shortest path between v_i, X_i; so Q_i is a path of $G[\Delta_{\mathcal{T}}(X_i)]$ for $i = 1, 2$. Let P be the union of Q_1, Q_2 and the edge v_1v_2. Then Q_1, Q_2 both have length at most d_0 and so P has length at most $2d_0 + 1$. Since $v_2 \notin \Delta_{\mathcal{T}}(X_1)$ and vice versa, it follows that the lengths of Q_1, Q_2 differ by at most one; and since the lengths sum to at least $\delta(\text{rk}(X_1) + \text{rk}(X_2)) \geq 2c$, both lengths are at least c.

To show that P is a passage, it remains to check that $\text{dist}_G(P, Y) > \delta(k + 1 + \text{rk}(Y))$ for each $Y \in \mathcal{T}$ with $Y \neq X_1, X_2$. To see this, let R be a shortest path from Y to $V(P)$, with an end $p \in V(P)$. From the symmetry we may assume that $p \in V(Q_1)$. Let P' be the subpath of P between X_1, p. Since $p \in \Delta_{\mathcal{T}}(X_1) \setminus \Delta_{\mathcal{T}}(Y)$, it follows that $|E(R)| \geq |E(P')|$. But the sum of their lengths is more than $\delta(\text{rk}(X_1) + \text{rk}(Y))$, and so R has length more than $\delta(\text{rk}(X_1) + \text{rk}(Y))/2 \geq \delta(k + 1 + \text{rk}(Y))$ since $\text{rk}(X_1) \leq k$. This proves 9.2. ∎

We call passages as in 9.2 *adjoinment passages.* (We will never need an adjoinment passage between two houses.) The eight graphs drawn in Figures 8 and 9 all represent configurations of forts, $\mathcal{T}$-communities and passages that we can prove do not appear if $\mathcal{T}$ is optimal; the vertices in F (drawn solid) represent forts, the other vertices (drawn hollow) represent pairwise disjoint $\mathcal{T}$-communities, and each edge represents a passage joining the forts or $\mathcal{T}$-communities represented by its ends. All edges represent adjoinment passages except possibly one, the "long edge" when it appears. Thus, for example, the first graph in Figure 8 shows a $\mathcal{T}$-community adjoining three forts. The first graph of Figure 9 shows a fort adjoining two forts and a $\mathcal{T}$-community C say; C is joined to another $\mathcal{T}$-community D by a passage, and D adjoins a fourth fort. If σ is a standard, we will often use the notation $\sigma = (\delta, \alpha, \beta)$ for the terms of σ without explicitly saying so.

9.3 *Let σ be a standard, let G be a graph, and let $\mathcal{T}$ be a kth-century realm in G, optimal for σ. There do not exist forts, pairwise disjoint $\mathcal{T}$-communities, and passages forming a configuration represented in any of the drawings in Figures 8 or 9.*

Proof. Suppose that these things do exist; so, three or four forts $X_1, \ldots, X_{m_1}$, at most four $\mathcal{T}$-communities $C_1, \ldots, C_{m_2}$, and at most seven passages $P_1, \ldots, P_{m_3}$, making a configuration as in the drawing. Choose the numbering such that X_1, X_2, X_3 are the three forts represented by the three leaves in the drawing, and P_i is the passage with an end in X_i for $i = 1, 2, 3$. Again for $i = 1, 2, 3$, let p_i be the end of P_i in X_i, let η_i exhibit H_k as a c-superfat minor of $G[X_i]$, let $J_i = \eta_i(H_k)$, and let Q_i be a path of $G[X_i]$ between p_i and $V(J_i)$. Choose a vertex r_i of the path $P_i \cup Q_i$ with $\operatorname{dist}_G(r_i, J_i) \le c + 1$ such that the subpath of $P_i \cup Q_i$ between r_i and J_i is maximal (thus, r_i might lie in $V(P_i)$ or in X_i). Let R_i be the Λ-shortest path in G between r_i and J_i. (Thus, R_i is not necessarily a path of $G[X_i] \cup P_i$.) Let S_i be the subpath of $P_i \cup Q_i$ between r_i and the end of $P_i \cup Q_i$ not in X_i. (See Figure 10.) From the choice of r_i,

$$\operatorname{dist}_G(S_i, J_i) = \operatorname{dist}_G(r_i, J_i) = |E(R_i)| = c + 1.$$

Let

$$W = V(S_1 \cup S_2 \cup S_3) \cup \bigcup_{4 \le i \le m_1} X_i \cup \bigcup_{1 \le i \le m_2} \Delta_{\mathcal{T}}(C_i) \cup \bigcup_{4 \le i \le m_3} V(P_i).$$

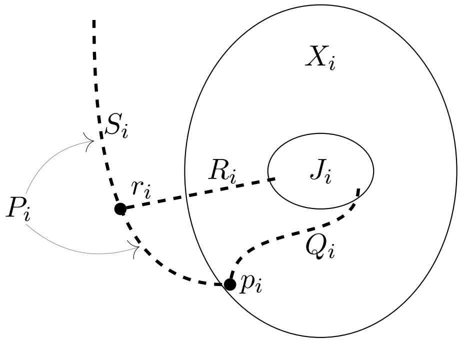

Figure 10: For the proof of 9.3. The vertex r_i might belong to Q_i.

(1) $G[W]$ *is connected, and* $\operatorname{dist}_G(W, J_i) = c + 1$ *for* $i = 1, 2, 3$.

$G[W]$ is clearly connected, since the corresponding configuration in Figures 8 or 9 remains connected when its leaves are deleted. Since $V(S_i) \subseteq W$ and $\operatorname{dist}_G(S_i, J_i) = c + 1$, it suffices to show that $\operatorname{dist}_G(W, J_i) \ge c + 1$ for $i = 1, 2, 3$. We claim:

- $\operatorname{dist}_G(S_i, J_i) \ge c + 1$ from the choice of r_i;
- $\operatorname{dist}_G(P_j, J_i) \ge \operatorname{dist}_G(P_j, X_i) > \delta(k + 1 + \operatorname{rk}(X_i)) = \delta(2k + 1) \ge c + 1$ for all $j \in \{1, \ldots, m_3\}$ with $j \ne i$, from the definition of a passage since P_j is not incident with X_i;

- $\operatorname{dist}_G(S_j, J_i) \geq c+1$ for $j = 1,2,3$ with $j \neq i$, since every vertex of S_j belongs either to P_j (and the claim follows from the previous bullet) or to X_j (and $\operatorname{dist}_G(X_j, X_i) > \delta(2k) \geq c+1$);
- $\operatorname{dist}_G(X_j, J_i) \geq \operatorname{dist}_G(X_j, X_i) > \delta(2k) \geq c+1$ for all $j \in \{1, \dots, m_1\}$ with $j \neq i$;
- $\operatorname{dist}_G(\Delta_{\mathcal{T}}(C_j), J_i) \geq \operatorname{dist}_G(\Delta_{\mathcal{T}}(C_j), X_i) \geq \operatorname{dist}_G(C_j, X_i)/2 \geq c+1$, since if $v \in \Delta_{\mathcal{T}}(C_j)$, then $\operatorname{dist}_G(v, C_j) \leq \operatorname{dist}_G(v, X_i)$ and

$$\operatorname{dist}_G(v, C_j) + \operatorname{dist}_G(v, X_i) \geq \operatorname{dist}_G(C_j, X_i) \geq 2c+2.$$

This proves (1).

Let $Z = W \cup X_1 \cup X_2 \cup X_3 \cup V(R_1 \cup R_2 \cup R_3)$.

(2) *Z has quasi-bound at most* $(9\alpha, \beta + 2d_0)$.

Let

$$A = \bigcup_{1 \leq i \leq m_1} X_i \cup \bigcup_{1 \leq i \leq m_2} V(C_i).$$

Since each of $X_1, \dots, X_{m_1}, V(C_1), \dots, V(C_{m_2})$ has quasi-line-width at most (α, β), so does A (because there are no edges between these sets). Since the boundary of each X_i and of each $V(C_i)$ has quasi-size at most (α, β), and $m_1 + m_2 \leq 8$, it follows that $\operatorname{bd}(A)$ has quasi-size at most $(8\alpha, \beta)$. Each vertex in $Z \setminus A$ has distance at most $(3d_0+1)/2 \leq 2d_0$ from some vertex in $\operatorname{bd}(A)$, and hence by 2.1, Z has quasi-line-width at most $(9\alpha, \beta + 2d_0)$, and $\operatorname{bd}(Z)$ has quasi-size at most $(8\alpha, \beta + 2d_0)$. This proves (2).

(3) *For each $Y \in \mathcal{T}$, either* $\operatorname{dist}_G(Z, Y) > \delta(k + 1 + \operatorname{rk}(Y))$, *or $Y \subseteq Z$.*

We assume that $Y \not\subseteq Z$, and so $Y \neq X_1, \dots, X_{m_1}$. Let M be a shortest path from Y to Z, and let z be its end in Z. Suppose that $z \in V(P_i)$ for some $i \in \{1, \dots, m_3\}$. Since P_i is a passage joining two houses or forts included in Z, and $Y \not\subseteq Z$, it follows that P_i is not incident with Y, and so $\operatorname{dist}_G(P_i, Y) > \delta(k + 1 + \operatorname{rk}(Y))$ from the definition of a passage, and the claim holds. So we assume that either z belongs to one of the forts X_i, or to one of the paths R_i, or to $\Delta_{\mathcal{T}}(X)$ for some house X in one of the $\mathcal{T}$-communities C_i. In either case, $z \in \Delta_{\mathcal{T}}(X)$ for some house or fort X included in Z (because for $i = 1,2,3$, every vertex of R_i has distance at most $c+1$ from X_i, and so $V(R_i) \subseteq \Delta_{\mathcal{T}}(X_i)$). Consequently

$$|E(M)| \geq \operatorname{dist}_G(X, Y)/2 > \delta(\operatorname{rk}(X) + \operatorname{rk}(Y))/2 \geq \delta(k + 1 + \operatorname{rk}(Y))$$

as required. This proves (3).

Let $\mathcal{T}'$ consist of the set of members of $\mathcal{T}$ that are not included in Z, together with Z, where Z is designated as a castle of $\mathcal{T}'$, and each other member of $\mathcal{T}'$ has the same class in $\mathcal{T}'$ that it has in $\mathcal{T}$. We claim that $\mathcal{T}'$ is a kth-century realm with standard σ.

To show this, we must check that the members of $\mathcal{T}'$ are pairwise vertex-disjoint buildings (which is true, since any member of $\mathcal{T}'$ that intersects Z is included in Z), and:

- For all $v \in V(G)$, there exists $X \in \mathcal{T}'$ such that $\mathrm{dist}_G(v, X) \le d_0$;
- If $X, Y \in \mathcal{T}'$ are distinct, then $\mathrm{dist}_G(X, Y) > \delta(\mathrm{rk}(X) + \mathrm{rk}(Y))$;
- Every fort or castle X of $\mathcal{T}'$ includes a c-superfat $H_{\mathrm{rk}(X)}$-minor of G; and
- Every castle of $\mathcal{T}'$ has quasi-bound at most $(9\alpha, \beta + 2d_0)$.

(The remaining conditions follow directly from the fact that $\mathcal{T}$ is a kth-century realm with standard σ.) The first is clear. For the second, we may assume that $X = Z$, so $\mathrm{rk}(X) = k + 1$. Since $Y \in \mathcal{T}'$ and therefore $Y \not\subseteq Z$, the second bullet follows from (3).

For the third, we only need check this when $X = Z$, and in that case the claim follows from 8.1 applied to W, J_1, J_2, J_3 and the geodesics R_1, R_2, R_3. The fourth bullet holds by (2). This proves that $\mathcal{T}'$ is a kth-century realm with standard σ, contrary to the optimality of $\mathcal{T}$. So there are no such $X_1, \dots, X_{m_1}, C_1, \dots, C_{m_2}, P_1, \dots, P_{m_3}$. This proves 9.3. ▮

Let $C \subseteq \mathcal{T}$. We need to talk about $\mathcal{T}$-communities that are subsets of C and maximal with this property. Let us call such sets *C-villages.* If $X_1, X_2 \in C$ are both forts, we say they *C-semiadjoin* if either X_1 adjoins X_2, or there is a C-village that adjoins them both. If $X \in C$ is a fort, we say X is *C-peripheral* if X C-semiadjoins at most one other fort in C. If D is a C-village, we say D is *C-peripheral* if D adjoins at most one fort in C, and any such fort is C-peripheral. (See Figure 11.) If $A \subseteq V(G)$, $\mathcal{T}[A]$ denotes the set of members of $\mathcal{T}$ included in A.

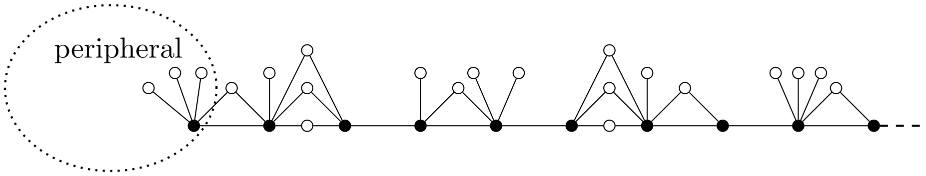

Figure 11: The black vertices represent forts in C, the others are C-villages, and edges are adjoinments.

An *integer interval* is a set I of integers such that if $a, b \in I$ then I contains all integers between a, b. We say (a, b) is an *end-set* of an integer interval I if $a, b \in I$, and $a - 1, b + 1 \notin I$ (and hence I is finite). If $\mathcal{T}$ is a realm, we denote the set of members of $\mathcal{T}$ that are houses or forts by $\mathcal{T}^0$.

9.4 *Let σ be a standard, let G be a graph, and let $\mathcal{T}$ be a kth-century realm in G, optimal for σ. Each $\mathcal{T}$-community adjoins at most two forts, and each fort $\mathcal{T}$-semiadjoins at most two other forts. Consequently, if C is an adjoin-connected subset of $\mathcal{T}^0$ that contains a fort, then the forts in C can be numbered as X_i ($i \in I$), where I is an integer interval, with the following properties:*

- *for each $i \in I$, if $i + 1 \in I$ then X_i C-semiadjoins X_{i+1}, and no other pair of forts in C $\mathcal{T}$-semiadjoin each other except possibly X_a, X_b, where (a, b) is an end-set of I and $b \ge a + 2$;*
- *each $\mathcal{T}$-community S with $S \subseteq C$ adjoins at most two forts in C (which therefore C-semiadjoin if there are two).*

Moreover, if C, D are disjoint adjoin-connected subsets of $\mathcal{T}^0$, and P is a passage joining some $X \in C$ and some $Y \in D$, then:

- *either X is a C-peripheral fort, or X belongs to a C-peripheral C-village, or D contains no forts; and*
- *either Y is a D-peripheral fort, or Y belongs to a D-peripheral D-village, or C contains no forts.*

Proof. Each $\mathcal{T}$-community adjoins at most two forts, because otherwise we have the configuration represented by the first drawing in Figure 8. Similarly, each fort $\mathcal{T}$-semiadjoins at most two forts, because otherwise we have a configuration represented by one of the other drawings in the same figure.

Let H be the graph with vertex set the set of forts in C, together with the set of all C-villages; two forts, or a fort and a C-village, are adjacent in H if they adjoin. (No two C-villages adjoin.) By 9.1, the forts in C can be numbered as in the theorem.

Finally, suppose that C, D are disjoint adjoin-connected subsets of $\mathcal{T}^0$, and P is a passage joining some $X \in C$ and some $Y \in D$. Suppose that X is not a C-peripheral fort and does not belong to a C-peripheral C-village, and there is a fort in D. Thus either

- X is a fort, and X C-semiadjoins two other forts in C; or
- X is a house, and the C-village containing X adjoins two forts in C; or
- X is a house, and the C-village containing X adjoins a fort in C that is not C-peripheral.

Moreover, the end of P in $V(D)$ either belongs to a fort of D, or it belongs to a D-village that adjoins a fort of D. This again gives one of the configurations represented in Figures 8 and 9 (the latter if P joins a house with a house), in contradiction to 9.3. This proves 9.4. ▮

We will need the following lemma about composing line-decompositions.

9.5 *Let $\mathcal{A}$ be a set of disjoint nonempty subsets of $V(G)$, with union W say, such that each $A \in \mathcal{A}$ has quasi-bound at most (a, b). Suppose also that there is a line-decomposition of $G[W]$ in which each bag is the union of at most k members of $\mathcal{A}$. Then $G[W]$ has quasi-line-width at most $((k+1)a, b)$.*

Proof. By hypothesis, there is a line-decomposition $(B'_t : t \in T)$ of $G[W]$ such that each bag is the union of at most k members of $\mathcal{A}$. Let Z be the union of the sets $\mathrm{bd}(A)$ $(A \in \mathcal{A})$, and define $B_t = B'_t \cap Z$ for each $t \in T$. Then $(B_t : t \in T)$ is a line-decomposition of $G[Z]$ such that each bag is the union of at most k sets of the form $\mathrm{bd}(A)$, and so has quasi-size at most (ka, b). For each $A \in \mathcal{A}$, since A is nonempty, there exists $r(A) \in T$ such that $B'_{r(A)} \cap A \neq \emptyset$, and hence $\mathrm{bd}(A) \subseteq B_{r(A)}$ (since $B_{r(A)}$ is a union of boundaries of members of $\mathcal{A}$ and the members of $\mathcal{A}$ are pairwise disjoint). By duplicating points of T, we may assume that the elements $r(A)$ $(A \in \mathcal{A})$ are all distinct; and also by duplicating all the points of T, we may assume that each $r(A)$ has a successor $s(A) \in T$ (that is, an element $s(A) \in T$ different from $r(A)$, such that $r(A) < s(A)$, and there is no $t \in T$ with $r(A) < t < s(A)$); and moreover, $B_{s(A)} = B_{r(A)}$. For each $A \in \mathcal{A}$, let $(C^A_t : t \in T^A)$ be a line-decomposition of $G[A]$ with quasi-width at most (a, b). By adding two new elements to T^A, we may assume that T^A has a maximum and minimum element; and so, by inserting T^A into T, we may assume that T^A equals $\{t \in T : r(A) \leq t \leq s(A)\}$, where $B_t = B_{r(A)}$ for each $t \in T^A$. For each $t \in T$, if $t \in T^A$ for some (necessarily unique) $A \in \mathcal{A}$, let $D_t = B_t \cup C_t^A$. If there is no such A, let $D_t = B_t$.

We claim that $(D_t : t \in T)$ is a line-decomposition of $G[W]$. If $v \in W$, choose $A \in \mathcal{A}$ with $v \in A$; then $v \in C_t^A$ for some $t \in T^A$, and hence $v \in D_t$. Next, suppose that $uv \in E(G[W])$. If there exists $A \in \mathcal{A}$ with $u, v \in A$, then $u, v \in C_t^A$ for some $t \in T^A$, and hence $u, v \in D_t$. If there is no such A, choose $A, A' \in \mathcal{A}$ with $u \in A$ and $v \in A'$. Then $u \in \mathrm{bd}(A)$ and $v \in \mathrm{bd}(A')$, and so uv is an edge of $G[Z]$. Choose $t \in T$ such that $u, v \in B_t$; then $u, v \in D_t$.

Finally, suppose that $r < s < t$ are elements of T, and $v \in D_r \cap D_t$. We need to show that $v \in D_s$. Choose $A \in \mathcal{A}$ with $v \in A$. If $v \notin Z$, then each of r, s, t belong to T^A, since no other bags contain v; and then, since $v \in C_r^A \cap C_t^A$, it follows that $v \in C_s^A \subseteq D_s$ as required. Now suppose that $v \in Z$. Then, for $p \in T$, v belongs to D_p if and only if $v \in B_p$; and so $v \in B_r \cap B_t \subseteq B_s \subset D_s$. This proves that $(D_t : t \in T)$ is a line-decomposition of $G[W]$. It is easy to check that each of its bags has quasi-size at most $((k+1)a, b)$. This proves 9.5. ▮

9.6 *Let σ be a standard, let G be a graph, let $\mathcal{T}$ be a kth-century realm in G, optimal for σ. Then $\Delta_{\mathcal{T}}(\mathcal{T}^0)$ has quasi-line-width at most $(10\alpha, \beta + d_0)$.*

Proof. Let $S = \Delta_{\mathcal{T}}(\mathcal{T}^0)$. Let J be the graph with vertex set the set of forts of $\mathcal{T}$ together with the set of all $\mathcal{T}$-villages, where we say $X, Y \in V(J)$ are adjacent if X adjoins Y (and consequently at least one of X, Y is a fort). From 9.4, each $\mathcal{T}$-village has degree at most two in J, and its neighbours in J are forts. By 9.4, J has line-width at most three. For each $X \in V(J)$, let $A(X) = \Delta_{\mathcal{T}}(X)$. Thus the sets $A(X)$ $(X \in V(J))$ are pairwise disjoint and have union S. Each $X \in V(J)$ has quasi-bound at most (α, β). Every vertex in $\Delta_{\mathcal{T}}(X)$ has distance at most d_0 from X, so by 2.1, it follows that $\mathrm{bd}(\Delta_{\mathcal{T}}(X))$ has quasi-size at most $(\alpha, \beta + d_0)$, and $\Delta_{\mathcal{T}}(X)$ has quasi-line-width at most $(2\alpha, \beta + d_0)$. By 9.5, S has quasi-line-width at most $(10\alpha, \beta + d_0)$. This proves 9.6. ▮

10 Governments

Let $0 \leq k \leq \ell - 1$, let σ be a standard, let G be a graph, and let $\mathcal{T}$ be a kth-century realm in G, optimal for σ. In the same way that we built a castle from three forts in 9.3, now we want to build higher-level objects called "palaces", but the construction regulations are more complicated. For $k + 1 \leq t \leq \ell$, a *palace (over $\mathcal{T}$)* of *rank* t is a building A in G with the following properties. If $t = k + 1$ then A is a castle of $\mathcal{T}$, and $\mathcal{P}_A = \emptyset$. If $t \geq k + 2$, then there is a set $\mathcal{P}_A$ of paths of $G[A]$, each of length at most $d_0 + 1$, and with $|\mathcal{P}_A| \leq 3^{t-k} - 3$, such that:

- every member of $\mathcal{T}$ is either a subset of A or disjoint from A;
- $A \subseteq \left(\bigcup_{X \in \mathcal{T}[A]} \Delta_{\mathcal{T}}(X)\right) \cup \left(\bigcup_{P \in \mathcal{P}_A} V(P)\right)$;
- A includes a c-superfat H_t-minor of G;
- there are at most $3^{t-k-1} - 1$ forts of $\mathcal{T}$ that are included in A and are $\mathcal{T}[A]$-peripheral;

- there are at most $(3^{t-k}-3)/2$ $\mathcal{T}[A]$-peripheral $\mathcal{T}[A]$-villages; and
- there are exactly 3^{t-k-1} castles of $\mathcal{T}$ included in A.

Define $\gamma = 8(\ell-k-1)3^{\ell-k-1}+1$. For $k+1 \le t \le \ell$, define $\pi_t = 3^{t-k-1}(10+\gamma)-\gamma$. So $\pi_{k+1}=10$, and $\pi_t = 3\pi_{t-1}+2\gamma$ for $t \ge k+2$. We say a *government* for $\mathcal{T}$ is a set $\mathcal{G}$ of pairwise disjoint buildings, each either a palace over $\mathcal{T}$, or a house or fort of $\mathcal{T}$, satisfying:

- If $A_1, A_2 \in \mathcal{G}$ are distinct, of ranks i, j respectively, then $\operatorname{dist}_G(A_1, A_2) > \delta(i+j)$.
- Each castle of $\mathcal{T}$ is a subset of some palace in $\mathcal{G}$.
- Every house or fort of $\mathcal{T}$ either belongs to $\mathcal{G}$ or is a subset of a palace in $\mathcal{G}$; we call the houses and forts that are not subsets of palaces *rebels.* (Thus, no castles are rebels.)
- Each palace $A \in \mathcal{G}$ has quasi-bound at most $(\pi_t\alpha, \beta+4d_0+1)$, where t is the rank of A.

If $A \in \mathcal{G}$, we denote its rank by $\operatorname{rk}(A)$. (Rebel houses and forts have the same rank in $\mathcal{G}$ as they do in $\mathcal{T}$.)

Thus, adding palaces is much like adding castles, but there is one important difference: the realm does not change. When a house or fort is used to make a castle, it is never going to be seen again, and we can forget it. But when a house, fort or castle is used to make a palace, it is not lost: it remains in the realm, and we may need it in the future, to help analyze properties of the palace.

Keeping $k, \sigma, G, \mathcal{T}$ fixed, we will consider different governments for $\mathcal{T}$. There is at least one, because $\mathcal{T}$ itself is a government for $\mathcal{T}$, since all castles have quasi-bound at most

$$(9\alpha, \beta+2d_0) \le (\pi_{k+1}\alpha, \beta+4d_0+1).$$

We call this the *self-government* of $\mathcal{T}$.

Let $\mathcal{G}$ be a government for $\mathcal{T}$. Thus, $\mathcal{G}$ is a sort of generalization of a realm, and once again we can use it to define Voronoi cells, using the same tie-breaker Λ as before. If $X \in \mathcal{G}$, let $\Delta_{\mathcal{G}}(X)$ be the corresponding Voronoi cell. Now we have two different Voronoi partitions, one defined by $\mathcal{T}$ and one defined by $\mathcal{G}$. We need to work with them both, so from now on, we write "$\mathcal{T}$-adjoin" or "$\mathcal{G}$-adjoin" to indicate which partition we are using. Since every member of $\mathcal{T}$ is a subset of a member of $\mathcal{G}$, it follows that if X is a rebel, then $\Delta_{\mathcal{G}}(X) \subseteq \Delta_{\mathcal{T}}(X)$; and consequently every $\mathcal{G}$-community is a $\mathcal{T}$-community. The converse is false: sets of rebels that are $\mathcal{T}$-communities need not be $\mathcal{G}$-communities.

10.1 *In the same notation, suppose that G does not contain H_ℓ as a c-fat minor. Let C be a $\mathcal{T}$-adjoin-connected set of rebels that contains at least one fort, and let $A \in \mathcal{G}$ be a palace. Let U be the set of vertices in $\Delta_{\mathcal{G}}(C)$ with a neighbour in $\Delta_{\mathcal{G}}(A)$. Then U has quasi-size at most*

$$\left(4 \cdot 3^{\ell-k-1}\alpha - 2\alpha, \beta+4d_0+1\right).$$

Proof. Let A have rank t; so $k+1 \le t \le \ell-1$ (since G does not contain H_ℓ as a c-fat minor). We say a subset of A is *special* if either it is a castle, or it is a $\mathcal{T}[A]$-peripheral fort, or it is a house in a $\mathcal{T}[A]$-peripheral $\mathcal{T}[A]$-village, or it is the vertex set of a member of $\mathcal{P}_A$.

(1) *For each $u \in U$, there is a path of length at most $2d_0 + 1$ between u and the boundary of some special subset of A.*

Since $u \notin A$, it suffices to show that there is a path of length at most $2d_0 + 1$ between u and some special subset of A. Choose $X \in C$ with $u \in \Delta_{\mathcal{G}}(X)$, and $v \in \Delta_{\mathcal{G}}(A)$ adjacent to u. Since $u \in \Delta_{\mathcal{G}}(X)$, the Λ-shortest path P between u, X is in $G[\Delta_{\mathcal{G}}(X)]$. Let Q be the Λ-shortest path between v, A (which is therefore a path of $G[\Delta_{\mathcal{G}}(A)]$). Let the ends of Q be v, q where $q \in A$. If q belongs to the vertex set of a member of $\mathcal{P}_A$, then the claim is true, so we may assume that there exists $X' \in \mathcal{T}[A]$ with $q \in \Delta_{\mathcal{T}}(X')$. If X' is a castle of $\mathcal{T}$, then X' is special and again the claim is true, so we assume that $\mathrm{rk}(X') \leq k$. Let R be the Λ-shortest path from q to X'. Thus, R has length at most d_0. (But R might not be contained in $G[A]$, and might contain vertices of Q different from q.) Let P' be a path between X, X' in the union of P, the edge uv, Q and R. Thus, P' has length at most $3d_0 + 1$. (See Figure 12.)

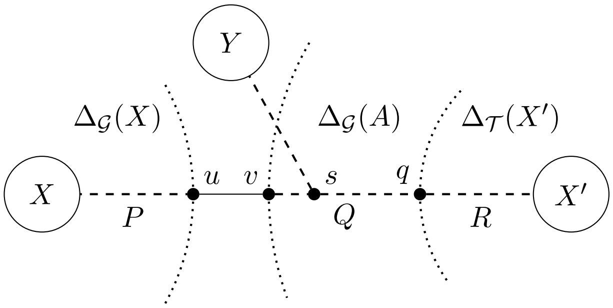

Figure 12: The passage. The picture is over-simplified, because R might contain more vertices of Q.

We claim that P' is a passage joining X, X'; and it remains to check that

$$\mathrm{dist}_G(P', Y) > \delta(k + 1 + \mathrm{rk}(Y))$$

for each $Y \in \mathcal{T}$ with $Y \neq X, X'$. We will show the stronger result that $\mathrm{dist}_G(Y, s) > \delta(k+1+\mathrm{rk}(Y))$ for each $s \in V(P \cup Q \cup R)$ and each such Y.

Suppose first that $s \in V(P)$. Since $s \in \Delta_{\mathcal{T}}(X)$, $\mathrm{dist}_G(s, Y) \geq \mathrm{dist}_G(s, X)$ and the two sum to at least

$$\mathrm{dist}_G(X, Y) > \delta(\mathrm{rk}(X) + \mathrm{rk}(Y)) \geq \delta(k + \mathrm{rk}(Y)).$$

Hence

$$\mathrm{dist}_G(Y, s) > \delta(k + \mathrm{rk}(Y))/2 \geq \delta(k + 1 + \mathrm{rk}(Y)),$$

as required. Similarly, if $s \in V(R)$, then

$$\mathrm{dist}_G(Y, s) > \delta(\mathrm{rk}(X') + \mathrm{rk}(Y))/2 \geq \delta(k + 1 + \mathrm{rk}(Y))$$

since $\mathrm{rk}(X') \leq k$, so we assume that $s \in V(Q)$. Write $\ell_1 = \mathrm{dist}_G(Y, s)$, and $\ell_2 = \mathrm{dist}_G(s, q)$, and $\ell_3 = \mathrm{dist}_G(X', q)$ for brevity. Since $q \in \Delta_{\mathcal{T}}(X')$, it follows that

$$\mathrm{dist}_G(Y, s) + \mathrm{dist}_G(s, q) \geq \mathrm{dist}_G(Y, q) \geq \mathrm{dist}_G(X', q),$$

and so $\ell_1 + \ell_2 \geq \ell_3$.

We recall that Q is the Λ-shortest path between v, A, and q is its end in A, and $s \in V(Q)$. Now there are three cases: Y is a rebel; Y is a subset of a palace in $\mathcal{G}$ different from A; and $Y \subseteq A$. But in each case, $\text{dist}_G(Y, s) \geq \text{dist}_G(s, q)$: if Y is a rebel or a subset of a different palace then because $s \in \Delta_{\mathcal{G}}(A)$, and if $Y \subseteq A$ then trivially. Thus, $\ell_1 \geq \ell_2$. It follows that

$$\ell_1 + (2\ell_1) + (\ell_1 + \ell_2) \geq \ell_1 + 2\ell_2 + \ell_3,$$

that is, $4\ell_1 \geq \ell_1 + \ell_2 + \ell_3$. Consequently,

$$\text{dist}_G(Y, s) \geq \text{dist}_G(Y, X')/4 > \delta(k + \text{rk}(Y))/4 \geq \delta(k + 1 + \text{rk}(Y)),$$

as required. (This is why we need the definition of "spacing" to be powers of 4, not of 2.) This proves that P' is a passage joining X, X'.

Now $X \in C$, and X' is a house or fort in $\mathcal{T}[A]$. Let D be the maximal subset of $\mathcal{T}[A]$ that contains X' and is $\mathcal{T}$-adjoin-connected. Since C is $\mathcal{T}$-adjoin-connected, and contains a fort by hypothesis, it follows from 9.4 that either X' is a D-peripheral fort, or X' belongs to a D-peripheral D-village. But every D-peripheral fort is $\mathcal{T}[A]$-peripheral, and every D-peripheral D-village is a $\mathcal{T}[A]$-peripheral $\mathcal{T}[A]$-village, from the maximality of D. Consequently X' is special. This proves (1).

The special houses in A can be partitioned into at most $(3^{t-k} - 3)/2$ $\mathcal{T}$-villages, and there are at most $3^{t-k-1} - 1$ special forts in A, and 3^{t-k-1} special castles, and $|\mathcal{P}_A| \leq 3^{t-k} - 3$. For each of these $\mathcal{T}$-villages, the union of the boundaries of its members has quasi-size at most (α, β); the boundary of each of the special forts has quasi-size at most (α, β); the boundary of each castle has quasi-size at most $(9\alpha, \beta + 2d_0)$, and the vertex set of each member of $\mathcal{P}_A$ has quasi-size at most $(1, d_0)$. Consequently, the union of the boundaries of the special subsets of A has quasi-size at most

$$\begin{aligned}&\left(\left(3^{t-k} - 3\right)(\alpha + 4)/2 + \left(3^{t-k-1} - 1\right)\alpha + 3^{t-k-1}(9\alpha), \beta + 2d_0\right)\\ \leq &\left(\left(4 \cdot 3^{t-k} - 2\right)\alpha, \beta + 2d_0\right)\end{aligned}$$

(since $\alpha \geq 12$). Hence, by (1), we deduce that U has quasi-size at most

$$\left(\left(4 \cdot 3^{t-k} - 2\right)\alpha, \beta + 4d_0 + 1\right).$$

Since $t < \ell$, this proves 10.1. ∎

11 Revolution

Again, let σ be a standard, let G be a graph that does not contain H_ℓ as a c-fat minor, let Λ be a tie-breaker, and let $\mathcal{T}$ be a kth-century realm in G, optimal for σ. Let $\mathcal{G}$ be a government for $\mathcal{T}$. It follows that all members of $\mathcal{G}$ have rank $< \ell$. For each rebel X, $\Delta_{\mathcal{G}}(X) \subseteq \Delta_{\mathcal{T}}(X)$, and so if two rebels $\mathcal{G}$-adjoin then they $\mathcal{T}$-adjoin each other; but whether two $\mathcal{T}$-adjoining rebels are $\mathcal{G}$-adjoining depends on the government.

Suppose that there is a palace A of $\mathcal{T}$ that does not belong to $\mathcal{G}$, that either includes or is disjoint from every member of $\mathcal{G}$. Let $\mathcal{G}'$ be obtained from $\mathcal{G}$ by removing all members of $\mathcal{G}$ that are included in A, and adding A itself. If $\mathcal{G}'$ is another government for $\mathcal{T}$, we say it is obtained from $\mathcal{G}$ by a *revolution*. Let us try to assemble the ingredients for a revolution.

We say a set C of rebels is *$\mathcal{G}$-village-closed* if the set of houses in C is a union of $\mathcal{G}$-villages; in other words, if no rebel house outside of C $\mathcal{G}$-adjoins a rebel house in C. A set C of rebels is a *cabal* if:

- C is $\mathcal{G}$-adjoin-connected and $\mathcal{G}$-village-closed;
- zero, one or two forts in C are designated as "leader forts"; and zero, one, two or three $\mathcal{G}$-villages included in C are designated as "leader $\mathcal{G}$-villages";
- every C-peripheral fort is a leader fort, and every C-peripheral C-village is a leader $\mathcal{G}$-village;
- if $X \in C$ $\mathcal{G}$-adjoins some rebel not in C, then either X is a leader fort, or X belongs to a leader $\mathcal{G}$-village;
- for some $j \in \{k+1, \dots, \ell-1\}$ there are three palaces $A_1, A_2, A_3 \in \mathcal{G}$ of rank j, such that for $1 \le i \le 3$, some member of C $\mathcal{G}$-adjoins A_i; and
- either $|C| = 1$, or C is a $\mathcal{G}$-village, or for each $j' \in \{k+1, \dots, \ell-1\}$ there are at most four palaces $A \in \mathcal{G}$ of rank j' such that some member of C $\mathcal{G}$-adjoins A.

We will show that if we can find a cabal, we can use it to make a revolution by fusing together some three palaces of $\mathcal{G}$ of the same rank to make a new palace of rank one bigger. The next result is used to control the quasi-bound of the new palace.

11.1 *With notation as above, suppose that C is a cabal. Then* $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$ *has quasi-size at most*

$$(16(\ell-k-1)3^{\ell-k-1}\alpha, \beta + 4d_0 + 1).$$

Proof. Since C is a cabal, there exists $j \in \{k+1, \dots, \ell-1\}$ and three palaces $A_1, A_2, A_3 \in \mathcal{G}$ of rank j, such that for $1 \le i \le 3$, some member of C $\mathcal{G}$-adjoins A_i. It follows that $k < j \le \ell - 1$, and in particular $k \le \ell - 2$. If $|C| = 1$, say $C = \{X\}$, then $\mathrm{bd}(X)$ has quasi-size at most (α, β), and so $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$ has quasi-size at most $(\alpha, \beta + d_0)$, since every vertex in $\mathrm{bd}(\Delta_{\mathcal{G}}(X))$ has distance at most d_0 from some vertex in $\mathrm{bd}(X)$. Similarly, if C is a $\mathcal{G}$-village, and hence a $\mathcal{T}$-community, then $\mathrm{bd}(V(C))$ has quasi-size at most (α, β), and hence again $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$ has quasi-size at most $(\alpha, \beta + d_0)$. Thus, we may assume that neither of these is true, and so for each $j \in \{k+1, \dots, \ell-1\}$ there are at most four palaces $A \in \mathcal{G}$ of rank j such that some member of C $\mathcal{G}$-adjoins A. Let $\mathcal{Q}$ be the set of all such palaces in $\mathcal{G}$; so $|\mathcal{Q}| \le 4(\ell - k - 1)$. If some $X \in C$ $\mathcal{G}$-adjoins some rebel Y not in C, then either X is a leader fort, or X belongs to a leader $\mathcal{G}$-village; and the boundary of each leader fort has quasi-size at most (α, β), and so does the boundary of the union of the members of each leader $\mathcal{G}$-village. Consequently, $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$ is the union of at most five sets of quasi-size at most $(\alpha, \beta + d_0)$, and at most $4(\ell - k - 1)$ further sets each with quasi-size at most

$$\left(4 \cdot 3^{\ell-k-1}\alpha - 2\alpha, \beta + 4d_0 + 1\right),$$

by 10.1. It follows that $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$ has quasi-size at most

$$\left(5\alpha + 4(\ell - k - 1)\left(4 \cdot 3^{\ell-k-1}\alpha - 2\alpha\right), \beta + 4d_0 + 1\right).$$

This proves 11.1. ∎

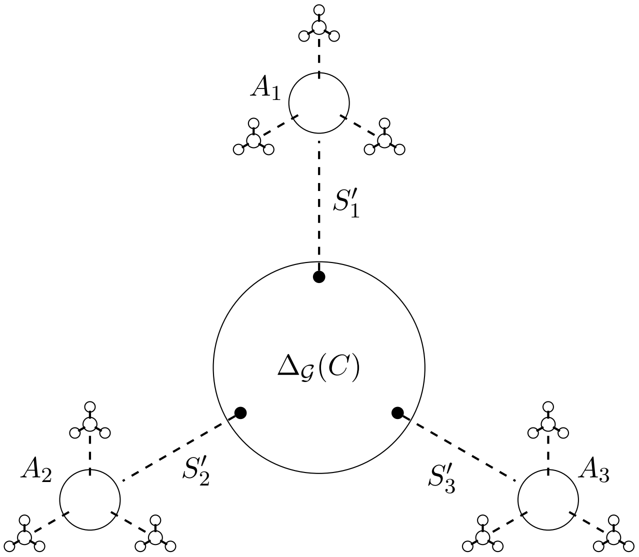

Figure 13: The headquarters of the cabal C. The three objects labelled A_i are palaces of rank j.

With notation as before, suppose that C is a cabal, and let j, A_1, A_2, A_3 be as in the fifth bullet in the definition of a cabal. For $i = 1, 2, 3$, choose $u_i \in \Delta_{\mathcal{G}}(C)$ with a neighbour $v_i \in \Delta_{\mathcal{G}}(A_i)$, and define P_i to be the path formed by the union of the edge u_iv_i and the Λ-shortest path in G between v_i, A_i. Hence, P_i has length at most $d_0 + 1$. Again for $i = 1, 2, 3$, let p_i be the end of P_i in A_i, let η_i exhibit H_j as a c-superfat minor of $G[A_i]$, let $J_i = \eta_i(H_j)$, and let Q_i be a path of $G[A_i]$ between p_i and $V(J_i)$. Choose a vertex r_i of the path $P_i \cup Q_i$ with $\mathrm{dist}_G(r_i, J_i) \leq c+1$ such that the subpath of $P_i \cup Q_i$ between r_i and J_i is maximal (thus, r_i might lie in $V(P_i)$ or in A_i). Let R_i be the Λ-shortest path in G between r_i and J_i. (Thus, R_i is not necessarily a path of $G[A_i] \cup P_i$; but, since it has length $c+1$ and has an end in J_i, all its vertices belong to $\Delta_{\mathcal{G}}(A_i)$.) Let S_i be the subpath of $P_i \cup Q_i$ between u_i, r_i. From the choice of r_i,

$$\mathrm{dist}_G(S_i, J_i) = \mathrm{dist}_G(r_i, J_i) = |E(R_i)| = c + 1.$$

Since $V(P_i) \subseteq \Delta_{\mathcal{G}}(A_i) \cup \{u_i\}$ and $V(Q_i) \subseteq A_i$, and $V(R_i) \subseteq \Delta_{\mathcal{G}}(A_i)$, and S_i is a subpath of $P_i \cup Q_i$, it follows that $V(P_i \cup Q_i \cup R_i \cup S_i) \subseteq \Delta_{\mathcal{G}}(A_i) \cup \{u_i\}$ for $i = 1, 2, 3$. Let

$$A = \Delta_{\mathcal{G}}(C) \cup A_1 \cup A_2 \cup A_3 \cup V(P_1 \cup R_1) \cup V(P_2 \cup R_2) \cup V(P_3 \cup R_3),$$

and define $\mathcal{P}_A$ to be the union of $\{P_1, P_2, P_3, R_1, R_2, R_3\}$ and the three sets $\mathcal{P}_{A_i}$ for $i = 1, 2, 3$ (taking $\mathcal{P}_{A_i} = \emptyset$ if A_i is a castle). It follows that

$$A \subseteq \Delta_{\mathcal{G}}(C) \cup \Delta_{\mathcal{G}}(A_1) \cup \Delta_{\mathcal{G}}(A_2) \cup \Delta_{\mathcal{G}}(A_3).$$

We call A the *headquarters* of the cabal C. (See Figure 13.) We will show that A is a new palace, and causes a revolution.

11.2 *With notation as above, let A be the headquarters of the cabal C. Then A is a palace over $\mathcal{T}$ of rank $j+1$.*

Proof. Certainly $G[\Delta_{\mathcal{G}}(C)]$ is connected, because C is $\mathcal{G}$-adjoin-connected, and $G[\Delta_{\mathcal{G}}(X)]$ is connected for each $X \in C$. Moreover, $G[A_i]$ is connected, and the path P_i joins $G[\Delta_{\mathcal{G}}(C)]$ and $G[A_i]$ for $i = 1,2,3$, and so $G[A]$ is connected.

The members of $\mathcal{P}_A$ have length at most d_0+1, since this is true for the paths in $\mathcal{P}_{A_i}$ for $i = 1,2,3$, and true for P_i, R_i for $i = 1,2,3$. Moreover,

$$|\mathcal{P}_A| = 6 + \sum_{1 \le i \le 3} |\mathcal{P}_{A_i}| \le 6 + 3(3^{j-k} - 3) = 3^{j+1-k} - 3.$$

We claim next that:

(1) $A \subseteq \left(\bigcup_{X \in \mathcal{T}[A]} \Delta_{\mathcal{T}}(X)\right) \cup \left(\bigcup_{P \in \mathcal{P}_A} V(P)\right)$.

For $i = 1,2,3$, since A_i is a palace and $\mathcal{P}_{A_i} \subseteq \mathcal{P}_A$, it follows that

$$A_i \subseteq \bigcup_{X \in \mathcal{T}[A_i]} \Delta_{\mathcal{T}}(X) \cup \bigcup_{P \in \mathcal{P}_{A_i}} V(P) \subseteq \bigcup_{X \in \mathcal{T}[A]} \Delta_{\mathcal{T}}(X) \cup \bigcup_{P \in \mathcal{P}_A} V(P).$$

Also, $\Delta_{\mathcal{G}}(C) \subseteq \bigcup_{X \in \mathcal{T}[A]} \Delta_{\mathcal{T}}(X)$ since $C \subseteq \mathcal{T}[A]$. And trivially $V(P_i \cup R_i) \subseteq \bigcup_{P \in \mathcal{P}_A} V(P)$ since $P_i, R_i \in \mathcal{P}_A$. This proves (1).

(2) *Every member of $\mathcal{T}$ is a subset of A or disjoint from A.*

Let $Y \in \mathcal{T}$ with $Y \cap A \neq \emptyset$, and choose $y \in Y \cap A$. Thus, by (1), either $y \in \Delta_{\mathcal{T}}(X)$ for some $X \in \mathcal{T}[A]$, or $y \in V(P)$ for some $P \in \mathcal{P}_A$. In the first case, $Y \cap \Delta_{\mathcal{T}}(X) \neq \emptyset$, and so $Y = X$ since $X, Y \in \mathcal{T}$, and hence $Y \subseteq A$. In the second case, either y belongs to some member of $\mathcal{P}_{A_i}$ for some $i \in \{1,2,3\}$, or $y \in V(P_i \cup R_i)$ for some $i \in \{1,2,3\}$. If y belongs to some member of $\mathcal{P}_{A_i}$ for some $i \in \{1,2,3\}$, then $Y \cap A_i \neq \emptyset$, and so $Y \subseteq A_i$ (because A_i is a palace) and so $Y \subseteq A$. Thus we may assume that

$$y \in V(P_i \cup R_i) \subseteq \Delta_{\mathcal{G}}(A_i) \cup \{u_i\}.$$

Hence either $Y \cap \Delta_{\mathcal{G}}(A_i) \neq \emptyset$, or $Y \cap \Delta_{\mathcal{G}}(C) \neq \emptyset$, and in either case it follows that $Y \subseteq A$. This proves (2).

To show that A is a palace, it remains to check that:

- A includes a c-superfat H_{j+1}-minor of G;
- there are at most $3^{j-k} - 1$ forts of $\mathcal{T}$ that are included in A and are $\mathcal{T}[A]$-peripheral;
- there are at most $(3^{j+1-k} - 3)/2$ $\mathcal{T}[A]$-peripheral $\mathcal{T}[A]$-villages; and

- there are exactly 3^{j-k} castles of $\mathcal{T}$ included in A.

The first follows from 8.1, applied with $W = \Delta_{\mathcal{G}}(C) \cup V(S_1 \cup S_2 \cup S_3)$ and the geodesics R_1, R_2, R_3. For the second, every $\mathcal{T}[A]$-peripheral fort included in A either belongs to C or is a $\mathcal{T}[A_i]$-peripheral fort included in some A_i. Since there are at most two in C (from the definition of a cabal) and at most $3^{j-1-k} - 1$ in A_i for $i = 1, 2, 3$, this proves the second bullet. The third and fourth are proved similarly. This proves 11.2. ▮

11.3 *In the same notation, let A be the headquarters of the cabal C, and $\mathcal{G}'$ be the union of $\{A\}$ and the set of members of $\mathcal{G}$ that are disjoint from A. Then $\mathcal{G}'$ is a government for $\mathcal{T}$.*

Proof. We need to check that:

- If $A' \in \mathcal{G}'$ with $A' \neq A$ then $\text{dist}_G(A, A') > \delta(\text{rk}(A') + j + 1)$.
- Each castle of $\mathcal{T}$ is a subset of some palace in $\mathcal{G}'$.
- Every house or fort of $\mathcal{T}$ either belongs to $\mathcal{G}'$ or is a subset of a palace in $\mathcal{G}'$.
- Each palace $A' \in \mathcal{G}'$ has quasi-bound at most $(\pi_t \alpha, \beta + 4d_0 + 1)$ where t is the rank of A'.

For the first bullet, let $A' \in \mathcal{G}'$ with $A' \neq A$. Let $v \in A$; we need to show that $\text{dist}_G(v, A') > \delta(\text{rk}(A') + j + 1)$. Since

$$v \in A \subseteq \Delta_{\mathcal{G}}(C) \cup \Delta_{\mathcal{G}}(A_1) \cup \Delta_{\mathcal{G}}(A_2) \cup \Delta_{\mathcal{G}}(A_3),$$

there exists $X \in \mathcal{G}$ with rank at most j, with $X \subseteq A$, and with $v \in \Delta_{\mathcal{G}}(X)$. Since $A', X \in \mathcal{G}$ and $v \in \Delta_{\mathcal{G}}(X)$, it follows that $\text{dist}_G(v, A') \geq \text{dist}_G(v, X)$, but

$$\text{dist}_G(v, A') + \text{dist}_G(v, X) > \delta(\text{rk}(A') + \text{rk}(X)),$$

so

$$\text{dist}_G(v, A') > \delta(\text{rk}(A') + \text{rk}(X))/2 > \delta(\text{rk}(A') + j + 1).$$

This proves the first bullet. The second holds since it holds for $\mathcal{G}$, and similarly so does the third. Finally, for the fourth bullet, it is only necessary to check the claim when $A' = A$. We know:

- A_1, A_2, A_3 all have quasi-bound at most $(\pi_j \alpha, \beta + 4d_0 + 1)$;
- $V(P_i), V(R_i)$ have quasi-bound at most $(1, d_0)$ for $i = 1, 2, 3$;
- $\Delta_{\mathcal{G}}(C)$ has quasi-line-width at most $(10\alpha, \beta + d_0)$, by 9.6; and
- $\text{bd}(\Delta_{\mathcal{G}}(C))$ has quasi-size at most $(16(\ell - k - 1)3^{\ell-k-1}\alpha, \beta + 4d_0 + 1)$ by 11.1.

Since there are no edges between any of the sets $A_1, A_2, A_3, \Delta_{\mathcal{G}}(C) \setminus \text{bd}(\Delta_{\mathcal{G}}(C))$, and $\pi_j \geq 10$, it follows that the union of these four sets also has quasi-line-width at most $(\pi_j \alpha, \beta + 4d_0 + 1)$. By adding

$$\text{bd}(\Delta_{\mathcal{G}}(C)) \cup V(P_1 \cup P_2 \cup P_3 \cup R_1 \cup R_2 \cup R_3)$$

to each of the bags of a corresponding line-decomposition, we deduce that A has quasi-line-width at most

$$(\pi_j\alpha + 16(\ell - k - 1)3^{\ell-k-1}\alpha + 6, \beta + 4d_0 + 1).$$

Its boundary $\mathrm{bd}(A)$ is a subset of the union of the boundaries of $A_1, A_2, A_3, \Delta_{\mathcal{G}}(C)$ and the set $V(P_1 \cup P_2 \cup P_3 \cup R_1 \cup R_2 \cup R_3)$, and so has quasi-size at most

$$(3\pi_j\alpha + 16(\ell - k - 1)3^{\ell-k-1}\alpha + 6, \beta + 4d_0 + 1).$$

Since

$$\pi_{j+1} = 3\pi_j + 2\gamma \geq 3\pi_j + 16(\ell - k - 1)3^{\ell-k-1} + 1$$

and $\alpha \geq 6$, it follows that A has quasi-bound at most $(\pi_{j+1}\alpha, \beta + 4d_0 + 1)$. This proves the fourth bullet, and consequently $\mathcal{G}'$ is a government. This proves 11.3. ▮

12 Stable government

With the century k fixed, and given a kth-century realm $\mathcal{T}$ that is optimal for some standard σ, we want to modify the self-government to make the best possible government. To do so, we say a government $\mathcal{G}_2$ *extends* a government $\mathcal{G}_1$ if:

- every rebel of $\mathcal{G}_1$ either is a rebel of $\mathcal{G}_2$, or is a subset of a palace of $\mathcal{G}_2$;
- every palace X_1 of $\mathcal{G}_1$ is a subset of a palace X_2 of $\mathcal{G}_2$, where $X_1 \subseteq X_2$, and either the rank of X_2 in $\mathcal{G}_2$ is greater than the rank of X_1 in $\mathcal{G}_1$, or $X_1 = X_2$ and the two ranks are equal.

Let us say a government $\mathcal{G}$ is *stable* if no other government extends $\mathcal{G}$. Since the self-government exists, it follows from Zorn's lemma that there is a stable government. If a government is stable, then by 11.2 and 11.3 there are no cabals.

12.1 *With notation as before, let $\mathcal{G}$ be a stable government for $\mathcal{T}$, and let C be a maximal $\mathcal{G}$-adjoin-connected set of rebels. Then $\Delta_{\mathcal{G}}(C)$ has quasi-bound at most*

$$\left(\ell\, 3^{\ell+1}\alpha, \beta + 4d_0 + 1\right).$$

Proof. By 9.6, $\Delta_{\mathcal{G}}(C) \subseteq \Delta_{\mathcal{T}}(C)$ has quasi-line-width at most $(10\alpha, \beta + d_0)$, so it remains to bound the quasi-size of $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$. If $k = \ell - 1$, then there are no castles in $\mathcal{T}$ and no palaces in $\mathcal{G}$ (because G does not contain H_ℓ as a c-fat minor), and so $\mathrm{bd}(\Delta_{\mathcal{G}}(C)) = \emptyset$ and the claim is true. So we may assume that $k \leq \ell - 2$.

If C is a $\mathcal{T}$-community, then, since $\mathrm{bd}(V(C))$ has quasi-size at most (α, β) and every vertex in $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$ has distance at most d_0 from $\mathrm{bd}(V(C))$, it follows that $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$ has quasi-size at most $(\alpha, \beta + d_0)$. So we may assume that C is not a $\mathcal{T}$-community.

We say that $C' \subseteq C$ is *dangerous* if C' is $\mathcal{G}$-adjoin-connected, and there exist $j > k$ and three members $A_1, A_2, A_3 \in \mathcal{G}$, all of rank j, such that for $1 \leq i \leq 3$, some member of C' $\mathcal{G}$-adjoins A_i. If C is dangerous, we want to choose a minimal subset D' of C that is still dangerous, and $\mathcal{G}$-village-closed, and $\mathcal{G}$-adjoin-connected,

(1) *C is not dangerous.*

Suppose that C is dangerous, but let us digress for a moment and consider what we need. What we would really like is to obtain a "good" dangerous subset C', where "good" means

- C' is $\mathcal{G}$-village-closed and $\mathcal{G}$-adjoin-connected,
- only a bounded number of palaces in the government $\mathcal{G}$-adjoin members of C';
- no rebel house or fort that is not in C' $\mathcal{G}$-adjoins a non-C'-peripheral fort in C, or a member of a non-C'-peripheral C'-village; and
- the number of C'-peripheral C'-villages is bounded. (There are automatically at most two C'-peripheral forts.)

The third condition is needed for 11.1, and the fourth condition is needed to prove 10.1.

C itself satisfies the first and third conditions. By simply choosing C' to be a minimal subset of C that is $\mathcal{G}$-village-closed, $\mathcal{G}$-adjoin-connected and dangerous, we can arrange the second bullet and fourth bullets, but that might wreck the third, so we need to be more cautious. Indeed, there are some cases where no good subset exists. For instance, if C consists of a single fort or a single $\mathcal{G}$-village, and $\mathcal{G}$-adjoins many palaces in $\mathcal{G}$, there is nothing we can do; or if $|C| = 6$, and consists of three forts and three $\mathcal{G}$-villages making a six-cycle under $\mathcal{G}$-adjoinment, and C is minimally dangerous, then again there is no good subset. This explains why cabals sometimes have "leader" terms that are not peripheral.

Let us continue the proof of (1). If some $\mathcal{G}$-village included in C is dangerous, then it is a cabal (designating itself as the only leader $\mathcal{G}$-village), a contradiction. So no $\mathcal{G}$-village included in C is dangerous. Thus we assume (for a contradiction) that C is dangerous and not a $\mathcal{T}$-community, and no $\mathcal{G}$-village in C is dangerous. Since C is $\mathcal{G}$-adjoin-connected, it is $\mathcal{T}$-adjoin-connected, and so we may number the forts in C as X_i $(i \in I)$ where I is an integer interval, as in 9.4. Since C is a maximal $\mathcal{G}$-adjoin-connected set of rebels, it follows that C is $\mathcal{G}$-village-closed.

For $i_1, i_2 \in I$ with $i_1 \le i_2$, let $C(i_1, i_2)$ be the union of $\{X_{i_1}, \ldots, X_{i_2}\}$ and all $\mathcal{G}$-villages that $\mathcal{G}$-adjoin one of $X_{i_1}, \ldots, X_{i_2}$. It follows that $C(i_1, i_2)$ is $\mathcal{G}$-village-closed. Since C is dangerous, we may choose i_1, i_2 with $i_2 - i_1$ minimal such that $C(i_1, i_2)$ is dangerous. (Possibly $i_1 = i_2$.)

Define D as follows:

- if $i_2 = i_1$, let $D = \{X_{i_1}\}$;
- if $i_2 = i_1 + 1$, let D be the union of $\{X_{i_1}, X_{i_2}\}$ and all $\mathcal{G}$-villages that $\mathcal{G}$-adjoin both of X_{i_1}, X_{i_2};
- if $i_2 \ge i_1 + 2$, let D be the union of $\{X_{i_1}, \ldots, X_{i_2}\}$ and all $\mathcal{G}$-villages that $\mathcal{G}$-adjoin one of $X_{i_1+1}, \ldots, X_{i_2-1}$.

In each case, D is $\mathcal{G}$-village-closed, and $\mathcal{G}$-adjoin-connected. Each member of $C(i_1, i_2) \setminus D$ is a house, and belongs to a $\mathcal{G}$-village that $\mathcal{G}$-adjoins one or both of X_{i_1}, X_{i_2} and none of $X_{i_1+1}, \ldots, X_{i_2-1}$.

Since $C(i_1, i_2)$ is dangerous, there exist $j > k$ and three members $A_1, A_2, A_3 \in \mathcal{G}$, all of rank j, such that for $1 \le i \le 3$, some member of $C(i_1, i_2)$ $\mathcal{G}$-adjoins A_i. For $i = 1, 2, 3$, A_i $\mathcal{G}$-adjoins either some member of D, or some $\mathcal{G}$-village included in $C(i_1, i_2) \setminus D$. So by adding to D at most three of the

$\mathcal{G}$-villages included in $C(i_1, i_2) \setminus D$, we can construct a set D' that is dangerous, $\mathcal{G}$-adjoin-connected, and $\mathcal{G}$-village-closed. Let us construct such a set D' by adding to D as few $\mathcal{G}$-villages as possible; so in particular, if D is dangerous then $D' = D$. We designate X_{i_1}, X_{i_2} as leader forts of D', and the (at most three) $\mathcal{G}$-villages with union $D' \setminus D$ as leader $\mathcal{G}$-villages of D'. We claim that D' is a cabal (which will be a contradiction). To show this, we must check that

- D' is $\mathcal{G}$-adjoin-connected, and $\mathcal{G}$-village-closed;
- at most two forts of D' are designated as leader forts, and at most three $\mathcal{G}$-villages of D' are designated as leader $\mathcal{G}$-villages;
- every D'-peripheral fort is a leader fort, and every D'-peripheral D'-village is a leader $\mathcal{G}$-village;
- if $X \in D'$ $\mathcal{G}$-adjoins some rebel not in D', then either X is a leader fort, or X belongs to a leader $\mathcal{G}$-village;
- for some $j \in \{k+1, \dots, \ell - 1\}$ there are three palaces $A_1, A_2, A_3 \in \mathcal{G}$ of rank j, such that for $1 \leq i \leq 3$, A_i $\mathcal{G}$-adjoins some member of D'; and
- either $|D'| = 1$, or D' is a $\mathcal{G}$-village, or for each $j' \in \{k+1, \dots, \ell - 1\}$ there are at most four palaces $A \in \mathcal{G}$ of rank j' such that A $\mathcal{G}$-adjoins a member of D'.

The first two bullets are clear. The third holds since no member of D is D-peripheral except possibly the forts X_{i_1}, X_{i_2}, and no D-village is D-peripheral. Hence the only D'-peripheral D'-villages are those (at most three) that we added.

For the fourth bullet, suppose that $X \in D'$ $\mathcal{G}$-adjoins some rebel $Y \notin D'$. If X is a fort, then $X = X_i$ for some $i \in \{i_1, \dots, i_2\}$; and since $Y \notin D$, it follows that $i \in \{i_1, i_2\}$ and so X is a leader fort. Now we assume that X is a house; let B be the $\mathcal{G}$-village of $\mathcal{G}$ that contains X. Since D' is $\mathcal{G}$-village-closed, it follows that $B \subseteq D'$. Since $Y \notin D'$, it follows that $Y \notin B$, and so B $\mathcal{G}$-adjoins Y and hence Y is a fort. Choose $i \in \{i_1, \dots, i_2\}$ such that B $\mathcal{G}$-adjoins X_i, with $i \notin \{i_1, i_2\}$ if possible. If $i \neq i_1, i_2$, then X_i $\mathcal{T}$-semiadjoins X_{i-1}, X_{i+1} and Y, contrary to 9.4. Hence we cannot choose $i \notin \{i_1, i_2\}$; so $B \not\subseteq D$, and therefore B is a leader $\mathcal{G}$-village. This proves the fourth bullet.

The fifth bullet holds since D' is dangerous. Finally, for the sixth bullet, suppose first that $D' \neq D$, and let B be a $\mathcal{G}$-village in $D' \setminus D$. Let $j' \in \{k+1, \dots, \ell - 1\}$. Since we added as few $\mathcal{G}$-villages to D as possible to make a dangerous set, $D' \setminus B$ $\mathcal{G}$-adjoins at most two members of $\mathcal{G}$ of rank j'; and since B is not dangerous, B also $\mathcal{G}$-adjoins at most two members of $\mathcal{G}$ of rank j'. Hence D' $\mathcal{G}$-adjoins at most four members of $\mathcal{G}$ of rank j' and the sixth bullet holds. So we may assume that $D' = D$, and so D is dangerous. If $i_1 \neq i_2$, then for every palace $A \in \mathcal{G}$, if A $\mathcal{G}$-adjoins some member of D then it also $\mathcal{G}$-adjoins a member of one of $C(i_1+1, i_2), C(i_1, i_2 - 1)$, and since $C(i_1 + 1, i_2)$ and $C(i_1, i_2 - 1)$ are not dangerous (by the minimality of $i_2 - i_1$), it follows that the sixth bullet holds. So we may assume that $i_1 = i_2$. Since $D = D'$, it follows that $D = \{X_{i_1}\}$ and so $|D| = 1$, and again the sixth bullet holds.

Hence D' is a cabal, a contradiction since $\mathcal{G}$ is a stable government. This proves (1).

Thus, C is not dangerous, and so for $k + 1 \leq j \leq \ell - 1$, there are at most two palaces $A \in \mathcal{G}$ with rank j such that A $\mathcal{G}$-adjoins some member of C. Let $\mathcal{Q}$ be the set of all such palaces in $\mathcal{G}$; so $|\mathcal{Q}| \leq 2(\ell - k - 1)$. But since C is a maximal $\mathcal{G}$-adjoin-connected set of rebels, it follows that for each $u \in \mathrm{bd}(\Delta_{\mathcal{G}}(C))$ there exists $A \in \mathcal{Q}$ such that u has a neighbour in $\Delta_{\mathcal{G}}(A)$; and so by 10.1, $\mathrm{bd}(\Delta_{\mathcal{G}}(C))$ has quasi-size at most

$$\left(8(\ell - k - 1)3^{\ell-k-1}\alpha, \beta + 4d_0 + 1\right),$$

and the theorem holds. This proves 12.1. ▮

13 Into a new century

Finally, we can move into a new century, and deduce the main theorem.

13.1 *Let $k < \ell$ be a century, let $\sigma = (\delta, \alpha, \beta)$ be a standard, and let G be a graph admitting a kth-century realm with standard σ, and not containing H_ℓ as a c-fat minor. Let τ be the canon*

$$(\delta(2\ell), \ell\, 3^{2\ell}\alpha, \beta + 4d_0 + 1).$$

Then G admits a $(k+1)$st-century τ-society.

Proof. Since G admits a kth-century realm with standard σ, it admits a kth-century realm, $\mathcal{T}$ say, that is optimal for σ. Since there is the self-government for $\mathcal{T}$, there is a stable government $\mathcal{G}$ for $\mathcal{T}$. Let $\mathcal{T}' = \mathcal{G}$, where every house or fort of $\mathcal{G}$ is designated as a house of $\mathcal{T}'$, and every palace of $\mathcal{G}$ is designated as a fort of $\mathcal{T}'$. If $X \in \mathcal{T}'$, we write $\mathrm{rk}'(X) = k$ or $k+1$ depending whether X is a house or fort of $\mathcal{T}'$. We claim that $\mathcal{T}'$ is a $(k+1)$st-century τ-society.

Let $\alpha' = \ell\, 3^{2\ell}\alpha$ and $\beta' = \beta + 4d_0 + 1$. We must check that

1. The members of $\mathcal{T}'$ are pairwise vertex-disjoint and induce connected subgraphs of G.
2. For all $v \in V(G)$, there exists $X \in \mathcal{T}'$ such that $\mathrm{dist}_G(v, X) \leq d_0$.
3. If $X, Y \in \mathcal{T}'$ are distinct then $\mathrm{dist}_G(X, Y)$ is more than $\delta(2\ell)$.
4. Every fort of $\mathcal{T}'$ includes a c-superfat H_{k+1}-minor of G.
5. Each fort of $\mathcal{T}'$ has quasi-bound at most (α', β').
6. For every $\mathcal{T}'$-village C, $\Delta_{\mathcal{T}'}(C)$ has quasi-bound at most (α', β').

Statement 1 is clear, and statement 2 holds since each member of $\mathcal{T}$ is a subset of some member of $\mathcal{T}'$. The third statement is clear.

The fourth statement holds since if X is a fort of $\mathcal{T}'$, then $X = A$ for some $A \in \mathcal{G}$; and since A has rank some $t > k$, A includes a c-superfat H_t-minor of G, and so also includes a c-superfat H_{k+1}-minor of G.

For statement 5, let X be a fort of $\mathcal{T}'$, and hence $X = A$ for some palace $A \in \mathcal{G}$. Since $\mathcal{G}$ is a government, A has quasi-bound at most $(\pi_t\alpha, \beta')$ (where t is the rank of A in $\mathcal{G}$). But π_t is at most

$$3^{\ell-k-2}(10 + 8(\ell-k-1)3^{\ell-k-1} + 1) = 8(\ell-k-1)3^{2\ell-2k-3} + 11 \cdot 3^{\ell-k-2} \leq \ell\, 3^{2\ell}.$$

We deduce that statement 5 holds.

Finally, for statement 6, let C be a $\mathcal{T}'$-village. Thus, C is a maximal $\mathcal{G}$-adjoin-connected set of rebels of $\mathcal{G}$, so the statement follows from 12.1. This proves 13.1. ▮

By combining 13.1 with 6.1 we deduce:

13.2 *Let $k \geq 0$ be a century with $k < \ell$, and let $\tau = (d, \alpha, \beta)$ and $\tau' = (d', \alpha', \beta')$ be canons, satisfying:*

$$\begin{aligned} \alpha &\geq 12 \\ d' &\geq 5c \\ d_0 &\geq 4^{2\ell} d' \\ d &= 4^{2\ell - 2k + 2} d' \\ \alpha' &= \ell\, 3^{2\ell} \alpha \\ \beta' &= \beta + 4d_0 + 1 \end{aligned}$$

Let G be a connected graph that does not contain H_ℓ as a c-fat minor. If G admits a civilized kth-century τ-society then it admits a $(k+1)$st-century τ'-society.

Proof. Define $\delta(t) = 4^{2\ell - t} d'$ for $0 \leq t \leq 2\ell$. Then δ is a spacing, since $\delta(2\ell) = d' \geq 5c$ and $\delta(0) \leq d_0$. Hence $\sigma = (\delta, \alpha, \beta)$ is a standard, because $\alpha \geq 12$. Since $d = \delta(2k - 2)$ if $k \geq 1$, and $d \geq \delta(0)$ if $k = 0$, 6.1 implies that G admits a kth-century realm with standard σ. By 13.1, G admits a $(k+1)$st-century τ'-society. This proves 13.2. ▮

In turn, by combining the previous result with 5.1, we obtain:

13.3 *Let $k \geq 0$ be a century with $k < \ell$, and let $\tau = (d, \alpha, \beta)$ and $\tau' = (d', \alpha', \beta')$ be canons, satisfying:*

$$\begin{aligned} \alpha &\geq 6 \\ d &\leq d_0 \\ d' &\geq 5c \\ d &\geq 4^{4\ell - 2k + 4} d' \\ \alpha' &= 2\ell\, 3^{2\ell} \alpha \\ \beta' &\geq \beta + 7d_0 \end{aligned}$$

Let G be a connected graph that does not contain H_ℓ as a c-fat minor. If G admits a kth-century τ-society then it admits a $(k+1)$st-century τ'-society.

Proof. It follows that $4^{2\ell} d' \leq d_0$. Let $d'' = 4^{2\ell - 2k + 2} d'$ (and so $d \geq 4^{2\ell + 2} d''$). Let $\alpha'' = 2\alpha$ (and so $\alpha' = \ell\, 3^{2\ell} \alpha''$). Let $\beta'' = \beta + 2d_0 + 4^{\ell+1} d''$. Since

$$\beta' \geq \beta + 7d_0 = \beta'' - 4^{\ell+1} d'' + 5d_0$$

and

$$4^{\ell+1} d'' = 4^{3\ell - 2k + 3} d' \leq 4^{3\ell - 2k + 3} 4^{-4\ell + 2k - 4} d \leq 4^{-\ell - 1} d_0 \leq d_0 - 1.$$

it follows that $\beta' \geq \beta'' + 4d_0 + 1$. Since G admits a kth-century τ-society and $d \geq 4^{\ell+2} d''$, it also admits a kth-century $(4^{\ell+2} d'', \alpha, \beta)$-society. By 5.1, G admits a civilized kth-century (d'', α'', β'')-society, since $\beta'' \geq \beta + 2d_0 + 4^{\ell+1} d''$. By 13.2, G admits a $(k+1)$st-century τ'-society. This proves 13.3. ▮

Now we can deduce the main result:

13.4 *For all integers $\ell \ge 1$ and $c \ge 0$ there exist α, β such that every graph that does not contain H_ℓ as a c-fat minor has quasi-line-width at most (α, β).*

Proof. It suffices to prove the result when $\ell \ge 4$ and $c \ge 2$. For $0 \le k \le \ell$, let us define $d_k = 5 \cdot 4^{5\ell(\ell-k)}c$ (note that this at last defines $d_0 = 5 \cdot 4^{5\ell^2}c$), and $\alpha_k = 6(2\ell\, 3^{2\ell})^k$, and $\beta_k = 1 + 7kd_0$, and $\tau_k = (d_k, \alpha_k, \beta_k)$. Let $\alpha = \alpha_\ell$ and $\beta = \beta_\ell$; we claim that α, β satisfy the theorem. By working with each component of G separately, we may assume that G is connected. By Zorn's lemma, there exists $S \subseteq V(G)$ maximal such that $\operatorname{dist}_G(u,v) > d_0$ for all distinct $u, v \in S$. Let $\mathcal{T}_0 = \{\{v\} : v \in S\}$ where each member of $\mathcal{T}_0$ is assigned to be a fort of $\mathcal{T}_0$. Then $\mathcal{T}_0$ is trivially a 0th-century τ_0-society. By ℓ applications of 13.3 we deduce that G admits an ℓth-century τ_ℓ-society $\mathcal{T}$. Since G does not contain H_ℓ as a c-fat minor, it follows that $\mathcal{T}$ has no forts, and, since G is connected, $\mathcal{T}$ has only one village C say, with $\Delta_{\mathcal{T}}(C) = V(G)$. Moreover, $\Delta_{\mathcal{T}}(C)$ has quasi-bound at most $(\alpha_\ell, \beta_\ell)$, from the definition of a society. This proves 13.4. ∎

Consequently, this proves 1.7, and hence 1.6, and together with 1.5 proves 1.4.

14 Graph searching

For finite graphs G, these results are related to graph searching. Intuitively, imagine that there is an infection loose in the graph; any vertex of the graph instantly becomes infected if it is adjacent to an infected vertex, unless a doctor is positioned on it. A search can be thought of as a sequence of positions of doctors in a procedure to eliminate the infection. (Or, equally, a sequence of moves by a band of cops to capture an invisible robber.)

Let G be a finite graph, let $X \subseteq V(G)$, and let F be a subset of $V(G)$ with $\operatorname{bd}(F) \subseteq X \subseteq F$. We say that (X, F) is a *split* of G. If (X, F) and (X', F') are splits, we say that (X, F) *justifies* (X', F') if $F \setminus X \subseteq F'$, and either $X \subseteq X'$ or $X' \subseteq X$. A *search* of G is a finite sequence $(X_1, F_1), \dots, (X_n, F_n)$ of splits of G such that

- $(X_1, F_1) = (\emptyset, V(G))$, and $(X_n, F_n) = (\emptyset, \emptyset)$; and
- (X_i, F_i) justifies (X_{i+1}, F_{i+1}) for $1 \le i \le n-1$.

We call the sets X_i the *bags* of the search. (In terms of infections, at the ith stage, X_i is the set of vertices being treated, and all infected vertices are in F_i. In cops and robbers language, at time i the cops occupy the vertices in X_i, and the robber is known to be within F_i.)

For instance, let $(B_1, \dots, B_n)$ be a path-decomposition of a finite graph G. Define $R_i = B_i \cup B_{i+1} \cup \dots \cup B_n$ for $1 \le i \le n$. Then the sequence

$$(\emptyset, R_1), (B_1, R_1), (B_1 \cap B_2, R_2), (B_2, R_2), (B_2 \cap B_3, R_3), (B_3, R_3), \dots, (B_{n-1} \cap B_n, R_n), (B_n, R_n), (\emptyset, \emptyset)$$

is a search.

A search $(X_1, F_1), \dots, (X_n, F_n)$ is *monotone* if $F_{i+1} \subseteq F_i$ and $F_{i+1} \cap X_i \subseteq X_{i+1}$ for $1 \le i < n$. The search above, derived from a path-decomposition, is monotone, and it is easy to see that every monotone search arises from a path-decomposition in this way, so monotone searches are essentially the same as path-decompositions.

One important parameter of a search $(X_1, F_1), \ldots, (X_n, F_n)$ is its *bag-size* $\max(|X_1|, \ldots, |X_n|)$. If we fix some number k, when is there a search with bag-size at most k? It was proved by Kirousis and Papadimitriou [10, 11], building on work of LaPaugh [12] (who proved the same thing for a slightly different kind of search), that:

14.1 *For each integer k and finite graph G, if there is a search with bag-size at most k, then there is a monotone search with bag-size at most k.*

Consequently, there is a search with bag-size at most k if and only if G has path-width at most $k-1$.

Here we are concerned with searches in which each bag has quasi-size at most (a,b), for some fixed (a,b). We say that a finite graph G is *(a,b)-searchable* if there is a search $(X_1, F_1), \ldots, (X_n, F_n)$ of G such that X_i has quasi-size at most (a,b) for $1 \le i \le n$. It is easy to prove that for all a, b, there exists $c, \ell \ge 2$ such that if G is (a,b)-searchable then G does not contain H_ℓ as a c-fat minor. (Proved in 14.5, below).

Our result 1.7 gives a converse:

14.2 *For all $c, \ell \ge 2$, there exist a, b such that if a finite graph G does not contain H_ℓ as a c-fat minor, then G is (a,b)-searchable (because it has bounded quasi-line-width).*

We want to make two comments. First, we also have a direct proof of 14.2 (for finite G), that we found before we found the proof of 1.7, and it is considerably simpler. The idea is, to replace the assertions that certain sets have bounded quasi-bound by the weaker assertions that these sets are *subsets* of sets with bounded quasi-bound. Then all our problems with communities go away, and so we don't need the sections about societies. Except the proof of 9.5 does not work any more; but it works if we replace quasi-line-width with being (a,b)-searchable with a, b bounded, and this is why we have to introduce graph searching. So, it would be really nice if we could find a direct proof of:

14.3 *For all $a, b \ge 1$ there exist $a', b' \ge 1$ such that if a finite graph is (a,b)-searchable then it has quasi-line-width at most (a', b').*

We know that this is true, because of the main theorem of this paper, but if a simple direct proof could be found, it would give a simpler proof of 1.7.

Second, how do a', b' depend on a, b in 14.3? In particular, can we take $a = a'$? Yes we can if $b = 0$, by 14.1, and in that case we can also take $b' = 0$. But what happens if $b > 0$? For instance, if $b = 1$, can we take $a' = a$?

On this topic, let us mention that there is an interesting conjecture of Chudnovsky, Gollin, Krnc, and Milanič [4] with a special case that is closely related to our results, the following:

14.4 Conjecture: *For every tree T there exists k such that if a finite graph G has no induced minor isomorphic to T, then there is a balanced separator in G that is the union of at most k balls of radius one.*

An "induced minor" is much the same as a 1-fat minor: G contains H as an *induced minor* if H can be obtained from G by vertex-deletion and edge-contraction, and the deletion of all resultant loops and parallel edges. A *balanced separator* in G is a cutset X such that each component of $G \setminus X$ contains at most $|V(G)|/2$ vertices. It is easy to see that if G is $(a,1)$-searchable for some bounded a then the desired balanced separator exists.

Finally, here is a proof of the claim we made earlier in this section:

14.5 *For all $a, b \geq 1$, no finite graph that contains H_{4a} as a $2b$-fat minor is (a, b)-searchable.*

Proof. Given integers a, b, let $c = 2b$, and let $H = H_{4a}$, with root r. Let G be a finite graph that contains H as a c-fat minor, and choose η that exhibits H as a c-fat minor of G.

Suppose that there is a search $(X_1, F_1), \ldots, (X_n, F_n)$ of G such that each X_i has quasi-size at most (a, b). For $1 \leq i \leq n$, let Y_i be the set of $h \in V(H)$ such that either

- $V(\eta(h)) \cap X_i \neq \emptyset$; or
- h is an end of some edge $f \in E(H)$ such that $V(\eta(f)) \cap X_i \neq \emptyset$.

There exists $Z_i \subseteq V(G)$ with $|Z_i| \leq a$ such that every vertex $x \in X_i$ satisfies $\mathrm{dist}_G(x, Z_i) \leq b$. Since η exhibits H as a c-fat minor of G, and $c = 2b$, it follows that if $x_1, x_2 \in V(H)$ are distinct vertices of H then $\mathrm{dist}_G(\eta(x_1), \eta(x_2)) > c$; and so for each $z \in Z_i$, there is at most one vertex $x \in V(H)$ with $\mathrm{dist}_G(z, \eta(x)) \leq b$. Similarly, there is at most one edge $f \in E(H)$ such that $\mathrm{dist}_G(z, \eta(f)) \leq b$, and if there is one of each type, then the edge and vertex are incident in H. Consequently $|Y_i| \leq 2a$.

(1) *For $1 \leq i \leq n$, if Q is a component of $H \setminus Y_i$, and $F_i \cap V(\eta(x)) \neq \emptyset$ for some vertex or edge x of Q, then $V(\eta(x)) \subseteq F_i \setminus X_i$ for every vertex or edge x of Q.*

Let R be the subgraph of G induced on the union of $\eta(x)$ over all vertices and edges x of Q. Thus R is connected since Q is connected. For each $v \in V(R)$, if $v \in X_i$, choose a vertex or edge x of Q with $v \in V(\eta(x))$; then $x \in Y_i$ (if x is a vertex) or one end of x is in Y_i (if x is an edge), and in either case some vertex of Q is in Y_i, contradicting that Q is a component of $H \setminus Y_i$. Thus $V(R) \cap X_i = \emptyset$. By hypothesis, $F_i \cap V(\eta(x)) \neq \emptyset$ for some vertex or edge x of Q. But $V(\eta(x)) \subseteq V(R)$, and so $V(R) \cap F_i \neq \emptyset$. Hence, $V(R) \subseteq F_i \setminus X_i$, since (X_i, F_i) is a split of G and R is connected and $V(R) \cap X_i = \emptyset$. This proves (1).

It follows from (1) that if Q is a component of $H \setminus Y_i$ and $F_i \cap V(\eta(x)) \neq \emptyset$ for some vertex or edge x of Q, then $V(\eta(x)) \subseteq F_i$ for every vertex or edge x of Q. For $1 \leq i \leq n$, let P_i be the union of Y_i and the vertex sets of all components Q of $H \setminus Y_i$ with the property that $F_i \cap V(\eta(x)) \neq \emptyset$ for some vertex or edge x of Q.

Now each (Y_i, P_i) is a split of H. We claim that:

(2) *$P_i \subseteq P_{i+1} \cup Y_i$ for $1 \leq i < n$.*

To see this, let $h \in P_i \setminus Y_i$; we need to show that $h \in P_{i+1}$, and may therefore assume that $h \notin Y_{i+1}$. Let Q_i, Q_{i+1} be the components of $H \setminus Y_i, H \setminus Y_{i+1}$ respectively that contain h. Since $h \in P_i$, it follows that $V(\eta(x)) \subseteq F_i \setminus X_i \subseteq F_{i+1}$ for every vertex or edge x of Q_i. Hence, $F_{i+1} \cap V(\eta(x)) \neq \emptyset$ for some vertex or edge x of Q_{i+1} (namely, $x = h$), and so $V(Q_{i+1}) \subseteq P_{i+1}$. This proves (2).

From (2), and since $(Y_1, P_1) = (\emptyset, V(H))$ and $(Y_n, P_n) = (\emptyset, \emptyset)$, the sequence $(Y_1, P_1), \ldots, (Y_n, P_n)$ is a search of H. But $H = H_{4a}$, and therefore has path-width at least $2a + 1$ (see [16]), and so is not $(2a + 1, 0)$-searchable, by a result of Kirousis and Papadimitriou [10, 11]. Yet $Y_1, \ldots, Y_n$ each have cardinality at most $2a$, a contradiction. This proves 14.5. ∎

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