Asymptotic strucure. I. Coarse tree-width

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Abstract

We prove that a graph G admits a tree-decomposition in which each bag is contained in the union of a bounded number of balls of bounded radius, if and only if G admits a quasi-isometry to a graph with bounded tree-width. (The “if” half is easy, but the “only if” half is challenging.) This generalizes a recent result of Berger and Seymour, concerning tree-decompositions when each bag has bounded radius.

1 Introduction

We need to begin with some definitions. Graphs in this paper may be infinite. If X is a vertex of a graph G, or a subset of the vertex set of G, or a subgraph of G, and the same for Y, then $\operatorname{dist}_G(X,Y)$ denotes the distance in G between X, Y, that is, the number of edges in the shortest path of G with one end in X and the other in Y. (If no path exists we set $\operatorname{dist}_G(X,Y) = \infty$.)

Let G, H be graphs, and let $\phi : V(G) \to V(H)$ be a map. Let $L, C \geq 0$; we say that ϕ is an *(L,C)-quasi-isometry* if:

- for all u, v in $V(G)$, if $\operatorname{dist}_G(u,v)$ is finite then $\operatorname{dist}_H(\phi(u), \phi(v)) \leq L \operatorname{dist}_G(u,v) + C$;
- for all u, v in $V(G)$, if $\operatorname{dist}_H(\phi(u), \phi(v))$ is finite then $\operatorname{dist}_G(u,v) \leq L \operatorname{dist}_H(\phi(u), \phi(v)) + C$; and
- for every $y \in V(H)$ there exists $v \in V(G)$ such that $\operatorname{dist}_H(\phi(v), y) \leq C$.

If $X \subseteq V(G)$, let us say the *diameter of X in G* is the maximum of $\operatorname{dist}_G(u,v)$ over all $u, v \in X$. A *tree-decomposition* of a graph G is a pair $(T, (B_t : t \in V(T)))$, where T is a tree (possibly infinite), and B_t is a subset of $V(G)$ for each $t \in V(T)$ (called a *bag*), such that:

- $V(G)$ is the union of the sets B_t ($t \in V(T)$);
- for every edge $e = uv$ of G, there exists $t \in V(T)$ with $u, v \in B_t$; and
- for all $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path of T between t_1, t_3, then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$.

The *width* of a tree-decomposition $(T, (B_t : t \in V(T)))$ is the maximum of the numbers $|B_t| - 1$ for $t \in V(T)$, or ∞ if there is no finite maximum; and the *tree-width* of G is the minimum width of a tree-decomposition of G. If T is a path, we call $(T, (B_t : t \in V(T)))$ a *path-decomposition*, and the *path-width* of G is defined analogously.

Our first result is an extension of a result of Berger and Seymour [1] (which can also be derived from a combination of results of Chepoi et al. [3]). They proved:

1.1 *For all r, if G is connected and admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t has diameter at most r in G, then G admits a $(1, 6r+1)$-quasi-isometry to a tree.*

This has a sort of converse, also proved in [1]: if G is connected and (L,C)-quasi-isometric to a tree then it admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that B_t has diameter at most $L(L+C+1)+C$ in G, for each $t \in V(T)$.

We will extend 1.1 from trees to graphs of bounded tree-width, as follows (although saying that this extends 1.1 is something of a stretch, because we do not know whether 1.2 holds with $L = 1$):

1.2 *For all k, r, there exist $L, C \geq 1$ such that if G admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each with diameter at most r in G, then G admits an (L,C)-quasi-isometry to a graph with tree-width at most k.*

A similar result (with weaker constants) was obtained independently by R. Hickingbotham [6], by applying a result of Dvořák and Norin [5].

Our proof obtains a quasi-isometry to a graph with a tree-decomposition indexed by a subdivision of the same tree T that indexed the tree-decomposition of G; and so if T is a path, we find a quasi-isometry to a graph with bounded path-width. Consequently:

1.3 *For all k, r, there exist $L, C \geq 1$ such that if G admits a path-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each with diameter at most r in G, then G admits an (L, C)-quasi-isometry to a graph with path-width at most k.*

A path-decomposition is essentially a sequence of sets of vertices satisfying the "betweenness" condition. There is a more general notion (see [2, 4, 8, 7]), where we replace the sequence by a family of subsets indexed by a linearly ordered set, giving what we call "line-width". Line-width and path-width are the same for finite graphs, but for infinite graphs they may be different. One could ask whether 1.3 works with line-width in place of path-width, but we do not know.

In 1.2, we start with a tree-decomposition in which each bag is the union of k bounded-radius balls, and we obtain a tree-decomposition in which each bag has size at most $k+1$: and one might hope that the final k in the statement of 1.2 should be $k-1$. Obviously not for $k = 1$; but not when $k \geq 2$ either. To see this when $k = 2$, let G be a cycle, with vertices v_1-$\cdots$-v_n-v_1 in order. For $1 \leq i \leq n-1$, let $B_{v_i} = \{v_i, v_{i+1}, v_n\}$, and let T be the tree $G \setminus \{v_n\}$. Then $(T, (B_t : t \in V(T)))$ is a tree-decomposition of G, and each of its bags is the union of two balls of bounded radius (one the singleton $\{v_n\}$ and the other consisting of two adjacent vertices). On the other hand, for all (L, C), if n is large enough then there is no (L, C)-quasi-isometry from G to a graph with tree-width at most 1. A similar example works for each value of $k \geq 2$ (take a $k \times k$ grid and subdivide each of its edges many times).

Again, 1.2 has a sort of converse, because if G admits an (L, C)-quasi-isometry to a graph with tree-width at most k, then G admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most $k+1$ sets each of bounded diameter — we will prove this in the next section. But if we start with a graph G that admits a quasi-isometry to a graph with tree-width at most k, and apply this converse, we obtain a tree-decomposition in which each bag is a union of $k+1$ sets of bounded diameter; and if we then apply 1.2, we obtain a quasi-isometry to a graph with tree-width at most $k+1$. Somewhere we went from tree-width k to tree-width $k+1$, and this is unsatisfying, at least on aesthetic grounds.

A way to get rid of it is to make a small tweak in the definition of tree-decomposition; say a *pseudo-tree-decomposition* $(T, (B_t : t \in V(T)))$ is the same as a tree-decomposition, except we relax the condition that every edge has both ends in some bag. Instead, we insist that for every edge uv, either some bag contains both u, v, or there is an edge st of T such that $B_s \setminus B_t = \{u\}$ and $B_t \setminus B_s = \{v\}$. Define *pseudo-tree-width* correspondingly (it differs from tree-width by at most one). We will prove a version of 1.2 with "tree-width at most k" replaced by "pseudo-tree-width at most $k-1$", and a version of 2.1 with "tree-width at most k" replaced by "pseudo-tree-width at most k", and the anomalous error of one is gone. More exactly, we will prove:

1.4 *For all k, r, there exist L, C such that if G admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each with diameter at most r in G, then G admits an (L, C)-quasi-isometry to a graph with pseudo-tree-width at most $k-1$.*

Conversely, for all $L, C \geq 1$, if G admits an (L, C)-quasi-isometry to a graph with pseudo-tree-width at most $k-1$, then G admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each of diameter at most $2L(L+C)+C$.

2 The proof of 1.4

Let us state the definition of pseudo-tree-width more formally. A *pseudo-tree-decomposition* of a graph G is a pair $(T, (B_t : t \in V(T)))$, where T is a tree, and B_t is a subset of $V(G)$ for each $t \in V(T)$ (called a *bag*), such that:

- $V(G)$ is the union of the sets B_t ($t \in V(T)$);
- for every edge $e = uv$ of G, either there exists $t \in V(T)$ with $u, v \in B_t$, or there is an edge $st \in E(T)$ such that $B_s \setminus B_t = \{u\}$ and $B_t \setminus B_s = \{v\}$; and
- for all $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path of T between t_1, t_3, then $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$.

The *width* of a pseudo-tree-decomposition $(T, (B_t : t \in V(T)))$ is the maximum of the numbers $|B_t|-1$ for $t \in V(T)$, or ∞ if there is no finite maximum; and the *pseudo-tree-width* of G is the minimum width of a pseudo-tree-decomposition of G. If T is a path, we call $(T, (B_t : t \in V(T)))$ a *pseudo-path-decomposition*, and the *pseudo-path-width* of G is defined analogously. When T is a finite path, we sometimes use the notation $(B_1, \dots, B_n)$ (with the usual meaning) in place of $(T, (B_t : t \in V(T)))$.

Before we prove the main part of 1.4, let us prove its (much easier) second part, the converse:

2.1 *If G admits an (L, C)-quasi-isometry to a graph with pseudo-tree-width at most $k-1$, then G admits a tree-decomposition $(T, (D_t : t \in V(T)))$ such that for each $t \in V(T)$, D_t is the union of at most k sets each of diameter at most $2L(L+C)+C$.*

Proof. Let H be a graph with pseudo-tree-width at most $k-1$, and let $(T, (B_t : t \in V(T)))$ be a pseudo-tree-decomposition of H with width at most $k-1$. Let ϕ be an (L, C)-quasi-isometry from a graph G to H. For each $h \in V(H)$, let X_h be the set of vertices $i \in V(H)$ such that $\text{dist}_H(h, i) \le L + C$. For each $t \in V(T)$, let D_t be the set of all vertices $v \in V(G)$ such that $\phi(v) \in X_h$ for some $h \in B_t$. We claim that $(T, (D_t : t \in V(T)))$ is a tree-decomposition of G satisfying the theorem. So we must check that:

- $\bigcup_{t \in V(T)} D_t = V(G)$;
- for every edge uv of G there exists $t \in V(T)$ with $\{u, v\} \in D_t$;
- for all $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path of T between t_1, t_3, then $D_{t_1} \cap D_{t_3} \subseteq D_{t_2}$; and
- for each $t \in V(T)$, D_t is the union of at most k sets each of diameter (in G) at most $2L(L+C)+C$.

For the first statement, let $v \in V(G)$; then $\phi(v) \in V(H)$, and so $\phi(v) \in B_t$ for some $t \in V(T)$. In particular, since $\phi(v) \in X_{\phi(v)}$, it follows that $v \in D_t$. This proves the first statement.

For the second statement, let $uv \in E(G)$, and choose $t \in V(T)$ with $\phi(v) \in B_t$. Since ϕ is an (L, C)-quasi-isometry, $\text{dist}_H(\phi(u), \phi(v)) \le L + C$, and so $\phi(u) \in X_{\phi(v)}$. It follows that $u, v \in D_t$. This proves the second statement.

For the third statement, let $t_1, t_2, t_3 \in V(T)$, such that t_2 lies on the path of T between t_1, t_3, and let $v \in D_{t_1} \cap D_{t_3}$. Hence for $i = 1, 3$, there exists $h_i \in B_{t_i}$ with $\phi(v) \in X_{h_i}$; let P_i be a path of H between $\phi(v), h_i$ of length at most $L + C$. Since $P_1 \cup P_3$ is a connected graph with vertices in B_{t_1} and in B_{t_3}, it also has a vertex in B_{t_2}, say h_2. Thus h_2 belongs to one of $V(P_1), V(P_3)$, and so $\mathrm{dist}_H(h_2, \phi(v)) \leq L + C$; and hence $\phi(v) \in X_{h_2}$, and therefore $v \in D_{t_2}$. This proves the third statement.

Finally, for the fourth statement, let $t \in V(T)$. For each $h \in B(t)$, let F_h be the set of all $v \in V(G)$ such that $\phi(v) \in X_h$. Thus D_t is the union of the sets F_h ($h \in B_t$), and there are $|B_t| \leq k$ such sets. We claim that each F_h has diameter at most $2L(L + C) + C$ in G. If $u, v \in F_h$, then each of $\phi(u), \phi(v)$ has distance at most $L + C$ from h, and so $\mathrm{dist}_H(\phi(u), \phi(v)) \leq 2(L + C)$. Since ϕ is an (L, C)-quasi-isometry, it follows that $\mathrm{dist}_H(u, v) \leq 2L(L + C) + C$. This proves the fourth statement, and so proves 2.1. ∎

To prove 1.4, we need the following lemma:

2.2 *Let G be a graph, and let A, B be disjoint subsets of $V(G)$ with union $V(G)$. Let $|A|, |B| \leq k$, and suppose that there are at most k edges between A, B. Then there is a pseudo-path-decomposition $(B_1, \ldots, B_n)$ of G with width at most $k - 1$ and with $A \subseteq B_1$ and $B \subseteq B_n$.*

Proof. We proceed by induction on $k+|A|+|B|$. If some vertex $a \in A$ has no neighbours in B, then from the inductive hypothesis, applied to $G \setminus \{a\}$, there is a pseudo-path-decomposition $(B_1, \ldots, B_n)$ of $G \setminus \{a\}$ with width at most $k - 1$ and with $A \setminus \{a\} \subseteq B_1$ and $B \subseteq B_n$. But then $(A, B_1, \ldots, B_n)$ satisfies the theorem. Thus we may assume that each vertex in A has a neighbour in B, and vice versa.

If every vertex in A has exactly one neighbour in B and vice versa, the result is true; so we assume that some vertex in A has at least two neighbours in B, and hence $|A| \leq k - 1$. Let $b \in B$ with a neighbour in A, and let G' be obtained by deleting b. In G', there are at most $k - 1$ edges between A and $B \setminus \{b\}$, and these two sets both have size at most $k - 1$. From the inductive hypothesis applied to G', there is a pseudo-path-decomposition $(C_1, \ldots, C_n)$ of G' with width at most $k - 2$ and with $A \subseteq C_1$ and $B \setminus \{b\} \subseteq C_n$. Define $B_i = C_i \cup \{b\}$ for $1 \leq i \leq n$; then $(B_1, \ldots, B_n)$ is a pseudo-path-decomposition of G satisfying the theorem. This proves 2.2. ∎

To prove the first part of 1.4, it suffices to prove it when G is connected (by working with each component of G separately); and it suffices to prove it when $r = 1$. To see the latter, let G be a connected graph that admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that for each $t \in V(T)$, B_t is the union of at most k sets each with diameter at most r in G. For each $t \in V(T)$, and each pair u, v of nonadjacent vertices of $G[B_t]$ with $\mathrm{dist}_G(u, v) \leq r$, add an edge joining u, v, and let G' be the resultant graph. Then $(T, (B_t : t \in V(T)))$ is a tree-decomposition of G', and for each $t \in V(T)$, B_t is the union of at most k cliques of G'. Moreover, the identity map is an $(r, 0)$-quasi-isometry between G, G'; and so if G' admits an (L, C)-quasi-isometry to a graph with pseudo-tree-width at most $k - 1$, then G admits an (rL, rC)-quasi-isometry to the same graph. Consequently, for given k, if L, C satisfy the theorem when $r = 1$, then rL, rC satisfy the theorem for general r. Hence it suffices to prove the following:

2.3 *For all k, if G is connected and admits a tree-decomposition $(T, (B_t : t \in V(T)))$ such that B_t is the union of at most k cliques for each $t \in V(T)$, then G admits a $(2k + 2, 2k - 1)$-quasi-isometry to a graph with pseudo-tree-width at most $k - 1$.*

Proof. Let $(T, (B_t : t \in V(T)))$ be a tree-decomposition of G such that for each $t \in V(T)$, B_t is the union of at most k cliques. Fix a root $r \in V(T)$ (arbitrarily). For each $t \in V(T)$, its *ancestors* are the vertices of the path of T between r, t, and its *strict ancestors* are its ancestors different from t. If s is an ancestor of t then t is a *descendant* of s, and descendants of t different from t are *strict descendants* of t. For $t \in V(T)$, its *height* is the length of the path of T between r, t.

We will recursively define a set of pairs, called "cores". Each core will be a pair (t, C) where $t \in V(T)$ and C is a subset of B_t inducing a non-null connected subgraph, and we will call t its *birthday*. The set of all cores with the same birthday will be given an arbitrary linear order called the "birth order", and if (t, C) precedes (t, C') in the birth order then we will say that (t, C) is an *elder sibling* of (t, C'), and (t, C') is a *younger sibling* of (t, C). Each core (t, C) will have a *spread* $S(t, C)$, which is the vertex set of a certain subtree of T with root t, defined below.

Here is the inductive definition. If there exists $t \in V(T)$ such that we have not yet defined the set of cores with birthday t, choose some such t with minimum height. We say $v \in B_t$ is *disqualified* if there is a core (s, C) such that s is a strict ancestor of t, and $t \in S(s, C)$, and either $v \in C$ or v has a neighbour in C. Let Z be the set of vertices in B_t that are not disqualified. For each component C of $G[Z]$, we define (t, C) to be a core; this defines the set of all cores with birthday t. Choose an arbitrary linear order, called the *birth order*, of the set of cores with birthday t. For each core (t, C), its *spread* $S(t, C)$ is the set of all $t' \in V(T)$ such that

- t' is a descendant of t;
- $C \cap B_{t'} \neq \emptyset$;
- $t' \in S(s, C')$ for every core (s, C') such that s is a strict ancestor of t and $t \in S(s, C')$; and
- $t' \in S(t, C')$ for every elder sibling (t, C') of (t, C).

This completes the inductive definition of the set of all cores. We see that the spread of every core includes the spread of all its younger siblings; and for any two cores, either their spreads are vertex-disjoint, or one is included in the other.

Two subsets $X, Y \subseteq V(G)$ are *anticomplete* if they are disjoint and there are no edges of G between them. We need, first:

(1) *If $(t_1, C_1), (t_2, C_2)$ are distinct cores and their spreads intersect, then C_1, C_2 are anticomplete.*

We may assume that $t_1 \neq t_2$. Since the spreads of $(t_1, C_1), (t_2, C_2)$ intersect, t_1, t_2 have a common descendant t_0 say, so one of t_1, t_2 is a strict ancestor of the other. Hence we may assume that t_1 is a strict ancestor of t_2, and therefore $t_2 \in S(t_1, C_1)$ since the spreads intersect. Since (t_2, C_2) is a core, it follows that for each $v \in C_2$, $v \notin C_1$ and v has no neighbour in C_1. Consequently, C_1, C_2 are anticomplete. This proves (1).

(2) *For each $t \in V(T)$, there are at most k cores (s, C) such that $t \in S(s, C)$.*

Let $(s_1, C_1), \ldots, (s_n, C_n)$ be the set of all cores whose spread contains t, and let $D_1, \ldots, D_m$ be cliques with union B_t, with $m \leq k$. The sets $C_1 \cap B_t, \ldots, C_n \cap B_t$ are nonempty, and by (1) they are pairwise anticomplete. Consequently, for $1 \leq i \leq n$, there exists $j_i \in \{1, \ldots, m\}$ such that $C_i \cap B_t$ contains a vertex of D_{j_i}; and if $i, i' \in \{1, \ldots, n\}$ are distinct, then $j_i \neq j_{i'}$, because $C_i \cap B_t$ and $C_{i'} \cap B_t$ are anticomplete and D_{j_i} is a clique. Thus $n \leq m \leq k$. This proves (2).

For each $v \in V(G)$, there exists $t \in V(T)$ with $v \in B_t$, and the set of such vertices t induces a subtree of T. In particular, there is a unique $t \in V(T)$ of minimum height with $v \in B_t$, and we call t the *source* of v. If t is the source of v, there might or might not exist $C \subseteq B_t$ with $v \in C$ such that (t, C) is a core. If there exists such C we say v is *central.* If there exists a core (t', C') such that t' is a strict ancestor of t and $t \in S(t', C')$ and v has a neighbour in C', we say v is *peripheral.* (Note that v cannot belong to C', from the definition of t.)

(3) *Every vertex $v \in V(G)$ is central or peripheral, and not both.*

Let t be the birthday of v. The first statement is clear from the definition of the set of cores with birthday t. For the "not both" part, suppose that v is central and peripheral; choose $C \subseteq B_t$ with $v \in C$ such that (t, C) is a core, and choose a core (t', C') such that t' is a strict ancestor of t and $t \in S(t', C')$ and v has a neighbour in C'. Since $t \in S(t, C) \cap S(t', C')$, and $v \in C$ has a neighbour in C', this contradicts (1). This proves (3).

For each $v \in V(G)$, we define a core $\phi(v)$ as follows. Let $t_1 \in V(T)$ be the source of v. If v is central, $\phi(v)$ is the core (t_1, C_1) with $v \in C_1$. Now assume v is peripheral. Hence there is a strict ancestor t_0 of t_1 and a core (t_0, C_0) such that $t_1 \in S(t_0, C_0)$, and v has a neighbour in C_0. Choose such t_0 of minimum height; and of all the cores (t_0, C_0) such that $t_1 \in S(t_0, C_0)$, and v has a neighbour in C_0, choose (t_0, C_0) with this property, as early as possible in the birth order. We define $\phi(v) = (t_0, C_0)$.

(4) *Let $v \in V(G)$, let $\phi(v) = (t_0, C_0)$, and let $t \in V(T)$, such that $v \in B_t$. Then exactly one of the following holds:*

- *v is peripheral, and $t \in S(t_0, C_0)$; or*
- *there is a core (t', C') with $t \in S(t', C')$ and $v \in C'$.*

If both statements hold, then since $t \in S(t_0, C_0)$ and $t \in S(t', C')$ and there is an edge between C_0, C' (because $v \in C'$ and has a neighbour in C_0), this contradicts (1). So not both hold. We prove that at least one holds by induction on the height of t. If there exists C with $v \in C$ such that (t, C) is a core, the claim is true, so we assume not. Hence, from the definition of cores, there is a core (t_2, C_2) with $t \in S(t_2, C_2)$, such that t_2 is a strict ancestor of t and v belongs to or has a neighbour in C_2. If $v \in C_2$, the claim holds, so we assume that $v \notin C_2$ and v has a neighbour in C_2.

Let t_1 be the source of v. Thus, t_0, t_1, t_2 all belong to the path of T between r, t, and t_0 is an ancestor of t_1. Suppose that either t_2 is a strict ancestor of t_0, or (t_2, C_2) is an elder sibling of (t_0, C_0); and hence v is peripheral, in both cases. Since v has a neighbour in C_2, this contradicts the definition of $\phi(v)$. So we assume that either t_2 is a strict descendant of t_0 or (t_2, C_2) is a younger sibling of (t_0, C_0).

If $t = t_1$ the result is true, so we assume that $t \neq t_1$. Let s be the parent of t; so s lies in the path of T between t_1, t, and therefore $v \in B_s$. From the inductive hypothesis, either v is peripheral and $s \in S(t_0, C_0)$, or there is a core (t', C') with $s \in S(t', C')$ and $v \in C'$.

Suppose the first holds. Since either t_0 is a strict ancestor of t_2, or (t_0, C_0) is an elder sibling of (t_2, C_2), and since $S(t_2, C_2)$ contains t and $t_2 \in S(t_0, C_0)$, it follows (from the second half of the definition of cores) that $S(t_2, C_2) \subseteq S(t_0, C_0)$. Thus $t \in S(t_0, C_0)$ and the claim is true.

So we assume the second holds, that is, there is a core (t', C') with $s \in S(t', C')$ and $v \in C'$. If $t \in S(t', C')$ the claim holds, so we assume not. Since t_2 is a strict ancestor of t and $t \in S(t_2, C_2)$, it follows that t_2 is an ancestor of s and $s \in S(t_2, C_2)$. But there is an edge between C_2, C', since $v \in C'$ and v has a neighbour in C_2; and so from (1), either $(t', C') = (t_2, C_2)$ or the spreads of (t', C') and (t_2, C_2) are disjoint. The first is impossible since $t \notin S(t', C')$ and $t \in S(t_2, C_2)$, and the second is impossible since s belongs to both spreads. This proves (4).

(5) *Let P be a path of T with one end r, and let $v \in V(G)$. Let $\phi(v) = (t_0, C_0)$. Let $\mathcal{C}(P, v)$ be the set of cores (t, C) such that $t \in V(P)$ and $v \in C$. Let the members of $\mathcal{C}(P, v)$ with birthday different from t_0 be $(t_1, C_1), \dots, (t_n, C_n)$, numbered such that $t_0, t_1, \dots, t_n$ have strictly increasing height. Then:*

- *$t_i \notin S(t_h, C_h)$ for $0 \le h < i \le n$;*
- *for $1 \le i \le n$, let s_i be the parent of t_i: then $s_i \in S(t_{i-1}, C_{i-1})$;*
- *$n \le k - 1$.*

The first bullet holds by (1), since $v \in C_i$ and either $v \in C_h$, or $h = 0$ and v has a neighbour in C_h.

For the second bullet, let t'_0 be the source of v. Thus t_0 is an ancestor of t'_0 (possibly $t'_0 = t_0$), and $t_1, \dots, t_n$ are strict descendants of t'_0 (to see that $t_1 \neq t'_0$, observe that this is trivially true if v is not central, and true if v is central since then $t_0 = t'_0$.) Let $1 \le i \le n$. If $v \notin B_{s_i}$, then $i = 1$ and $t_i = t'_0$, which is impossible. So $v \in B_{s_i}$. If $s_i \in S(t_0, C_0)$, then $t_{i-1} \in S(t_0, C_0)$, and so $i = 1$ by the first bullet of (5) (because otherwise $t_{i-1} \notin S(t_0, C_0)$) and the claim is true. So we assume that $s_i \notin S(t_0, C_0)$. From (4), there is a core (t', C') with $s_i \in S(t', C')$ and $v \in C'$. Hence $(t', C') = (t_h, C_h)$ for some $h \in \{0, \dots, i-1\}$. If $h < i - 1$, then $t_{i-1} \in S(t_h, C_h)$, contradicting the first bullet of (5). Thus $h = i - 1$ and the claim holds.

For the third bullet, we may assume that $n \ge 1$. For $0 \le i \le n$ define $g(i)$ to be the number of cores (t, C) such that t is a strict ancestor of t_i and $t_i \in S(t, C)$. We will prove by induction on i that $g(i) \le k - i - 1$. Since there is a core (t_0, C_0), it follows that $g(0) \le k - 1$ by (2). Inductively, suppose that $1 \le i \le n$, and $g(i-1) \le k - (i-1) - 1$. Let A_{i-1} be the set of all cores (t, C) such that t is a strict ancestor of t_{i-1} and $t_{i-1} \in S(t, C)$; and let A_i be the set of all cores (t, C) such that t is a strict ancestor of t_i and $t_i \in S(t, C)$. Thus $g(i-1) = |A_{i-1}|$ and $g(i) = |A_i|$. We claim that $A_i \subseteq A_{i-1}$. Let $(t, C) \in A_i$, and suppose that $(t, C) \notin A_{i-1}$. Thus t is a strict ancestor of t_i, and a descendant of t_{i-1}. Since $t_i \notin S(t_{i-1}, C_{i-1})$, and $C_{i-1} \cap B_{t_i} \neq \emptyset$ (because it contains v), the definition of $S(t_{i-1}, C_{i-1})$ implies that there is a core (d, D) such that d is a strict ancestor of t_{i-1}, and $t_{i-1} \in S(d, D)$, and $t_i \notin S(d, D)$. But this contradicts the definition of the spread of (t, C), since d is a strict ancestor of t_{i-1} and $t_i \in S(t, C)$.

Consequently $A_{i-1} \subseteq A_i$ for $1 \le i \le n$. But for $1 \le i \le n$, since $C_{i-1} \cap B_{t_i} \neq \emptyset$ and yet $t_i \notin S(t_{i-1}, C_{i-1})$, there is a core (d, D) such that d is a strict ancestor of t_{i-1}, and $t_{i-1} \in S(d, D)$, and $t_i \notin S(d, D)$. But then $(d, D) \in A_{i-1} \setminus A_i$, and so $g(i) leg(i-1) - 1 \le (k - (i-1) - 1) - 1 = k - i - 1$. This proves the third bullet and so proves (5).

Next we construct a graph J. Its vertex set is the set of all triples (s, t, C) where (t, C) is a core and s is in its spread. Consequently s is a descendant of t for all vertices (s, t, C) of J. If $(s_1, t_1, C_1), (s_2, t_2, C_2) \in V(J)$ are distinct, they are adjacent in J if either:

- $s_1 = s_2$ and $\text{dist}_G(C_1, C_2) \leq 3$, or
- s_1, s_2 are adjacent in T and $C_1 \cap C_2 \neq \emptyset$.

In particular, if $(s, t, C) \in V(J)$ and $s \neq t$, let s' be the parent of s; then $(s', t, C) \in V(J)$ is adjacent in J to $(s, t, C) \in V(J)$, and edges of this type are called *green* edges. All edges of J that are not green are called *red*. We will eventually show that there is a $(2k + 2, 2k - 1)$-quasi-isometry from G to the graph obtained from J by contracting all green edges. But first we prove some properties of J.

(6) *J has pseudo-tree-width at most $k - 1$.*

For each $s \in V(T)$, let A_s be the set of all $(s, t, C) \in V(J)$. Thus the sets A_s $(s \in V(T))$ are pairwise disjoint and have union $V(J)$. Let $s, t \in V(T)$ where s is the parent of t. There may be edges of J between A_s and A_t, but we claim that there are at most k such edges. Choose a set $\mathcal{F}$ of at most k cliques with union B_s. For each edge $e \in E(J)$ between A_s, A_t, we define $F_e \in \mathcal{F}$ as follows. Let the ends of e be $(s, s_1, C_1) \in V(J)$ and (t, t_1, D_1). Then $C_1 \cap D_1 \neq \emptyset$; choose $F_e \in \mathcal{F}$ that contains a vertex in $C_1 \cap D_1$. We claim that $F_{e_1} \neq F_{e_2}$ for all distinct edges e_1, e_2 between A_s, A_t. To see this, let e_i have ends $(s, s_i, C_i) \in V(J)$ and (t, t_i, D_i) for $i = 1, 2$. Either $(s_1, C_1) \neq (s_2, C_2)$ or $(t_1, D_1) \neq (t_2, D_2)$. In the first case, C_1, C_2 are anticomplete by (1); so no clique intersects both C_1, C_2; and so $F_{e_1} \neq F_{e_2}$. In the second case, D_1, D_2 are anticomplete by (1); so no clique intersects both D_1, D_2; and so $F_{e_1} \neq F_{e_2}$. Since $|\mathcal{F}| \leq k$, this proves that there are at most k edges of J between A_s, A_t.

Let $f = st$ be an edge of T, where s is the parent of t. From 2.2, since $|A_s|, |A_t| \leq k$ by (2), there is a pseudo-path-decomposition $(B^f_1, \ldots, B^f_{n(f)})$ of $J[A_s \cup A_t]$ with width at most $k - 1$ and with $A_s \subseteq B^f_1$ and $A_t \subseteq B^f_{n(f)}$. This defines $n(f)$, for each edge f of T. Subdivide each edge $f \in E(T)$ $n(f)$ times, making a tree T'. Define $C_t = B_t$ for each $t \in V(T)$. For each $f = st \in E(T)$ where s is the parent of t, let $s, u_1, \ldots, u_{n(f)}, t$ be the vertices in order of the path formed by subdividing f, and define $C_{u_i} = B^f_i$ for $1 \leq i \leq n(f)$. This defines a pseudo-tree-decomposition of J with width at most $k - 1$, and so proves (6).

The function ϕ does not map into $V(J)$, since $\phi(v)$ is a pair, not a triple. For each $v \in V(G)$, define $\psi(v) = (t, t, C)$ where $\phi(v) = (t, C)$.

(7) *Let $v \in V(G)$, and let (t, C) be a core with $v \in C$. Then there is a path of J between $\psi(v)$ and (t, t, C) with at most $k - 1$ red edges.*

Let P be the path of T between r, t, and define $(t_0, C_0), \ldots, (t_n, C_n)$ as in (5). By the second bullet of (5), for $0 \leq i < n$, there is a path of J from $(t_{i-1}, t_{i-1}, C_{i-1})$ to (t_i, t_i, C_i) in which all edges are green except the last; and since $n \leq k - 1$ (again by (5)), and $(t, C) = (t_n, C_n)$, this proves (7).

(8) *Let $v_1, v_2 \in V(G)$ be adjacent. Then there is a path of J between $\psi(v_1), \psi(v_2)$ using at most k red edges.*

Let $\psi(v_i) = (t_i, t_i, C_i)$ for $i = 1, 2$, and let t'_i be the source of v_i for $i = 1, 2$. Since v_i belongs to or has a neighbour in C_i, for $i = 1, 2$, and $v_1v_2 \in E(G)$, it follows that $\text{dist}_G(C_1, C_2) \le 3$. There exists $s \in V(T)$ with $v_1v_2 \in B_s$, since v_1v_2 is an edge; and by choosing s of minimum height we may assume that s is the source of one of v_1, v_2, say v_2, and so $s = t'_2$.

A *green path* of J means a path of J containing only green edges. Suppose that $t_2 \in S(t_1, C_1)$. Conequently there is a green path of J between (t_1, t_1, C_1) and (t_2, t_1, C_1), with vertex set all the triples (t, t_1, C) such that t is in the path of T between t_1, t_2, in order. Since there is a (red) edge of J between (t_2, t_1, C_1) and (t_2, t_2, C_2) (from the definition of J, since $\text{dist}_G(C_1, C_2) \le 3$), the claim is true. Thus we may assume that $t_2 \notin S(t_1, C_1)$. In particular, t_2 is a strict descendant of t'_1.

Since t_2 is in the path of T between t'_1, t'_2, and $v_1 \in B_{t'_1} \cap B_{t'_2}$, it follows that $v_1 \in B_{t_2}$. Since $t_2 \notin S(t_1, C_1)$, (4) implies that there is a core (d, D) with $t_2 \in S(d, D)$ and $v_1 \in D$. Thus (t_1, t_1, C_1) is joined to (d, d, D) by a path of J with only $k-1$ red edges, by (7); (d, d, D) is joined to (t_2, d, D) by a green path; and (t_2, d, D) is adjacent to (t_2, t_2, C_2) via a red edge, since $\text{dist}_G(C_2, D) \le 2$ (because v_2 has a neighbour in both). This proves (8).

(9) *For each core (t, C), $G[C]$ has diameter at most $2k - 1$.*

$G[C]$ has no stable set of size $k + 1$ (because C can be partitioned into at most k cliques), and therefore $G[C]$ has no induced path with $2k + 1$ vertices. Since it is connected, it has diameter at most $2k - 1$. This proves (9).

(10) *If (s_1, t_1, C_1) and $(s_, t_2, C_2)$ are joined by a green path of J, and $v_1 \in C_1$ and $v_2 \in C_2$, then* $\text{dist}_G(v_1, v_2) \le 2k - 1$.

Any two vertices of J joined by a green edge have the same second and third coordinates, and so $t_1 = t_2$ and $C_1 = C_2$. Consequently $v_1, v_2 \in C_1$, and the result follows from (9). This proves (10).

(11) *Let $v_1, v_2 \in V(G)$, and suppose P is a path of J between $\psi(v_1), \psi(v_2)$ containing at most n red edges. Then* $\text{dist}_G(v_1, v_2) \le (2k + 2)n + 2k - 1$.

If $n = 0$ the result follows from (10), so we assume that $n \ge 1$. Let P have ends b_0 and a_{n+1}, and let the red edges of P be $a_1b_1, a_2b_2, \dots, a_nb_n$ in order, numbered such that there there is a green subpath of P between b_i, a_{i+1} for $0 \le i \le n$. For $1 \le i \le n$, define α_i, β_i as follows: let $a_i = (s, t, C)$ and $b_i = (s', t', C')$ say; choose $\alpha_i \in C$ and $\beta_i \in C'$ with distance at most three in G. (This is possible from the definition of red edges.) Let $\beta_0 = v_1$ and $\alpha_{n+1} = v_2$. Thus $\text{dist}_G(\alpha_i, \beta_i) \le 3$ for $1 \le i \le n$; and $\text{dist}_G(\beta_i, \alpha_{i+1}) \le 2k - 1$ by (10). Consequently $\text{dist}_G(v_1, v_2) \le (2k + 2)n + 2k - 1$.

(12) *For each $j \in J$, there exists $v \in V(G)$ such that there is a path of J between j and $\psi(v)$ using at most $k - 1$ red edges.*

Let $j = (s, t, C)$, and choose $v \in C \cap B_s$. There is a green path between j and (t, t, C); and by (7), since $v \in C \subseteq B_t$, there is a path between (t, t, C) and $\psi(v)$ containing at most $k - 1$ red edges. This proves (12).

Let H be obtained from J by contracting all green edges. Thus each vertex of H is formed by identifying all the vertices (s, t, C) for a fixed core (t, C), and so we can identify $V(H)$ with the set of all cores in the natural way. From (6), and since contraction does not increase pseudo-tree-width, H has pseudo-tree-width at most $k-1$, and from (8), (11), (12), the function ψ is a $(2k+2, 2k-1)$-quasi-isometry from G to H. This proves 2.3 and hence (with 2.1) proves 1.4. ∎

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