Open Problems for the 2025 Barbados Graph Theory Workshop

Collected by Julien Codsi

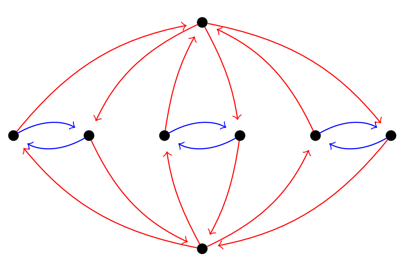
Problem 1. (From Paul Seymour)

Take a digraph drawn in the plane without crossings, and let O be a point of the plane not used by the drawing. For each edge $e = uv$, let $w(e)$ be the angle the line OP turns through as P follows the edge e from $P = u$ to $P = v$. Suppose that $w(e) > 0$ for each edge. Then, obviously, for every directed cycle, the sum of $w(e)$ over the edges of the cycle equals 360 degrees.

Is that the only way to make such a digraph? Not in general, but what about strongly 2-connected digraphs? Suppose that G is a strongly 2-connected digraph and for every edge e, $w(e)$ is some real number, such that for every directed cycle, the sum of $w(e)$ over the edges of the cycle equals 1. Does G have to come from a planar drawing in this way? In particular, does G have to be planar?

From Stéphan: This looks like some problem we looked at with Adrian and Pierre where vertices are on the unit cycle and every circuit has winding number one. We started with a longest cycle and tried to perform ear decomposition if I remember well, but got nowhere. A spinoff question: Is it true that minimum out-degree 3 implies the existence of a cycle with winding number >1?

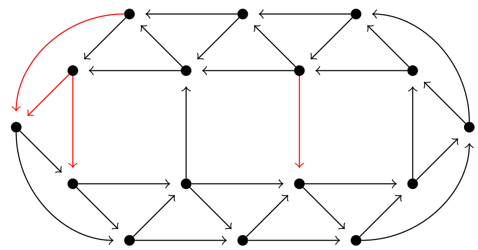
From Daniel Carter (and Paul Seymour): The answer to the first question ("Does G have to come from a planar drawing in this way?") is *no*. Here is a counterexample:



A weighting is given by giving all of the red edges weight 1/4 and all the blue edges weight 1/2. But one can check there is no way to draw this graph in the plane minus the origin without crossings so all edges go strictly clockwise. I can come up with examples with no digons as well. Of course, this is planar, so the latter statement "does G have to be planar?" is still open. Also the former statement may be true with more assumptions.

(From Paul Seymour) I had better modify this: assume the digraph is strongly 2-connected and the underlying undirected graph is 3-connected. Is it true then?

From Daniel Carter: Still no. Here is how to generate many counterexamples. Let f be a vector field on S^2 with isolated zeros. Project onto $\mathbb{R}^2$ so one of the zeros gets mapped to ∞. The rest of the zeros are either "whorls" or "saddles", and there is one more whorl than saddle by Poincaré–Hopf. We can arrange so they are mapped onto the x-axis in the order *whorl, saddle, whorl, saddle, …, whorl.* Draw a ray going left from the first whorl, and draw a segment between consecutive saddle-whorl pairs for the rest (matching the saddles to the whorl immediately to their right), arranging so this ray and these segments are orthogonal to the vector field everywhere. Draw a planar graph on this so that nothing touches any of the zeros of the vector field, and the tangent vector at each point along each edge is never orthogonal to the vector field. Orient the edges in this graph according to the vector field. We claim the resulting digraph is weightable by putting weight 1 on any edge crossing the ray or any of the line segments we drew earlier and 0 on all other edges. This is again a consequence of Poincaré–Hopf. Starting with a large enough planar graph and at least one saddle allows you to obtain strongly 2-connected, weakly 3-connected counterexamples. In the example below, give black edges weight 0 and red edges weight 1.



So here is the current set of conjectures: (1) If G is strongly 2-connected and weightable, then G is planar; (2) If G is strongly 2-connected, weakly 3-connected, and weightable, then there is a drawing of G in the plane and vector field of the form described above so that G is given by the above construction.

I believe I can prove that (2) implies (1). It would be nice to know if (1) also implies (2); I think I can prove this for graphs whose planar drawing has just two faces bounded by cycles, corresponding to the case of no saddles. Raphael notes that it might be possible to use a result of Guenin and Thomas to prove that strongly 2-connected, weakly 3-connected, weightable, and planar implies strongly planar, which could be a helpful intermediate step to showing (1) implies (2).

Problem 2. (From Paul Seymour)

What are the digraphs in which every directed cycle has length four? I know the answer if they all have length three (see `https://www.arxiv.org/abs/2410.13008`), but I can't do length four. What if they all have length some nice big prime number, is that easier?

Problem 3. Tree-treewidth (from David Wood)

Given a graph parameter f, Liu–Norin–Wood defined a new graph parameter tree-f as follows. For a graph G, let tree-$f(G)$ be the minimum integer k such that G has a tree-decomposition $(B_x : x \in V(T))$ such that $f(G[B_x]) \leqslant k$ for each node $x \in V(T)$. Seymour introduced tree-χ a few years ago. Tree-α has recently been extensively studied. Liu–Norin–Wood recently introduced tree-treewidth. They showed that any proper minor-closed class has bounded tree-treewidth (in fact, graphs excluding a fixed odd minor have bounded tree-pathwidth). The proof uses the Graph Product Structure Theorem, so the bounds are large. It would be interesting to prove this without using the Graph Minor Structure Theorem and with better bounds. In particular, what is the maximum of tree-$\operatorname{tw}(G)$ for a K_t-minor-free graph G? Liu–Norin–Wood gave a lower bound of $\Omega(t^2)$, and asked whether K_t-minor-free graphs G have tree-$\operatorname{tw}(G) \in O(t^2)$? Natural tools to use for answering this question include chordal partitions and sparse covers [a,b,c].

Problem 4. (tw,had)-bounded classes (from David Wood)

Let $\operatorname{tw}(G)$ be the treewidth of a graph G. The Hadwiger number $\operatorname{had}(G)$ of a graph G is the maximum integer t such that K_t is a minor of G. We have $\operatorname{had}(G) \leqslant \operatorname{tw}(G) + 1$ for every graph G. Campbell–Davies–Distel–Frederickson–Gollin–Hendrey–Hickingbotham–Wiederrecht–Wood–Yepremyan asked for what classes of graphs is a reverse inequality true?

To this end, they defined a hereditary graph class $\mathcal{G}$ to be *(tw,had)-bounded* if there is a function f such that $\text{tw}(G) \leqslant f(\text{had}(G))$ for every graph $G \in \mathcal{G}$, and $\mathcal{G}$ is *linearly (tw,had)-bounded* if there is a constant c such that $\text{tw}(G) \leqslant c\,\text{had}(G)$ for every $G \in \mathcal{G}$. Campbell et al. conjectured that every (tw,had)-bounded hereditary graph class is linearly (tw,had)-bounded. This conjecture is true for several examples, including circle graphs and outer-string graphs (on surfaces with the vertices on a bounded number of cuffs).

Problem 5. Polynomial / Linear Grid Minor Theorem (from David Wood)

The following two results are the state-of-the-art bounds in Robertson and Seymour's Grid Minor Theorem: Every graph with treewidth at least $ck^9 \operatorname{polylog} k$ has a $k \times k$ grid minor [Chuzhoy–Tan]. Every graph with treewidth at least $t^{ct^2}k$ has a K_t minor or a $k \times k$ grid minor [Kawarabayashi–Kobayashi]. Does the Grid Minor Theorem hold with polynomial dependence on t and linear dependence on k? That is, does every graph with treewidth at least $t^c k$ have a K_t minor or a $k \times k$ grid minor?

Problem 6. Undecidability of graph class containment (from Daniel Carter)

Let S and S' be two families of graphs defined by finitely many forbidden (induced) subgraphs, (induced) minors, and/or (induced) topological minors. (We say H is an induced minor of G if it may be formed from G by vertex deletions and edge contractions. We say H is an induced topological minor of G if a subdivision of H is an induced subgraph of G.) Is it decidable to determine if $S \subseteq S'$? In an upcoming paper, Maria Chudnovsky, Nicolas Trotignon and I prove this problem *is* decidable if either (a) S is defined by forbidding just subgraphs, minors, and/or topological minors (and S' is unrestricted), or (b) S' is defined by forbidding just (induced) subgraphs and/or induced topological minors (and S is unrestricted). We suspect the general case is undecidable.

Towards that end, we prove the following problem is undecidable. Let H be a given graph with four special vertices a_1, a_2, b_1, b_2, and two special subsets of vertices U_1 and U_2, and $\mathcal{F}$ a finite set of graphs. Is it possible to add two induced paths P_1, P_2 to H, P_i going from a_i to b_i, so that the resulting graph is $\mathcal{F}$-free, and where the neighborhood of each vertex in P_i is restricted to lie in $U_i \cup P_i \cup P_{3-i}$? This seems to be close to the desired result. On the other hand, we prove the analogous problem is decidable with any number of paths if the vertices on each path P_i to be added have their neighborhoods restricted to $U_i \cup P_i$. This was used to get the two special cases (a) and (b) above.

Similar methods can be used to prove the following problem is undecidable. Let H be a graph and $\mathcal{F}$ a finite set of graphs. Is there an $\mathcal{F}$-free Hamiltonian graph containing H?

Problem 7. Erdős–Hajnal for digraphs (from Tung Nguyen)

For two digraphs G, H, we say that H is an *induced subdigraph* of G if there exists $S \subseteq V(G)$ for which H is isomorphic to $G[S]$; and say that G is *H-free* if it has no H induced subdigraph. A digraph is *acyclic* if it contains no (induced) subdigraph which is a directed cycle; and the *dichromatic number* of a digraph G is the least integer $k \geqslant 0$ for which $V(G)$ can be partitioned into k vertex subsets each inducing an acyclic subdigraph. The following conjecture was due to Harutyunyan and McDiarmid and first appeared as Conjecture 1.1 in a paper of Harutyunyan, Le, Newman, and Thomassé:

> *For every tournament H, there exists $c > 0$ such that every H-free digraph G has an acyclic induced subdigraph with at least $|G|^c$ vertices.*

(In the cited paper above this was stated for all forbidden digraphs H; but the conjecture only makes sense when H is a tournament since one can take G to be a tournament.) Another formulation of this conjecture says that there exists $\tau > 0$ such that every H-free digraph G

has dichromatic number at most $|G|^{1-\tau}$. The conjecture contains the tournament version of the Erdős–Hajnal conjecture (which is known to be equivalent to the original EH conjecture), but it is unknown if the other direction holds.

Even though a bound of $2^{c\sqrt{\log|G|}}$ (in place of $|G|^c$) is known, it is currently unknown whether such a bound can be substantially improved when H is the cyclic triangle C_3. In more detail, given the recent work on the EH conjecture, can a bound of $2^{c\sqrt{\log|G|\log\log|G|}}$ be proven in this case?

Moreover, in order to gain a better understanding of C_3-free digraphs, Harutyunyan, Le, Newman, and Thomassé proved (Theorem 1.7 in the paper) that every C_3-free digraph G has dichromatic number at most $35^{\alpha(G)-1}\alpha(G)! = \alpha(G)^{O(\alpha(G))}$ where $\alpha(G)$ is the usual stability number of G, and they conjectured (Conjecture 1.6) that a polynomial bound on $\alpha(G)$ is true. Can a singly exponential bound be done in this direction?

Problem 8. Clustered colouring and girth (from David Wood)

An (improper vertex) colouring of a graph has *clustering* k if each monochromatic component has at most k vertices. Here, a *monochromatic component* is a maximal connected subgraph of vertices all assigned the same colour. Brianski–Hickingbotham–Wood recently studied clustering colouring with respect to girth. Here are some questions that arose from their work:

1. Are graphs with girth at least some absolute constant and with bounded treewidth 2-colourable with bounded clustering? Here the clustering may depend on the treewidth. The answer is "yes" for graphs with treewidth 2 and girth at least 5. In general, girth at least 7 is needed for a positive answer. Three colours is achievable for girth at least 5 by a more general result of Liu and Wood. This question remains open with "treewidth" replaced by "pathwidth".

2. Are K_t-minor-free graphs with girth at least some absolute constant 3-colourable with bounded clustering (depending on t). Girth at least 5 is needed for a positive answer. Four colours is achievable for girth at least 5 by a more general result of Liu and Wood.

3. Are K_t-minor-free graphs with girth at least some absolute constant 2-colourable with bounded clustering (depending on t)? Girth at least 7 is needed for a positive answer. The answer is "yes" for graphs embeddable in a fixed surface. The answer is "yes" if the girth is $\Omega(\log t)$.

Problem 9. Clustered 3-colouring planar graphs (from David Wood)

Linial–Matoušek–Sheffet–Tardos defined $f_c(n)$ to be the minimum integer such that every n-vertex planar graph is c-colourable with clustering $f_c(n)$. The 4-Colour Theorem says that $f_c(n) = 1$ for every $c \geqslant 4$. Linial et al. showed that $f_2(n) = \Theta(n^{2/3})$ and $\Omega(n^{1/3}) \leqslant f_3(n) \leqslant O(n^{1/2})$. Determining $f_3(n)$ is wide open.

As discussed by Campbell–Gollin–Hendrey–Lesgourgues–Mohar–Tamitegama–Tan–Wood, a natural approach for improving the upper bound on $f_3(n)$ is to apply the Planar Graph Product Structure Theorem of Dujmović–Joret–Micek–Morin–Ueckerdt–Wood, which says that every planar graph is contained in $H \boxtimes P$ for some graph H of treewidth at most 8 and some path P. The idea would be to 3-colour $H \boxtimes P$ with small clustering, thus determining a 3-colouring of G. Two issues arise, however. First, it may be that $|V(H \boxtimes P)|$ is significantly larger than $|V(G)|$ (the best bounds are $|V(H)| \leqslant |V(G)|$ and $|V(P)| \leqslant O(|V(G)|^{(1+\epsilon)/2})$ by Hendrey–Wood). So clustering $O(|V(H \boxtimes P)|^{\beta})$ does not immediately imply clustering $O(|V(G)|^{\beta})$. Second, Campbell et al. showed there are graphs H with treewidth 3 and there are paths P such that every 3-colouring of $H \boxtimes P$ has clustering $\Omega(|V(H \boxtimes P)|^{3/7})$. So (even ignoring the first issue) the best upper bound on $f_3(n)$ that one could hope for using this method is $f_3(n) \leqslant O(n^{3/7})$, which is between the known bounds mentioned above.

Islands provide another approach: what is the least function $g(n)$ such that every n-vertex planar graph G has a set S of at most $g(n)$ vertices, such that every vertex in S has at most two neighbours in $V(G) \setminus S$? A greedy colouring algorithm shows that $f_3(n) \leqslant g(n)$. I would be surprised if $g(n) \in o(n^{1/2})$, but it is worth considering.

The above lower bound is established as follows. Fix an integer $k \geqslant 2$. Let $F_1, \dots, F_k$ be disjoint fans each with k^2 vertices. Let w_i be the centre of F_i. Let G be obtained by adding one vertex v adjacent to every vertex in $F_1 \cup \dots \cup F_k$. So G has $k^3 + 1$ vertices. Suppose that G is 3-colourable with clustering less than k. Say v is red. Since v is adjacent to every other vertex, some F_i has no red vertex. Say w_i is blue. So the path $F_i - w_i$ has less than k blue vertices, and no red vertices. Deleting the blue vertices from $F_i - w_i$ leaves at most k components, all of which are green. At least one such component has at least $\lceil (k^2 - 1)/k \rceil = k$ vertices, which is a contradiction. So every 3-colouring has clustering at least $k \approx n^{1/3}$. It would be very interesting to find a better lower bound.

Problem 10. The sunny side of χ-boundedness (from Nicolas Trotignon)

Call a *sun* any graph obtained from a cycle by adding a pending edge at each vertex. The length of a sun is the length of the cycle. Are sun-free graphs χ-bounded?

By "-free" we mean that we exclude all suns as induced subgraphs. Here are some remarks. The triangle-free case is of interest and can be rephrased as: is there an upper bound on the chromatic number of graphs with no triangle and no sun of length at least 4 (as induced subgraphs)?

It has been observed by several researchers (though I could not find a reference) that shift graphs are triangle-free graphs of arbitrarily large chromatic number that contain no sun of length at least 5. So, excluding suns of length 4 is essential.

Remark from James Davies: This becomes easy if you forbid $K_{t,t}$ using degree-boundedness arguments. In the general case, I think I can find vertex sets $A = \{a_1, \dots, a_t\}$, $B = \{b_1, \dots, b_t\}$, C so that $G[C]$ has large chromatic number, A cover C, and $G[A \cup B]$ is the matching $a_1b_1, \dots, a_tb_t$. It feels like this might be useful, particularly if there is a way to also make B anti-complete to C.

Problem 11. El-Zahar–Erdős for triangle-free graphs (from Tung Nguyen)

A conjecture of El-Zahar and Erdős says that for every two integers $w, t \geq 1$, there exists $d \geq 1$ such that every graph G with $\omega(G) \leq w$ and $\chi(G) \geqslant d$ contains two disjoint anticomplete vertex subsets A and B such that $\chi(G[A]), \chi(G[B]) \geqslant t$. In particular this remains open when $w = 2$, that is when G is triangle-free. Is there a counterexample to the following ultra-strong strengthening? There exists $C > 0$ such that for every integer $t \geq 1$, every triangle-free graph G with $\chi(G) \geqslant Ct$ contains two disjoint anticomplete vertex subsets A and B with $\chi(G[A]), \chi(G[B]) \geqslant t$. In the case of K_4-free graphs a counterexample to this ultra-strong version can be built by random graphs.

Raphael Steiner: The triangle-free process gives a counterexample. In fact, the following argument can also be used to show that the smallest value $d = d(w,t)$ satisfying El-Zahar-Erdős, if it exists, satisfies $d(2,t) \geqslant \Omega(t^{4/3})$. The triangle-free process is the following random process to generate a triangle-free graph on n vertices: Start from the empty graph G_0 on n vertices. For $i \geqslant 1$, while there is still a non-edge in G_{i-1} that can be added without creating a triangle, uniformly at random pick one such non-edge among all addable non-edges and add it to G_{i-1} to create G_i. Addable non-edges are also called *open*, and non-addable non-edges are called *closed*. Let G be the eventual outcome of the process, which is always triangle-free. See also `https://www.sciencedirect.com/science/article/pii/`

S0001870809000620. In this article Bohman proves the following fact about this random graph G: W.h.p. it has independence number at most $O(\sqrt{n \ln n})$, and hence chromatic number at least $\chi := \Omega(\sqrt{\frac{n}{\ln n}})$. Now, suppose towards a contradiction that Tung's statement holds with some constant C. This would mean that there exist anticomplete disjoint sets $A, B \subseteq V(G)$ such that $\chi(G[A]), \chi(G[B]) \geqslant \frac{\chi}{C} = \Omega(\sqrt{\frac{n}{\ln n}})$. Since it is well-known (see e.g. the intro of https://doi.org/10.1137/21M1437573 for some history) that every x-vertex triangle-free graph has chromatic number at most $O(\sqrt{\frac{x}{\ln x}})$, the above inequality implies that $|A|, |B| \geqslant \varepsilon n$ where $\varepsilon > 0$ is a constant depending solely on C. Now, what is left to prove is that w.h.p. as $n \to \infty$ the random graph G does not contain two disjoint anticomplete sets of size at least εn. So fix any pair A, B of disjoint subsets of $V(G)$ of size at least εn. For technical reasons, we would like to estimate the probability of the following event $\mathcal{E} := \{G$ has no edges between A and B and $\Delta(G) \leq B\sqrt{n \ln n}\}$, for some large enough constant $B > 0$. I claim that $\mathbb{P}[\mathcal{E}] \leq \exp(-\Omega(n^{1.49}))$ for any fixed pair (A, B). The probability of $\mathcal{E}$ occuring can be bounded as follows: For every $i \geqslant 0$ let $\mathcal{E}_i$ be the event that G_i has no edges between A and B and that $\Delta(G_i) \leq B\sqrt{n \ln n}$ (to let this make sense for every $i \geqslant 0$, set $G_i := G$ for every $i > e(G)$). We then have

$$\mathbb{P}[\mathcal{E}] = \prod_{i \geqslant 1} \mathbb{P}[\mathcal{E}_i | \mathcal{E}_{i-1}].$$

Note that a non-edge that is closed in G_{i-1} must have both endpoints in the neighborhood of some vertex. Hence, for every $i \geqslant 1$, if $\mathcal{E}_{i-1}$ is satisfied then the number of open non-edges in G_{i-1} between A and B is at least

$$|A||B| - \sum_{v \in V(G_{i-1})} \frac{d_{G_{i-1}}(v)^2}{2} \geqslant \varepsilon^2 n^2 - \Delta(G_{i-1}) e(G_{i-1}) \geqslant \varepsilon^2 n^2 - B\sqrt{n \ln(n)}(i-1).$$

In particular, for $1 \leq i \leq n^{1.49}$, conditional on $\mathcal{E}_{i-1}$, the chance that the newly added edge when creating G_i from G_{i-1} does not go between A and B is at most $1 - \varepsilon^2 + o(1)$. This implies $\mathbb{P}[\mathcal{E}_i | \mathcal{E}_{i-1}] \leq 1 - \varepsilon^2 + o(1)$ for $1 \leq i \leq n^{1.49}$. Hence, the above product gives $\mathbb{P}[\mathcal{E}] \leq (1 - \varepsilon^2 + o(1))^{n^{1.49}} = \exp(-\Omega(n^{1.49}))$, as desired. Let $\mathcal{F}$ be the event that G contains two anticomplete sets of size εn and that $\Delta(G) \leq B\sqrt{n \ln n}$. By a union bound over all choices of A and B (at most 4^n), we find that $\mathbb{P}[\mathcal{F}] \leq 4^n \exp(-\Omega(n^{1.49})) = o(1)$. Hence, with high probability, G has no anticomplete sets of size εn, or satisfies $\Delta(G) > B\sqrt{n \ln n}$. However, by Bohman's result, the latter outcome happens with probability $o(1)$, since $\alpha(G) \geqslant \Delta(G)$ as G is triangle-free. Hence, with high probability, G has no anticomplete sets of size εn. This completes the argument.

From Paul Seymour: I don't understand, please would you explain more? RS = Raphael Steiner? What's the triangle free process? Linear in what?

From Tung (to RS): I guess the triangle-free process you mentioned is the one in arXiv:0806.4375, but I'm not sure about the pseudorandom properties comparable to an ER random graph of the same density like you said; this seems to require nontrivial calculations to back up. But at least I haven't been able to build a counterexample from the ER random graphs. Also, the problem asks for "linear-chromatic", not "linear-sized"; and I think it is well-known that sparse ER random graphs give graphs with arbitrarily large girth and no linear-sized anticomplete pairs. And I am not aware of any relative quick way to go from "linear-chromatic" to "linear-sized". It would be very nice if there's such a way. Thanks!

From Daniel Carter: See "Dynamic concentration of the triangle-free process" by Bohman–Keevash for where the intuition RS mentions is formulated more precisely. We expect the triangle-free process to give a graph that "looks like" $G(n, p)$ with $p = \Theta(\sqrt{\log n / n})$, except with no triangles. Such a graph has $\chi = \Theta(\sqrt{n / \log n})$, but this graph also has anticomplete sets of size $\Omega(\sqrt{n/\log n})$, so RS's simple back-of-the-envelope calculation doesn't work. However, we need anticomplete sets much larger than this in order to get anticomplete sets of chromatic number that large, so the construction probably still works.

From Tung (to Raphael and Daniel): That's great! Thanks for the explanation! So that means the statement I proposed is very false in fact, beyond what I had hoped for.

Problem 12. Special cases of Hadwiger's conjecture (from Vaidyanathan Sivaraman)

Hadwiger's conjecture (1943) states that $h(G) \geqslant \chi(G)$ ($h(G)$ denotes the number of vertices in a largest complete graph that is a minor of G, and $\chi(G)$ is the chromatic number of G). Prove Hadwiger's conjecture for

1. Graphs all of whose induced subgraphs H satisfy $\chi(H) - \omega(H) \leq 2$
2. (P_5, C_5)-free graphs
3. (P_5, P_5^c)-free graphs
4. $ISK4$-free graphs (these are graphs not containing a subdivision of K_4 as an induced subgraph)

Remark from Nicolas Trotignon: A way to prove Hadwiger conjecture for ISK4-free graphs would be to prove a conjecture of Lévêque, Maffray and Trotignon stating that ISK4-free graphs are 4-colorable. The best upper bound so far is due to Ngoc Khang Le who proved that ISK4-free graphs are 24-colorable. Chudnovsky, Liu, Schaudt, Spirkl, Trotignon and Vušković proved that (ISK4, triangle)-free graphs are 3-colorable.

From Tung: Is it known whether $\{P_5, \overline{P_5}\}$-free graphs satisfy linear Hadwiger?

From Daniel Carter: Here is another interesting class: $(K_2 \cup 2K_1, K_3 \cup K_1)$-free graphs (i.e. complements of line graphs of triangle-free graphs). If Hadwiger holds here then it also holds in $(\text{fork}, \overline{\text{fork}})$-free graphs by the structure theorem of Chudnovsky, Cook, and Seymour (it is easy to see that all of the decompositions and the other basic classes pose no issue for Hadwiger).

Problem 13. Pivot-minors (from Jim Geelen)

Sang-il Oum conjectured that excluding a bipartite circle graph as a pivot-minor bounds rank-width. I would be very happy to work on that conjecture with anyone who is interested.

Here is another, likely easier, problem on pivot minors. Let's say that two graphs G_1 and G_2, on the same vertex set, are at *distance* $\leq d$ if G_2 is a pivot-minor of a graph obtained from G_1 by adding at most d vertices. The set of graphs at distance at most d from bipartite graphs form a pivot-minor-closed class. Can we find a qualitative characterization for this class? Is it enough to exclude K_n and the direct sum of n triangles? That is, if G is a graph that contains neither K_n nor the direct-sum of n triangles as a pivot-minor, is G at distance at most $f(n)$ from a bipartite graph?

Problem 14. Regular chromatic thresholds (from Stéphan Thomassé)

An application of the Dense Neighborhood Lemma gives that for every ε, every K_5-free $0.7n+\varepsilon n$-regular graph on n vertices has bounded chromatic number (joint work with Romain Bourneuf and Pierre Charbit).

The threshold 0.7 is best possible, but can we drop the εn term?

Also, do $n/4$-regular triangle-free graphs have bounded χ? This is true for $n/4 + \varepsilon n$ from a result of O'rourke, and the 1/4-threshold is best possible.

Problem 15. Domination in tournaments (from Stéphan Thomassé)

An easy application of the Dense Neighborhood Lemma gives that every tournament with fractional chromatic number k (fractional cover by transitive tournaments) has domination number at most $O(k^k)$. (joint work with Romain Bourneuf and Pierre Charbit).

**** Can the bound be improved to poly(k)?

From Tung: Could you explain what the Dense Neighborhood Lemma is? Is that your result saying every tournament with huge chromatic number contains a vertex whose out-neighbourhood induces a tournament with large chromatic number? Or maybe your stronger theorem (which implies the above) stating that every tournament with huge domination number contains a bounded-size subtournament with large chromatic number?

From Stéphan: This is a variation of VC-dimension, easier to explain on the blackboard friday. I'm extremely interested if you have an alternative proof that bounded fractional chi implies bounded domination.

Note that this directly implies an already known fact (that we proved with Ararat Harutyunyan, Tien Nam Le and Hehui Wu): if the out-neighborhoods of a tournament T have chromatic number at most k, then $\chi(T) \leq f(k)$. Indeed T has fractional χ at most $2k$ since Farkas Lemma provides a fractional 2-cover of $V(T)$ by its out-neighborhoods.

**** What is the best bound for f?

From Tung: Just a small reminder from last year's problem list that it is open whether $f(k) = 2k$ suffices.

Let us introduce a silly parameter generalizing fractional chi: let us say that a tournament has *fractional VC-dimension* at most k if we can find a fractional cover of its vertices by tournaments of VC-dimension at most k, which total weight is at most k.

We can prove (via DNL) that bounded fractional VC-dimension implies bounded domination.

**** Is it useful for anything?

A possible application: Call T_k the tournament obtained by iterating k times substitutions of directed 3-cycles.

**** Is it true that T_k-free tournaments have bounded fractional VC-dimension?

From Tung: A related question (perhaps easier?): Is it true that S_k-free tourmanents have bounded fractional VC-dimension? Here S_k, inductively, is obtained by substituting a copy of S_{k-1} for two vertices of the directed 3-cycle; and S_1 is the one-vertex tournament.

From Stéphan: This is definitely a better toy problem.

Problem 16. Asymptotic Dimension of Fat Minor Free Graphs (from Robert Hickingbotham)

The following conjecture is due to Bonamy–Bousquet–Esperet–Groenland–Liu–Pirot–Scott.

Let G be a graph and H be a subgraph of G. The *weak-diameter* of H (in G) is $\max\{dist_G(u,v) : u,v \in V(H)\}$. For integers $c, d \geqslant 0$, we say that G is (c,d)-*weak diameter colourable* if there exists an (improper) c-colouring of G such that each monochromatic component has weak-diameter at most d.

For a positive integer K, a K-*fat minor model* of a graph H in a graph G is a collection $(B_v : v \in V(H)) \cup (P_e : e \in E(H))$ of connected sets in G such that

- $V(B_v) \cap V(P_e) \neq \emptyset$ whenever v is an end of e in H; and
- for any pair of distinct $X, Y \in \{B_v : v \in V(H)\} \cup \{P_e : e \in E(H)\}$ not covered by the above condition, we have $dist_G(X,Y) \geqslant K$.

If G contains a K-fat minor model of H, then we say that H is a K-*fat minor* of G.

Conjecture: There exist functions c, d such that every graph G with no K-fat H-minor is $(c(H), d(H, K))$-weak diameter colourable.

Problem 17. Additive Distortion Quasi-Isometries (from Robert Hickingbotham)
Nguyen–Scott–Seymour conjectured the following:

Conjecture: For all $k, L, C \in \mathbb{N}$ there exists $C' \in \mathbb{N}$ such that if ϕ is an (L, C)-quasi-isometry from a graph G to a graph H where $\text{tw}(H) \leq k$, then there is a function $w\colon E(H) \to \mathbb{N}_0$ such that the same function ϕ is a $(1, C')$-quasi-isometry from G to the weighted graph (H, w).

Here's a weakening of the conjecture (potentially due to Scott).

Conjecture: For all $k, L, C \in \mathbb{N}$ there exists $d, C' \in \mathbb{N}$ such that if ϕ is an (L, C)-quasi-isometry from a graph G to a graph H where $\text{tw}(H) \leq k$, then there are functions $w_H\colon E(H) \to \mathbb{N}_0$ and $w_G\colon E(G) \to \mathbb{N}_0$ such that each component of $w_G^{-1}(0)$ has weak-diameter at most d and the same function ϕ is a $(1, C')$-quasi-isometry from the weighted graph (G, w_G) to the weighted graph (H, w_H).

Problem 18. Asymptotic dimension of k-gap planar graphs (from Robert Hickingbotham)
Let $k \geqslant 0$. A graph G is *k-gap planar* if it has a drawing in the plane such that each crossing is charged to one of the two edges involved and every edge has at most k crossings charged to it. Does this class of graphs have bounded asymptotic dimension? Some comments on this:

- The class of k-gap planar graphs has polynomial expansion https://www.sciencedirect.com/science/article/pii/S0304397518303670.
- It suffices to consider the case $k = 1$ since every k-gap planar graph is quasi-isometric to a 1-gap planar graph (by subdividing each edge k times).
- A graph is *k-planar* if it has a drawing in the plane such that each edge is in at most k crossings. The class of k-planar graphs has bounded asymptotic dimension due to having bounded layered treewidth https://ems.press/journals/jems/articles/10594301. However, the class of k-gap planar graphs does not have bounded layered treewidth (there exist 1-gap planar graphs with radius 1 and arbitrarily large complete graph minors https://epubs.siam.org/doi/10.1137/22M1542854).
- Asymptotic dimension can be defined via weak-diameter colouring https://ems.press/journals/jems/articles/10594301. A warm-up question to this problem is to first determine the weak-diameter chromatic number for the class of k-gap-planar graphs.

Problem 19. Burling number of Burling graphs (from Linda Cook and James Davies)
Let $B_1, B_2, B_3, \ldots$ be the usual sequence of Burling graphs. See e.g. [13] for a definition of the Burling sequence. A graph is a Burling graph if it is an induced subgraph of some B_i. We define the Burling number $B(G)$ of a graph G to be equal to the maximum i such that G contains B_i as an induced subgraph.

Note that for each i, B_i is a fixed graph, so for each fixed k there is a polynomial time algorithm that tests whether or not $B(G) = k$ (by looking at all subsets of size $|B_k|$ and $|B_{k+1}|$).

1. Is there a polynomial time algorithm for determining $B(G)$? What if G is a Burling graph?

2. Is there a polynomial time approximation algorithm? To be more precise, is there some increasing function f and a polynomial time algorithm that outputs some s so that $s \leq B(G) \leq f(s)$. What if G is a Burling graph?

By recent work, a positive answer to any of these questions would give a polynomial time approximation algorithm for the chromatic number of Burling graphs.

Note that for general graphs we have no idea whether there is a polynomial time approximation algorithm for chromatic number and is attributed to Alon from 1993. (listed as problem 10.2 in Jensen and Toft). See also a nice related previous Barbados Problem of Maria from 2016 [4]) We hope that this paper of Pawel and Bartosz on MIS in Burling Graphs is helpful [13]. Some nice papers showing NP-Hardness of detecting certain induced subgraphs: From Nicolas and Maffray [10] from Bienstock [2]. We do not know whether the decision problem "Is $B(G) = k$?" on input G and k is even in NP.

Problem 20. Cycles in vertex-transitive digraphs (from Raphael Steiner)

Let $f(n)$ be the largest integer such that every connected vertex-transitive digraph on n vertices contains a directed cycle of length at least $f(n)$. For graphs, denote the analogous function by $g(n)$.

The corresponding problem for undirected graphs is well-researched (Lovász conjecture), and the currently best lower bound is $g(n) \geqslant \Omega(n^{13/21})$. For digraphs it seems little is known. Can you prove a non-trivial lower bound on $f(n)$? In particular, is there $\varepsilon > 0$ such that $f(n) \geqslant n^{\varepsilon}$? An open problem of Alspach asks whether $f(n) \leq n - \omega(1)$. Erdős and Trotter proved that $f(n) \leq n - 1$ for some n. Looking at their construction and determining the maximum length of a directed cycle in it could be a first step to understand the problem of Alspach.

We (Bućic, Hendrey, Mohar, Steiner, Yepremyan) now have positive answers to these questions.

Problem 21. Odd Girth (from Stéphan Thomassé)

Let $\mathrm{og}(n)$ be the minimum value k such that every graph with n vertices and odd girth at least k has chromatic number at most 8. We can prove that $\mathrm{og}(n)$ is at most $16n^{2/3}$ (not fantastic, but $o(n)$. This has been probably investigated somewhere...).

**** What is the best bound for $\mathrm{og}(n)$?

**** Can we replace 8 by 3 and still get $o(n)$?

We are interested in $\mathrm{og}(n)$, with Romain Bourneuf and Pierre Charbit, since we can show that every triangle-free graph with minimum degree at least $n/3 - n/8\mathrm{og}(n)$ has chromatic number less than some fixed value.

From Raphael: This old paper by Kierstead, Szemerédi and Trotter seems highly relevant. It implies that $\mathrm{og}(n) = O(n^{1/7})$. Also it shows that the answer to the second question is positive, one can get even $O(\sqrt{n})$ for 3 instead of 8.

From Stéphan: Thank you, very nice! It implies then that triangle free and minimum degree $n/3 - n^{1-\varepsilon}$ implies bounded chi (and this is I think matched by some constructions). Sorry for not having dug enough biblio...

Problem 22. Holetown problem (From Nicolas Trotignon)

Problem 1: Decide in polytime if an input graph contains induced cycles of three different lengths.

Problem 2, possibly more difficult: Decide in polytime if an input graph contains holes of three different lengths, where a hole is an induced cycle of length at least 4.

There exists a polytime algorithm that decides if a graph has two holes of different lengths, see [7]. This is why Problem 1 is maybe easier. Because if there are triangles, we may reduce

Problem 1 to decide whether a graph has holes of two different lengths, and when there is no triangle, the structure of graphs with answer "no" might be possible to describe.

Problem 23. Independent sets in triangle-free graphs (from Matija Bucić)

Is it true that in a triangle-free graph of independence number α there are at most $2^{O(\alpha)}$ maximum independent sets?

We asked this in a recent paper with Maria Chudnovsky and Julien Codsi where we looked at versions of this question for *maximal* instead of maximum. In that case the answer is no since the triangle-free process produces graphs with $\alpha^{O(\alpha)}$ many maximal independent sets. The question would have some loose implications to determining Ramsey number $R(3,k)$ very precisely (it is in a sense an easier prerequisite I am told by Will Perkins), this is currently known up to a factor of 4.

David: The very recent paper "Triangle-free graphs with the fewest independent sets" by Pjotr Buys, Jan van den Heuvel and Ross Kang may or may not be relevant.

Problem 24. Discrete X-Ray reconstruction problem (from Matija Bucić)

Given a properly edge coloured n-vertex graph G with average degree $d \geqslant O(\log n)$ show that one can find a subset of vertices U and a partition of the colours into two sets C_1, C_2 such that $G[U]$ is connected using only edges of colors in C_1 and only using edges of colors in C_2.

This question is due to Matoušek, Přívětivý, and Škovroň from 2006 [11]. Its solution would provide a solution to a nice discrete geometry question via a linear algebraic reduction they prove. It is also interesting from some subsampling expanders point of view. As an example that degree $\log n$ is needed look at the hypercube graph with the usual coloring. In [11] they prove degree $O(\log n \log\log n)$ is enough. If it helps you may assume (for the original geometry problem purposes) that each color class is a perfect matching.

Problem 25. Eigenvalues of regular graphs (from Matija Bucić)

Let $d > 1$ be a fixed integer. Prove that in any d-regular connected graph on n vertices the multiplicity of the second largest eigenvalue of its adjacency matrix is $o(n/\log\log n)$.

This was explicitly asked recently by McKenzie, Rasumussen and Srivastava [12] but traces its origins to [8]. It would have nice implications towards equiangular lines problem but is nice in its own right imo.

Problem 26. Chromatic number vs fractional chromatic number (from Sang-il Oum)

Which hereditary class $\mathcal{C}$ of graphs has a function f such that $\chi(G) \leq f(\chi_f(G))$ for all graphs $G \in \mathcal{C}$?

Any χ-bounded class would have such property. Can we extend any of interesting theorems on χ-boundedness to such graph classes?

Are there some graph classes not χ-bounded but has such a function?

From Tung: I have also asked this question to some people in the community (including Paul and Louis if I remember correctly), in particular in the case where $\mathcal{C}$ is defined by excluding a given induced forest (focusing on the Gyárfás–Sumner conjecture). In this special case I suspect that f can be even taken to be a polynomial; so we could reduce polynomial Gyárfás–Sumner to the fractional version (which seems much more amenable given the recent work on Erdős–Hajnal).

Problem 27. Shortest cycle with modularity constraints (from Chun-Hung Liu)

Fix integers ℓ and m with $0 \leq \ell < m$ and $m \geqslant 2$. Can we find a shortest cycle of length ℓ mod m in the input graph G in polynomial time?

The cases $(\ell, m) = (1, 2)$ and $(0, 2)$ correspond to the problem on finding the odd girth and even girth, respectively, and they are known to be polynomial time solvable. For any $m \geqslant 3$ and nonzero ℓ, whether there is a polynomial time algorithm that simply tests the existence of a cycle of length ℓ mod m was an old open problem; a positive answer was recently provided by Youngho Yoo and I via providing a polynomial time algorithm to test the existence of a subdivision of an arbitrary fixed graph with group expressable constraints. It is possible that the algorithm for testing the existence of such a cycle can be used as a black box to find the shortest one.

Problem 28. Near faithful-universal graphs (from Alexandra Wesolek)

A host graph H is called *subgraph-universal* for a family of graphs $\mathcal{G}$ if every graph $G \in \mathcal{G}$ appears as a subgraph of H. For host graphs that are trees, the first result from Gol'dberg and Livshits from 1968 shows that there exists a subgraph-universal tree for n-vertex trees on $n^{O(\log(n))}$ vertices. Their construction was shown to be tight up to a polynomial factor in n by Chung, Graham and Coppersmith. We can often find a much smaller subgraph universal graph for n-vertex trees if we weaken the conditions of the host graph.

Let $\mathcal{G}_k(n)$ be the family of n-vertex graphs with no C_k-minor. Does there exists a function f such that for every $k \geqslant 1$ there exists a subgraph-universal graph with no $C_{f(k)}$-minor and $n^{O_k(1)}$ vertices for $\mathcal{G}_k(n)$?

We showed that there exists an outerplanar universal graph of treewidth 3 for the class of trees in Bergold-Iršič-Lauff-Orthaber-Scheucher-Wesolek. It contains large cycles. In another yet unpublished work we show that there exists a subgraph universal graph of treewidth $3k-1$ and $n^{O_k(1)}$ vertices for treewidth k graphs [1].

Problem 29. Defective colouring of blowups (from Sergey Norin)

This recent problem is due to Guo-Kang-Zwaneveld. The *d-defective chromatic number $\chi_{\Delta,d}(G)$ of a graph G* is the least k such that there exists a partition $V_1, \dots, V_k$ of $V(G)$ with $\Delta(G|V_i) \leq d$ for every $1 \leq i \leq k$.

Conjecture (Guo, Kang, Zwaneveld): For every graph G and every integer $d \geqslant 0$

$$\chi(G) = \chi_{\Delta,d}(G \boxtimes K_{d+1}).$$

Update (Sergey Norin and Raphael Steiner): This conjecture is false. Further update: $\chi(G) \leq 2\chi_d(G \boxtimes K_{d+1})$ always holds, and the constant in this estimate cannot be improved below 30/29, not even asymptotically. It is easy to see that $\chi(G) \geqslant \chi_{\Delta,d}(G \boxtimes K_{d+1})$. Guo-Kang-Zwaneveld show that the conjecture holds when $\chi_{\Delta,d}(G \boxtimes K_{d+1}) \leq 3$ (and so when $\chi(G) \leq 4$). The proof is very short, but the next case appears to be significantly harder:

Question: Let G be a graph, $d \geqslant 0$ be an integer and let $G' = G \boxtimes K_{d+1}$. Suppose that there exists a partition $V_1, \dots, V_4$ of $V(G')$ such that $\Delta(G'|V_i) \leq d$ for $1 \leq i \leq 4$. Is it true that $\chi(G) \leq 4$?

Problem 30. Analogue of tree decompositions for 2-complexes (from Sergey Norin)

Let G and H be graphs. An *H-decomposition of G* is a function $\beta : V(H) \to 2^{V(G)}$ assigning to every vertex of H a subset of $V(G)$ such that

- for every $uv \in E(G)$ there exists $t \in V(H)$ such that $u, v \in \beta(t)$,
- for every $v \in V(G)$ the subgraph of H induced by $\{t \in V(H) \mid v \in \beta(t)\}$ is connected and non-null.

Question: What is the "right" extension of this definition to 2-dimensional simplicial complexes G and H?

In particular, we seem to have a reasonable notion of a "bramble" on a 2-dimensional simplicial complex. (See Eppstein-Hickingbotham-Merker-Norin-Seweryn-Wood.) For graphs, brambles are dual to tree-decompositions, in a sense that a high-order bramble is an obstruction to a low-width tree-decomposition. It would be nice to preserve this property, i.e. to have brambles with large hitting number to be (the only?) obstructions to decompositions over a contractible 2-complex with small overlap.

Problem 31. Linear versus Centered Chromatic Number (from Pat Morin)

A *centered colouring* of a graph G is a vertex colouring with the property that every connected subgraph H of G contains a vertex whose colour is different from that of all other vertices in H. The *centered chromatic number* $\chi_{\text{cen}}(G)$ of G is the minimum number k of colours for which G has a a centered colouring using k colours. A *linear colouring* of G is a vertex colouring with the property that every path P in G contains a vertex whose colour is different from that of all other vertices in P. The *linear chromatic number* $\chi_{\text{lin}}(G)$ of G is the minimum number k of colours for which G has a a linear colouring using k colours.

Obviously, $\chi_{\text{lin}}(G) \leq \chi_{\text{cen}}(G)$ (since every path in G is a connected subgraph of G). On the other hand, Bose-Dujmović-Houdrouge-Javarsineh-Morin show that $\chi_{\text{cen}}(G) \leq \tilde{O}(\chi_{\text{lin}}^{10}(G))$. Is it true that $\chi_{\text{cen}}(G) \in O(\chi_{\text{lin}}(G))$? Czerwiński-Nadara-Pilipczuk bravely conjecture that $\chi_{\text{cen}}(G) \leq 2\,\chi_{\text{lin}}(G)$.

Problem 32. Some Erdős-Pósa Problems (from Kevin Hendrey)

A guest class $\mathcal{H}$ has the *Erdös-Pósa property* in a host class $\mathcal{G}$ if there is a function f such that for every $G \in \mathcal{G}$ and every positive integer k, one of the following holds:

- G contains k disjoint members of $\mathcal{H}$, or
- there is a set S of at most $f(k)$ vertices hitting all members of $\mathcal{H}$ in G.

Famously, Erdős and Pósa proved that every graph contains either many disjoint cycles or a small set hitting all cycles, so the class of cycles has the Erdős-Pósa property in the class of all graphs.

A cycle C in a graph G is *geodesic* for all $v, w \in V(C)$, the distance from v to w in C is the same as the distance from v to w in G.

1. Do geodesic cycles have the Erdős-Pósa property in the class of all graphs?
2. Do induced cycles of length 0 mod 3 have the Erdős-Pósa property in the class of all graphs?
3. Do pairs of linked cycles have the Erdős-Pósa property in the class of graphs embedded in 3-space?

I believe the first two questions are due to O-joung Kwon, and the third is due to Jinha Kim.

Some Remarks on Kevin's #2 from Linda

- A weaker version of problem 2 coming from of O-Joung Kwon, Eunjung Kim and Tony Hyunh is stated as Problem 8 in `https://web.math.princeton.edu/~tunghn/2022bbda.pdf`

- Problem 2 is from Jihna Kim who was motivated by concerns in topological combinatorics (graphs with no induced cycle of length $0 \mod 3$ have nice properties [9, 6]). However already weaker versions are enough for her application: see the MATRIX Workshop 2023 Open Problem list Problem 18 (and also in the comments) `https://dimag.ibs.re.kr/home/cook/open-problems/`. If you are interested in 2, I would strongly recommend contacting her :)

Problem 33. Minimal monotone class with unbounded chromatic number (from Bartosz Walczak)

A class of graphs $\mathcal{G}$ is *monotone* if it is closed under (non-induced) subgraphs. Does there exist a monotone class $\mathcal{G}$ of graphs with unbounded chromatic number which is minimal with that property, that is, for every proper monotone subclass $\mathcal{H}$ of $\mathcal{G}$, the graphs in $\mathcal{H}$ have bounded chromatic number?

A negative answer follows from the well-known conjecture of Erdős and Hajnal which asserts that for any c and g there is c' such that every graph with chromatic number at least c' contains a subgraph with girth at least g and chromatic number at least c. The latter conjecture is known to be true for $g = 4$ by a result due to Rödl, but it is open for $g \geq 5$. A weaker variant of the conjecture with "girth" replaced by "odd girth", stated by Mohar and Wu, also implies a negative answer, but it is also open. By Rödl's result, it suffices to focus on classes of graphs $\mathcal{G}$ that are triangle-free.

Problem 34. Fractional degree list packing (from Evelyne Smith-Roberge)

This is a problem of Cambie, Cames van Batenburg, Davies, and Kang that I like (it is Problem 29 in [3]). First, some background: given a graph G with list assignment L where $|L(v)| = k$ for all $v \in V(G)$, an L-packing is a list of k L-colourings where for each vertex v and each colour $c \in L(v)$, v gets assigned colour c in exactly one of the colourings in our list. The smallest such k such that an L-packing exists no matter L is the *list packing number* of G. If our lists are different sizes, we cannot ask for an L-packing, but we can still ask for something analogous: a probability distribution on the L-colourings of G where for each v and each $c \in L(v)$, the probability that v gets colour c is $\frac{1}{|L(v)|}$. This is a *fractional L-packing*.

Consider now a graph G with list assignment L where $|L(v)| = \deg(v)$ for all $v \in V(G)$ (call this a *degree list assignment*). The question is: if G has a fractional L-packing no matter the degree list assignment L, what does G look like? (I don't know of any examples of such G, so as far as I know it is possible the answer is G *does not exist.* (ESR: actually there are already some even in the first paper where list packing is introduced, but the paper doesn't talk about this degree list assignment so I did not think of them. Apologies!))

Raphael Steiner: I can show that all bipartite graphs with minimum degree at least $C \log n$ have this property. So these graphs do exist and there are in fact many.

(ESR: Woohoo! Thanks. I figured there must be *some* such graphs, but I somehow hadn't considered looking at graphs whose list chromatic number was way below what the degree list assignment would give) RS: A generalization of the above argument shows that for some absolute constant $C > 0$ the following holds: For every graph G on n vertices, we have $\chi_\ell^\bullet(G) \leq C\chi(G) \log n$, where the first quantity is the fractional list packing number. (This is also implied by a result in the paper Packing List-Colourings; the characterization is the goal! Again, I apologize – my parenthetical comment above was definitely known previously, and implied by results I definitely should have thought of when writing this before)

NB: In the same paper linked above, Cambie et al. show that if we work instead with list assignments L where $|L(v)| = \deg(v) + 1$ for all v, *every* graph is fractionally packable.

A helpful, related question: suppose G is a 2-connected graph. Suppose I can precolour any single vertex in G any way I like no matter the degree list assignment L, and always have that precolouring extend to a colouring of G. What does G look like?

We could also ask the same two questions for *correspondence colouring* (DP-colouring). It would be extra interesting if the correspondence colouring questions had different answers than the list colouring questions.

Problem 35. We are still interested in 4-critical t-perfect graphs (communicated by Linda Cook, joint with Maria, James, Jane and Sang-il)

This was presented other workshops this year so I don't think we present it again, but for those who weren't there here is the question.

We recently showed that t-perfect graphs [1] have chromatic number at most 200,000. It is open whether they are 4-colorable. There are only two known minimal examples of 4-critical t-perfect graphs (due to Paul Seymour, Monique Laurent and due to Yohann Benchetrit) and they are both small graphs. This seems silly, and we would like more (ideally an infinite list of them). See [5] for more background on the problems.

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