Research Statement

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1 Introduction

My research interests are matrix/non-commutative inequalities, and their relationship with the geometry of Banach spaces and with operator theory. Matrix inequalities can be easy to state as a generalization of classic real/vector inequalities, but the methods to prove them are very different from the classical setting. Many simple looking inequalities have lead to remarkable discoveries. As a simple example, while the Minkowski (triangle) inequality $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ for $f, g \in L_p(0,1)$ is relatively straightforward to prove, proving the triangle inequality for matrices with the Schatten-p norm $\|\cdot\|_{S_p} : M_n(\mathbb{C}) \to \mathbb{R}_+$,

$$\|A\|_{S_p} = \left(\operatorname{tr}(A^*A)^{p/2}\right)^{1/p},$$

requires a good understanding of the behaviour of the singular values of matrices. My research is inspired by the driving question: what about inequalities that are as yet out of our reach?

The main motivating example in my work has been the Schatten-p analogue of Hanner's inequality, which states that

$$\|A+B\|_{S_p}^p + \|A-B\|_{S_p}^p \leq (\|A\|_{S_p} + \|B\|_{S_p})^p + |\|A\|_{S_p} - \|B\|_{S_p}|^p \tag{1}$$

whenever $p \geq 2$ and $A, B \in S_p$. This inequality was proven for L_p by Hanner [Han56], and for S_p for $p \geq 4$ by Ball, Carlen, and Lieb [BCL94]. While working on the case $p \geq 2$, I discovered a novel structural result for Hermitian matrices which implies this inequality as a corollary.

Theorem 1.1. *Let n be a positive integer and $A, B \in M_n(\mathbb{C})$ be Hermitian. Then, there exists a positive measure $\mu_{A,B}$, the* tracial joint spectral measure*, on $\mathbb{R}^2$, such that the following is true:*

For a nice enough $f : \mathbb{R} \to \mathbb{R}$, there is a closely associated function $H(f) : \mathbb{R} \to \mathbb{R}$, such that the trace of $H(f)(xA + yB)$ for any numbers $x, y \in \mathbb{R}$ can be computed solely in terms of x, y, the function f, and the measure $\mu_{A,B}$, as follows

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by)\, \mathrm{d}\mu_{A,B}(a, b). \tag{2}$$

For the case of $f(t) = |t|^p$ the function $H(f)$ is a positive rescaling of f, and this can be used to prove Hanner's inequality (1) for every $p \geq 2$. Theorem 1.1 has implications that go far beyond merely proving Hanner's inequality. As another example of a corollary of it, one can deduce a conceptually new proof of a conjecture of Bessis–Moussa–Villani [BMV75], which was proven by Stahl in the celebrated work [Sta13].

In my future research, I plan to explore further properties and applications of the tracial joint spectral measures.

Question 1.2. *Does the theory of tracial joint spectral measures extend beyond the setting of Hermitian matrices?*

Question 1.3. *What tools are needed to get past the main limitation of tracial joint spectral measures: they can only be used to prove inequalities for linear combinations of two matrices?*

One of my projects has been investigating, with Vijay Bhattiprolu, Grothendieck's inequality, introduced in the seminal work [Gro53]. In our ongoing work, we make the conjecture that the 3-dimensional Grothendieck constant $K_G(3) = 1/\log(2) \approx 1.4427$. This is based on a careful analysis and modification of the argument of Braverman–Makarychev–Makarychev–Naor [BMMN13]. Aside from promising numerical testing of our conjecture, we have also been investigating ways in which one can generalize the arguments used by Krivine to establish the value of the 2-dimensional constant $K_G(2) = \sqrt{2}$.

Computer-aided investigations have been a big part of my research projects, and they were instrumental in the discovery of tracial joint spectral measures. Mathematical packages like Mathematica and traditional programming languages such as Python have provided invaluable help to me in stress testing inequalities and discovering patterns and identities. I hope to explore creative ways to use computers for discovery and visualization for my future projects.

The remaining two sections contain a detailed exposition of my past research and future research projects.

2 My research

My main contributions to the study of matrix inequalities are the following results.

Theorem 2.1. *If A and B are elements of Schatten-p for some $p \geq 1$, then their span is linearly isometric to a subspace of $L_p(0,1)$.*

As a consequence, to prove an inequality that depends on the S_p norms of linear combinations of two matrices, it suffices to consider real diagonal matrices. This provides a powerful new tool for deducing non-commutative inequalities from their commutative counterparts. In particular, it resolves a conjecture of Ball–Carlen–Lieb on the Schatten-p version of Hanner's inequality (1) for $p \geq 2$, given that Hanner proved it for vectors in [Han56].

I first proved Theorem 2.1 for p equals $3, 4$, and 6 in [Hei22].

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with non-negative kth derivative. For any $A, B \in M_n(\mathbb{C})$ Hermitian, consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by*

$$F(t) = \operatorname{tr} f(tA + B).$$

Then, if k is even, F has non-negative kth derivative. The same is true for odd k if we additionally assume that A is positive semidefinite.

This result is well known for k equals 1 and 2. For k equals 3 and 4, I proved it in [Hei22].

Theorem 2.2 is a vast generalization of a celebrated result of Stahl [Sta13] (formerly the BMV conjecture) on the completely monotone nature of the function $t \mapsto \operatorname{tr} \exp(-tA+B)$. Indeed, one obtains Stahl's theorem by applying Theorem 2.2 to the function $f(t) = \exp(-t)$.

The proof of both Theorem 2.1 for $p = 3$, and Theorem 2.2 for $k = 4$, follow from the following result, proven in [Hei22].

Theorem 2.3. *Let $A, B \in M_n(\mathbb{C})$ be Hermitian. Then the function*

$$t \mapsto \operatorname{tr} |tA + B|^3$$

has non-negative 4th derivative.

The proof of this result is based on a rather miraculous identity, expressing the fourth derivative as a sum of clearly non-negative terms. Despite serious efforts, I was unable to find a similar identity for higher derivatives.

The cases p equals 4 and 6 follow similarly from equally miraculous trace identities that express the embedding condition in terms of the non-negativity of certain trace polynomials. These trace polynomials are then given a "sum of squares certificate" to establish their non-negativity. This approach does not generalize easily for larger even integers p, and might in fact be impossible. Indeed, the recent refutation of the Connes embedding conjecture [JNV$^+$21] has the following interesting consequence, proven by Klep and Schweighofer in [KS08]: there exist polynomials $p(X, Y)$ of matrices such that the trace $\operatorname{tr} p(X, Y)$ is non-negative for every Hermitian X and Y, but the non-negativity of $\operatorname{tr} p(X,Y)$ cannot be exhibited by a "sum of squares certificate" of a certain type.

The general cases of Theorems 2.1 and 2.2 follow from a totally different approach, namely a new structural result for Hermitian matrices. Given two Hermitian matrices A and B, I introduce in [Hei23] a positive measure $\mu_{A,B}$, which I call the *tracial joint spectral measure*.

Theorem 2.4. (= Theorem 1.1) *Let n be a positive integer and $A, B \in M_n(\mathbb{C})$ be Hermitian. Then, there exists a positive measure $\mu_{A,B}$ on $\mathbb{R}^2$ such that the following is true:*

Fix any measurable function f on $\mathbb{R}$ such that for any $M > 0$,

$$\int_{-M}^{M} \left| \frac{f(t)}{t} \right| \mathrm{d}t < \infty.$$

Define a function $H(f) : \mathbb{R} \to \mathbb{R}$ by

$$H(f)(x) = \int_0^1 \frac{1-t}{t} f(xt)\, \mathrm{d}t.$$

Then, for any $x, y \in \mathbb{R}$, we have

$$\operatorname{tr} H(f)(xA + yB) = \int_{\mathbb{R}^2} f(ax + by)\, \mathrm{d}\mu_{A,B}(a,b). \tag{3}$$

If one applies Theorem 2.4 to the function $f(t) = |t|^p$, we have $H(f) = f/(p(p+1))$, and hence

$$\operatorname{tr} |xA + yB|^p = p(p+1) \int_{\mathbb{R}^2} |ax + by|^p\, \mathrm{d}\mu_{A,B}(a,b).$$

This identity can be interpreted as an embedding of the span of A and B to $L_p(\mu_{A,B})$. Strikingly, the embedding is proportional to an isometry simultaneously for every $p > 0$. From the embedding for the Hermitian case, one can deduce the embedding for the S_p case.

Similarly, applying Theorem 2.4 to the function $f(t) = t_+^{k-1}$ implies Theorem 2.2.

I also give an explicit expression for $\mu_{A,B}$.

Theorem 2.5. *Let n, A, B, and $\mu_{A,B}$ be as in Theorem 2.4. Denote by $\mu_c = \mu_{c,A,B}$ and $\mu_s = \mu_{s,A,B}$ the continuous and singular parts of $\mu_{A,B}$ w.r.t. the Lebesgue measure m_2 on $\mathbb{R}^2$. We assume some linear combination of A and B is invertible. Then, the continuous part μ_c is given by*

$$\frac{\mathrm{d}\mu_c}{\mathrm{d}m_2}(a,b) = \frac{1}{2\pi} \sum_{i=1}^{n} \left| \Im \left(\lambda_i \left(\left(I - \frac{aA + bB}{a^2 + b^2} \right) (bA - aB)^{-1} \right) \right) \right|. \tag{4}$$

Furthermore, if A is invertible and $A^{-1}B$ has n distinct eigenvalues, the singular part μ_s satisfies

$$\mu_s(\varphi) = \sum_{v \in E(A^{-1}B)} \int_0^1 \frac{1-t}{t} \varphi\left(\langle Av, v\rangle t, \langle Bv, v\rangle\right) \mathrm{d}t. \tag{5}$$

where $E(C)$ denotes a set of normalized eigenvectors of a matrix $C \in M_n(\mathbb{C})$ and φ is a smooth function with compact support that does not contain 0.

See Figure 1 for some illustrations of tracial joint spectral measures.

3 Current and future projects

3.1 Properties of tracial joint spectral measures

In my upcoming work titled "Properties of tracial joint spectral measures", I will further examine the properties of tracial joint spectral measures.

One naturally asks to what extent tracial joint spectral measures can be generalized beyond matrices, especially given that it was proven in [Hei23] that the embedding result Theorem 2.1 extends to the infinite dimensional setting S_p.

Question 3.2. *Given two compact self-adjoint operators A and B in S_p, does there exist a measure $\mu_{A,B}$ satisfying the conditions of Theorem 2.4? How about the setting of von Neumann algebras with n.s.f. trace?*

More specifically, one can ask if the isometric embedding result generalizes to non-commutative L_p spaces in the sense of Haagerup (see [PX03]).

Question 3.3. *Is every 2-dimensional subspace of a non-commutative L_p space linearly isometrically embeddable into L_p?*

Question 3.4. *Is there an explicit formula similar to equation (4) in these infinite dimensional settings?*

As observed in [Hei23], the tracial joint spectral measure $\mu_{A,B}$ only depends on the so called Kippenhahn polynomial

$$p_{A,B}(x, y, z) = \det(zI + xA + yB). \tag{6}$$

Such polynomials are hyperbolic (of degree n) in the sense of Gårding [Går59], meaning that for every fixed x and y, the polynomial in z is of degree n and only has real roots. It was shown by Helton and Vinnikov in [HV07] that any hyperbolic polynomial with $p(0, 0, 1) = 1$ arises as in (6) from some pair of Hermitian matrices. This associates to any hyperbolic polynomial a tracial joint spectral measure μ_p with the defining property

$$\int_0^1 \log|p(1, tx, ty)| \frac{1-t}{t}\, \mathrm{d}t = \int_{\mathbb{R}^2} \log|1 + ax + by|\, \mathrm{d}\mu_p(a, b). \tag{7}$$

Question 3.5. *Given a hyperbolic polynomial p, can one see without the introduction of A and B that a measure satisfying (7) exists?*

3.6 Random tracial joint spectral measures

While the illustrations in Figure 1 are of small matrices, one would be interested in seeing figures of measures $\mu_{A,B}$ when A and B are, say, Hermitian 100×100 matrices. For such sizes, estimation of the density with (4) becomes algorithmically infeasible, but one can still approximate the boundary curve. In my upcoming work, I prove that this boundary coincides with the Kippenhahn curve [Kip51], an algebraic curve the boundary of which coincides with the joint numerical range

$$W(A, B) := \{(\langle Av, v\rangle, \langle Bv, v\rangle) \mid |v| = 1\}.$$

For two random Gaussian Hermitian matrices, one obtains the curve in Figure 2. The following patterns can be observed.

1. The convex hull of the support of $\mu_{c,A,B}$, i.e. the joint numerical range of A and B, approximates a disk of radius $2\sqrt{n} = 20$. This is proven in [CGLZ14], where joint numerical ranges of random matrices are investigated.
2. The Kippenhahn curve seems to consist of several circle-like curves, at least away from the origin.
3. The singular part has $10 \sim \sqrt{n}$ segments, in superficial agreement with the fact that a random real characteristic polynomial p should have $\sim \sqrt{\deg(p)}$ real roots, see [EKS94]. These segments seem to be close to the origin.

Question 3.7. *Is there a way to formalize these observations for some random matrix ensembles? Do random tracial joint spectral measures converge, properly normalized, to a universal limiting distribution?*

3.8 Beyond two dimensions

In [Hei22], I proved that Theorem 2.1 cannot be generalized to more than two matrices. This implies, in particular, that there is no natural analogue of Theorem 2.4 for more than two matrices, i.e. any such $\mu_{A,B,C}$ would sometimes not be a positive measure.

Question 3.9. *While $\mu_{A,B,C}$ is not a positive measure, can it still shed light on matrix inequalites involving more than two matrices? Can it be used to characterize inequalities governing three dimensional subspaces of S_p?*

Consider the generalized roundness inequality introduced by Enflo in [Enf69]. We say that a metric space has generalized roundness p if, for any positive integer n and points $x_1, x_2, \ldots, x_n, y_1, \ldots, y_n$, one has

$$\sum_{1\le i<j\le n} d(x_i,x_j)^p + \sum_{1\le i<j\le n} d(y_i,y_j)^p \le \sum_{1\le i,j\le n} d(x_i,y_j)^p.$$

Enflo proved that L_p has generalized roundness p for $1 \le p \le 2$, and used this to prove that L_p spaces are not uniformly homeomorphic for $1 \le p \ne q \le 2$.

Schatten-p spaces don't have generalized roundness, but one can still ask for the best constant C_p in the inequality

$$\sum_{1\le i<j\le n} \|A_i - A_j\|_{S_p}^p + \sum_{1\le i<j\le n} \|B_i - B_j\|_{S_p}^p \le C_p \sum_{1\le i,j\le n} \|A_i - B_j\|_{S_p}^p.$$

It turns out that even the case $p = 1$ here is difficult. Naor and Oleskiewicz [NO20] verified that $C_1 \ge \sqrt{2}$, and answering a question of theirs, I proved that C_1 is strictly less than the trivial bound 2. While the constant C_1 is still unknown, I have made the conjecture that the following stronger inequality, which would imply that $C_1 = \sqrt{2}$, is true:

$$\sum_{1\le i<j\le n} \|A_i - A_j\|_{S_1}^2 + \sum_{1\le i<j\le n} \|B_i - B_j\|_{S_1}^2 \le 2 \sum_{1\le i,j\le n} \|A_i - B_j\|_{S_1}^2. \tag{8}$$

Note that this inequality is unknown even for L_1, as it doesn't directly reduce to a scalar inequality. I have proved this inequality in non-trivial special cases for L_1 based on miraculous identities that I don't currently know how to generalize.

Question 3.10. *Is the inequality (8) true? Is there an analogue of (8) for S_p for some $p > 1$?*

3.11 Beyond linear

Theorem 2.1 only treats inequalities between linear combinations of matrices. If one introduces non-linear functions or the product structure, more complicated behaviour arises; new tools are needed.

As the first example, for $p, q \ge 1$ consider the Mazur map, $M_{p,q} : L_p(0,1) \to L_q(0,1)$, defined by

$$M_{p,q}(f) = f|f|^{\frac{p-q}{q}}.$$

This map was used my Mazur [Maz29] to show that $L_p(0,1)$ and $L_q(0,1)$ are homeomorphic, and his argument further implies that the balls B_{L_p} and B_{L_q} are uniformly homeomorphic. The natural analogue for Schatten classes is also true, as proven by Raynaud [Ray02].

From the work of Mazur it also follows that the Mazur map is $\min(p/q, 1)$-Hölder from $B_{L_p} \to B_{L_q}$, and a modification of his argument gives the optimal bounds

$$\|M_{p,q}(f) - M_{p,q}(g)\|_{L_q} \le \begin{cases} 2^{q/p-1}\|f-g\|_{L_p}^{p/q} & p \le q \\ \frac{p}{q}\|f-g\|_{L_p} & p \ge q. \end{cases} \tag{9}$$

For Schatten classes, Ricard [Ric15] proved that $M_{p,q}$ is still $\min(p/q, 1)$-Hölder, but the optimal bounds (as in (9)) are not known. It was proven by Jocić [Joc97] that if $q \ge 2p$, the estimate (9) extends to S_p. Jocić asked whether the same is true for $p < q < 2p$, but I have found a counterexample for every $p = 1 < q < 2$, which will appear in forthcoming work.

Question 3.12. *What are the Lipschitz/Hölder constants of Mazur maps $M_{p,q} : B_{S_p} \to B_{S_q}$?*

As the second example, by a celebrated result of Potapov and Sukochev [PS11], for any $1 < p < \infty$, there exists a constant c_p, such that for any 1-Lipschitz function and any two Hermitian matrices A and B, one has

$$\|f(A) - f(B)\|_{S_p} \leq c_p \|A - B\|_{S_p}. \tag{10}$$

It was later shown by Caspers–Montgomery-Smith–Potapov–Sukochev [CMSPS14] that $c_p \sim p^2/(p-1)$, but the exact value of the constant c_p is only known for $p = 2$ (with $c_2 = 1$).

Question 3.13. *What is the best constant c_p in the inequality (10)?*

As the third example, consider the improvement of the triangle inequality for L_p due to Carlen–Frank–Ivanisvili–Lieb [CFIL21] (see also [IM20]), namely for $p \geq 2$

$$\|f + g\|_p^p \leq \left(1 + \frac{2^{2/p}\|fg\|_{p/2}}{(\|f\|_p^p + \|g\|_p^p)^{2/p}}\right)^{p-1} \left(\|f\|_p^p + \|g\|_p^p\right) \tag{11}$$

It was also proven in [CFIL21] that the natural generalization of (11) to S_p is true for positive definite A and B when p is a power of 2.

Question 3.14. *Is the S_p-analogue of (11) true for $p \geq 2$?*

3.15 Grothendieck's inequality

The d-dimensional Grothendieck constant $K_G(d)$ is the smallest constant such that for any $A \in M_{n\times m}(\mathbb{R})$, the following holds

$$\sup_{v_i, w_j \in S^{d-1}} \sum_{i=1}^n \sum_{j=1}^m A_{i,j} \langle v_i, w_j \rangle \leq K_G(d) \sup_{\varepsilon_i, \delta_j \in \{\pm 1\}} \sum_{i=1}^n \sum_{j=1}^m A_{i,j} \varepsilon_i \delta_j.$$

Grothendieck proved [Gro53] that the constants $K_G(d)$ are bounded and that

$$K_G := \sup_{d\geq 1} K_G(d) = \lim_{d\to\infty} K_G(d) \leq \sinh\left(\frac{\pi}{2}\right), \tag{12}$$

with K_G the Grothendieck constant. This bound was later improved to $\pi/(2\log(1+\sqrt{2}))$ by Krivine [Kri79], who also conjectured this to be optimal. Both Grothendieck's and Krivine's proofs are based on the identity

$$\mathbb{E}_{u\in S^{d-1}}\Big[\operatorname{sign}(\langle v, u\rangle)\operatorname{sign}(\langle w, u\rangle)\Big] = \frac{2}{\pi}\arcsin\left(\langle v, w\rangle\right),$$

where the expectation is taken with the Haar measure on S^{d-1}. This identity is often interpreted as a hyperplane rounding algorithm: by rounding a unit vector v to $\operatorname{sign}(\langle v, u\rangle)$ for a random u, one in expectation preserves the inner product structure on the sphere up to the function $2/\pi \arcsin$. Grothendieck and Krivine use different arguments to get rid of this function, with Krivine pushing the hyperplane rounding to a logical limit of sorts. It was however proved by Braverman–Makarychev–Makarychev–Naor [BMMN13] that Krivine's bound is not optimal, and that more complicated rounding schemes improve the bound.

As observed by Tsirelson [Tsi85], the Grothendieck constants $K_G(d)$ play a central role in quantum mechanics, but despite much interest, only the value $K_G(2) = \sqrt{2}$ [Kri79] is known. For other $K_G(d)$ and K_G, only lower and upper bounds are known [Kri79, HLZ+15, DBV17, DIB+23]. Besides K_G itself, the constant $K_G(3)$ in particular is of great interest for its direct relationship with the non-locality of the Werner state [AGT06]. The current best bounds are $1.4367 \leq K_G(3) \leq 1.4546$ due to [DIB+23].

Vijay Bhattiprolu and I conjecture that $K_G(3) = 1/\log(2) \approx 1.4427$ based on a modification of the argument of [BMMN13]. Roughly speaking, it is proven in [BMMN13] prove that the aforementioned hyperplane rounding algorithm can be improved by a carefully chosen perturbation. This perturbation emerges from a counterexample to a conjecture of König [KÖ1] identifying the worst possible matrix A in Grothendieck's inequality (along with the respective vectors v_i, w_j and signs ε_i, δ_j). We observe that the natural analogue of this perturbation argument fails for $d = 3$, and a variant of König's conjecture then suggests that $K_G(3) = 1/\log(2)$. We have convincing evidence for the conjecture based on numerical analysis of discretizations. We are also optimistic that we can generalize the arguments Krivine used to prove that $K_G(2) = \sqrt{2}$ for the $d = 3$ case.

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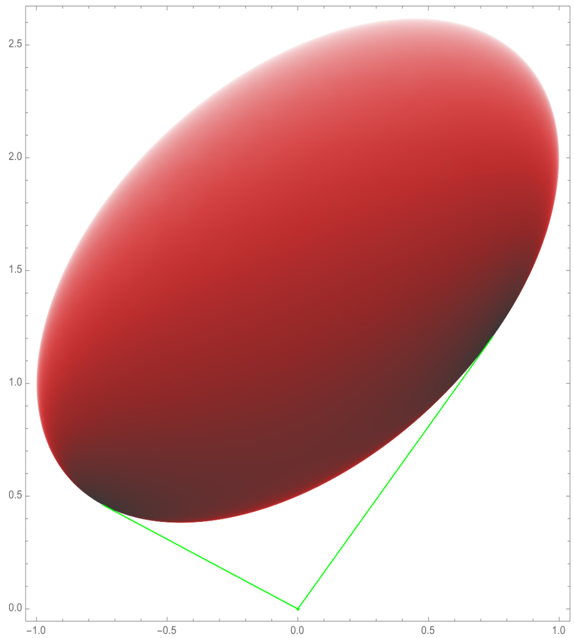
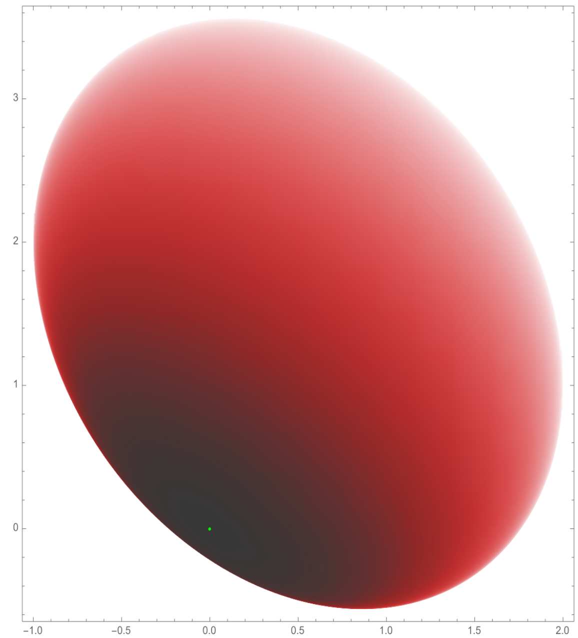
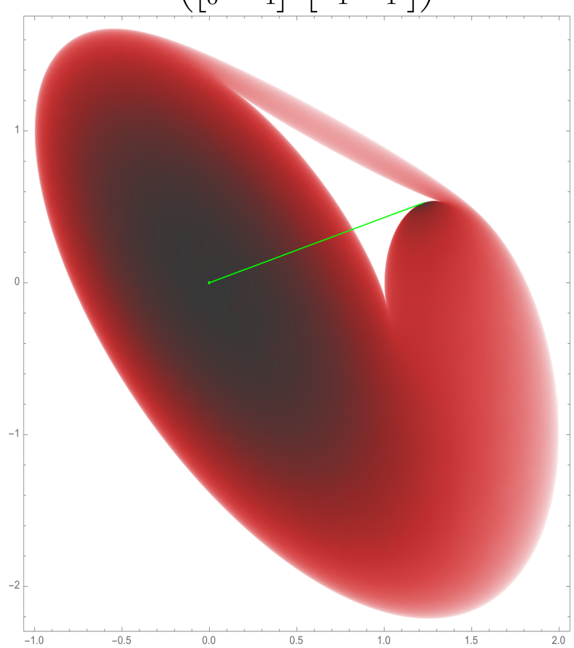


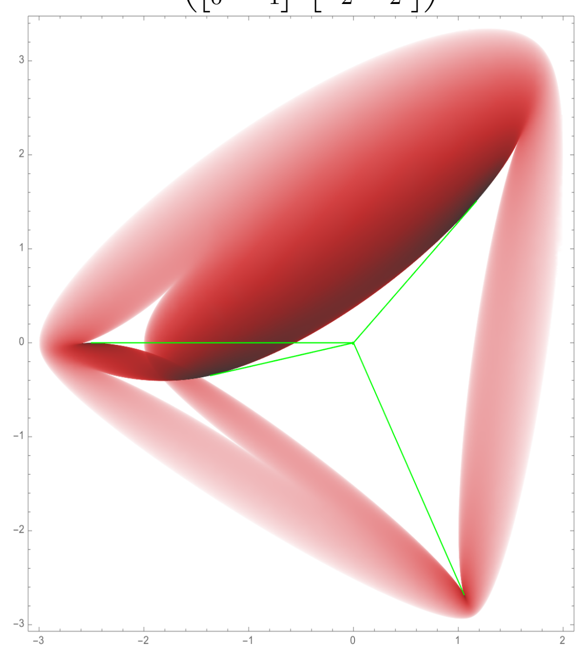
(1) $\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\right)$.



(2) $\left(\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}1 & -2\\ -2 & 2\end{bmatrix}\right)$.



(3) $\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & 1\\ 1 & -1 & -1\\ 1 & -1 & 1\end{bmatrix}\right)$.



(4) $\left(\begin{bmatrix}-3 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\end{bmatrix}, \begin{bmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 0 & -1\\ 0 & 0 & -2 & -2\\ -1 & -1 & -2 & 2\end{bmatrix}\right)$.

Figure 1: Four illustrations of the measures $\mu_{A,B}$ for the pairs of matrices (A, B) listed below the pictures. The horizontal and vertical axes correspond to a and b respectively. The density of $\mu_{c,A,B}$ is represented with the color running from white (zero) to black (infinity) through red. The green line segments depict the support of the singular part.

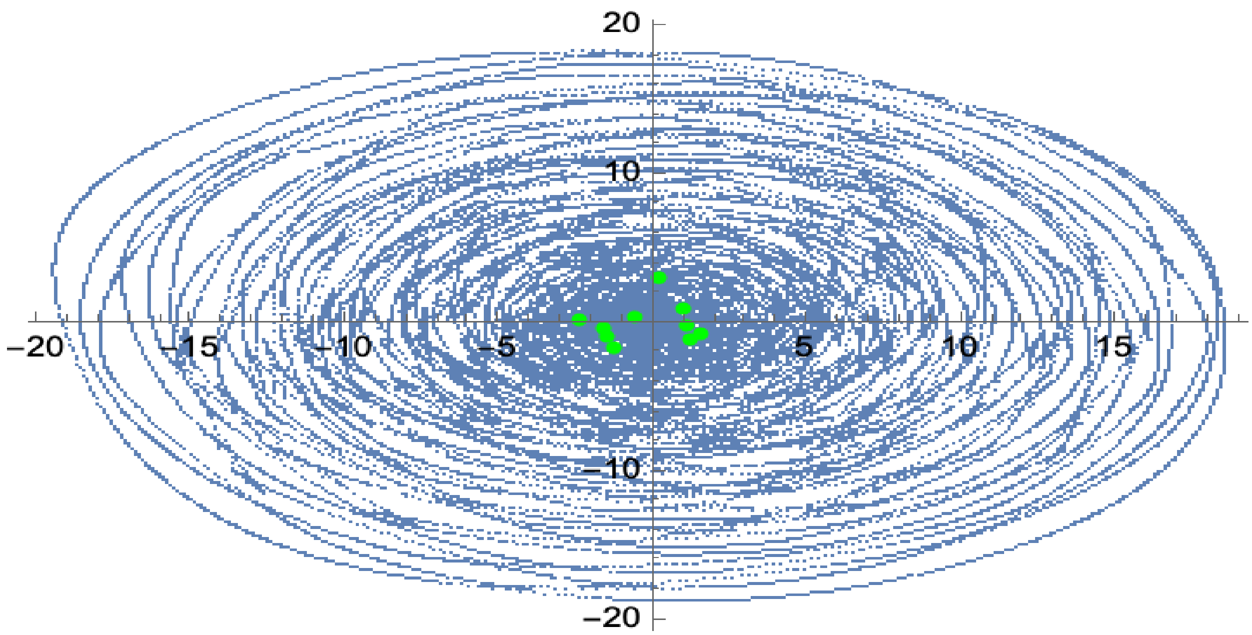


Figure 2: Representation of the Kippenhahn curve of two Gaussian 100×100 matrices in blue. In green are the endpoints of the segments of the singular part $\mu_{s,A,B}$. The Hermitian matrices A and B are the real and imaginary parts of a random complex matrix with iid complex Gaussian entries.