Math 355 - Spring 2019 - Homework 10

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Due: May 8

1. For $M$ an $n$-dimensional $C^\infty$-manifold, show that the tangent bundle $TM$ to a $C^\infty$-manifold is itself a $2n$-dimensional $C^\infty$-manifold.

2. We explained how the “chain rule” $\frac{\partial}{\partial x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\partial}{\partial \tilde{x}^j}$ holds for tangent vectors when we change coordinates. For 1-forms, show that $dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j$ holds. Hint: given a (finite dimensional) vector space, if we consider the change of basis matrix $a_{ij}$ defined by $v^i = \sum_j a_{ij} \tilde{v}^j$, suppose that $v^i_*$ are the canonical basis for the dual vector space $V^*$, i.e., $v^i_*(v_j) = \delta^i_j$ (i.e., the function which is 1 when $i = j$ and 0 otherwise), how does $a_{ij}$ relate to the change of basis for $v^i_*$ to $\tilde{v}^j_*$?

3. Given a Riemannian metric $g$ on a $C^\infty$-manifold, explain why we can write $g$ in local coordinates $\{x^i\}_{i=1}^n$ using the metric coefficients $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ as $g = g_{ij} dx^i \otimes dx^j$.

Suppose that $\{\tilde{x}^i\}$ is a new set of coordinates on the same chart. Compute how the metric coefficients $\tilde{g}_{ij}$ change and check that you still have the formula $g = \tilde{g}_{ij} d\tilde{x}^i \otimes d\tilde{x}^j$.

4. Suppose that $f : \Omega \to \mathbb{R}$ is a $C^\infty$-function on $\Omega \subset \mathbb{R}^n$. Define the graph of $f$ by

$\text{graph}(f) := \{(x, f(x)) : x \in \Omega\}$

(a) Show that the chart $\pi : \text{graph}(f) \to \Omega, \pi(x, z) \mapsto x$ makes $M := \text{graph}(f)$ into a $C^\infty$-manifold (with the induced topology as a subset of $\mathbb{R}^{n+1}$).

(b) For $p \in M$, set $V_p := \{\gamma'(0) : \gamma : (-\epsilon, \epsilon) \to M, \gamma(0) = p\}$ (where we consider $C^\infty$ maps $\gamma$, thought of as maps into $\mathbb{R}^{n+1}$). Define a map $\Phi : V_p \to T_pM$ as follows: for $f \in C^\infty(M)$, set $\Phi(\gamma'(0)) := \frac{d}{dt}|_{t=0} f \circ \gamma(t)$.

Check (i) $\Phi$ is well defined (independent of the choice of $\gamma$) and (ii) $\Phi$ is an isomorphism.

(c) The usual inner product on $\mathbb{R}^{n+1}$ induces an inner product on $V_p$. Show that this defines a Riemannian metric on $M$ (via $\Phi$). Compute the metric coefficients for the chart $\pi$ described above.