1 Solution

As a warmup, we first prove the following:

**Proposition 1.** Suppose that \( A \subset \mathbb{R}^2 \) is a regular surface. Then \( A \) is open as a subset of \( \mathbb{R}^2 \).

**Proof.** For \( p \in A \), choose a parametrization \( x : U \rightarrow A \cap V \) where \( V \subset \mathbb{R}^3 \) is an open set and \( p \in x(U) \). Set \( q = x^{-1}(p) \). Since \( z(u,v) \equiv 0 \), we must have that

\[
\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0.
\]

Hence, for \( F(u,v) := (x(u,v), y(u,v)) \), we have that \( F \) is differentiable on \( U \) and \( dF_q \) is invertible. Thus, the inverse function theorem guarantees that there is \( p \in U' \subset U \) and \( V' \subset \mathbb{R}^2 \) so that \( F|_{U'} : U' \rightarrow V' \) is a diffeomorphism. This implies, in particular that \( V' \subset F(U) \). This implies that \( F(U) \) is open.

To solve the full problem, we try the same strategy, except use the fact that \( S \) is a regular surface to say that it’s locally a graph:

**Proposition 2.** Suppose that \( A \subset S \) is a subset of a regular surface. The set \( A \) is a regular surface if and only if \( A = U \cap S \) for \( U \) an open set in \( \mathbb{R}^3 \).

**Proof.** We will only prove that if \( A \) is a regular surface then \( A = U \cap S \) for \( U \) open in \( \mathbb{R}^3 \) (the other direction is easier). Note that this is equivalent to proving that \( A \) is an open subset of \( S \) (where \( S \), as usual, is given the subspace topology).

Choose \( p \in A \subset S \). Because \( S \) is a regular surface, we can assume that there is an open set \( W \subset \mathbb{R}^3 \) with \( S \cap W \) the graph of \( z = f(x,y) \). Choose a parametrization \( x : U \rightarrow A \cap V \) with \( p \in x(U) \). We claim that

\[
\frac{\partial(x,y)}{\partial(u,v)}(q) \neq 0 \quad (*)
\]

for \( q = x^{-1}(p) \). Given (\(^\ast\)) we can complete the proof as follows. Set \( F(u,v) := (x(u,v), y(u,v)) \). Applying the inverse function theorem as in the previous proof, we find that \( F|_{U'} : U' \rightarrow V' \) is a diffeomorphism. We claim that this implies that \( x(U') \subset A \) is open as a subset of \( S \). To see this, we note that (because \( S \cap W \) is a graph)

\[
x(U') = \pi^{-1}(V') \cap \underbrace{W \cap S}_{\text{open} \subset \mathbb{R}^3}
\]

where \( \pi(x,y,z) = (x,y) \). This concludes the proof, except we must justify (\(^\ast\)). We prove this below. \( \square \)

**Lemma 1.** The claim (\(^\ast\)) holds.

**Proof 1.** Recall that we are assuming that \( A \) and \( S \) are regular surfaces \( A \subset S \). We claim that \( T_pA = T_pS \). Any differentiable curve \( \alpha : (-\epsilon, \epsilon) \rightarrow A \) with \( \alpha(0) = p \) is also a differentiable curve in \( S \). This gives \( T_pA \subset T_pS \). Since both are 2-dimensional, they are equal.

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Now, if \((*)\) fails, it would mean that the map \(F := \pi \circ x\) had \(dF_q\) singular. This would mean that there was a curve \(\beta : (-\epsilon, \epsilon) \to U\) with \(\beta(0) = q\) so that \(dF_p(\beta'(0)) = 0\) but \(\beta'(0) \neq 0\). Note that \(\alpha(t) := x \circ \beta(t)\) has \(0 \neq \alpha'(0) \in T_p S\). By assumption \(d\pi_p(\alpha'(0)) = 0\), which means \(\alpha'(0)\) is completely in the \(z\)-direction. This is absurd, since the tangent plane to a graph \(S \cap W\) cannot be completely vertical. To prove this, note that \(\tilde{x}(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v}, f(\tilde{u}, \tilde{v}))\) is a parametrization of \(S \cap W\), so

\[
\frac{d\tilde{x}}{dR^2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ f_{\tilde{u}} & f_{\tilde{v}} \end{pmatrix}
\]

so \(d\tilde{x}(\mathbb{R}^2)\) is not vertical.

Proof 2. We can also give a proof that does not explicitly use the notion of the tangent plane. Replacing \(U\) by \(U \cap x^{-1}(W)\) (which is still an open set since \(x\) is continuous), we can assume that \(x(U) \subset W\). Because \(S \cap W\) is a graph and \(x(U) \subset S \cap W\), we see that

\[
z(u, v) = f(x(u, v), y(u, v))
\]

Thus, we find that

\[
d\tilde{x} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ f_xx_u + f_yy_u & f_xx_v + f_yy_v \end{pmatrix}.
\]

Because the third row is a linear combination of the other two, and we know that some \(2 \times 2\)-minor of this matrix is nonsingular, we see that it must be the minor formed by the first two rows. This completes the proof.

2 Invariance of domain

Note that in general, if \(f\) is a homeomorphism from \(A \to B \subset C\) (where \(A, B, C\) are topological spaces), then one cannot conclude that \(B\) is open! Because \(A\) is open in itself, the homeomorphism property implies that \(B\) is open in the subspace topology it inherits from \(A\). But this was already automatic from the fact that \(B = C \cap B\) and \(C\) (the entire set) is always open. So, in the previous problem, one cannot argue that \("U\) is open and \(x\) is a homeomorphism so \(x(U)\) is open."

We remark that it turns out that with a lot more machinery, one can justify the previous claim \textit{in our setting} (but this is not how one should solve the above homework problem). Indeed, the following is true but to prove it requires nontrivial tools from topology:

**Theorem 1** (Invariance of domain). \(\text{Suppose that } U \subset \mathbb{R}^n\) is open and \(f : U \to \mathbb{R}^n\) is an injective continuous map. Then \(f(U)\) is open and \(f\) is a homeomorphism onto its image.

Note that Proposition 1 above essentially contains the proof of this in the \textit{very special case} when \(f\) is differentiable and \(df_p\) is non-singular for all \(p \in U\).

Note also that the invariance of domain is false for other topological spaces! A simple example is the shift map \(f : \ell^\infty(N) \to \ell^\infty(N)\) (the Banach space of bounded sequences with the supremum norm), i.e., \(f((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots)\). The domain is the entire space (and thus open) but the image of \(f\) is not open.