3. Triangulated structure

3.1 Lagrangian connect sum and mapping cones

Let \( f_\varepsilon : [0, \varepsilon] \to [0, \varepsilon] \) have graph:

\[
\begin{align*}
\gamma = f_\varepsilon (x) \\
\text{area} = \varepsilon
\end{align*}
\]

Consider the Lagrangian submanifold

\[
L_\varepsilon := \left \{ p:= \frac{3}{2} \begin{pmatrix} f_\varepsilon (1) & 1 \end{pmatrix} \right \} \subset (\mathbb{R}^n \times \mathbb{R}^n, \omega_{std})
\]

Note \( L_\varepsilon \setminus B_\varepsilon (0) \cong (\mathbb{R}^n \times 0 \cup 0 \times \mathbb{R}^n) \setminus B_\varepsilon (0) \).

Let \( L_1 \) and \( L_2 \) be Lagrangians in \( M \), meeting at \( p \). We can identify a nbhd of \( p \) with a nbhd of \( 0 \times \mathbb{R}^n \times \mathbb{R}^n \) so that \( L_1 \) corresponds to \( \mathbb{R}^n \times 0 \) and \( L_2 \) to \( 0 \times \mathbb{R}^n \). Gluing in a copy of \( L_\varepsilon \) gives the Lagrangian connect sum \( L_1 \#_\varepsilon L_2 \).

It's independent of choices up to Ham. isotopy.

So if it's embedded ( \( \Leftrightarrow L_1 \cap L_2 = \{p\} \) ), this is a new object of the Fukaya category, independent of choices made in its...
construction up to quasi-isomorphism.

Claim: In $D^b \text{Fuk}(X,\omega)$,

$$L_1 \#_\varepsilon L_2 \cong \text{Cone} \left( L_2 \xrightarrow{T^{-\varepsilon}_p} L_1 \right).$$

This follows from work of FOoo, studying the moduli space of $J$-hol. discs with boundary on $L_1 \#_\varepsilon L_2$.

For example, let $L_3$ be another Lagrangian. We may assume (perturbing $L_3$ by an element of Ham) that $L_1 \cap L_2 \cap L_3 = \emptyset$. Then for $\varepsilon > 0$ sufficiently small,

$$L_3 \cap (L_1 \#_\varepsilon L_2) = (L_3 \cap L_1) \cup (L_3 \cap L_2).$$

We claim

$$\text{hom}^*(L_3, L_1 \#_\varepsilon L_2) \cong \text{hom}^*(L_3, L_2 \xrightarrow{T^{-\varepsilon}_p} L_1)$$

as cochain complexes (where both hom-spaces are taken in the $A_\infty$ category of twisted complexes). Indeed, recalling that

$$\text{hom} \left( L_3, L_2 \xrightarrow{T^{-\varepsilon}_p} L_1 \right) := \text{hom}(L_3, L_2) \oplus \text{hom}(L_3, L_1)$$

we see that there is already a canonical identification of the underlying vector spaces.

Now let $q, r \in L_3 \cap (L_1 \#_\varepsilon L_2)$. 

F000’s result says roughly that the moduli space of $J$-hol. discs with boundary on $L_1 \#_\varepsilon L_2$ is isomorphic to the space of $J$-hol. discs with boundary on $L_1 \cup L_2$, which are allowed to ‘switch’ from $L_2$ to $L_1$ at $p$ (but not in the reverse direction). In particular, the moduli space of strips is:

<table>
<thead>
<tr>
<th>$q \in \mathcal{L}$</th>
<th>$r \in \mathcal{L}$</th>
<th>$\mathcal{M}(q,r)<em>{L_1 #</em>\varepsilon L_2}$</th>
<th>$\cong$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_3 \cap L_1$</td>
<td>$L_3 \cap L_1$</td>
<td>$\mathcal{M}(q,r)_{L_1}$</td>
<td></td>
</tr>
<tr>
<td>$L_3 \cap L_2$</td>
<td>$L_3 \cap L_2$</td>
<td>$\mathcal{M}(q,r)_{L_2}$</td>
<td></td>
</tr>
<tr>
<td>$L_3 \cap L_1$</td>
<td>$L_3 \cap L_2$</td>
<td>$\emptyset$</td>
<td></td>
</tr>
<tr>
<td>$L_3 \cap L_2$</td>
<td>$L_3 \cap L_1$</td>
<td>$\mathcal{M}(q,p,r)_{L_1,L_2}$</td>
<td></td>
</tr>
</tbody>
</table>

picture:

So the differential on $\text{hom}(L_3, L_1 \#_\varepsilon L_2)$ is:

$\text{hom}(L_3, L_2) \oplus \text{hom}(L_3, L_1)$

$\text{mod}(T^{-\varepsilon}p, o)$
This coincides with the differential on
\[ \text{hom} \left( L_3, \frac{L_2}{T^{-p}} \to L_1 \right). \]

3.2 (Split-) Generation

**Defn:** The objects \( L_1, \ldots, L_k \) are said to generate the \( A_\infty \) category \( A \) if every object of \( A \) is quasi-isomorphic in \( \text{Tw} A \) to a twisted complex built from copies of \( L_i \).

Equivalently: every object of \( A \) is q.i. to one built from \( L_i \) by taking iterated cones and shifts.

They split-generate if every object of \( A \) is quasi-isomorphic in \( \text{Tw} A \) to a direct summand of such.

**E.g.** \( X = T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \)

\[ L_1 = \{ x = 0 \} \quad L_2 = \{ y = 0 \}. \]

**Ex:** You can obtain curves in any homology class on \( T^2 \), by taking iterated Lagrangian connect sums (= iterated cones) of \( L_1 \) & \( L_2 \).

However, \( L_1 \) & \( L_2 \) do not generate \( \text{Fuk} (T^2) \). To see this, let
\[ \theta \in \Omega^1 (T^2 \setminus \{ \frac{1}{2}, \frac{3}{2} \}) \] be such that
\[ d\theta = \omega, \quad \theta/L_1 = \theta/L_2 = 0. \]

For any twisted complex \((\oplus L_i, \alpha)\) in \(Tw(\text{Fuk}(T^2))\), define
\[
F(\oplus L_i, \alpha) := \sum_{i} \int L_i \theta \in \mathbb{R}/\mathbb{Z}
\]

**Lem. (Abouzaid):** If two twisted complexes \(L, K\) are quasi-isomorphic, then
\[
F(L) = F(K).
\]

Furthermore,
\[
F(\text{Cone}(L \xrightarrow{\epsilon} K)) = F(K) - F(L).
\]

**Cor:** \(L_1 \& L_2\) generate the subcategory of \(\text{Fuk}(T^2)\) consisting of balanced curves: i.e., those for which \(\int_0 L \in \mathbb{Z}\).

Note that in each homology class, there is exactly one Hamiltonian isotopy class of curves with each value of \(\int_0 L \in \mathbb{R}/\mathbb{Z}\).

**Lem:** \(L_1 \& L_2\) split - generate \(\text{Fuk}(T^2)\).

To see how this fixes the problem in an example, consider the following two curves:

\[
L_1 = \{x = 0\}
\]

\[
L_2 = \{y = -\frac{1}{2}x\}
\]

which are generated by \(L_1\) and \(L_2\).
Cone \((L_3 \xrightarrow{T_{\varepsilon_1} p_1 + T_{\varepsilon_2} p_2} L_1)\) is quasi-isomorphic to:

which is Hamiltonian isotopic to

\[ y = \frac{1}{4} + \varepsilon_2 - \varepsilon_1 \]

\[ y = -\frac{1}{4} + \varepsilon_2 - \varepsilon_1 \]

So \(L_1 \& L_2\) split-generate \(\{ y = \frac{1}{4} + \varepsilon_2 - \varepsilon_1 \}\).

By tuning \(\varepsilon_1 \& \varepsilon_2\) we can split-generate curves in the homology class of \(L_2\), but with any value of \(\varepsilon_0\).

**Thm (Abouzaid - FOOO):** The torus fibre \(L\) split-generates \(\text{Fuk}(\mathbb{C}P^2)_{3T_v^3}\).

In particular, recalling

\[ \text{HF}(L, L) \cong \text{Mat}_{2 \times 2}(\Lambda), \]

\[ D^\pi \text{Fuk}(\mathbb{C}P^2)_{3T_v^3} \cong D^\pi(\Lambda). \]