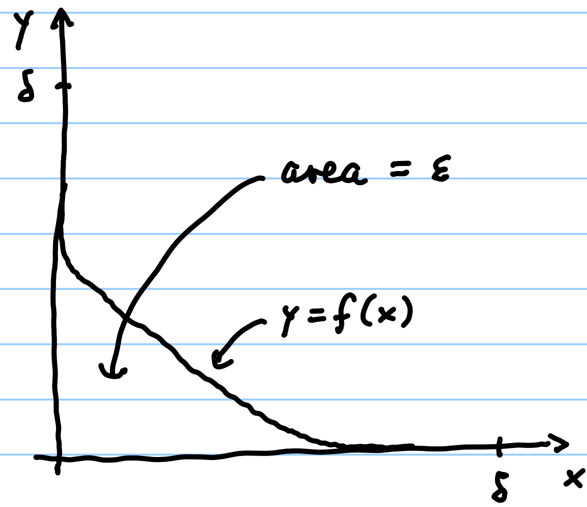
3. Triangulated structure

Note Title 6/18/2016

3.1 Lagrangian connect sum and mapping cones

Let $f_\varepsilon : [0,\delta] \to [0,\delta]$ have graph:



Consider the Lagrangian submanifold

$$L_\varepsilon := \left\{ p_i = \frac{\partial}{\partial q_i} f_\varepsilon(\|q\|) \right\} \subset (\mathbb{R}^n \times \mathbb{R}^n, \omega_{std})$$

Note $L_\varepsilon \setminus B_\delta(0) \cong (\mathbb{R}^n \times 0 \cup 0 \times \mathbb{R}^n) \setminus B_\delta(0)$.

Let L_1 and L_2 be Lagrangians in M, meeting at p. We can identify a nbhd of p with a nbhd of $0 \in \mathbb{R}^n \times \mathbb{R}^n$ so that L_1 corresponds to $\mathbb{R}^n \times 0$ and L_2 to $0 \times \mathbb{R}^n$. Gluing in a copy of L_ε gives the <u>Lagrangian connect sum</u> $L_1 \#_\varepsilon L_2$.
It's independent of choices up to Ham. isotopy.

So if it's embedded ($\Leftrightarrow L_1 \cap L_2 = \{p\}$), this is a new object of the Fukaya category, independent of choices made in its construction up to quasi-isomorphism.

<u>Claim</u>: In $D^b \mathrm{Fuk}(X,\omega)$,

$$L_1 \#_\varepsilon L_2 \simeq \mathrm{Cone}\left(L_2 \xrightarrow{T^{-\varepsilon} p} L_1\right).$$

This follows from work of FOOO, studying the moduli space of J-hol. discs with boundary on $L_1 \#_\varepsilon L_2$.

For example, let L_3 be another Lagrangian. We may assume (perturbing L_3 by an element of Ham) that $L_1 \cap L_2 \cap L_3 = \phi$. Then for $\varepsilon > 0$ sufficiently small,

$$L_3 \cap (L_1 \#_\varepsilon L_2) = (L_3 \cap L_1) \cup (L_3 \cap L_2).$$

We claim

$$\hom^{\cdot}(L_3, L_1 \#_\varepsilon L_2) \cong \hom^{\cdot}(L_3, L_2 \xrightarrow{T^{-\varepsilon} p} L_1)$$

as cochain complexes (where both hom-spaces are taken in the A_∞ category of twisted complexes). Indeed, recalling that

$$\hom\left(L_3, L_2 \xrightarrow{T^{-\varepsilon} p} L_1\right) := \hom(L_3, L_2) \oplus \hom(L_3, L_1)$$

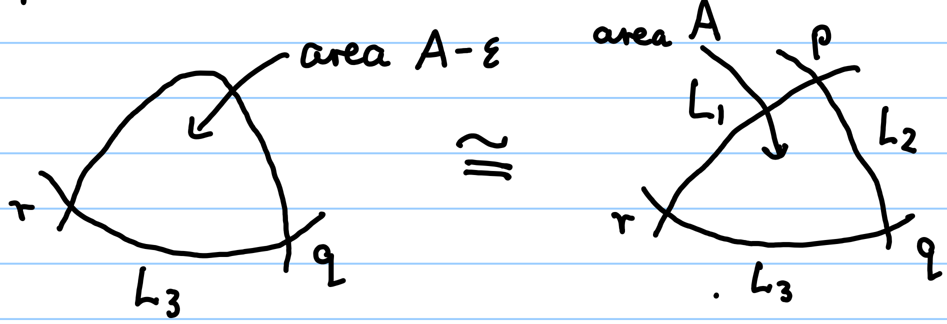
we see that there is already a canonical identification of the underlying vector spaces.

Now let $q, r \in L_3 \cap (L_1 \#_\varepsilon L_2)$.

FOOO's result says roughly that the moduli space of J-hol. discs with boundary on $L_1 \#_\varepsilon L_2$ is isomorphic to the space of J-hol. discs with boundary on $L_1 \cup L_2$, which are allowed to 'switch' from L_2 to L_1 at p (but not in the reverse direction). In particular the moduli space of strips is:

$q \in$	$r \in$	$\mathcal{M}(q,r)_{L_1 \#_\varepsilon L_2} \cong$
$L_3 \cap L_1$	$L_3 \cap L_1$	$\mathcal{M}(q,r)_{L_1}$
$L_3 \cap L_2$	$L_3 \cap L_2$	$\mathcal{M}(q,r)_{L_2}$
$L_3 \cap L_1$	$L_3 \cap L_2$	ϕ
$L_3 \cap L_2$	$L_3 \cap L_1$	$\mathcal{M}(q,p,r)_{L_1, L_2}$

picture:



So the differential on $\hom(L_3, L_1 \#_\varepsilon L_2)$ is:

$$\begin{tikzcd} \hom(L_3, L_2) \arrow[loop below, "\partial"] \arrow[r, bend right, "m^2(T^{-\varepsilon}p, \cdot)"'] & \oplus & \hom(L_3, L_1) \arrow[loop below, "\partial"] \end{tikzcd}$$

This coincides with the differential on

$$\hom_{Tw(Fuk(X,\omega))}\left(L_3,\ L_2 \xrightarrow{T^{-\varepsilon}p} L_1\right).$$

3.2 (Split-) Generation

Defn: The objects $L_1, \dots, L_k$ are said to *generate* the A_∞ category $\mathcal{A}$ if every object of $\mathcal{A}$ is quasi-isomorphic in $Tw\,\mathcal{A}$ to a twisted complex built from copies of L_i.

Equivalently: every object of $\mathcal{A}$ is q.i. to one built from L_i by taking iterated cones and shifts.

They *split-generate* if every object of $\mathcal{A}$ is quasi-isomorphic in $Tw\,\mathcal{A}$ to a direct summand of such.

E.g. $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$

$$L_1 = \{x=0\} \qquad L_2 = \{y=0\}.$$

Ex: You can obtain curves in any homology class on T^2, by taking iterated Lagrangian connect sums (= iterated cones) of L_1 & L_2.

However, L_1 & L_2 do not generate $Fuk(T^2)$. To see this, let $\theta \in \Omega^1(T^2 \setminus \{\frac{1}{2}, \frac{1}{2}\})$ be such that

$$d\theta = \omega, \quad \theta|_{L_1} = \theta|_{L_2} = 0.$$

For any twisted complex $(\oplus L_i, \alpha)$ in $Tw(Fuk(T^2))$, define

$$F(\oplus L_i, \alpha) := \sum_i \int_{L_i} \theta \in \mathbb{R}/\mathbb{Z}$$

Lem (Abouzaid): If two twisted complexes $\mathcal{L}, \mathcal{K}$ are quasi-isomorphic, then

$$F(\mathcal{L}) = F(\mathcal{K}).$$

Furthermore,

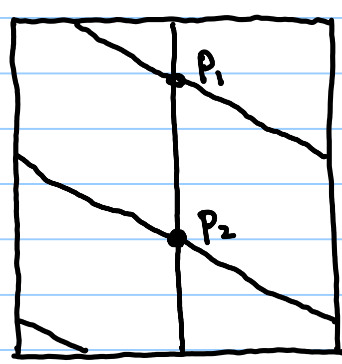
$$F(\mathrm{Cone}(\mathcal{L} \xrightarrow{c} \mathcal{K})) = F(\mathcal{K}) - F(\mathcal{L}).$$

Cor: L_1 & L_2 generate the subcategory of $Fuk(T^2)$ consisting of balanced curves: i.e., those for which $\int_\theta L \in \mathbb{Z}$.

Note that in each homology class, there is exactly one Hamiltonian isotopy class of curves with each value of $\int_\theta L \in \mathbb{R}/\mathbb{Z}$.

Lem: L_1 & L_2 split-generate $Fuk(T^2)$.

To see how this fixes the problem in an example, consider the following two curves:

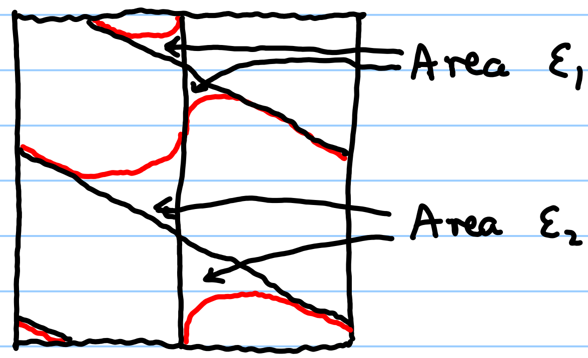


$L_1 = \{x = 0\}$

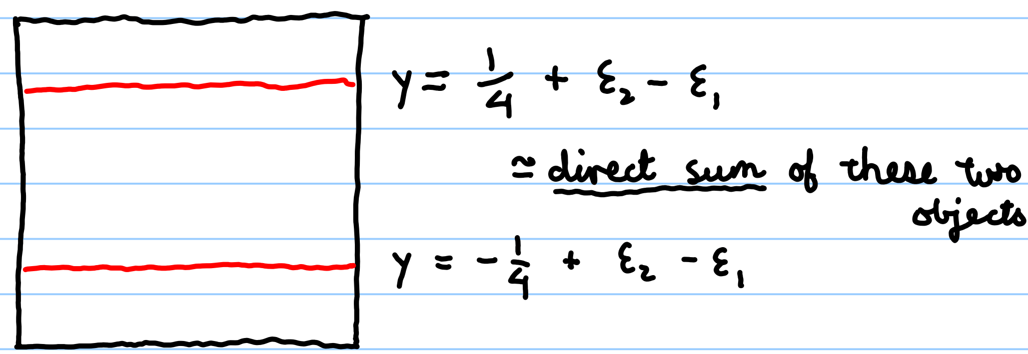
$L_3 = \{y = -\frac{1}{2}x\}$

which are generated by L_1 and L_2.

$\mathrm{Cone}\left(L_3 \xrightarrow{T^{-\varepsilon_1}p_1 + T^{-\varepsilon_2}p_2} L_1\right)$ is quasi-isomorphic to:



which is Hamiltonian isotopic to



So L_1 & L_2 split-generate $\{y = \frac{1}{4} + \varepsilon_2 - \varepsilon_1\}$.

By tuning ε_1 & ε_2 we can split-generate curves in the homology class of L_2, but with any value of $\int\theta$.

Thm (Abouzaid-FOOO): The torus fibre L split-generates $\mathrm{Fuk}(\mathbb{CP}^2)_{3T^{1/3}}$.

In particular, recalling
$HF(L,L) \cong \mathrm{Mat}_{2\times 2}(\Lambda)$,

$$D^\pi \mathrm{Fuk}(\mathbb{CP}^2)_{3T^{1/3}} \cong D^\pi(\Lambda).$$