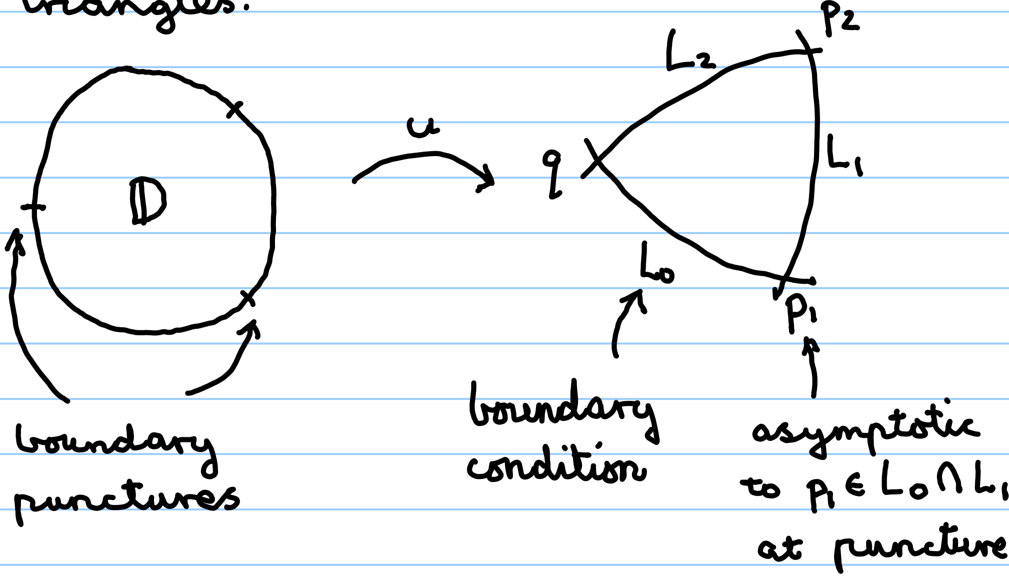
2. Product structure

2.1 The product

Now we consider J-holomorphic triangles:



$$\bar{\partial}_J u = 0 \Leftrightarrow Du \circ j = J \circ Du$$

complex structure on $\mathbb{D}$

almost-complex structure on M.

Again we have a moduli space $\mathcal{M}(p_1, p_2, q; \beta, J)$ of such J-hol. triangles u in homotopy class $[u] = \beta$.

Again we allow $J = J_z$ to depend on $z \in \mathbb{D}$; then for generic J_z, this moduli space is a smooth manifold of dimension $i(\beta)$.

We define a map

$$m^2 : CF(L_1, L_2) \otimes_\Lambda CF(L_0, L_1) \to CF(L_0, L_2)$$

$$m^2(p_2, p_1) := \sum_{q,\beta} \#\mathcal{M}(p_1, p_2, q, \beta, J) \cdot T^{\omega(\beta)} q$$

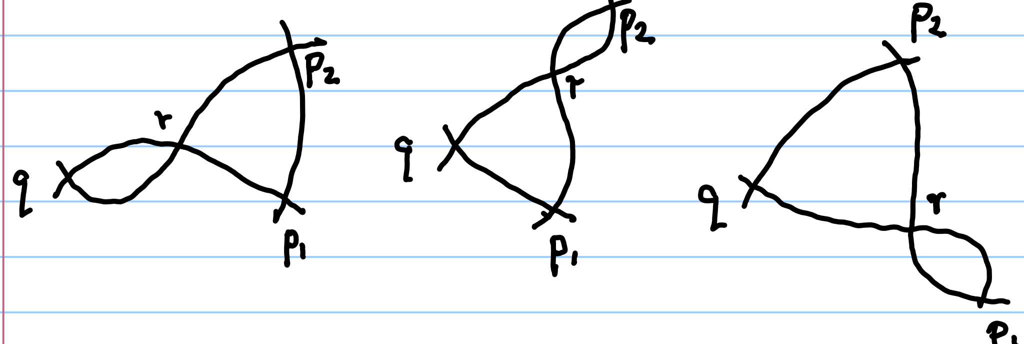
interpreted as before.

<u>Lem</u>: If the L_i are graded, then

$$i(\beta) = \deg(q) - \deg(p_1) - \deg(p_2).$$

In particular, because we only count the 0-dimensional part of the moduli space, $i(\beta) = 0$, so m^2 <u>respects</u> the grading.

To work out what algebraic relations m^2 satisfies, we look at the Gromov compactification of the 1-dimensional component of the moduli space. Again one can ensure it is a compact 1-mfld with boundary. Its boundary points correspond to



(where we again assume $\omega|_{\pi_2(M, L_i)} = 0$ to rule out disc and sphere bubbling).

The fact their signed count is 0 means

$$\partial m^2(p_2, p_1) + m^2(\partial p_2, p_1) + m^2(p_2, \partial p_1) = 0$$

so m^2 defines a map

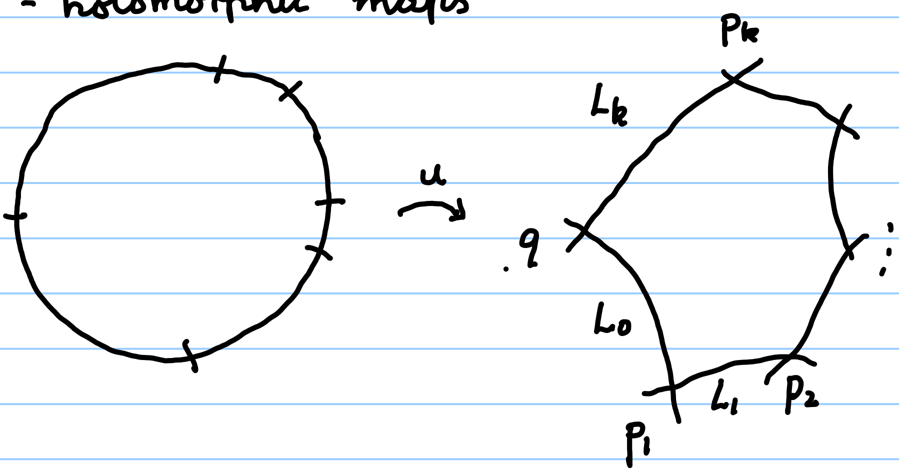
$$HF(L_1, L_2) \otimes HF(L_0, L_1) \to HF(L_0, L_2).$$

2.2 A_∞ products

More generally, we define the moduli space

$$\mathcal{M}(p_1, \ldots, p_k, q, \beta, J_z)$$

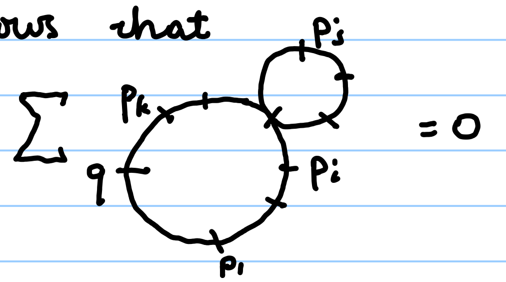
of J-holomorphic maps



(up to biholomorphism). Counting the 0-dim'l pieces of $\mathcal{M}$ defines a map

$$m^k : CF(L_{k-1}, L_k) \otimes \ldots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$$

of degree $2-k$; counting the boundary points of the Gromov compactification of the 1-dimensional component of $\mathcal{M}$ shows that



$$\Leftrightarrow \sum m^*(p_k \ldots, m^*(p_j, \ldots), p_i, \ldots, p_1) = 0$$

We recognise these as the A_∞ relations.

We would like to define the Fukaya category $\mathrm{Fuk}(M, \omega)$ to have

- objects $L \subset M$ Lagrangian, graded, $\omega|_{\pi_2(M,L)} = 0$ (and spin, if we want to use a Novikov field of characteristic $\neq 2$).
- morphism spaces
  $$\hom^\bullet(L_0, L_1) := CF^\bullet(L_0, L_1)$$
- A_∞ structure maps m^k.

And we would like to define the Donaldson-Fukaya category $H^\bullet(\mathrm{Fuk}(M,\omega))$ to be its cohomological category: i.e., the one with morphism spaces $HF^\bullet(L_0, L_1)$. It's an honest Λ-linear $\mathbb{Z}$-graded category (one has to check it has identity morphisms: $e \in H^0(L) \cong HF^0(L,L)$).

But there is a problem: we only defined $CF^\bullet(L_0, L_1)$ when $L_0 \pitchfork L_1$. We could try to define some full subcategory of $\mathrm{Fuk}(M, \omega)$ whose objects intersect transversely. But even this won't work: $CF^\bullet(L, L)$ won't be defined.

The key to resolving this is to use Hamiltonian isotopy invariance: we choose

$\psi \in \mathrm{Ham}(M)$ so that $\psi(L_1) \pitchfork L_0$, and define

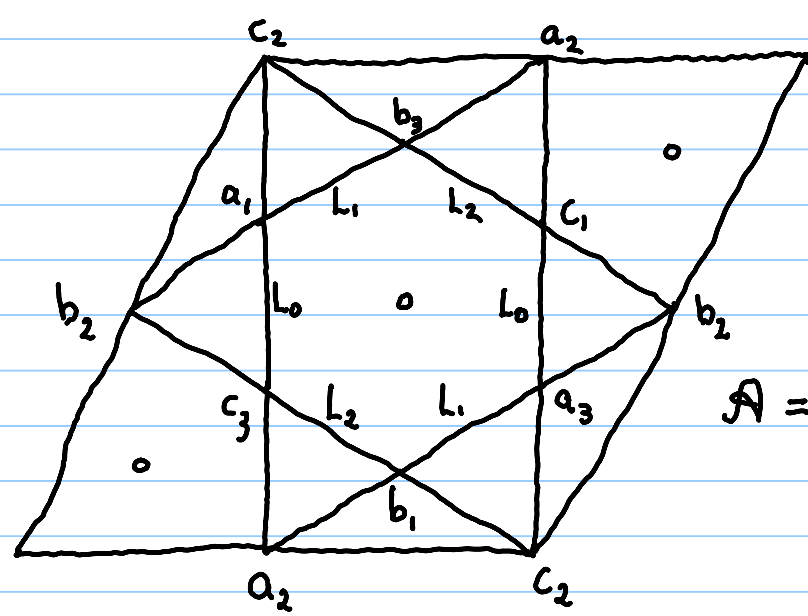
$$HF^\bullet(L_0, L_1) := HF^\bullet(L_0, \psi(L_1)).$$

By invariance, the result is independent of ψ. In fact, L and $\psi(L)$ will be isomorphic (resp. quasi-isomorphic) objects of the Donaldson-Fukaya (resp. Fukaya) category.

Defining the Donaldson-Fukaya category in this way is not too bad. Defining the Fukaya category itself is a bit more delicate because $CF^\bullet(L_0, \psi(L_1))$ depends on ψ, but it can be done; the final result is independent of choices up to <u>A_∞ quasi-equivalence</u>.

2.3 <u>Example: 3-punctured torus</u>

$$M = T^2 \setminus \{3 \text{ points}\}$$



$\mathcal{A}$ = full subcategory of $\mathrm{Fuk}(M, \omega)$ with objects L_i.

Ex: For appropriate gradings of L_i (relative to the fibrewise universal cover $\widetilde{\mathcal{G}M} \longrightarrow \mathcal{G}M$ which extends to T^2) we have

$$HF^\bullet(L_0, L_1) = \Lambda\langle a_1, a_2, a_3\rangle \quad \text{in deg} = 0$$

$$HF^\bullet(L_1, L_2) = \Lambda\langle b_1, b_2, b_3\rangle \quad \text{in deg} = 0$$

$$HF^\bullet(L_0, L_2) = \Lambda\langle c_1, c_2, c_3\rangle \quad \text{in deg} = 0$$

It follows that the morphism spaces in the other direction are in degree 1, and we also know $HF^\bullet(L_i, L_i) \cong H^\bullet(S^1)$. by Floer's theorem.

Note that the product has degree -0: so we can consider the non-full subcategory of the Donaldson-Fukaya category with morphism spaces

$$HF^0(L_i, L_j)$$

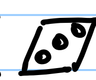
which is simpler to describe.

We see that the non-trivial products are:

$$b_j \cdot a_i = T^A c_k \quad \text{for } \{i, j, k\} = \{1, 2, 3\}$$

(and the identity morphisms in $HF^0(L, L)$ act as they should).

Thus we can give a quiver presentation

of $H^0(\mathcal{A})$ for this $\mathcal{A} \subset \mathrm{Fuk}($  $)$:

$$\begin{tikzcd} L_0 \arrow[r, bend left, "a_1"] \arrow[r, "a_2"] \arrow[r, bend right, "a_3"'] & L_1 \arrow[r, bend left, "b_1"] \arrow[r, "b_2"] \arrow[r, bend right, "b_3"'] & L_2 \end{tikzcd}$$

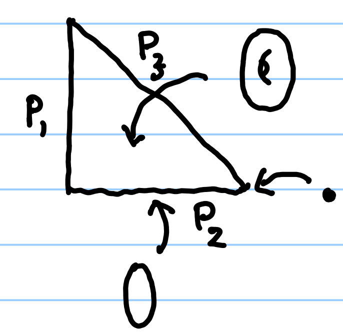
$b_i a_j = b_j a_i \quad \forall \ i \neq j$

$b_i a_i = 0 \quad \forall \ i.$

You may recognise this from $\mathrm{Coh}(\mathbb{P}^2)$!

2.4 <u>Example: $\mathbb{CP}^2$</u>

Recall $(\mathbb{CP}^2, \omega_{FS})$ is toric, with moment map $\pi : \mathbb{CP}^2 \to \Delta$



ϵ Lagrangian torus fibre

$\pi^{-1}(P_i) = D_i = \{z_i = 0\} \subset \mathbb{CP}^2.$

Let $L_p = \pi^{-1}(p)$, $p \in \mathrm{int}(\Delta)$.

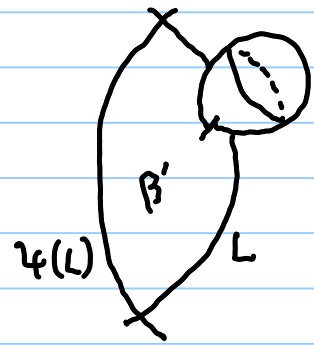
<u>Lem</u>: If $u \in \pi_2(M, L_p)$ then

$$\omega(u) = \sum_i (u \cdot D_i)\, d(p, P_i).$$

d — <u>affine</u> distance.

Oh no! $\omega(u) \neq 0$, so $\partial^2 \neq 0$ because we might have spheres and discs bubbling off, so $HF(L, L)$ undefined?

Sphere bubbles:



$\beta = \beta' + \beta''$

$i(\beta) = 2$ ($\Rightarrow$ after quotient by $\mathbb{R}$ get 1-dim'l $\mathcal{M}$)

Lem: $i(\beta) = i(\beta') + 2c_1(\beta'')$.

Pf: Ex.

β'' is represented by a non-constant holomorphic sphere

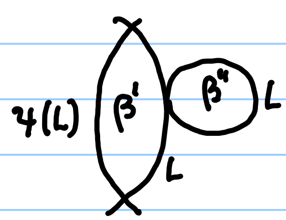
$\Rightarrow$ degree > 0

$\Rightarrow c_1(\beta'') = 3 \cdot \text{degree} \geq 3$

$$\Rightarrow i(\beta') = i(\beta) - 2c_1(\beta'') \leq 2 - 6 \leq -4$$

$\Rightarrow$ there is no such J-hol. strip, by regularity!

Disc bubbles



$\beta = \beta' + \beta''$ $\quad i(\beta) = 2$

Lem: $i(\beta) = i(\beta') + \mu_L(\beta'')$ ← Maslov index

Pf: Ex.

$\Rightarrow i(\beta') = i(\beta) - \mu_L(\beta'') = 2 - \mu_L(\beta'')$.

If $\mu_L(\beta'') \geq 3$, then we are saved by regularity.

If $\mu_L(\beta'') \leq 1$ we are in trouble.

<u>Lem</u>: $\mu_L(\beta'') = \sum_i 2(\beta'' \cdot D_i)$.

$\beta'' \cdot D_i \geq 0$ (as J-hol. curves intersect J-hol. divisors positively)

$\omega(\beta'') > 0 \Rightarrow \beta'' \cdot D_i > 0$ for some i

$\Rightarrow \mu_L(\beta'') \geq 2$, with equality iff $\beta'' \cdot D_i = \delta_{ij}$ for some j: then $\mu_L(\beta'') = 2$.

In this case, $i(\beta') = 2 - 2 = 0 \Rightarrow$ this is a <u>constant</u> strip (recall the $\mathbb{R}$-action by translation: an $\mathbb{R}$-action on a 0-mfld is trivial, so the strip is invariant under translation, hence constant).

<u>Defn</u>: Let

$$\mathcal{M}_1(L, \beta, J) := \{u : (\mathbb{D}, \partial \mathbb{D}) \to (M, L) : \bar{\partial}_J u = 0\} / \mathrm{Aut}(\mathbb{D}, +1).$$

<u>Lem</u>: For regular J,

$$\dim \mathcal{M}_1(L, \beta, J) = n + \mu(\beta) - 2$$

We have the evaluation map at $+1 \in \partial \mathbb{D}$:

$$ev : \mathcal{M}_1(L, \beta, J) \longrightarrow L$$

$$ev(u) := u(+1).$$

If $\mu(\beta) = 2$ this is a map of n-mflds so has degree $n_\beta \in \mathbb{Z}$.

Defn: $\lambda(L) := \sum_{\mu(\beta)=2} n_\beta T^{\omega(\beta)} \in \Lambda$

Then we have $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$

$$\partial^2(p) = \big(\lambda(L_0) - \lambda(L_1)\big) \cdot p$$

Another way to express this is to introduce a curvature term in the Fukaya category:

$$m^0 := \lambda(L) \cdot e_L \in CF(L, L)$$

for all L, so the second A_∞ relation reads

$$m^1(m^1(p)) + m^2(m^0, p) + m^2(p, m^0) = 0$$

$$\Leftrightarrow \partial^2(p) + \lambda(L_1) \cdot p - \lambda(L_0) \cdot p = 0.$$

In particular, if $\lambda(L_0) = \lambda(L_1)$, we can define $HF(L_0, L_1)$. This is true in particular for $L_0 = L_1$.

More generally, we can define

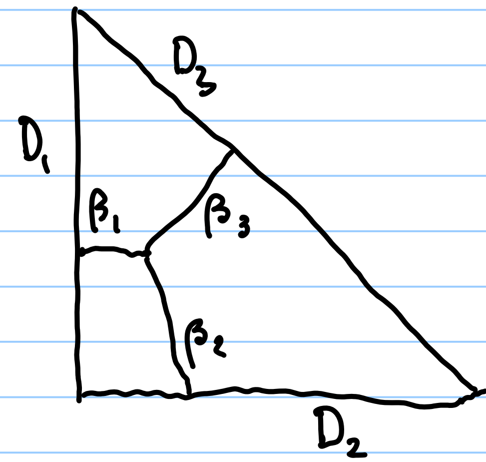
$Fuk(M, \omega)_\lambda$: objects are $L \subset M$ with $\lambda(L) = \lambda$.

So we get a separate Fukaya category for each $\lambda \in \Lambda$.

(It is not $\mathbb{Z}$-graded: $\mu_L \neq 0$. But if L_i are oriented we can define a $\mathbb{Z}_2$-grading: $|p|$ = sign with which L_0 intersects L_1 at p.)

Note: this relied on the non-existence of J-holomorphic discs with $\mu_L \leq 1$, which we established for fibres of the moment map using ad-hoc methods. Another general way of ruling these out is to require that L be orientable ($\Rightarrow \mu_L$ even) and monotone: $\mu = \tau\omega$ on $\pi_2(M, L)$ for $\tau > 0$. J-hol. discs have $\omega(u) > 0$ so this ensures $\mu(u) \geq 2$.

E.g. For $\mathbb{CP}^2$, we have 3 homotopy classes of $\mu = 2$ discs:



We claim $n_{\beta_i} = 1$:

$[u] = \beta_1 \Rightarrow u \cap D_2 = u \cap D_3 = \emptyset$.

So $u < \{z_2 z_3 \neq 0\} \cong \mathbb{C} \times \mathbb{C}^*$

and $\partial u \in L = S^1 \times S^1$

$\Rightarrow \pi_1 \circ u : (\mathbb{D}, \partial \mathbb{D}) \to (\mathbb{C}, S^1)$

$\pi_2 \circ u : (\mathbb{D}, \partial \mathbb{D}) \to (\mathbb{C}^*, S^1)$

both J-holomorphic (assuming π_i are J-holomorphic)

$\Rightarrow u(z) = (e^{i\theta} z, e^{i\theta'})$ for some $e^{i\theta}, e^{i\theta'} \in S^1$.

The boundary of this family of discs sweeps out L exactly once, so $n_{\beta_1} = 1$.

Thus $\lambda(L_p) = T^{d_1(p)} + T^{d_2(p)} + T^{d_3(p)}$, where $d_i(p)$ = affine distance to P_i

$$\Rightarrow d_1(p_1, p_2) = p_1$$

$$d_2(p_1, p_2) = p_2$$

$$d_3(p_1, p_2) = 1 - p_1 - p_2.$$

We can choose Ψ so that

$$CF^\bullet(L_p, \Psi(L_p)) \cong CM^\bullet(L_p) \cong \Lambda^\bullet(A)$$

where $A := H^1(L_p, \Lambda) \cong \Lambda^{\oplus 2}$.

Defn (FOOO): Let L be a torus fibre in a Fano toric variety. Then there's a function

$$\mathfrak{P}_p \in \Lambda[\![A]\!]$$

called the <u>disc potential</u>, such that

$$\sum_{i=0}^{\infty} m^i(a, a, \ldots, a) = \mathfrak{P}_p(a) \cdot e \quad \text{in } CF(L_p, L_p).$$

It is well-defined up to right-equivalence:

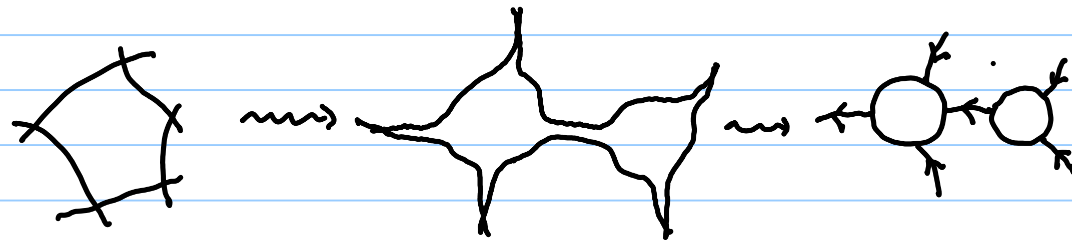
$$\mathfrak{P}_p \longmapsto \mathfrak{P}_p \circ \phi$$

where $\phi : A \xrightarrow{\sim} A$ is a formal diffeomorphism with $D\phi(0) = \mathrm{Id}$
(i.e., $\phi(a) = \sum_{i \geq 1} \phi_i(a, \ldots, a)$, $\phi_1 = id$).

<u>Thm</u> (FOOO): The disc potential is equal to

$$\mathfrak{P} = \sum_{\mu(\beta)=2} n_\beta T^{\omega(\beta)} \exp(a(\partial\beta))$$

Idea: If $\psi_\varepsilon(L)$ is a pushoff of L by some Morse function εf, and we let $\varepsilon \to 0$, then J-holomorphic polygons degenerate to J-holomorphic discs connected by Morse flowlines of f:



This allows one to compute A_∞ products in terms of of J-holomorphic discs with boundary on L, with certain incidence conditions on marked points on the boundary.

Indeed, FOOO define

$$CF^\bullet(L,L) := C^{\text{sing}}_{n-\bullet}(L)$$

some version of 'smooth singular chains'

They consider

$$\mathcal{M}_{k+1}(L,\beta,J) := \left\{ (z_0,\dots,z_k) \in \partial\mathbb{D} \text{ cyclically ordered},\ u:(\mathbb{D},\partial\mathbb{D}) \to (M,L),\ \bar\partial_J u = 0 \right\} / \mathrm{Aut}(\mathbb{D})$$

with its evaluation maps

$$ev_i : \mathcal{M}_{k+1} \longrightarrow L.$$

They define (roughly):

$$m^k : CF^\bullet(L,L)^{\otimes k} \longrightarrow CF^\bullet(L,L)$$

$$m^k(c_k, \ldots, c_1) := \sum_\beta (ev_0)_* \big(u \in \mathcal{M}_{k+1}(L, \beta, J) : ev_i(u) \in c_i \big).$$

(how to choose these chains and achieve transversality is a difficult business!).

(The above theorem is not immediate - note the appearance of denominators $\frac{1}{i!}$, because $\mathcal{M}$ ends up being an orbifold)

Note : $\mathfrak{P}_p(0) = \lambda(L_p)$.

Lem: $m^1(u_i) = \dfrac{\partial \mathfrak{P}}{\partial u_i}(0) \cdot e$

Pf: Easy exercise from the definition of $\mathfrak{P}$.

Cor: $HF(L_p, L_p) \neq 0 \iff \dfrac{\partial \mathfrak{P}_p}{\partial u_1}(0) = \dfrac{\partial \mathfrak{P}_p}{\partial u_2}(0) = 0.$

Pf: $\Leftarrow$: We claim $[e] \neq 0$ in $HF(L,L)$. Indeed, $\partial e = 0$, and e can only be killed by something in $H^1(L)$ (by $\mathbb{Z}_2$-grading), but $\partial|_{H^1(L)} = 0$ by hypothesis, so $e \notin \operatorname{im} \partial$.

$\Rightarrow$: $\dfrac{\partial \mathfrak{P}}{\partial u_i} \neq 0 \Rightarrow e = \partial\left(\left(\dfrac{\partial \mathfrak{P}}{\partial u_i}\right)^{-1} \cdot u_i\right)$

$\Rightarrow [e] = 0$

$\Rightarrow HF(L,L) \cong 0$

because $HF(L,L)$ is a unital algebra and $[e]$ is the unit. $\square$

Let's see what this gives us for $\mathbb{CP}^2$:

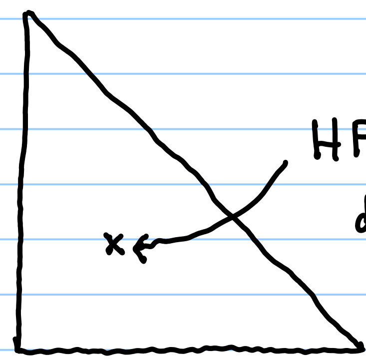
$$\mathfrak{P}_p(u_1, u_2) = T^{p_1} e^{u_1} + T^{p_2} e^{u_2} + T^{1-p_1-p_2} e^{-u_1-u_2}$$

$$\frac{\partial \mathfrak{P}_p}{\partial u_i}(0) = T^{p_i} - T^{1-p_1-p_2} = 0 \qquad \text{for } i = 1, 2$$

$$\Longleftrightarrow \quad p_1 = p_2 = 1 - p_1 - p_2$$

$$\Longleftrightarrow \quad p_1 = p_2 = \tfrac{1}{3}.$$

So there's only one p for which $HF(L_p, L_p) \neq 0$: the fibre over the centroid:



$HF(L_p, L_p) \neq 0 \Rightarrow L_p$ not displaceable by Ham. isot.

In fact, all other fibres can be displaced from themselves by Hamiltonian isotopies, by the method of 'probes'.

Finally, for p = centroid

Lem: $m_2(u_i, u_j) + m_2(u_j, u_i) = \dfrac{\partial^2 \mathfrak{P}}{\partial u_i \partial u_j}(0) \cdot e$

$$\Rightarrow \quad HF(L_p, L_p) \cong \mathrm{Cliff}(\mathrm{Hess}_0(\mathfrak{P}_p))$$

$$\cong \mathrm{Mat}_{2\times 2}(\Lambda).$$

(exercise: $\mathrm{Hess}_0(\mathfrak{P}_p)$ is non-degenerate).

Clifford algebras of (non-degenerate) quadratic forms are intrinsically formal: any A_∞ algebra whose cohomology algebra is a Clifford algebra, is quasi-isomorphic to the Clifford algebra with vanishing higher-order products. So there's no more 'information' hiding in the A_∞ structure, in this case.