1. Lagrangian Floer cohomology

Note Title 6/6/2016

1.1 Floer's work on the Arnol'd conjecture

(M, ω) compact symplectic manifold

$L \subset M$ compact Lagrangian submanifold

$\psi \in \mathrm{Ham}(M, \omega)$ (i.e., $\exists\, H : [0,1] \times M \longrightarrow \mathbb{R}$

$\omega(\cdot, X_t) = dH_t$

ψ_t = flow of X_t

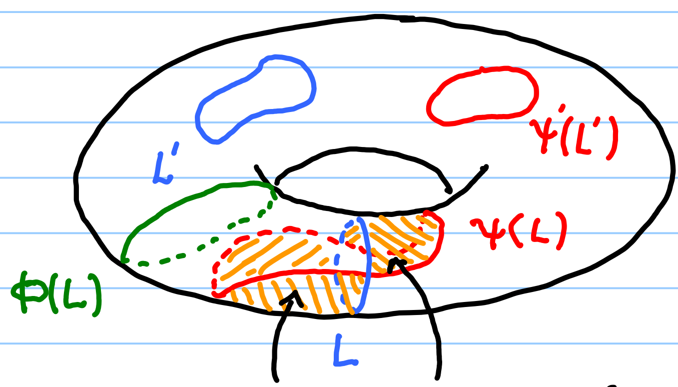
$\psi = \psi_1$)

Say L' Ham. isotopic to L if $\exists\, \psi \in \mathrm{Ham}(M, \omega)$ s.t. $L' = \psi(L)$.

Thm (Floer): Suppose that L' Ham. isot. to L, $L \pitchfork L'$, and $\omega|_{\pi_2(M,L)} = 0$. Then

$$|L \cap L'| \geq \sum_i \dim H^i(L; \mathbb{Z}_2).$$

E.g.



$\text{Area}_1 = \text{Area}_2$ (Hamiltonian $\Rightarrow$ flux $= 0$)

$|L \cap \psi(L)| = 2 = \dim H^0(L; \mathbb{Z}_2)$

$|L' \cap \psi'(L')| = 0$ $\quad (\omega|_{\pi_2(M,L)} \neq 0)$

$|L \cap \Phi(L)| = 0$ $\quad (\Phi$ = rotation $\in \mathrm{Symp}(M) \setminus \mathrm{Ham}(M))$.

How did Floer do it? He associated to L_0, L_1 (transverse Lagrangians) a cochain complex $(CF(L_0, L_1), \partial)$, whose cohomology is denoted $HF(L_0, L_1)$, such that

A. If L_i' is Ham. isotopic to L_i $(i=0,1)$ then
$$HF(L_0, L_1) \cong HF(L_0', L_1')$$
(Hamiltonian isotopy invariance)

$\rightsquigarrow$ can define $HF(L_0, L_1)$ for L_i <u>not</u> transverse.

B. $CF(L_0, L_1)$ is freely generated (over a certain field extension Λ of $\mathbb{Z}_2$) by $L_0 \cap L_1$.

C. $HF(L, L) \cong H^{\bullet}(L; \Lambda)$.

These properties suffice to prove Floer's theorem. Note: A+B $\Rightarrow$ if L_1 can be displaced from L_0 by a Ham. isotopy, then $HF(L_0, L_1) = 0$.

1.2 <u>The definition</u>

For any field $\mathbb{K}$, define

$$\Lambda_{\mathbb{K}} := \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in \mathbb{K},\ \lambda_i \in \mathbb{R},\ \lim_{i \to \infty} \lambda_i = +\infty \right\}.$$

This is a field extension of $\mathbb{K}$, called the <u>Novikov field over $\mathbb{K}$</u>. We will denote $\Lambda := \Lambda_{\mathbb{Z}_2}$

When $L_0 \pitchfork L_1$, we define

$$CF(L_0, L_1) := \Lambda \langle L_0 \cap L_1 \rangle$$

(a Λ-vector space which comes with a basis indexed by the finite set $L_0 \cap L_1$).

The differential is given by

$$\partial p := \sum_{\substack{q \in L_0 \cap L_1, \\ \beta}} \# \mathcal{M}(p, q, \beta, J) \cdot T^{\omega(\beta)} q$$

where $\mathcal{M}(p, q, \beta, J)$ is a set we now define.

It depends on a choice of almost-complex structure J compatible with ω (i.e. $J \in End(TM)$, $J^2 = -Id$, $\omega(\cdot, J\cdot)$ is a Riemannian metric).

$\widehat{\mathcal{M}}(p, q, J)$ is the set of <u>J-holomorphic strips</u>, which are smooth maps

$$u : \underset{s}{\mathbb{R}} \times \underset{t}{[0,1]} \longrightarrow M$$

satisfying

$$\frac{\partial u}{\partial s} + J(u) \frac{\partial u}{\partial t} = 0 \qquad \text{(J-hol. curve eqn.)}$$

$$u(s, 0) \in L_0, \quad u(s, 1) \in L_1 \qquad \text{(boundary condns)}$$

$$\lim_{s \to +\infty} u(s, t) = p, \quad \lim_{s \to -\infty} u(s, t) = q.$$

$\widehat{\mathcal{M}}(p,q,\beta,J)$ is the set of strips u with $[u] = \beta \in \pi_2(M, L_0, L_1)$. The quantity

$$\omega(\beta) = \int u^* \omega \underset{\text{ex.}}{=} \iint \left|\frac{\partial u}{\partial s}\right|^2 ds\, dt \geq 0$$

is also called the <u>energy</u> $E(u)$.
Equality $(\omega(\beta) = 0) \Leftrightarrow u$ is <u>constant</u>.

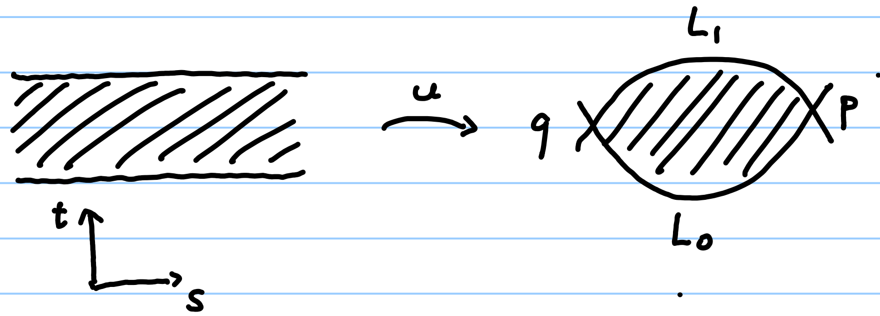
Finally, $\mathbb{R}$ acts on $\widehat{\mathcal{M}}(p,q,\beta,J)$, via

$$a \cdot u(s,t) := u(s+a, t),$$

and we define

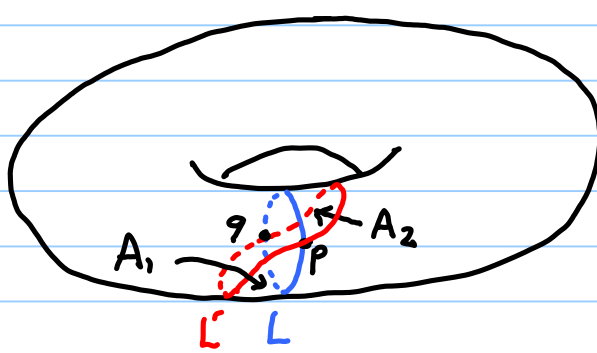
$$\mathcal{M}(p,q,\beta,J) := \widehat{\mathcal{M}}(p,q,\beta,J)/\mathbb{R}.$$

Picture:



We haven't shown ∂ is well-defined, but nevertheless let's illustrate with an example.

E.g.



$CF(L, L') = \Lambda\langle p, q\rangle, \quad \partial p = (T^{A_1} + T^{A_2})\cdot q$

So, if $A_1 = A_2$, $\partial = 0$ and

$HF(L, L') \cong \Lambda\langle p, q\rangle.$

Riemann mapping thm $\Rightarrow$ only strips are the obvious ones

If $A_1 \neq A_2$ ($\varphi \notin \mathrm{Ham}$) then WLOG $A_1 < A_2$ and

$$(T^{A_1} + T^{A_2})^{-1} = T^{-A_1}(1 + T^{A_2 - A_1} + T^{2(A_2 - A_1)} + \dots)$$

so $T^{A_1} + T^{A_2}$ is invertible, hence

$HF(L, L') \cong 0$

<u>Remark</u>: Because $E(u) \geq 0$ for any u, we could define $HF(L_0, L_1; \Lambda_0)$ where Λ_0 is the Novikov <u>ring</u>

$$\Lambda_0 \subset \Lambda$$

$$\Lambda_0 := \{\sum a_i T^{\lambda_i} \mid \lambda_i \geq 0 \ \forall i\}.$$

If $A_1 \neq A_2$ in the previous example, we would get

$$HF(L, L'; \Lambda_0) \cong \Lambda_0 / T^{A_1} + T^{A_2} \neq 0.$$

However L' is Hamiltonian displaceable from L in this case; hence $HF(L_0, L_1; \Lambda_0)$ no longer has the Hamiltonian isotopy invariance property.

1.3 Transversality

What does '$\# \mathcal{M}(p, q, \beta, J)$' mean?

It means the defining PDE is a <u>Fredholm problem</u>: when J is <u>regular</u>, $\widehat{\mathcal{M}}(p, q, \beta, J)$ has the structure of a manifold of dimension $i(\beta)$ (we'll come back to $i(\beta)$ later).

We would like to say J is 'generically' regular, but for this to be true one needs to change the definition a bit. We make J <u>domain-dependent</u>: i.e., we consider a family J_t, $t \in [0, 1]$, and modify the J-hol. curve equation to

$$\frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = 0.$$

Then, 'generic' J_t is regular by a result of Floer-Hofer-Salamon.

We define $\# \mathcal{M}(p, q, \beta, J_t) := 0$ if $i(\beta) \neq 1$, and if $i(\beta) = 1$ then $\# \mathcal{M}(p, q, \beta, J_t)$ is the count of points in the 0-dim'l mfld (it is only defined when J_t is regular). We will address why this count is finite in the next section.

<u>Note</u>: We made J_t depend on t but <u>not</u> on s: that would have meant there was no $\mathbb{R}$-action for us to quotient $\widehat{\mathcal{M}}$ by to get $\mathcal{M}$.

More explanations about transversality:

$$\begin{array}{rcl} & & \mathcal{E} \\ & & \downarrow \big) \bar\partial_J \\ \widehat{\mathcal{M}}(p, q, \beta, J_t) = \bar\partial_{J_t}^{-1}(0) & \subset & \mathcal{B} \end{array}$$

$\mathcal{B}$ = Banach manifold of maps
$u: \mathbb{R} \times [0,1] \longrightarrow M$, $[u] = \beta$,
with boundary conditions on L_0, L_1,
converging to p, q as $s \to \pm\infty$

$\mathcal{E}$ = Banach vector bundle of sections of
$\Omega^{0,1} \otimes u^* TM$.

This equips $\widehat{\mathcal{M}}(p, q, \beta, J_t)$ with a topology.

$\bar\partial_{J_t}$ is Fredholm, i.e., its linearization $D_{\bar\partial_{J_t}, u}$
is Fredholm. The index

$$i(D_{\bar\partial_{J_t}, u}) := \dim \ker(D_{\bar\partial_{J_t}, u}) - \dim \operatorname{coker}(D_{\bar\partial_{J_t}, u})$$
$$= i(\beta)$$

turns out only to be a function of the homotopy class β (we'll come back to it).

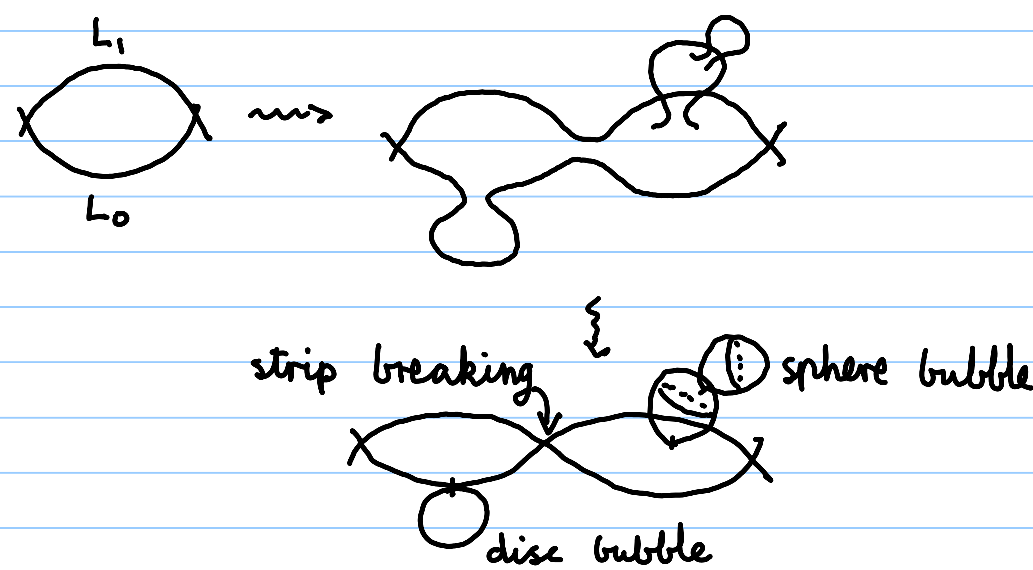
In particular, if $\bar\partial_J$ is transverse to the 0-section at $u \in \bar\partial_{J_t}^{-1}(0)$ $\Leftrightarrow$ $D_{\bar\partial_{J_t}, u}$ is surjective, then in a nbhd of u, $\widehat{\mathcal{M}}(p, q, \beta, J_t)$ looks like an $i(\beta)$-dimensional manifold. If this is true for all u then

$\widehat{\mathcal{M}}(p, q, \beta, J_t)$ is an $i(\beta)$-dim'l mfld. In this situation we say J_t is <u>regular</u>.

1.4 <u>Gromov compactness</u>

Gromov's compactness theorem says that any sequence of J-holomorphic curves $\{u_n\}$ with bounded energy $E(u_n) \leq E$

has a subsequence which 'converges' to a nodal tree of holomorphic curves:



The idea is that energy can 'concentrate' at points in the domain: if it concentrates at an interior point you get a sphere bubble, at a boundary point you get a disc bubble, at a boundary puncture you get strip breaking.

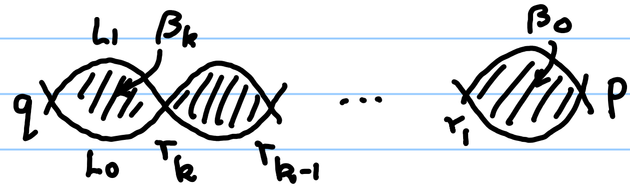
no 'pointless' constant components

So, taking the union of all nodal <u>stable</u> trees in a given homotopy class gives a compact topological space $\overline{\mathcal{M}}(p, q, \beta, J_t)$, the <u>Gromov compactification</u> of $\mathcal{M}(p, q, \beta, J_t)$.

If we assume that $\omega|_{\pi_2(M, L_i)} = 0$ $(\Rightarrow \omega|_{\pi_2(M)} = 0)$, then any J-holomorphic sphere or disc u has $\omega(u) = 0 \Rightarrow E(u) = 0 \Rightarrow u =$ constant.

I.e., only strip-breaking occurs. So

$$\overline{\mathcal{M}}(p, q, \beta, J_t) = \coprod_{\Sigma \beta_i = \beta} \mathcal{M}(p, r_1, \beta_0, J_t) \times \dots \times \mathcal{M}(r_k, q, \beta_k, J_t)$$



<u>Lem</u>: $i(\Sigma \beta_i) = \Sigma\, i(\beta_i)$.

<u>Cor</u>: If J_t is regular, $\omega|_{\pi_2(M, L_i)} = 0$, then

- $\mathcal{M}(p, q, \beta, J_t)$ is a compact 0-mfld if $i(\beta) = 1$.
- $$\overline{\mathcal{M}}(p, q, \beta, J_t) = \mathcal{M}(p, q, \beta, J_t) \sqcup \coprod_{\substack{r \in L_0 \cap L_1 \\ \beta_0 + \beta_1 = \beta \\ i(\beta_0) = i(\beta_1) = 1}} \mathcal{M}(p, r, \beta_0, J_t) \times \mathcal{M}(r, q, \beta_1, J_t)$$

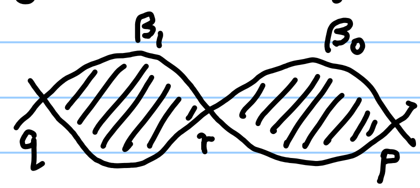
  if $i(\beta) = 2$.

Pf: If $i(\beta) = 0$ then $\mathcal{M}(p, q, \beta, J_t)$ consists only of trivial solutions (because $\mathbb{R}$ acts on the 0-mfld $\widehat{\mathcal{M}}(p, q, \beta, J_t)$, so the action must be trivial, so all strips are constant along their length), which do not contribute to the Gromov compactification, by definition.

1.5 Gluing

If J_t is regular, $i(\beta_0) = i(\beta_1) = 1$, then

a gluing theorem shows that there's a neighbourhood of



$\in \overline{\mathcal{M}}(p, q, \beta, J_t)$

which is homeomorphic to $[0, \delta)$.

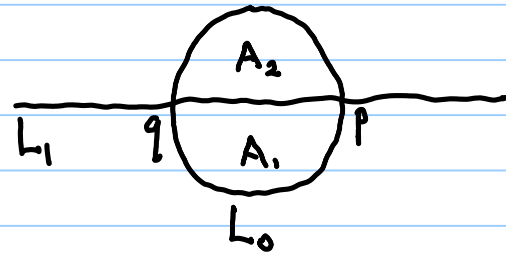
Thus, if $i(\beta) = 2$ then $\overline{\mathcal{M}}(p, q, \beta, J_t)$ is a compact 1-mfld with boundary; and its boundary points are as above.

It follows that the number of boundary points is 0 (mod 2). The boundary points are in one-to-one correspondence with the summands of the coefficient of q in ∂p. Thus one proves that $\partial^2 = 0$.

This completes the construction of

$$HF(K, L) := H^{\bullet}(CF(K, L), \partial).$$

E.g. The assumption $\omega|_{\pi_2(M,L)} = 0$ can be relaxed, but not completely removed:

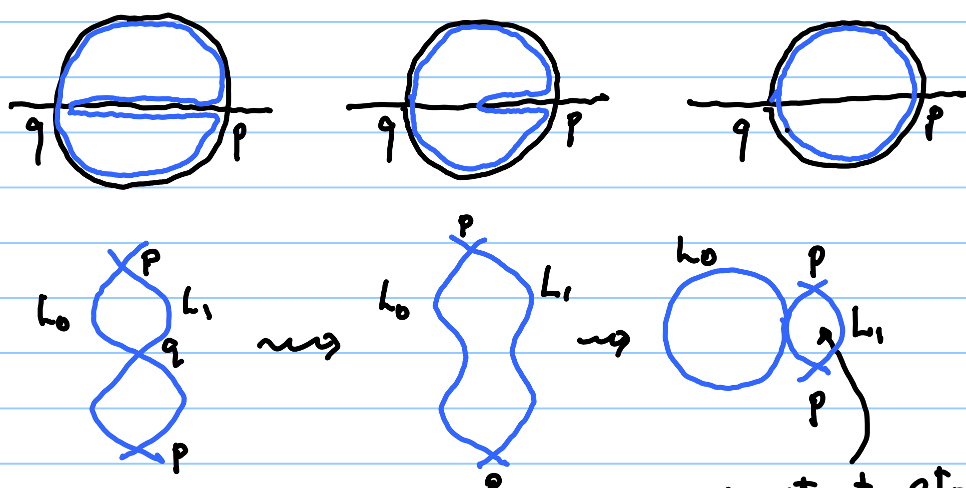


$$\partial p = T^{A_1} q, \quad \partial q = T^{A_2} p$$

(Riemann mapping thm).

$$\Rightarrow \partial^2 p = T^{A_1 + A_2} q \neq 0.$$

How does the disc bounded by L_0 interfere with the proof $\partial^2 = 0$?
We have a 1-dim'l moduli space with 2 boundary points, but only one corresponds to ∂^2:



constant strip at p.

Remark: If L_0 and L_1 are spin, we can equip the moduli spaces $\mathcal{M}(p, q, \beta, J_t)$ with natural orientations. Thus instead of counting points in 0-dimensional moduli spaces modulo 2, we can count them with signs. This allows us to define $(CF(L_0, L_1; \Lambda_{\mathbb{K}}), \partial)$ for char $\mathbb{K} \neq 2$.

We refer to Auroux' notes for the proof of Hamiltonian isotopy invariance, and to the exercises for $HF(L, L) \cong H^*(L)$.

1.6 Gradings

Define $\mathcal{G}(n) := \{$ Linear Lagrangian subspaces $L \subset (\mathbb{C}^n, \omega_{std}) \}$

the Lagrangian Grassmannian (a smooth manifold).

Lem: $\mathcal{G}(n) \cong U(n)/O(n)$, and it follows that
$$\pi_1(\mathcal{G}(n)) \cong \mathbb{Z}.$$

This iso. is called the Maslov index, and denoted μ.

This can be extended to a Maslov index for paths: let

$P\mathcal{G}(n) := \{$ cts maps $p: [0,1] \to \mathcal{G}(n)$, $p(0) \pitchfork p(1) \}$.

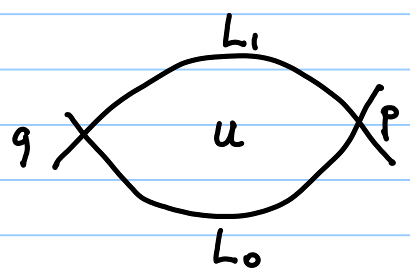
Then $\mu : P\mathcal{G}(n) \to \mathbb{Z}$ is the unique continuous map s.t.

- $\mu(\rho_1 \times \rho_2) = \mu(\rho_1) + \mu(\rho_2)$
- $\mu(e^{i\pi N t}) = |N| + 1$ in the 1-dimensional case.

(it follows that

$\mu(\rho \# \rho') = \mu(\rho) + \mu(\rho')$ for $\rho' \in \pi_1(\mathcal{G}(n))$).

Now let β be a homotopy class of strips:



Trivialize: $u^* TM \cong \mathbb{R} \times [0,1] \times \mathbb{C}^n$.
as complex vector bundle.

Choose a path ρ_p from $T_p L_0$ to $T_p L_1$,
and a path ρ_q from $T_q L_0$ to $T_q L_1$.

Concatenating we obtain a loop

$\tilde{\rho} : S^1 \to \mathcal{G}(n)$.

Lem: $i(\beta) = \mu(\tilde{\rho}) - \mu(\rho_p) + \mu(\rho_q)$,

where recall $i(\beta)$ is the Fredholm index of the linearized $\bar{\partial}$ operator.

Now consider

$$\begin{array}{ccc} \mathcal{G}(n) & \hookrightarrow & \mathcal{G}M \\ & & \downarrow \\ & & M \end{array}$$

$\mathcal{G}M$ ← bundle of Lagrangian subspaces of TM.

Let $\widetilde{\mathcal{G}M} \to \mathcal{G}M$ be a fibrewise universal cover of $\mathcal{G}M$: i.e.,

$$\begin{array}{ccc} \widetilde{\mathcal{G}(n)} & \hookrightarrow & \widetilde{\mathcal{G}M} \\ & & \downarrow \\ & & M. \end{array}$$

Such $\widetilde{\mathcal{G}M}$ need not exist (it exists iff $2c_1(TM) = 0$), and if it does exist it need not be unique.

(Such a fibrewise universal cover is specified by a choice of nowhere-vanishing section of $(\Lambda^n_{\mathbb{C}}(T^*M))^{\otimes 2}$ (a quadratic <u>holomorphic volume form</u>): such a section determines a map

$$\eta : \mathcal{G}M \longrightarrow S^1$$

by analogy with $\det^2$, and

$$\widetilde{\mathcal{G}M} \cong \{(\ell, \theta) \in \mathcal{G}M \times \mathbb{R} : \eta(\ell) = e^{i\theta}\}.)$$

Now any Lagrangian $L \subset M$ comes with a canonical lift $L \hookrightarrow \mathcal{G}M$ (given by its tangent spaces).

<u>Defn</u>: A <u>grading</u> of L (with respect to $\widetilde{\mathcal{G}M}$) is a choice of lift

$$L \longrightarrow \widetilde{\mathcal{G}M}.$$

The obstruction to the existence of a grading is the composition

$$\pi_1(L) \longrightarrow \pi_1(\mathcal{G}M) \longrightarrow \mathbb{Z}$$

classifies cover $\widetilde{\mathcal{G}M}$.

This element of $\mathrm{Hom}(\pi_1(L), \mathbb{Z}) \cong H^1(L; \mathbb{Z})$ is called the <u>Maslov class</u> and denoted μ_L. It does not depend on the choice of $\widetilde{\mathcal{G}M}$.

When $\widetilde{\mathcal{G}M}$ comes from a quadratic volume form as above, we have a map

$$\begin{tikzcd} L \arrow[r] \arrow[rr, bend right, "\varphi"'] & \mathcal{G}M \arrow[r, "\eta"] & S^1 \end{tikzcd}$$

and a grading is equivalent to a lift to a map $\tilde{\varphi} : L \to \mathbb{R}$.

<u>Defn</u>: Suppose L_0 and L_1 are equipped with gradings. Then for each $p \in L_0 \cap L_1$, there's a unique homotopy class of paths ρ from

T_pL_0 to T_pL_1 which lift to a path from $\widetilde{T_pL_0}$ to $\widetilde{T_pL_1}$ in $\widetilde{\mathcal{G}M}$.

We define

$$\deg(p) := \mu(\rho).$$

This equips $CF(L_0, L_1)$ with a $\mathbb{Z}$-grading.

<u>Lem</u>: ∂ has degree $+1$ with respect to this grading.

<u>Proof</u>: Exercise (use the previous Lemma and the fact that any J-hol. strip contributing to ∂ lies in a homotopy class β with $i(\beta)=1$).