EXERCISES FOR MINI-COURSE ON THE FUKAYA CATEGORY

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1. Combinatorial Floer homology

Let $L_0, L_1 \subset \Sigma$ be transversely intersecting, simple, closed, homotopically non-trivial curves in a symplectic 2-manifold (Σ, ω).

(1) Give a purely combinatorial definition of the Floer cochain complex $(CF^\bullet(L_0, L_1), \partial)$ over $\Lambda_{\mathbb{Z}_2}$. In particular, prove that $\partial^2 = 0$.
(2) Now do the same, but over $\Lambda_{\mathbb{K}}$ where $\mathbb{K}$ is an arbitrary field (not necessarily of characteristic 2).
(3) How far can the conditions on the L_i be relaxed? Can they be immersed curves, or nullhomotopic curves?
(4) More generally, let $L_0, \ldots, L_k$ be transversely-intersecting curves as above. Give a purely combinatorial definition of the A_∞ structure map

$$m^k : CF(L_{k-1}, L_k) \otimes \ldots \otimes CF(L_0, L_1) \to CF(L_0, L_k),$$

and prove that the maps defined in this way satisfy the A_∞ relations (start by working over a Novikov field of characteristic 2 as before, then generalize to arbitrary characteristic).

References for this section include [Che02], [dSRS14], [Sei08, §13b].

We will consider gradings in this case in §4.

2. The action functional (see [Auroux, §1.1])

Let L_0, L_1 be Lagrangian submanifolds of (M, ω). Define the path space

$$\mathcal{P} := \{\gamma \in C^\infty([0,1], M) : \gamma(0) \in L_0, \gamma(1) \in L_1\}.$$

We can think of this as an infinite-dimensional manifold with tangent space

$$T_\gamma \mathcal{P} = \{\eta \in \Gamma(\gamma^* TM) : \eta(0) \in T_{\gamma(0)} L_0, \eta(1) \in T_{\gamma(1)} L_1\}.$$

If J is an almost-complex structure on M compatible with ω, i.e., such that $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric, then we can define a Riemannian metric $g_{\mathcal{P}}$ on $\mathcal{P}$ by setting

$$g_{\mathcal{P}}(\eta,\xi) := \int_0^1 g(\eta_t,\xi_t)dt.$$

Now, we define a function $\mathcal{A}$ on $\tilde{\mathcal{P}}$, the universal cover of $\mathcal{P}$, by

$$\mathcal{A}(\gamma,\Gamma) := -\int_\Gamma \omega.$$

Here $\gamma \in \mathcal{P}$ and $\Gamma : [0,1]\times[0,1]$ is a homotopy between γ and a fixed base point in the connected component of $\mathcal{P}$. This is called the *action functional*.

Remark: To make this sort of thing more than formal one would need to pick some regularity class of paths to set up relevant Banach manifolds etc. However, attempts to make rigorous parts of this exercise about gradient flowlines would still run into severe analytical difficulties; in fact, existence of bubbling shows that such difficulties are not accidental. Reformulating the gradient flow equation as a Cauchy-Riemann PDE allows one to analyze the bubbles and is one of the insights underlying Floer theory.

(1) Prove that the critical points of the action functional are (lifts of) constant paths, i.e., intersection points between L_0 and L_1.
(2) Let $u : \mathbb{R} \to \tilde{\mathcal{P}}$ be a Morse flowline of the action functional with respect to the Riemannian metric $g_{\mathcal{P}}$. I.e., $u'(s) = \nabla\mathcal{A}$ where $g_{\mathcal{P}}(\nabla\mathcal{A},\cdot) = d\mathcal{A}$, and $\lim_{s\to\infty} u(s) = p$, $\lim_{s\to-\infty} u(s) = q$. Prove that such Morse flowlines u are in one-to-one correspondence with (lifts of) J-holomorphic strips connecting p and q. Thus, Lagrangian Floer cohomology can be thought of as a version of Morse cohomology for the action functional.
(3) Suppose that (M,ω) is exact (i.e., $\omega = d\alpha$), and L_0 and L_1 are exact Lagrangians (i.e., $\alpha|_{L_i} = dh_i$ for $i = 0,1$).
  (a) Show that $\mathcal{A}(\tilde{p}) = \mathcal{A}(p)$ does not depend on the lift $\tilde{p}$ of p.
  (b) Show that for any holomorphic strip u from p to q, we have
  $$\omega(u) = \mathcal{A}(q) - \mathcal{A}(p).$$
  (c) Show that in this case, one can define $CF(L_0,L_1;\mathbb{Z}_2)$ by counting holomorphic strips with coefficients in $\mathbb{Z}_2$ rather than Λ (i.e., show that one doesn't get any infinite sums); and that the map
  $$(CF(L_0,L_1;\mathbb{Z}_2),\partial)\otimes_{\mathbb{Z}_2}\Lambda \to (CF(L_0,L_1;\Lambda),\partial)$$
  $$p \mapsto T^{\mathcal{A}(p)}\cdot p$$
  is an isomorphism of cochain complexes.

Remark: The last exercise shows that in the exact case, the use of the Novikov field is unnecessary. For that reason one often simply defines $CF(L_0, L_1)$ with $\mathbb{Z}_2$ coefficients.

Remark: A compact symplectic manifold can never be exact (because $\omega^{\wedge top}$ must represent a non-zero class in the top degree of cohomology). All steps in the construction of Floer cohomology of closed Lagrangian submanifolds of a non-compact symplectic manifold work as in the compact case, except for compactness: one needs to impose some extra conditions on the manifold to ensure that the holomorphic curves remain confined inside a compact region, typically by some version of the maximum principle.

3. Gromov Compactness

(1) Let J be the standard complex structure on $\mathbb{CP}^1$.
   (a) Find a sequence of J-holomorphic maps $u_n : \mathbb{CP}^1 \to \mathbb{CP}^1$ which converge to a nodal sphere.
   (b) Now let $L_0, L_1 \subset \mathbb{CP}^1$ be simple closed curves (i.e., Lagrangians) which intersect, transversely. Find a sequence of J-holomorphic strips with boundary on L_0 and L_1 which converges to a non-constant J-holomorphic strip with a disc bubble attached at a boundary point.

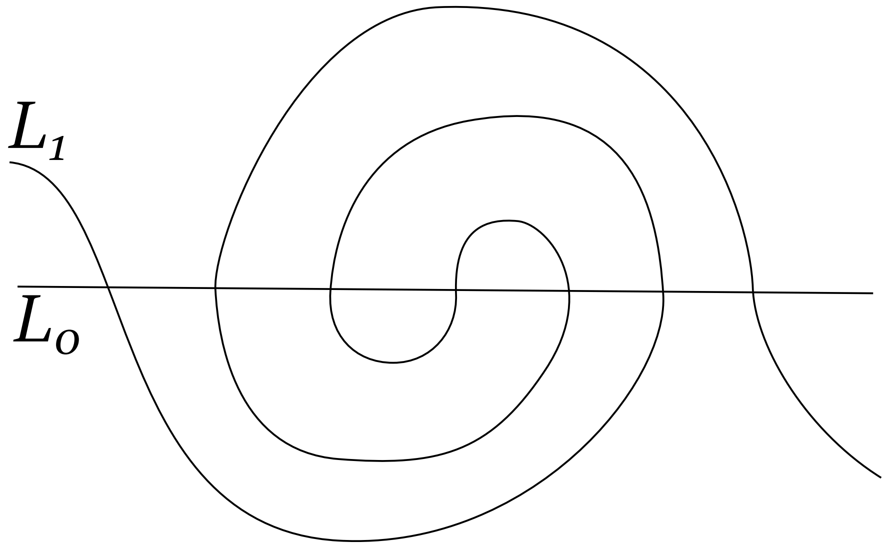
(2) Let J be the standard integrable complex structure on $\mathbb{CP}^2$.
   (a) Consider the J-holomorphic curves (without boundary)

$$u_\epsilon : \mathbb{CP}^1 \to \mathbb{CP}^2$$
$$u_\epsilon([z : w]) = [z^2 : \epsilon z w : w^2].$$

   What is the limit of this sequence of curves as $\epsilon \to 0$? As $\epsilon \to \infty$?
   (b) Use the above example to construct a sequence of J-holomorphic discs $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{CP}^2, \mathbb{RP}^2)$ whose limit is two discs joined at a boundary node.

4. Grading

(1) Prove that the Maslov index for paths is uniquely characterized by the properties stated in lecture.
(2) Prove the claim made in the lecture: the Floer differential has degree +1.
(3) Work out the graded Floer complex $CF^\bullet(L_0, L_1)$ for the following two Lagrangians in the plane:



(4) Let $L \subset M$ be a Lagrangian such that the image of $\pi_1(L, *_L)$ in $\pi_1(M, *)$ is trivial. An *anchoring* for L is a homotopy class of paths γ in M from $*$ to $*_L$ (see [FOOO]). Now suppose that L_0 and L_1 are anchored; for each $p \in L_0 \cap L_1$, assign a grading $deg(p) \in \pi_1(M, *)$ by following the chosen paths from $*$ to $*_{L_0}$, to p (inside L_0), to $*_{L_1}$ (inside L_1), to $*$. Prove that the A_∞ products m^k respect this grading, in the sense that

$$deg(m^k(p_1, \ldots, p_k)) = \sum_{i=1}^{k} deg(p_i).$$

(5) Show that there is a natural isomorphism of cochain complexes

$$(CF^\bullet(L_0, L_1), \partial) \cong (CF^{n-\bullet}(L_1, L_0)^\vee, \partial^\vee).$$

This isomorphism (or more precisely, the induced isomorphism on the cohomology level) is called *Poincaré duality*.

5. Holomorphic discs and displacement energy.

Let ϕ be a compactly supported Hamiltonian symplectomorphism of (M, ω). Define the Hofer norm of ϕ to be the infimum, over all Hamiltonians generating ϕ, of their time-averaged variations; that is define

$$\rho(\phi) = \inf_{H_t} \int_0^1 (\max H_t - \min H_t)\, dt.$$

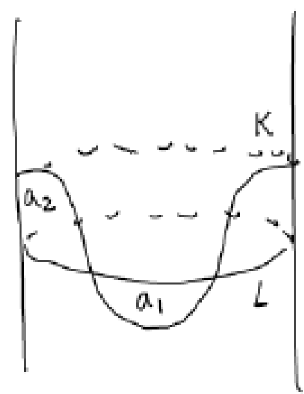
where the infimum is taken over all compactly supported $H_t : [0,1] \times M \to \mathbb{R}$ with $\phi^1 = \phi$ (this was proved to be a metric on $Ham^c(M, \omega)$ by Lalonde and McDuff [McDuff-Lalonde]).

We also define the displacement energy of $K, L \subset M$ to be the infimum of Hofer norms of all Hamiltonian symplectomorphisms disjoining K and L from each other:

$$e(K, L) = \inf_{\{\phi | \phi(K) \cap L = \emptyset\}} \rho(\phi).$$

When $K = L$ then $e(K) \coloneqq e(K, K)$ is known as the displacement energy of K.

(1) Prove $e(K, L) = e(L, K)$.
(2) Consider K and L two (Lagrangian) curves in the cylinder:



Suppose the areas of discs between K and L are a_1 and a_2 (with $a_1 < a_2$). Compute $e(K, L)$.
(3) In general, for two Lagrangians K and L for which the Floer cohomology is well defined, give a bound on $e(K, L)$ in terms of this Floer cohomology (over the Novikov *ring*), and give an idea of a proof.

6. Morse trajectories and holomorphic strips (see [Auroux, §1.6]).

Consider a Morse function f on smooth manifold N. Given a metric g such that the pair (f, g) satisfy the Morse-Smale transversality condition, one obtains a Morse-Witten complex generated by critical points of f and with differential counting gradient trajectories of f.

Let L_0 be the zero section in $T^\star N$, and L_1 the graph in $T^\star N$ of σdf for a fixed small real number σ.

(1) Show that the Hamiltonian $H = \sigma f \cdot \pi$ isotopes L_0 to L_1.
(2) Show that L_0 and L_1 intersect transversely at the critical points of f.
This identifies the generators of Morse-Witten and Floer complexes in this setting.
(3) Use g to get a compatible almost complex structure for $T^\star N$, and then identify $T_{(p,q)}(T^\star N) = \mathbb{C} \otimes T_q N$, with vertical subbundle purely imaginary and horizontal purely real. Show that this gives a trivialization of $(\Lambda^n_{\mathbb{C}}(T^\star T^\star N))^2$ as in the lecture.
(4) Show that L_0 is graded (by a constant lift!), and we can use the Hamiltonian isotopy above to grade L_1.
(5) Show that Maslov and cohomological Morse gradings of all $p \in L_0 \cap L_1 = crit(f)$ coincide.
This identifies Morse-Witten and Floer complexes as graded vector spaces.
(6) Recall that Floer equation perturbs the Cauchy-Riemann holomorphicity condition by adding a Hamiltonian term X_H:

$$\frac{\partial u}{\partial s} + J(t,u)\left(\frac{\partial u}{\partial t} - X_H(t,u)\right) = 0$$

Show that $\tilde{u}(s,t) = (\phi^t_H)^{-1}(u(s,t))$ solves the perturbed Floer equation if and only if $u(s,t)$ solves the unperturbed version. The two versions (one perturbing the boundary and one perturbing the equation) are therefore equivalent.
(7) Using the compatible J from above and Hamiltonian perturbation $-H$, see that $u(s,t) = \gamma(s)$ is a solution for perturbed equation whenever γ is a Morse flowline of f.
(8) Rescale the generators $p \to T^{\sigma f(p)} p$ to identify (modulo regularity issues which we ignore) Morse and Floer complexes over the Novikov ring.
Remark: In the general case of a compact Lagrangian submanifold L in a symplectic manifold (M,ω), under the assumption that $[\omega] \cdot \pi_2(M,L) = 0$ energy estimates imply that, for a sufficiently small Hamiltonian perturbation, the pseudo-holomorphic strips that determine the Floer cohomology $HF^\star(L,L)$ must all be contained in a small tubular neighborhood of L, so that the calculation of Floer cohomology reduces to the above, and we get that $HF^\star(L,L) = H(L,\Lambda)$ (this is originally due to Floer in the exact case for $K = \mathbb{Z}_2$, [Floer]).

7. The Stasheff associahedra and holomorphic discs

The Stasheff associahedron K_k [Stasheff] is a polyhedron whose faces F_T are indexed by planar trees T with $k+1$ labeled, cyclically ordered semi-infinite leaves (with no finite leaves, every vertex of degree at least 3, and up to planar isomorphism, of course); and $F_{T_1} \subset F_{T_2}$ if and only if T_2 is obtained from T_1 by contracting some edges.

(1) Draw K_3, K_4 and K_5. Describe the natural action of $\mathbb{Z}/k+1$ on K_k in these examples (the natural action arises by rotating the planar trees, aka cyclically shifting the labels).
(2) Let $\mathcal{R}_k := \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 < \ldots < x_k\}/Stab_\infty(SL(2,\mathbb{R}))$, where the subgroup $Stab_\infty(SL(2,\mathbb{R}))$ is the stabilizer of infinity inside $SL(2,\mathbb{R})$, viewed as Möbius transformations, acts by translations and (positive) scalings. Show that there is a well-defined embedding
$$i : \mathcal{R}_k \hookrightarrow [0,\infty]^{\binom{k}{3}},$$
where the coordinate indexed by the subset $\{\ell < m < n\} \subset \{1, \ldots, k\}$ is given by the corresponding *cross ratio* (see [McDuff-Salamon, Appendix D.1]), $(x_n - x_m)/(x_m - x_\ell)$.
(3) Define $\bar{\mathcal{R}}_k$ to be the closure of the image of i. This is the *Deligne–Mumford compactification* of $\mathcal{R}_k$. Prove that it is isomorphic to K_k in a way that preserves the natural stratifications.
(4) Now consider the analogous construction, but where we do not impose an ordering condition on the x_i: call the resulting topological space $\bar{\mathcal{R}}_k^{unord}$. Is the resulting i_{unord} an embedding? Identify the topological space $\bar{\mathcal{R}}_4^{unord}$.

The significance of this for Fukaya categories is as follows. We regard $\mathbb{R}$ as the boundary of the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : im(z) \geq 0\}$. Any complex disc with $k+1$ ordered boundary marked points is biholomorphic to $\mathbb{H} \cup \{\infty\}$, with the first boundary marked point corresponding to ∞ and the remaining ones corresponding to points $x_1 < \ldots < x_k \in \mathbb{R} = \partial\mathbb{H}$. Thus $\mathcal{R}_k$ is isomorphic to the moduli space of complex discs with $k+1$ boundary marked points, which are precisely the domains of the J-holomorphic maps whose count defines an A_∞ structure map m^k in the Fukaya category.

The codimension-1 boundary components of the Gromov compactification of a one dimensional moduli space of J-holomorphic discs are of two types. Firstly, the complex structure on the domain may degenerate: i.e., it may approach a point in the boundary $\bar{\mathcal{R}}_k \setminus \mathcal{R}_k \cong K_k \setminus int(K_k)$. Since we are considering a one-dimensional moduli space, generically it will avoid the faces of codimension 2 and only run into the faces of codimension 1. These correspond precisely to the terms in the A_∞ relations which do not involve $m^1 = \partial$. Secondly, we may have concentration of energy at a boundary puncture, as happened in the proof that $\partial^2 = 0$: these boundary points correspond to the remaining terms in the A_∞ relations.

8. Floer theory in toric varieties

(1) Calculate the disc potential function for Lagrangian torus fibres in the following cases:
   (a) $\mathbb{CP}^1$.
   (b) $\mathbb{CP}^1 \times \mathbb{CP}^1$.

(2) Hence work out which torus fibres have non-vanishing Floer cohomology in these cases.
(3) In the case of $\mathbb{CP}^1$, verify your answer to the previous question using the combinatorial model for $HF^\bullet(L, L) \cong HF^\bullet(L, \psi(L))$ from §1.
(4) Prove that every torus fibre in $\mathbb{CP}^1$ with vanishing self Floer cohomology (i.e., such that $HF^\bullet(L, L) \cong 0$) can be displaced by a Hamiltonian isotopy.
(5) In fact, prove that every torus fibre in $\mathbb{CP}^1$ with vanishing self Floer cohomology can be displaced by a Hamiltonian isotopy supported in $\mathbb{CP}^1 \setminus 0$ or $\mathbb{CP}^1 \setminus \infty$ (but not both).
(6) Use the previous exercise to show that every torus fibre in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with vanishing self Floer cohomology can be displaced by a Hamiltonian isotopy.
(7) Use the same idea to show that every torus fibre in $\mathbb{CP}^2$ with vanishing self Floer cohomology can be displaced by a Hamiltonian isotopy.

This method of displacing torus fibres is called the method of 'probes' [McD11].

9. The elliptic curve

This exercise is about the Fukaya category of an elliptic curve; be warned that it is more computationally intensive than the other exercises! It is based on [Zas05].

Let $(T, \omega) = (\mathbb{R}^2/\mathbb{Z}^2, \tau \cdot dx \wedge dy)$ be a symplectic torus of area τ. Consider the Lagrangians $L_k = \{y = kx\}$ for $k \in \mathbb{Z}$. Observe that the isomorphism

$$T \to T$$
$$(x, y) \mapsto (x, y + x)$$

takes L_k to L_{k+1} for all k. Therefore, we can define a map

$$HF^\bullet(L_0, L_j) \otimes HF^\bullet(L_0, L_k) \xrightarrow{\mathrm{id} \otimes T^j} HF^\bullet(L_0, L_j) \otimes HF^\bullet(L_j, L_{j+k}) \xrightarrow{m^2} HF^\bullet(L_0, L_{j+k}).$$

(1) Show that this product defines a bigraded algebra $\mathcal{A}_{\bullet,\bullet}$, where
$$\mathcal{A}_{j,k} := HF^j(L_0, L_k).$$
(2) Show that the corresponding graded algebra
$$\mathcal{C}_k := \mathcal{A}_{0,3k}$$
is isomorphic to $\Lambda[X_0, X_1, X_2]/p(X_0, X_1, X_2)$, where $|X_i| = 1$ and
$$p_\tau(X_0, X_1, X_2) = a_\tau X_0^3 + b_\tau X_1^3 + c_\tau X_2^3 + d_\tau X_0 X_1 X_2.$$
Compute $a_\tau, b_\tau, c_\tau, d_\tau$.
(3) Calculate the j-invariant of the elliptic curve defined by $\{p_\tau = 0\}$, as a function of τ.

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