Lecture 3:
Examples of $A_\infty$ categories we will consider:

E.g. Let $D \subset X$ be an ample divisor, $\omega$ a Kähler form on $X$ with $\omega|_{X\setminus D} = d\alpha$.

Define an $A_\infty$ cat. $\text{Fuk}(X, D)$ over $K_A := C((Q))$

- $\text{Obj} = \text{closed exact Lagrangians } L \subset X \setminus D$

  \[ \text{holomorphic discs with } u \cdot D = 0 \]

  (also equipped with grading and spin structure).

  \[ \text{Mor} = C(L_0, L_1) := K_A \langle L_0 \cap L_1 \rangle \]

- Structure maps: coefficient of $Q^d \cdot a_0$ in $m^s(a_1, \ldots, a_s)$ is

\[ \# \left( \text{holomorphic discs with } u \cdot D = 0 \right) \]

$^3 \#$ means we count the points in the 0-dimensional component of the moduli space of such curves.

The $A_\infty$ relations hold because the boundary points of the 1-dimensional component of this moduli space are configurations
and \# (boundary points of compact 1-mfld = 0)

Note: \(|L_0 \cap L_1| < \infty \Rightarrow \text{Fuk}(X,D)\) is proper.

Note: Use intersection points with D to stabilize domain, as in John's talks.

(Note: \text{Fuk}(X)\) defined in same way, but weight discs \(w\) by \(Q^w = Q((Q^R))\). This would force us to work over this field in \(V^A(X),\ V^B(Y)\). I don't know how to define monodromy weight filtration in this world.)

E.g. Let \(Y = \text{smooth projective} / \mathbb{M}_{g,b}\) formal punctured disc \(\text{Spec} \mathbb{K}_b\).
Define an \(A_{\infty}\) category \(D^b\text{Coh}(Y)\) over \(\mathbb{K}_b\).

Objects = bounded below complexes of injective quasi-coherent sheaves whose cohomology sheaves are bounded and coherent.

Morphisms = \(\mathcal{C}(\xi^*, \xi^*) := \text{Hom}(\xi^*, \xi^*)\)

\(m^1(f) = d_{\xi^*} \circ f \pm f \circ d_{\xi^*}\)

\(m^2(f, g) = g \circ f\), \(m^3 = 0\).

This is an \(A_{\infty}\) category (in fact a DG category, because \(m^3 = 0\)). It also is proper.
Goal: Given an $A_{\infty}$ cat. $\mathcal{C}$ over $K := \text{Spec } \mathbb{C}[[q]]$, define a polarized pre-VSHS

$$V(\mathcal{C}) = (\mathcal{H}C.(\mathcal{C}), \triangledown, (\cdot, \cdot))$$

over $\mathcal{M} := \text{Spec } K$.

Prove $V(\text{Fuk}(X, D)) \cong V^A(X)$

$$V(D^b \text{Coh}(Y)) \cong V^B(Y).$$

Defn: Let $\mathcal{C} = A_{\infty}$ cat. Define

$$\mathcal{C}C^*.(\mathcal{C}) := \bigoplus_{L_i} \mathcal{C}(L_0, L_1, \ldots, L_s, L_0)$$

$$b: \mathcal{C}C^*.(\mathcal{C}) \otimes$$

$$b([a_0 [... | a_s])] := \sum_{\psi_0} a_0 [... | m^*(...) | ... | a_s]$$

$$+ \sum_{\psi_s} m^*(..., a_s, a_0, ...) [...].$$

Lem: $b^2 = 0$.

Defn: $HH^*(\mathcal{C}) := H^*(\mathcal{C}C^*.(\mathcal{C}), b)$.

Now we define a new $A_{\infty}$ category $\mathcal{C}_e$:

$$\mathcal{C}_e(L_0, L_1) := \begin{cases} \mathcal{C}(L_0, L_1) & \text{for } L_0 \neq L_1, \\ \mathcal{C}(L_0, L_1) \otimes K\langle e \rangle & \text{for } L_0 = L_1. \end{cases}$$
\[ m_2(e, a) = m_2(a, e) = a \quad \forall a \]
\[ m_i(\ldots, e, \ldots) = 0 \quad \forall i \neq 0. \]

The inclusion \( CC_e(\varepsilon) \hookrightarrow CC_e(\varepsilon_e) \) is a quasi-iso.

\( D := \langle a_0[\ldots le!\ldots] \rangle \) is an acyclic subcomplex of \( CC_e(\varepsilon_e) \); henceforth we quotient by it on chain level.

**Defn**: \( B : CC_e(\varepsilon_e) \to \) Connes B-operator

\[ B(a_0[a_1, \ldots a_5]) := \sum_{\epsilon_c} e[a_1, \ldots a_5]. \]

**Lem**: \( Bb + Bb = 0, \quad B^2 = 0. \) Connes-Tsygan differential

**Defn**: \( CC_e(\varepsilon) := (CC_e(\varepsilon_e) \otimes K[u], b + uB) \)

\( \text{complete w.r.t. } u \)-adic filtration.

\( HC_e(\varepsilon) := H^*(CC_e(\varepsilon)) \) (Kevin's \( V^5 \))

\( \text{`negative cyclic homology'} \)

\( CP_e(\varepsilon) := (CC_e(\varepsilon_e) \otimes K[u^{\pm 1}], b + uB) \)

\( HP_e(\varepsilon) := H^*(CP_e(\varepsilon)) \) (Kevin's \( V_{tate} \))

\( \text{`periodic cyclic homology'} \)
The $u$-adic filtration on $\text{CC}^\cdot(\mathcal{C})$ is complete by construction. The corresponding spectral sequence

$$\text{HH}^\cdot(\mathcal{C}) \otimes K[u] \Rightarrow \text{HC}^\cdot(\mathcal{C})$$

is the Hodge-de Rham spectral sequence. If $\mathcal{C}$ is smooth and proper/compact, it degenerates at $E_2$ page by Kalinin’s proof of Kontsevich-Soibelman’s conjecture [Kal 16] (we assume $\mathcal{C}$ is $\mathbb{Z}$-graded).

Compare Tony’s talk.

$\text{HC}^\cdot(\mathcal{C})$ is an $\mathcal{O}_M[u]$-module, where $M = \text{Spec } K$. This is the $\mathcal{E}$ in our pre-VSHS.

Next we define the connection, following Getzler [Get 93]. It has the form

$$\nabla : TM \otimes \text{CC}^\cdot(\mathcal{C}) \longrightarrow u^{-1} \text{CC}^\cdot(\mathcal{C})$$

$$\nabla_u(\alpha) = u(\alpha) - u^{-1} b^1(u(m^\ast), \alpha) - B^1(u(m^\ast), \alpha).$$

To define $u(\alpha)$ we need a $K$-basis for $\text{CC}^\cdot$. We obtain one by choosing a $K$-basis for all $\mathcal{C}(L_0, L_1)$. 
We define

\[ b^1(\nu(m^*), a_0[\ldots|a_s]) := \sum_{cyc} m^*(\ldots, \nu(m^*)(\ldots), \ldots, a_0, \ldots)[\ldots|a_i] \]

Note: again we need our choice of \( K \)-basis in morphism spaces, in order to define

\[ \nu(m^*)(a_1, \ldots, a_s) := \nu(m^*(a_1, \ldots, a_s)) - \sum m^*(\ldots, \nu(a_i), \ldots) \]

(note: \( b^1(-,-) : C^* \circ \cdot \circ C^* \xrightarrow{\nu} C^* \circ \cdot \circ C^* \) induces module structure of \( HH^*(C) \) over \( HH^*(C) \)).

\[ B^1(\nu(m^*), a_0[\ldots|a_s]) := \sum_{cyc} e[\ldots|\nu(m^*)(\ldots)|\ldots|a_0|\ldots] \]

Getzler proves that

- \( [\nabla_{b+uB}, b+uB] = 0 \Rightarrow \) well-defined on homology

- \( \nabla \) is flat on \( HP^*(C) \) (automatic in our case

\[ \text{since base is 1-dim'f}. \]

The induced connection on \( HC_0^-(C) \) is the Getzler-Gauss-Manin connection. It depends on the choice of \( K \)-basis on the chain level, but not on the homology level. This is the connection \( \nabla \) in our pre-VSHS.
Finally we introduce the polarization \((\cdot, \cdot)\), following Costello, Kontsevich-Soibelman, Shklyarov [Shk 07]. If \(\mathcal{C}(L_0, L_i)\) is finite-dimensional for all \(L_i\) (which is stronger than properness = \(H^0 \mathcal{C}(L_0, L_i)\) f.d.), we can define

\[
(a_0[a_1, \ldots, a_s], b_0[b_1, \ldots, b_t]) = \sum T_T (m^*(a_i, \ldots, a_s), m^*(a_j, \ldots, - b_k, \ldots, b_0, \ldots), b_l, \ldots).
\]

This induces a pairing

\[
\mathcal{H}C^-(\mathcal{C}) \times \mathcal{H}C^-(\mathcal{C}) \rightarrow \mathbb{K}[u].
\]

One can show it is covariantly constant for Getzler’s connection.

The same formula defines a pairing on

\[
\mathcal{H}H^*(\mathcal{C}) = \mathcal{H}C^-(\mathcal{C})/u \cdot \mathcal{H}C^-(\mathcal{C})
\]

if H_{dR} degen. holds.

Shklyarov proves that, if \(\mathcal{C}\) is smooth and proper, \(\mathcal{H}H^*(\mathcal{C})\) is finite-dimensional and the pairing is nondegenerate.

Putting all of these ingredients together, we have defined
a pre-VSHS $(\mathcal{H}C^{-}(\mathcal{C}), \nabla)$. If $\mathcal{C}$ is proper we have a polarization. If $\mathcal{C}$ is furthermore smooth, it is a polarized pre-VSHS (although we don't need Kaledin's theorem for our main result).
III. Open-closed map

Now we want to compare $H^\infty(Fuk)$ with $V^A$.

We define a map

$OC: CC_* (Fuk(X,D)) \rightarrow H^* (X; K_A)$

$OC\langle a_0, [a_1, ..., [a_5]] \rangle$ is defined by considering the moduli space of holomorphic discs:

This is some finite-dimensional moduli space $M$; evaluation at $\bullet$ defines a cycle $ev_* M$; the contribution to $OC\langle a_0, [... \rangle$ is

$Q^{u*D} \cdot ev_* M$. 
Lem: \( \partial \circ \mathrm{OC} = \mathrm{OC} \circ \partial. \)

\[ \begin{align*}
\text{Pf:} \quad & \partial \circ \mathrm{OC}(a_0[\ldots]) = \text{ev}_* \partial M \\
& = \text{ev}_* \left( \mathrm{OC}(\ldots) + \mathrm{M}(\ldots) \right) \\
& = \mathrm{OC} \left( \sum a_i \ldots m^*(\ldots) \ldots \right) \\
& + \sum m^*(\ldots a_0 \ldots) \right) \\
& = \mathrm{OC} \circ \partial \left( a_0[\ldots] \right)
\end{align*} \]

\( \Rightarrow \ \mathrm{OC} \) defines a map \( \mathrm{HH}_* (\text{Fuk}(X, D)) \rightarrow \mathrm{H}_*(X) \).

We extend to \( \mathrm{OC}: \mathrm{CC}_* (C_e) \rightarrow \mathrm{H}_*(X) \) by regarding \( e \) as a marked point on the boundary of our disc, at which there is no constraint.

"Lem:" \( \mathrm{OC} \circ \partial = 0 \)

"Pf:" \( \mathrm{OC}(\partial(a_0[\ldots])) = \sum \text{ev}_* \left( \ldots a_0 \ldots \right) \)

this chain factors through a space
of lower dimension, by forgetting \( \theta \), so it is degenerate.

Thus we have a map

\[
OC : HC_* (Fuk(X,D)) \rightarrow H^*(X;K_A)[[u]] \cap V^A(X)
\]

Remark: In [FOOD10] the authors construct this map by proving \( OC \circ B \). In our technical setup, it is difficult to arrange for this to hold on the nose, so we actually construct

\[
OC = OC + u \cdot OC_1 + u^2 OC_2 + \ldots
\]

with \( \partial C \circ (b + uB) = \partial \circ OC \).