EXPONENTIAL SUMS, REE GROUPS AND SUZUKI GROUPS: CONJECTURES

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Contents

1. Introduction

In our earlier work [Ka-RLSFM], we exhibited some very simple one parameter families of exponential sums which gave rigid local systems on the affine line in characteristic p whose geometric (and usually, arithmetic) monodromy groups were $SL(2,q)$, and we exhibited other such families giving $SU(3,q)$. [Here q is a power of the characteristic p, and p is odd.] What we did there made essential use of the work [Gross] of Dick Gross. That work of Gross also discussed the Suzuki and Ree groups, and Gross asked us if those groups entered into our picture.

Our object here is to produce very simple one parameter families of exponential sums on the affine line $\mathbb{A}^1$ in characteristic two (for Suzuki) and three (for Ree) which give rigid local systems whose geometric and arithmetic monodromy groups are the Suzuki and Ree groups respectively. For simplicity, let us discuss the situation over $\overline{\mathbb{F}_p}$, p respectively $2,3$. By the solution [Ray] of the Abhyankar Conjecture, we know there exist local systems on $\mathbb{A}^1/\overline{\mathbb{F}_p}$ whose monodromy groups are the Suzuki and Ree groups respectively. Gross constructs local systems on $\mathbb{G}_m/\overline{\mathbb{F}_p}$ with these groups as monodromy groups. Gross further tells us that explicit Kummer pullbacks of his local systems give local systems on

$\mathbb{A}^1/\overline{\mathbb{F}_p}$ with these same monodromy groups. We expect that our local systems coincide with these Kummer pullbacks, cf. Section 4.

Our earlier work had two parts. The first was a theorem of Kubert which assured us that certain one parameter families of exponential sums gave (rigid) local systems whose geometric and arithmetic monodromy groups were finite. The second was to show that these finite groups were as asserted above.

In the discussion below of Suzuki and Ree groups, **both** of these parts are absent, replaced instead by conjectures. In Section 4, we explain what should be the relation between the families which will be the subject of our conjectures and the local systems on $\mathbb{G}_m$ constructed by Gross in the Suzuki and Ree cases.

We begin with the Suzuki groups. For $q_0 = 2^n, n \geq 1$, a power of 2, and $q := 2q_0^2$, the Suzuki group $Sz(q)$ is a simple group of order $q^2(q^2+1)(q-1)$. Its lowest dimensional nontrivial irreducible (complex) representations have dimension $q_0(q-1)$. There are two of these. Their complex-conjugate trace functions take values in the ring $\mathbb{Z}[i]$ of Gaussian integers. The values assumed are $\{-q_0 i, -q_0, -1, 0, 1, q_0 i, q_0(q-1)\}$, cf. the end of Suzuki's paper [Suz].

The first problem we consider here is to exhibit, for each $q_0 = 2^n, n \geq 1$, a pair of complex conjugate rigid local system of rank $q_0(q-1)$ on the affine line $\mathbb{A}^1/\mathbb{F}_q$ in characteristic 2 whose geometric and arithmetic monodromy groups are the group $Sz(q)$, in each of its two complex-conjugate irreducible representations of dimension $q_0(q-1)$.

We will write down explicit "candidate" pairs of complex conjugate rigid local system of rank $q_0(q-1)$ on $\mathbb{A}^1/\mathbb{F}_q$, about which we make two conjectures. The first is that their monodromy groups are finite. The second is that these finite monodromy groups are the asserted Suzuki groups $Sz(q)$.

We now turn to the Ree groups. For $q_0 = 3^n, n \geq 1$, a power of 3, and $q = 3q_0^2$, the Ree group $\mathrm{Ree}(q)$ is a simple group of order $q^3(q^3+1)(q-1)$. Its lowest dimensional nontrivial irreducible (complex) representation has dimension q^2-q+1. It is orthogonally self-dual. Its trace function take values $\mathbb{Z}$. The values assumed are $\{1-q, -1, 0, 1, 3, q^2-q+1\}$, cf. the end of Ward's paper [Ward].

The second problem we consider here is to exhibit, for each $q_0 = 3^n, n \geq 1$, a rigid local system of rank q^2-q+1 on the affine line $\mathbb{A}^1/\mathbb{F}_q$ in characteristic 3 whose geometric and arithmetic monodromy groups are the group $\mathrm{Ree}(q)$, in its irreducible representation of dimension q^2-q+1.

We have explicit candidate rigid local systems for the Ree groups, $\mathrm{Ree}(3^{2n+1}), n \geq 1$. Exactly as in the Suzuki case, we first conjecture that their monodromy groups are finite, and second that these finite monodromy groups are the asserted Ree groups.

2. The Suzuki case

We take $\psi : (\mathbb{F}_2, +) \to \pm 1$ to be the nontrivial additive character of $\mathbb{F}_2$. We recall that the p-Witt vectors, with $p = 2$, of length two have $W_2(\mathbb{F}_2) \cong \mathbb{Z}/4\mathbb{Z}$, by the explicit isomorphism $[a, b] \mapsto a^2 + 2b$. We denote by $\psi_2 : W_2(\mathbb{F}_2) \cong \mathbb{Z}/4\mathbb{Z} \to \mu_4(\mathbb{Z}[i])$ the faithful additive character $[a, b] \mapsto i^{a^2+2b}$, i.e., the faithful additive character of $\mathbb{Z}/4\mathbb{Z}$ given by $n \mapsto i^n$. This allows us to define the usual Artin-Schreier sheaf $\mathcal{L}_{\psi(x)}$ on $\mathbb{A}^1/\mathbb{F}_2$, and the Artin-Schreier-Witt sheaf $\mathcal{L}_{\psi_2([x,0])}$ on $\mathbb{A}^1/\mathbb{F}_2$.

We now turn to the definition of our candidate local systems. Recall that $q_0 = 2^n, n \geq 1$ and $q = 2q_0^2$. We denote by

$$t(q) := q - 2q_0 + 1$$

the order of the Coxeter torus in $Sz(q)$, cf. [Gross]. Thus $t(8) = 5, t(32) = 25, t(128) = 113$. We also denote by

$$d(q) := q_0(q - 1)$$

the degree of the lowest dimensional nontrivial irreducible complex representations of $Sz(q)$. Thus $d(8) = 14, d(32) = 124, d(128) = 1016$. For each $q = 2q_0^2 = 2^{2n+1}$, we define the polynomial $f_q(x) \in \mathbb{F}_2[x]$ as follows.

$$f_8(x) := x^{15},$$

$$f_{32}(x) := x^{125} + x^{3*25},$$

$$f_{128} := x^{1017} + x^{5*113} + x^{3*113}.$$

The general pattern is that for $q = 2^{2n+1}$,

$$f_q(x) := x^{1+d(q)} + \sum_{1 \leq m \leq n-1} x^{(2^m+1)t(q)}.$$

We now define, for each $q = 2^{2n+1}$ a rank one lisse local system on $A^1/\mathbb{F}_2$

$$\mathcal{S}_q(x) := \mathcal{L}_{\psi_2([x^{t(q)},0])} \otimes \mathcal{L}_{\psi(f_q(x))} = \mathcal{L}_{\psi_2([x^{t(q)},f_q(x)])},$$

the last equality simply because in Witt vector addition, $[a, b] = [a, 0] + [0, b]$. We then form its Fourier Transform

$$\mathcal{F}_q := FT_\psi(\mathcal{S}_q).$$

This is a rigid local system on $\mathbb{A}^1/\mathbb{F}_2$, lisse of rank $d(q)$, pure of weight one. Its trace function is given at time $t \in k$, $k/\mathbb{F}_2$ a finite extension, by the formula

$$\mathrm{Trace}(Frob_{t,k}|\mathcal{F}_q) = -\sum_{x\in k} \psi_2(\mathrm{Trace}_{W_2(k)/W_2(\mathbb{F}_2)}([x^{t(q)}, f_q(x) + tx])).$$

We then twist this local system by a suitable constant field twist.

$$\mathcal{G}_q := \mathcal{F}_q \otimes \beta_q^{-deg}.$$

For $q = 2^{2n+1}$, we define

$$\beta_q := 1 + i \text{ if } n \text{ is } odd,$$

$$\beta_q := 1 - i \text{ if } n \text{ is } even.$$

The conjugate local system begins with

$$\mathcal{L}_{\psi_2([x^{t(q)}, f_q(x)+x^{2t(q)}])},$$

and twists its FT_ψ by the complex conjugate of β_q^{-1}. [The point here is that for p-Witt vectors with $p = 2$, Witt vector addition is given by

$$[a, b] + [A, B] = [a + A, b + B - aA].$$

Thus for Witt vectors with entries in $\mathbb{F}_2$-algebras, we have

$$-[a, b] = [a, b + a^2],$$

and hence the two input rank one local systems have complex conjugate traces in $\mathbb{Z}[i]$. After Fourier Transform, their trace functions continue to be complex-conjugate, because ψ takes integer (in fact ± 1) values. After twisting, the trace functions remain complex conjugates of each other, but a priori now take values in $\mathbb{Z}[1/2][i]$.]

Conjecture 2.1. *For each For $q = 2^{2n+1}, n \geq 1$, the trace function of the local system $\mathcal{G}_q$ on $\mathbb{A}^1/\mathbb{F}_2$ takes values in $\mathbb{Z}[i]$.*

One knows[Ka-ESDE, 8.14.4] that the conjectured integrality of traces implies that both the geometric and arithmetic monodromy groups of $\mathcal{G}_q$ are finite.

Conjecture 2.2. *For each For $q = 2^{2n+1}, n \geq 1$, after pullback to $\mathbb{A}^1/\mathbb{F}_q$, the geometric and arithmetic monodromy groups of $\mathcal{G}_q$ are the group $Sz(q)$ in one of its irreducible complex representations of degree $d(q) = q_0(q-1)$.*

Remark 2.3. The Galois group $Gal(\mathbb{F}_q/\mathbb{F}_2)$ acts on $Sz(q)$ by conjugating the matrix entries (thinking of $Sz(q)$ as a subgroup of $Sp(4, q)$, cf. [Wil1]). This action preserves the isomorphism class of each of the two irreducible representations of dimension $q_0(q-1)$ (simply because the Galois group has odd order $2n+1$, so it can only act trivially on a set with two elements). Therefore each of these representations extends (in several ways, the indeterminacy being that we can twist any such extension by character of the cyclic group $Gal(\mathbb{F}_q/\mathbb{F}_2)$) to a representation of the semidirect product group $Sz(q) \rtimes Gal(\mathbb{F}_q/\mathbb{F}_2)$. It is reasonable to expect that for $\mathcal{G}_q$ on $\mathbb{A}^1/\mathbb{F}_2$, its arithmetic monodromy group is the image of this semidirect product in one of the extended representations. However, the trace function of our $\mathcal{G}_q$ on $\mathbb{A}^1/\mathbb{F}_2$ has values in $\mathbb{Q}(i)$ (and conjecturally in $\mathbb{Z}[i]$). So if our expectation is correct, there should be a unique extended representation whose trace have values in $\mathbb{Z}[i]$.

3. The Ree case

We take $\psi : (\mathbb{F}_3, +) \to \mu_3(\mathbb{Z}[\zeta_3])$ to be the one of the two nontrivial additive character of $\mathbb{F}_3$. We denote by $\chi_2 : \mathbb{F}_3^\times \to \pm 1$ the quadratic character of $\mathbb{F}_3^\times$, extended to all of $\mathbb{F}_3$ by defining $\chi_2(0) := 0$. Thus we may speak of the Artin-Schreier sheaf $\mathcal{L}_\psi$ on $\mathbb{A}^1/\mathbb{F}_3$, and of the extension by zero to $\mathbb{A}^1/\mathbb{F}_3$ of the Kummer sheaf $\mathcal{L}_{\chi_2}$ on $\mathbb{G}_m/\mathbb{F}_3$, which we will denote $j_!\mathcal{L}_{\chi_2}$ ($j : \mathbb{G}_m \subset \mathbb{A}^1$ denoting the inclusion).

We now turn to the definition of our candidate local systems. Recall that $q_0 = 3^n, n \geq 1$ and $q = 3q_0^2$. We denote by

$$t(q) := q - 3q_0 + 1$$

the order of the Coxeter torus in Ree(q), cf. [Gross]. Thus $t_{27} = 19, t_{243} = 217$. We also denote by

$$d(q) := q^2 - q + 1$$

the degree of the lowest dimensional nontrivial irreducible complex representation of Ree(q). Thus $d(27) = 703, d(243) = 58807$.

For each $q = 3q_0^2 = 3^{2n+1}$, we define the polynomial $f_q(x) \in \mathbb{F}_3[x]$ as follows.

$$f_{27}(x) := x^{703} + 2x^{11*19} + 2x^{7*19} + 2x^{19},$$
$$f_{243}(x) := x^{58807} + 2x^{29*217} + 2x^{19*217} + 2x^{217}.$$

The general pattern is

$$f_q(x) := x^{d(q)} + 2x^{(3^{n+1}+2)t(q)} + 2x^{(2*3^n+1)t(q)} + 2x^{t(q)}.$$

We then define, for each $q = 3^{2n+1}$ a rank one lisse local system on $\mathbb{G}_m/\mathbb{F}_3$

$$\mathcal{R}_q(x) := \mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(f_q(x))}.$$

We then form the Fourier Transform of its extension by zero,

$$\mathcal{F}_q := FT_\psi(j_!\mathcal{R}_q).$$

This is a rigid local system on $\mathbb{A}^1/\mathbb{F}_3$, lisse of rank $d(q)$, pure of weight one. Its trace function is given at time $t \in k$, $k/\mathbb{F}_3$ a finite extension, by the formula

$$\mathrm{Trace}(Frob_{t,k}|\mathcal{F}_q) = -\sum_{x \in k^\times} \chi_2(\mathrm{Norm}_{k/\mathbb{F}_3})(x))\psi(\mathrm{Trace}_{k/\mathbb{F}_3}(f_q(x) + tx)).$$

We then twist this local system by a suitable constant field twist.

$$\mathcal{G}_q := \mathcal{F}_q \otimes \beta^{-deg},$$

with

$$\beta := \psi(1) - \psi(-1) = 1 + 2\zeta_3 \text{ for } \zeta_3 := \psi(1)$$

the quadratic Gauss sum over $\mathbb{F}_3$.

Conjecture 3.1. *For each For $q = 3^{2n+1}, n \geq 1$, the trace function of the local system $\mathcal{G}_q$ on $\mathbb{A}^1/\mathbb{F}_3$ takes values in $\mathbb{Z}$.*

Conjecture 3.2. *For each For $q = 3^{2n+1}, n \geq 1$, after pullback to $\mathbb{A}^1/\mathbb{F}_q$, the geometric and arithmetic monodromy groups of $\mathcal{G}_q$ are the group* Ree(q) *in its irreducible complex representation of degree $d(q) = q^q - q + 1$.*

Remark 3.3. The Galois group $Gal(\mathbb{F}_q/\mathbb{F}_3)$ acts on Ree(q) by conjugating the matrix entries (thinking of Ree(q) as a subgroup of $G_2(q)$, cf. [Wil2]. This action (necessarily) preserves the isomorphism class of the unique irreducible representation of dimension $q^2 - q + 1$. Therefore this representation extends (in several ways, the indeterminacy being that we can twist any such extension by character of the cyclic group $Gal(\mathbb{F}_q/\mathbb{F}_3)$) to a representation of the semidirect product group Ree$(q) \rtimes Gal(\mathbb{F}_q/\mathbb{F}_3)$. It is reasonable to expect that on $\mathbb{A}^1/\mathbb{F}_3$, the arithmetic monodromy group is the image of this semidirect product in one of the extended representations. However, the trace function of our $\mathcal{G}_q$ on $\mathbb{A}^1/\mathbb{F}_3$ has values in $\mathbb{Q}$ (and conjecturally in $\mathbb{Z}$). So if our expectation is correct, there should be a unique extended representation whose trace have values in $\mathbb{Z}$.

4. Comments on the conjectures: relation to the work of Gross

For G either $Sz(q)$ or Ree(q), Gross [Gross] constructs a G-torsor on $\mathbb{G}_m/\mathbb{F}_q$ whose inertia group at 0 is the Coxeter torus in G. When we push out this torsor by one of the representations of G with which we are concerned, we get a local system Gr_q on $\mathbb{G}_m/\mathbb{F}_q$ of rank $d(q)$ whose geometric and arithmetic monodromy groups are G, in that representation. When we pull back Gr_q by the $t(q)$'th power map, the resultng Kummer pullback is lisse on (i.e., has a unique lisse extension to) $\mathbb{A}^1/\mathbb{F}_q$. It is this local system which we conjecture to be our $\mathcal{G}_q$.

So one test that our local system $\mathcal{G}_q$ (or equivalently $\mathcal{F}_q$) should pass is that its restriction to $\mathbb{G}_m$ descend through the $t(q)$'th power map on t-space. It is to insure this that our polynomial $f_q(x)$ is a polynomial in $x^{t(q)}$. [Notice that in both the Ree and Suzuki cases, $t(q)$ divides the degree of $f_q(x)$. In the Suzuki case, we have

$$d(q) + 1 = t(q)(q_0 + 1),$$

while in the Ree case we have

$$d(q) = t(q)(q + 3q_0 + 1).]$$

To see the descent, consider first the Suzuki case. Write the formula for the trace at a nonzero t,

$$\mathrm{Trace}(Frob_{t,k}|\mathcal{F}_q) = -\sum_{x\in k} \psi_2(\mathrm{Trace}_{W_2(k)/W_2(\mathbb{F}_2)}([x^{t(q)}, f_q(x) + tx]),$$

and make the substitution $x \mapsto x/t$. Then the trace is

$$-\sum_{x\in k} \psi_2(\mathrm{Trace}_{W_2(k)/W_2(\mathbb{F}_2)}([x^{t(q)}/t^{t(q)}, f_q(x/t) + x]),$$

in which only powers of $t^{t(q)}$ appear.

In the Ree case, remember that $t(q) = q - 3q_0 + 1$ is odd, so the formula for the trace at a nonzero t is

$$\mathrm{Trace}(Frob_{t,k}|\mathcal{F}_q) = -\sum_{x\in k^\times} \chi_2(\mathrm{Norm}_{k/\mathbb{F}_3})(x^{t(q)}))\psi(\mathrm{Trace}_{k/\mathbb{F}_3}(f_q(x)+tx)).$$

Then the same substitution $x \mapsto x/t$ gives the trace as

$$-\sum_{x\in k^\times} \chi_2(\mathrm{Norm}_{k/\mathbb{F}_3})(x^{t(q)}/t^{t(q)}))\psi(\mathrm{Trace}_{k/\mathbb{F}_3}(f_q(x/t) + x)),$$

in which only powers of $t^{t(q)}$ appear.

Our conjecture also has consequences for the local system Gr_q on $\mathbb{G}_m$. Recall that the local system $\mathcal{G}_q$ is a rigid local system on $\mathbb{A}^1$ (because it is the Fourier Transform of a rank one local system), cf. [Ka-RLS, 2.0.2 and 3.0.1]. If $\mathcal{G}_q$ is the $t(q)$'th power pullback of Gr_q, then Gr_q must itself be rigid, which does not seem obvious. Here is the argument, due to Lei Fu.

Lemma 4.1. (Lei Fu) *Over an algebraically closed field, let $j : U \subset \mathbb{P}^1$ be the inclusion of a dense open set, $\mathcal{A}$ a $\overline{\mathbb{Q}_\ell}$-local system ($\ell$ invertible in k). Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a finite morphism (i.e. a nonconstant rational* *function) which is generically étale, and which makes $V := f^{-1}(U)$ a finite stale covering of U. Call this map*

$$f_U : V \to U.$$

Denote by $J : V \subset \mathbb{P}^1$ the inclusion. Then we have the inequality of dimensions

$$dim H^1(\mathbb{P}^1, j_\star \mathcal{A}) \leq \dim H^1(\mathbb{P}^1, J_\star f_U{}^\star \mathcal{A}).$$

Proof. We have the commutation relation

$$f \circ J = j \circ f_U.$$

Because f is a finite morphism, we have

$$H^1(\mathbb{P}^1, J_\star f_U{}^\star \mathcal{A}) = H^1(\mathbb{P}^1, f_\star J_\star f_U{}^\star \mathcal{A}) = H^1(\mathbb{P}^1, j_\star f_{U\star} f_U{}^\star \mathcal{A}).$$

Now use the fact that $\overline{\mathbb{Q}_\ell}_U$ is a direct factor of $f_{U\star}\overline{\mathbb{Q}_\ell}_V$, and hence that $\mathcal{A}$ is a direct factor of $f_{U\star} f_U{}^\star \mathcal{A}$. Then $j_\star \mathcal{A}$ is a direct factor of $j_\star f_{U\star} f_U{}^\star \mathcal{A}$, and the assertion on dimensions is obvious. □

Corollary 4.2. *If $\mathcal{A}$ is irreducible and $f_U{}^\star \mathcal{A}$ is both irreducible and rigid, then $\mathcal{A}$ is rigid.*

Proof. Apply the lemma to $End(\mathcal{A})$ and the $t(q)$'th power map, remembering that for an irreducible $\mathcal{A}$, rigidity is the vanishing of $H^1(\mathbb{P}^1, j_\star End(\mathcal{A}))$. □

Another consequence for Gr_q is that $Swan_\infty(Gr_q)$ must be $q_0 + 1$ in the Suzuki case, and $q + 3q_0 + 1$ in the Ree case (simply because pullback by the $t(q)$'th power map multiplies $Swan_\infty$ by $t(q)$, and $Swan_\infty(\mathcal{G}_q)$ is $d(q) + 1$ in the Suzuki case, and is $d(q)$ in the Ree case).

In neither the Suzuki nor the Ree case do we have a conceptual explanation for the precise form of the polynomial $f_q(x)$, which we found through computer experiments. These experiments are, at least so far, compatible with the idea that our local systems $\mathcal{G}_q$ do have integral traces, that their traces are among the (very few, seven for the Suzuki groups and six for the Ree groups) traces of elements of the group G in the representation in question, and that the traces of our local systems are distributed like the traces of random elements of these groups. Caveat Emptor.

5. Numerical data: the Suzuki case

Recall that $q_0 = 2^n, q = 2^{2n+1}, n \geq 1$. In either irreducible representation of $Sz(q)$ of dimension $q_0(q-1)$, the traces attained are

$$\{-q_0 i \ , \ -q_0 \ , \ -1 \ , \ 0 \ , \ 1 \ , \ q_0 i \ , \ q_0(q-1)\}.$$

Thanks to Frank Luebeck [Lu], using CHEVIE [CH], the fraction of elements in $Sz(q)$ with these traces is known to be

$$\mathrm{Frac}_{\mathrm{Sz}}(q) :=$$

$$\{\frac{1}{2q}, \frac{1}{q^2}, \frac{q_0(q_0-1)}{2(q-2q_0+1)}, \frac{q_0^2-1}{q-1}, \frac{q_0(q_0+1)}{2(q+2q_0+1)}, \frac{1}{2q}, \frac{1}{q^2(q^2+1)(q-1)}\}.$$

We calculated, for our local systems $\mathcal{G}_q$, the traces of Frobenius obtained at the $\mathbb{F}_{q^d}$ points of $\mathbb{A}^1$ for various q and some low d. We tabulate the multiplicity with which each trace occurs, in the order listed above. Below this tabulation, we list the "expected" multiplicities, by which we that we round the table of rational numbers

$$\mathrm{Frac}_{\mathrm{Sz}}(\mathrm{q})q^d.$$

For $Sz(8)$ we have

$$d = 2,$$
$$\text{Found} = \{6, 1, 13, 27, 15, 2, 0\}$$
$$\text{Expected} = \{4, 1, 13, 27, 15, 4, 0\}.$$

$$d = 3,$$
$$\text{Found} = \{28, 9, 109, 219, 111, 36, 0\}$$
$$\text{Expected} = \{32, 8, 102, 219, 118, 32, 0\}.$$

$$d = 4,$$
$$\text{Found} = \{240, 40, 845, 1755, 975, 240, 1\}$$
$$\text{Expected} = \{256, 64, 819, 1755, 945, 256, 0\}.$$

$$d = 5,$$
$$\text{Found} = \{2080, 520, 6605, 14043, 7503, 2016, 1\}$$
$$\text{Expected} = \{2048, 512, 6554, 14043, 7562, 2048, 1\}.$$

For $Sz(32)$, we have

$$d = 2,$$
$$\text{Found} = \{20, 1, 246, 495, 250, 12, 0\}$$
$$\text{Expected} = \{16, 1, 246, 495, 250, 16, 0\}.$$

For $Sz(128)$, we have

$$d = 2,$$
$$\text{Found} = \{72, 1, 4060, 8127, 4086, 56, 0\}$$
$$\text{Expected} = \{64, 1, 4060, 8127, 4068, 64, 0\}.$$

For $Sz(512)$, we have

$$d = 1,$$
$$\text{Found} = \{1\ ,\ 0\ ,\ 136\ ,\ 255\ ,\ 120\ ,\ 0\ ,\ 0\}$$
$$\text{Expected} = \{8\ ,\ 1\ ,\ 123\ ,\ 248\ ,\ 125\ ,\ 8\ ,\ 0\}.$$

For $Sz(2048)$, we have

$$d = 1,$$
$$\text{Found} = \{0\ ,\ 0\ ,\ 496\ ,\ 1023\ ,\ 528\ ,\ 1\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 0\ ,\ 512\ ,\ 1023\ ,\ 512\ ,\ 0\ ,\ 0\}.$$

For $Sz(8192)$, we have

$$d = 1,$$
$$\text{Found} = \{0\ ,\ 0\ ,\ 259\ ,\ 507\ ,\ 233\ ,\ 1\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 0\ ,\ 250\ ,\ 500\ ,\ 250\ ,\ 0\ ,\ 0\}.$$

6. Numerical data: the Ree case

Recall that $q_0 = 3^n, q = 3^{2n+1}, n \geq 1$. In the irreducible representation of $\mathrm{Ree}(q)$ of dimension $q(q-1)$, the traces attained are

$$\{1 - q\ ,\ -1\ ,\ 0\ ,\ 1\ ,\ 3\ ,\ q(q-1)\}.$$

Thanks to Frank Luebeck [Lu], using CHEVIE [CH], the fraction of elements in $\mathrm{Ree}(q)$ with these traces is known to be

$$\mathrm{Frac}_{\mathrm{Ree}}(q) :=$$
$$\{\frac{1}{q^3}\ ,\ \frac{3}{8}\ ,\ \frac{q(q-2)}{3(q^2-q+1)}\ ,\ \frac{q^3+q^2-4}{4q^2(q-1)}\ ,\ \frac{q-3}{24(q+1)}\ ,\ \frac{1}{q^3(q^3+1)(q-1)}\}.$$

We calculated, for our local systems $\mathcal{G}_q$, the traces of Frobenius obtained at the $\mathbb{F}_{q^d}$ points of $\mathbb{A}^1$ for various q and some low d. We tabulate the multiplicity with which each trace occurs, in the order listed above. Below this tabulation, we list the "expected" multiplicities, by which we mean that we round the table of rational numbers

$$\mathrm{Frac}_{\mathrm{Ree}}(q)q^d.$$

What we found, to our complete astonishment, is that for each of the first five Ree groups $\mathrm{Ree}(3^{2n+1})$ with $1 \leq n \leq 5$, calculating traces for our $\mathcal{G}_q$ over $\mathbb{F}_q$ (i.e. with $d = 1$), we get **perfect** agreement with what is "expected", better than with some higher values of d when we could compute them. We have no explanation of this. Here is the data.

For $\mathrm{Ree}(27)$ we have

$$d = 1,$$
$$\text{Found} = \{0\ ,\ 10\ ,\ 9\ ,\ 7\ ,\ 1\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 10\ ,\ 9\ ,\ 7\ ,\ 1\ ,\ 0\}.$$

$$d = 2,$$
$$\text{Found} = \{0\ ,\ 280\ ,\ 225\ ,\ 196\ ,\ 28\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 273\ ,\ 233\ ,\ 196\ ,\ 26\ ,\ 0\}.$$

$$d = 3,$$
$$\text{Found} = \{1\ ,\ 7381\ ,\ 6300\ ,\ 5298\ ,\ 703\ ,\ 0\}$$
$$\text{Expected} = \{1\ ,\ 7381\ ,\ 6300\ ,\ 5298\ ,\ 703\ ,\ 0\}.$$

$$d = 4,$$
$$\text{Found} = \{28\ ,\ 199108\ ,\ 170325\ ,\ 143052\ ,\ 18928\ ,\ 0\}$$
$$\text{Expected} = \{27\ ,\ 199290\ ,\ 170091\ ,\ 143052\ ,\ 18980\ ,\ 0\}.$$

For Ree(243) we have

$$d = 1,$$
$$\text{Found} = \{0\ ,\ 91\ ,\ 81\ ,\ 61\ ,\ 10\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 91\ ,\ 81\ ,\ 61\ ,\ 10\ ,\ 0\}.$$

$$d = 2,$$
$$\text{Found} = \{0\ ,\ 22204\ ,\ 19521\ ,\ 14884\ ,\ 2440\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 22143\ ,\ 19601\ ,\ 14884\ ,\ 2420\ ,\ 0\}.$$

For Ree($3^7 = 2187$) we have

$$d = 1,$$
$$\text{Found} = \{0\ ,\ 820\ ,\ 729\ ,\ 547\ ,\ 91\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 820\ ,\ 729\ ,\ 547\ ,\ 91\ ,\ 0\}.$$

For Ree($3^9 = 19683$) we have

$$d = 1,$$
$$\text{Found} = \{0\ ,\ 7381\ ,\ 6561\ ,\ 4921\ ,\ 820\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 7381\ ,\ 6561\ ,\ 4921\ ,\ 820\ ,\ 0\}.$$

For Ree($3^{11} = 177147$) we have

$$d = 1,$$
$$\text{Found} = \{0\ ,\ 66430\ ,\ 59049\ ,\ 44287\ ,\ 7381\ ,\ 0\}$$
$$\text{Expected} = \{0\ ,\ 66430\ ,\ 59049\ ,\ 44287\ ,\ 7381\ ,\ 0\}.$$

7. Appendix: the limiting cases $Sz(2)$ and Ree(3)

In this section, we see what happens if we apply our constructions in the "$q_0 = 1$" case.

The group $Sz(2)$ is isomorphic to the $ax + b$ group over $\mathbb{F}_5$. It has a pair of complex conjugate linear characters of order 4.

Lemma 7.1. *For $Sz(2)$, our local system $\mathcal{G}_2$ has arithmetic and geometric monodromy groups equal to the image μ_4 of $Sz(2)$ in either of its one-dimensional representations of order 4.*

Proof. For $Sz(2)$, i.e. the case $q = 2, q_0 = 1$, we have $d(2) = q_0(q-1) = 1$ and $t(2) = q - 2q_0 + 1 = 1$. The polynomial $f_2(x)$ is x^2, the input local system is

$$\mathcal{S}_2(x) := \mathcal{L}_{\psi_2([x,0])} \otimes \mathcal{L}_{\psi(x^2)} \cong \mathcal{L}_{\psi_2([x,0])} \otimes \mathcal{L}_{\psi(x)},$$

and our local system $\mathcal{F}_2$ is $\mathcal{F}_2 := FT_\psi(\mathcal{S}_2)$. The twisting factor β_2 is $1 - i$. The local system $\mathcal{G}_2$ is lisse of rank $d(2) = 1$, so is geometrically of finite order on $\mathbb{A}^1/\mathbb{F}_2$, of some 2-power order N. Hence for some $\gamma \in \overline{\mathbb{Q}_\ell}^\times$ we have

$$\mathcal{G}_2^{\otimes N} \cong \overline{\mathbb{Q}_\ell} \otimes \gamma^{deg}.$$

But at the points $t = 0$ and $t = 1$ in $\mathbb{A}^1(\mathbb{F}_2)$, we have the equalities

$$\text{Trace}(Frob_{t=0,\mathbb{F}_2}(\mathcal{G}_2) = -(1-i)/\beta_2 = -1,$$
$$\text{Trace}(Frob_{t=1,\mathbb{F}_2}(\mathcal{G}_2) = -(1+i)/\beta_2 = -i.$$

The first shows that $\gamma = 1$, i.e., $\mathcal{G}_2$ is arithmetically of finite order. As $\mathcal{G}_2$ has traces in $\mathbb{Q}(i)$ which are roots of unity, it has traces in μ_4 The second equality shows that the arithmetic monodromy group is all of μ_4. The two inequalities together show that the ratio of the traces at the two $\mathbb{F}_2$ points is i, and hence that the geometric monodromy group is all of μ_4. ☐

The Ree group Ree(3) is not simple, but its derived group, of index 3, is the simple group $SL(2,8)$. Indeed, Ree(3) is Aut($SL(2,8)$), the semidirect product of $SL(2,8)$ with $Gal(\mathbb{F}_8/\mathbb{F}_2)$, this latter group acting entrywise on $SL(2,8)$. For Ree(3), with $q = 3, q_0 = 1$, we have

$$d(3) = 3^2 - 3 + 1 = 7, \;\; t(3) = 3 - 3 + 1 = 1, \;\; f_3(x) = x^7 + 2x^5 + 2x^3 + 2x.$$

With this input data, we form the local system

$$\mathcal{F}_3 := FT_\psi(\mathcal{L}_{\chi_2(x)} \otimes \mathcal{L}_{\psi(f_3(x))}),$$

and its twist

$$\mathcal{G}_3 := \mathcal{F}_3 \otimes \beta^{-deg}$$

for β the quadratic Gauss sum over $\mathbb{F}_3$.

The Ree group Ree(3) has three irreducible representations of degree 7, precisely one of which is orthogonally self dual. Its traces are $\{-2, -1, 0, 1, 7\}$. From the character tables in Magma, where Ree(3) is ChevalleyGroup("2G",2,3) [Mag, 65.2.1, p. 1881], we see that the fraction of elements of Ree(3) with these traces is $\{1/27, 3/8, 1/7, 4/9, 1/1512\}$ (which are also the fractions Lübeck's table gives setting $q = 3$).

Computer experiments support the following conjecture.

Conjecture 7.2. *For* Ree(3)*, the local system* $\mathcal{G}_3$ *has integer traces, and its geometric and arithmetic monodromy groups on* $\mathbb{A}^1/\mathbb{F}_3$ *are the group* Ree(3) *in its orthogonal seven-dimensional irreducible representation.*

Here is some data for Ree(3).

$$d = 6,$$

$$\text{Found} = \{21\ ,\ 280\ ,\ 112\ ,\ 315\ ,\ 1\}$$

$$\text{Expected} = \{27\ ,\ 273\ ,\ 104\ ,\ 324\ ,\ 0\}.$$

$$d = 7,$$

$$\text{Found} = \{84\ ,\ 820\ ,\ 301\ ,\ 981\ ,\ 1\}$$

$$\text{Expected} = \{81\ ,\ 820\ ,\ 312\ ,\ 972\ ,\ 1\}.$$

$$d = 8,$$

$$\text{Found} = \{252\ ,\ 2440\ ,\ 949\ ,\ 2916\ ,\ 4\}$$

$$\text{Expected} = \{243\ ,\ 2460\ ,\ 937\ ,\ 2916\ ,\ 4\}.$$

$$d = 9,$$

$$\text{Found} = \{729\ ,\ 7381\ ,\ 2812\ ,\ 8748\ ,\ 13\}$$

$$\text{Expected} = \{729\ ,\ 7381\ ,\ 2812\ ,\ 8748\ ,\ 13\}.$$

$$d = 10,$$

$$\text{Found} = \{2160\ ,\ 22204\ ,\ 8401\ ,\ 26244\ ,\ 40\}$$

$$\text{Expected} = \{2187\ ,\ 22143\ ,\ 8436\ ,\ 26244\ ,\ 39\}.$$

Given that Ree(3) contains $SL(2, 8)$ as a normal subgroup of index three, it is natural to wonder what happens if we take our local system $\mathcal{G}_3$ and pull it back by the Artin-Schreier covering $t \mapsto t^3 - t$ of $\mathbb{A}^1/\mathbb{F}_3$. Computer experiments support the following conjecture.

Conjecture 7.3. *The pullback local system*

$$\mathcal{H}_3 := [t \mapsto t^3 - t]^\star \mathcal{G}_3$$

on $\mathbb{A}^1/\mathbb{F}_3$ *has integer traces, it has* $G_{geom} = SL(2,8)$ *and it has* $G_{arith} =$ Ree(3) *in its orthogonal seven-dimensional irreducible representation. If we pull back* $\mathcal{H}_3$ *to* $\mathbb{A}^1/\mathbb{F}_{3^3}$, *it has* $G_{geom} = G_{arith} = SL(2,8)$ *in the unique irreducible seven dimensional representation of* $SL(2,8)$ *with integer traces.*

Remark 7.4. We are surprised to find, at least conjecturally, $SL(2,8)$ occurring "naturally" in characteristic 3 rather than 2.

If this conjecture is correct, it has the following equidistribution consequence, cf.[Ka-Sar, 9.7.13]. The group $SL(2,8)$ has three cosets inside Ree(3). Fix an isomorphism of the quotient $\mathrm{Ree}(3)/SL(2,8) \cong \mathbb{Z}/3\mathbb{Z}$. If we let $d \to \infty$ through d's in a fixed congruence class mod 3, the traces we find should become equidistributed according to the distribution of traces of our given representation on elements of the corresponding coset of $SL(2,8)$. Looking at the character table of Ree(3), we see that

(0) In coset 0, the subgroup $SL(2,8)$, the traces attained are

$$\{-2, -1, 0, 1, 7\},$$

and the fraction of elements with these traces is

$$\{1/9, 1/8, 3/7, 1/3, 1/504\}.$$

[These are also the traces, and fractions of occurrence, in the unique irreducible seven dimensional representation of $SL(2,8)$ with integer traces.]

($\pm$ 1) In each of the two cosets 1 and -1, the traces attained are $\{-1, 1\}$, and the fraction of elements with these traces is $\{1/2, 1/2\}$.

So when we compute traces of the pullback local system $\mathcal{H}_3$ over degree d extensions of $\mathbb{F}_3$ with $d \neq 0$ mod 3, we expect to find only $\{-1, 1\}$ as traces, and approximately equal occurrences of each. When we compute over degree d extensions with $3|d$, we expect to find the traces $\{-2, -1, 0, 1, 7\}$ in approximate fractions $\{1/9, 1/8, 3/7, 1/3, 1/504\}$.

Here is some data for $\mathcal{H}_3$, first for the coset 0, the group $SL(2,8)$.

$$d = 3,$$
$$\text{Found} = \{3\ ,\ 3\ ,\ 12\ ,\ 9\ ,\ 0\}$$
$$\text{Expected} = \{3\ ,\ 3\ ,\ 12\ ,\ 9\ ,\ 0\}.$$

$$d = 6,$$

$$\text{Found} = \{63\ ,\ 84\ ,\ 336\ ,\ 243\ ,\ 3\}$$
$$\text{Expected} = \{81\ ,\ 91\ ,\ 312\ ,\ 243\ ,\ 1\}.$$

$$d = 9,$$
$$\text{Found} = \{2187\ ,\ 2460\ ,\ 8436\ ,\ 6561\ ,\ 39\}$$
$$\text{Expected} = \{2187\ ,\ 2460\ ,\ 8436\ ,\ 6561\ ,\ 39\}.$$

Here is data for extensions $\mathbb{F}_{3^d}$ with $d \neq 0 \bmod 3$, where what we expect is equal occurrences of -1 and 1.

$$d = 1, \quad \text{Found} = \{3, 0\}.$$
$$d = 2, \quad \text{Found} = \{6, 3\}.$$
$$d = 4, \quad \text{Found} = \{36, 45\}.$$
$$d = 5, \quad \text{Found} = \{108, 135\}.$$
$$d = 7, \quad \text{Found} = \{1053, 1134\}.$$
$$d = 8, \quad \text{Found} = \{3240, 3321\}.$$
$$d = 10, \quad \text{Found} = \{29646, 29403\}.$$

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