Searching for thin groups

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Sarnak got me interested in the question of thinness.

Here is what comes up googling "sarnak photo":

We wanted to find, over $\mathbb{C}$, a one-parameter family of curves of genus $g \geq 2$ whose integer monodromy group $\Gamma \subset Sp(2g, \mathbb{Z})$ was Zariski dense in $Sp(2g)$ but thin, i.e., of infinite index in $Sp(2g, \mathbb{Z})$.

We failed miserably, and now we wonder if such families exist.

There are thin, Zariski dense subgroups of $Sp(4, \mathbb{Z})$ which occur in the monodromy of one-parameter families of three-dimensional varieties. In the known examples, there is a local monodromy which is a unipotent Jordan block of full size 4. In families of curves, any local monodromy has all its Jordan blocks of size at most 2, so we can't hope somehow to “transport" these examples to the curve setting.

On the other hand, I am told that Stephen Humphries has constructed Zariski dense thin subgroups of $Sp(4, \mathbb{Z})$ generated by finitely many transvections (unipotent pseudoreflections). Can these subgroups be realized as monodromy?

Here are three explicit one-parameter families of genus $g \geq 2$ curves one might wonder about. For all three, it is known that the monodromy is Zariski dense in $Sp(2g)$.

The first family, parameter t, is

$$y^2 = (x^{2g} - 1)(x - t),$$

or, more generally,

$$y^2 = f_{2g}(x)(x - t),$$

for any chosen polynomial $f_{2g}(x)$ of degree $2g$ having all distinct roots. Here the (finite) bad values of t are the roots of f, and at each the local monodromy is a transvection.

The second family is

$$y^2 = f_{2g+1}(x) - t,$$

for $f_{2g+1}(x)$ any chosen Morse polynomial of degree $2g + 1$ (i.e. its derivative f' has $2g$ distinct zeroes, and f separates these zeroes; equivalently f as a map of $\mathbb{C}$ to itself has $2g$ distinct critical values). Here the bad values of t are the critical values of f, and at each the local monodromy is a transvection.

The third family is

$$y^2 = x^{2g+1} + ax + b,$$

over the parameter space which is the curve in a, b space of equation

$$(n-1)^{n-1}a^n + n^n b^{n-1} = 1.$$

This parameter curve is the complement of a single point ∞ in a (projective smooth) curve of genus $(n-1)(n-2)/2$, and ∞ is the only bad point.

Sadly, it turns out that in all these families, the monodromy group is of finite index in $Sp(2g, \mathbb{Z})$.

What are the tools that allow one to prove this?

The first tool is a 1979 theorem of A'Campo: Over the parameter space Config_{2g+1} consisting of all $2g+1$-tuples $(a_1, a_2, ..., a_{2g+1})$ of complex numbers which are pairwise distinct ($a_i \neq a_j$ if $i \neq j$), the family of curves

$$y^2 = \prod_{i=1}^{2g+1} (x - a_i)$$

has (an explicitly known) monodromy group of finite index in $Sp(2g, \mathbb{Z})$

The second tool is (a special case of) the Margulis normal subgroup theorem: if $\Gamma \subset Sp(2g, \mathbb{Z})$, $g \geq 2$, is a subgroup of finite index, and if $\Gamma_1 \subset \Gamma$ is a **normal** subgroup of Γ, then either Γ_1 is itself of finite index in $Sp(2g, \mathbb{Z})$, or Γ_1 is $\{1\}$ or it is ± 1.

How do we get normal subgroups? We use the low end of the long exact homotopy sequence of a (Serre) fibration

$$E \to B, \quad \text{fibre } F,$$

which ends, if the fibre is connected, with

$$... \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1.$$

The key observation is that the image of $\pi_1(F)$ in $\pi_1(E)$ is a **normal** subgroup of $\pi_1(E)$, simply because it is the kernel of the map $\pi_1(E) \to \pi_1(B)$.

Let's apply these two tools to the first family. The map from $E := \mathit{Config}_{2g+1}$ to $B := \mathit{Config}_{2g}$ which forgets the last coordinate a_{2g+1} is a fibration. The fibre F over a chosen point $(b_1, b_2, ..., b_{2g})$ is the space

$$\mathbb{C} \setminus \{b_1, b_2, ..., b_{2g}\}$$

over which we have the one parameter family

$$y^2 = f_{2g}(x)(x - t)$$

for $f_{2g}(x) := \prod_{i=1}^{2g}(x - b_i)$.

Here the image of $\pi_1(F)$ is normal in $\pi_1(E)$, which maps onto a subgroup Γ of finite index in $Sp(2g, \mathbb{Z})$ by A'Campo. So the monodromy of our one-parameter family is a normal subgroup Γ_1 of Γ. Since we already know it is Zariski dense, it cannot be either $\{1\}$ or ± 1, so it is finite index in $Sp(2g, \mathbb{Z})$.

A variant of A'Campo's theorem concerns the family we get when we multiply out $\prod(x - a_i)$ and write it as

$$\prod(x - a_i) = x^{2g+1} + \sum_{i=1}^{2g+1} (-1)^i s_i x^{2g+1-i}.$$

Now we view the s_i as parameters, and consider the family

$$y^2 = x^{2g+1} + \sum_{i=1}^{2g+1} (-1)^i s_i x^{2g+1-i}$$

over the open set *InvertibleDiscrim* in this $\mathbb{C}^{2g+1}$ where the discriminant is nonzero. For this family we have the same monodromy group, still finite index in $Sp(2g, \mathbb{Z})$.

We now invoke one more tool. Suppose U is a smooth connected quasiprojective variety over $\mathbb{C}$ of dimension $d \geq 1$, and $V \subset U$ is a nonempty Zariski open set. Then $\pi_1(V)$ maps onto $\pi_1(U)$. [Here is a way to see this which I learned from Griffiths. It is obvious if $d = 1$. If $d \geq 2$, we reduce to the $d = 1$ curve case by successively using the Lefschetz theorem that for a general hyperplane H, $\pi_1(U \cap H)$ maps onto $\pi_1(U)$, and $\pi_1(V \cap H)$ maps onto $\pi_1(V)$

We apply this to the open set $\mathit{Morse} \cap \mathit{InvertibleDiscrim} \subset \mathit{InvertibleDiscrim}$ consisting of those polynomials which have all distinct roots and are Morse polynomials. The map

$$f(x) \mapsto f(x) - f(0)$$

maps

$$\mathit{Morse} \cap \mathit{InvertibleDiscrim} \to \mathit{Morse}_0,$$

the target being the space of monic Morse polynomials of degree $2g+1$ with vanishing constant term. This map is again a fibration, and the fibre over a Morse polynomial f is the one parameter family $f(x) - t$. So the one parameter family

$$y^2 = f(x) - t$$

has monodromy group either finite index in $Sp(2g, \mathbb{Z})$, or contained in ± 1. By its known Zariski density, we can rule out this second possibility.

To treat the third case, we infer from the second case that if f is Morse, then the two parameter family

$$y^2 = f(x) + ax + b$$

a fortiori has monodromy of finite index in $Sp(2g, \mathbb{Z})$ (since this is already true on the subfamily $a = 0$). Now for any polynomial $g(x)$ of degree $2g + 1$, $g(x) + ax$ will be Morse for all but finitely many a. [For x^{2g+1}, $x^{2g+1} + ax$ is Morse for any nonzero a.]

So for the two parameter family

$$y^2 = x^{2g+1} + ax + b,$$

the parameter space is the open set $\mathbb{A}^2[1/\Delta]$ of a, b space where the discriminant

$$\Delta = (n-1)^{n-1}a^n + n^n b^{n-1}$$

is nonzero. What happens here is that the map

$$\Delta : \mathbb{A}^2[1/\Delta] \to \mathbb{G}_m$$

is a fibration, its fibre over $\delta \in \mathbb{C}^\times$ is the curve

$$(n-1)^{n-1}a^n + n^n b^{n-1} = \delta,$$

and on each of these fibres we combine Margulis with the known Zariski density to win.

Is there any conceptual reason to think we cannot get Zariski dense thin subgroups of $Sp(2g, \mathbb{Z})$ as monodromy of families of curves (or of abelian varieties)? In the known thin $Sp(4)$ examples, there are four successive nonzero Hodge numbers $(1, 1, 1, 1)$. In the Fuchs-Meiri-Sarnak thin orthogonal examples in odd dimensions $2k + 1$, the Hodge numbers are $k, 1, k$. Perhaps having Hodge numbers g, g somehow wards off being thin in Sp(2g)?

Much remains to be done.

Happy Birthday, Peter, and many more.