Sato–Tate Equidistribution of Kurlberg–Rudnick Sums

Nicholas M. Katz

Summary We prove equidistribution results for certain exponential sums that arise in the work of Kurlberg–Rudnick on "cat maps". We show (Theorems 1 and 2) that suitable normalizations of these sums behave like the traces of random matrices in SU(2). We also show that as a suitable parameter varies, the corresponding sums are statistically independent (Theorems 3 and 4). The main tools are Deligne's Equidistribution Theorem, the Feit–Thompson Theorem, the Goursat–Kolchin–Ribet Theorem, and Laumon's Theorem of Stationary Phase.

Introduction, and Statement of Results

Fix a finite field k of **odd** characteristic p and cardinality q, a nontrivial $\mathbb{C}$–valued additive character ψ of k,

$$\psi : (k, +) \to \mathbb{C}^\times,$$

and a nontrivial $\mathbb{C}$–valued multiplicative character of $k^\times$,

$$\chi : (k, \times) \to \mathbb{C}^\times.$$

We extend χ to a function on all of k by defining $\chi(0) := 0$.

Kurlberg–Rudnick [Kur–Rud], in their study of "cat maps", encounter the the $\mathbb{C}$–valued function $H(\psi, \chi)$ on k defined by

$$H(\psi, \chi)(t) := \Sigma_{x \text{ in } k}\, \psi(x^2 + tx)\chi(x).$$

It will be convenient to consider a "normalized" version $F(\psi, \chi)$ of this function. Denote by χ_{quad} the quadratic character of $k^\times$. Recall that for any nontrivial χ, the Gauss sum $G(\psi, \chi)$ is defined by

$$G(\psi, \chi) := \Sigma_{x \text{ in } k}\, \psi(x)\chi(x).$$

It is well known that $|G(\psi, \chi)| = \mathrm{Sqrt}(q)$.

Denote by $A(\psi, \chi)$ the complex constant of absolute value q defined as the product

$$A(\psi, \chi) := \chi(-1/2)(-G(\psi, \chi))(-G(\psi, \chi_{quad})).$$

Choose a square root $B(\psi, \chi)$ of $1/A(\psi, \chi)$. With this choice, we define the $\mathbb{C}$–valued function $F(\psi, \chi)$ on k by

$$F(\psi, \chi)(t) := -H(\psi, \chi)(t)(\psi(t^2/8)B(\psi, \chi)).$$

Theorem 1 Notations as above, the function $F(\psi, \chi)$ on k takes real values which lie in the closed interval [–2, 2].

For each t in k, denote by $\theta(\psi, \chi)(t)$ in $[0, \pi]$ the unique angle for which

$$F(\psi, \chi)(t) = 2\cos(\theta(\psi, \chi)(t)).$$

Denote by μ_{ST} the Sato–Tate measure $(2/\pi)\sin^2(\theta)d\theta$ on $[0, \pi]$. Denote by $\{S_n\}_{n \geq 1}$ the orthonormal basis of $L^2([0, \pi], \mu_{ST})$ given by

$$S_n(\theta) := \sin(n\theta)/\sin(\theta).$$

We interpret $[0, \pi]$ as the space of conjugacy classes in the group SU(2), by mapping A in SU(2) to the unique $\theta(A)$ in $[0, \pi]$ for which trace(A) = $2\cos(\theta(A))$. Then the Sato–Tate measure becomes the measure induced on conjugacy classes by the (total mass one) Haar measure on SU(2). The function $S_n(\theta)$ becomes the character of the unique n–dimensional irreducible representation of SU(2). From this interpretation, and the representation theory of SU(2), we see that $S_{n+1}(\theta)$ is a monic polynomial with integer coefficients P_n of degree n in $S_2(\theta) = 2\cos(\theta)$. Moreover, the sequence $\{S_{n+1}\}_{n\geq 0}$ is obtained from the sequence $\{(2\cos(\theta))^n\}_{n\geq 0}$ by applying Gram–Schmid orthonormalization. The **Chebychev polynomials of the second kind**, U_n, defined by

$$U_n(\cos(\theta)) = S_{n+1}(\theta),$$

are thus related to our P_n by

$$U_n(u) = P_n(2u).$$

The representation theoretic interpretation of the functions S_n shows that have the integration formula

$$\int_{[0, \pi]} S_n d\mu_{ST} = \delta_{n,1}.$$

So if we expand a continuous $\mathbb{C}$–valued function f on $[0, \pi]$ into its "representation–theoretic fourier series"

$$f = \Sigma_{n\geq 1} a_n S_n,$$

then its integral against Sato–Tate measure is given by

$$\int_{[0, \pi]} f d\mu_{ST} = a_1.$$

Interlude: review of equidistribution

We now recall some basic notions of equidistribution. Given a compact Hausdorf space X and a Borel probability measure μ on X, a sequence of Borel probability measures μ_i on X is said to converge "weak *" to μ if for every continuous $\mathbb{C}$–valued function f on X, we have the integration formula

$$\int_X f d\mu = \lim_{i\to\infty} \int_X f d\mu_i.$$

If this integration formula holds for a set of test functions f_n whose finite $\mathbb{C}$–linear combinations are uniformly dense in the space of all continuous functions on X, then it holds for all continuous functions f.

In many applications, the measures μ_i arise as follows. For each i, one is given a nonempty finite set X_i, and a map $\theta_i : X_i \to X$

of sets. One takes for μ_i the average of the Dirac delta measures $\delta_{\theta_i(x)}$ as x runs over X_i:

$$\mu_i := (1/|X_i|)\Sigma_{x \text{ in } X_i} \delta_{\theta_i(x)}.$$

More concretely, for any continuous $\mathbb{C}$–valued function f on X,

$$\int_X f d\mu_i = (1/|X_i|)\Sigma_{x \text{ in } X_i} f(\theta_i(x)).$$

In this situation, if the measures μ_i converge weak * to μ, we will say that the points $\theta_i(x)$, as x varies in X_i, are "approximately equidistributed" in X for the measure μ.

In our applications below, (X, μ) will first be $([0, \pi], \mu_{ST})$, and the test functions will be the functions $S_n(\theta)$. Later (X, μ) will be the r–fold self product of $([0, \pi], \mu_{ST})$ with itself, and the test functions will be the r–fold products

$$S_{n_1, n_2,...,n_r}(\theta_1,...., \theta_r) = \Pi_j S_{n_j}(\theta_j).$$

Thus, concretely, a sequence of Borel probability measures μ_i on $[0, \pi]$ converges weak * to the Sato–Tate measure μ_{ST} if and only if

$$\lim_{i \to \infty} \int_{[0, \pi]} S_n d\mu_i = 0, \text{ for each } n \geq 2.$$

[The point is that $\int_{[0, \pi]} S_n d\mu_{ST} = 0$ for $n \geq 2$, while S_1 is the constant function 1, and so each $\int_{[0, \pi]} S_1 d\mu_i$ and $\int_{[0, \pi]} S_1 d\mu_{ST}$ is 1.]

Similarly, for any $r \geq 1$, a sequence of Borel probability measures μ_i on $[0, \pi]^r$ converges weak * to the Sato–Tate measure $(\mu_{ST})^r$ if and only if for each r–tuple $(n_1, ..., n_r)$ of strictly positive integers with $\Sigma_j n_j \geq r+1$, we have

$$\lim_{i \to \infty} \int_{[0, \pi]^r} (\Pi_j S_{n_j}(\theta_j)) d\mu_i = 0.$$

Return to Statement of Results

Given a finite field k of odd characteristic, and a pair (ψ, χ) as above, we view the formation of the angle $\theta(\psi, \chi)(t)$ as defining a map from k to $[0, \pi]$. We form the corresponding probability measure $\mu(k, \psi, \chi)$ on $[0, \pi]$, defined by

$$\mu(k, \psi, \chi) := (1/q)\Sigma_{t \text{ in } k} \delta_{\theta(\psi, \chi)(t)},$$

i.e. for any $\mathbb{C}$–valued continuous function f on $[0, \pi]$, we have

$$\int_{[0, \pi]} f d\mu(k, \psi, \chi) := (1/q)\Sigma_{t \text{ in } k} f(\theta(\psi, \chi)(t)).$$

Theorem 2 Take any sequence of data (k_i, ψ_i, χ_i) in which each k_i has characteristic at least 7, and in which $q_i := \mathrm{Card}(k_i)$ is strictly increasing. Then the angles $\{\theta(\psi_i, \chi_i)(t)\}_{t \text{ in } k_i}$ are approximately equidistributed in the interval $[0, \pi]$ with respect to the Sato–Tate measure μ_{ST}, in the sense that as $i \to \infty$, the measures $\mu(k_i, \psi_i, \chi_i)$ tend weak * to the Sato–Tate measure μ_{ST}. More precisely, for any integer $n \geq 2$, and any datum (k, ψ, χ) with k of characteristic at least 7 we have the estimate

$$\left|\int_{[0, \pi]} S_n d\mu(k, \psi, \chi)\right| = \left|(1/q)\Sigma_{t \text{ in } k} S_n(\theta(\psi, \chi)(t))\right| \leq n/\mathrm{Sqrt}(q).$$

Another way to state this last result is in terms of "semi–circle measure"

$$\mu_{scir} := (2/\pi)\mathrm{Sqrt}(1 - u^2)du$$

on the closed interval [–1, 1], which corresponds to Sato–Tate measure on [0, π], via u := cos(θ). By means of this change of variable, the functions $\{S_{n+1}(\theta)\}_{n\geq 0}$ become the Chebychev polynomials of the second kind $\{U_n(u)\}_{n\geq 0}$, and the measure $\mu(k, \psi, \chi)$ on [0. π] becomes the measure $\nu(k, \psi, \chi)$ on [–1, 1] defined by

$$\nu(k, \psi, \chi) := (1/q)\Sigma_{t \text{ in } k}\, \delta_{F(\psi, \chi)(t)/2},$$

i.e. for any $\mathbb{C}$–valued continuous function f on [–1, 1], we have

$$\int_{[-1, 1]} f d\nu(k, \psi, \chi) := (1/q)\Sigma_{t \text{ in } k}\, f(F(\psi, \chi)(t)/2).$$

Theorem 2 bis Take any sequence of data (k_i, ψ_i, χ_i) in which each k_i has characteristic at least 7, and in which $q_i := \mathrm{Card}(k_i)$ is strictly increasing. Then the real numbers $\{F(\psi, \chi)(t)/2\}_{t \text{ in } k}$ are approximately equidistributed in the interval [–1, 1] with respect to the semicircle measure μ_{scir}, in the sense that as $i \to \infty$, the measures $\nu(k_i, \psi_i, \chi_i)$ tend weak * to the semicircle measure μ_{scir}. More precisely, for any integer n ≥ 1, and any datum (k, ψ, χ) with k of characteristic at least 7, we have the estimate

$$\left|\int_{[-1, 1]} U_n d\nu(k, \psi, \chi)\right| = \left|(1/q)\Sigma_{t \text{ in } k}\, U_n(F(\psi, \chi)(t)/2)\right| \leq (n+1)/\mathrm{Sqrt}(q).$$

In the next theorem, we consider several χ's simultaneously. Fix an integer r ≥ 1. Given a finite field k of odd characteristic, a nontrivial additive character ψ of k, and r distinct nontrivial multiplicative characters $\chi_1, \chi_2, ..., \chi_r$ of $k^\times$, we define a map from k to $[0, \pi]^r$ by

$$t \mapsto \theta(\psi, \chi\text{'s})(t) := (\theta(\psi, \chi_1)(t), \theta(\psi, \chi_2)(t), ..., \theta(\psi, \chi_r)(t)).$$

We form the corresponding probability measure $\mu(k, \psi, \chi\text{'s})$ on $[0, \pi]^r$, defined by

$$\mu(k, \psi, \chi\text{'s}) := (1/q)\Sigma_{t \text{ in } k}\, \delta_{\theta(\psi, \chi\text{'s})(t)}$$

i.e. for any $\mathbb{C}$–valued continuous function f on [0, π], we have

$$\int_{[0, \pi]^r} f d\mu(k, \psi, \chi\text{'s}) := (1/q)\Sigma_{t \text{ in } k}\, f(\theta(\psi, \chi\text{'s})(t)).$$

Theorem 3 Fix r ≥ 1. Take any sequence of data $(k_i, \psi_i, \chi_i\text{'s})$ in which each k_i has characteristic at least 7, and in which $q_i := \mathrm{Card}(k_i)$ is strictly increasing. Then the r–tuples of angles

$$\{\theta(\psi_i, \chi_i\text{'s})(t)\}_{t \text{ in } k_i}$$

are approximately equidistributed in $[0, \pi]^r$ with respect to $(\mu_{ST})^r$, in the sense that as $i \to \infty$, the measures $\mu(k_i, \psi_i, \chi_i\text{'s})$ tend weak * to to $(\mu_{ST})^r$. More precisely, for any r tuple of strictly positive integers $(n_1, n_2, ..., n_r)$ with $\Sigma_j n_j \geq r+1$, and any datum $(k, \psi, \chi\text{'s})$ with k of characteristic at least 7,we have the estimate

$$|\int_{[0,\pi]^r} S_{n_1,n_2,\ldots,n_r} d\mu(k, \psi, \chi\text{'s})| = |(1/q)\Sigma_{t \text{ in } k} S_{n_1,n_2,\ldots, n_r}(\theta(\psi, \chi\text{'s})(t))| \le (\Pi_i n_i)/\mathrm{Sqrt}(q).$$

In terms of semicircle measure, the measure $(\mu_{ST})^r$ on $[0, \pi]^r$ becomes the measure $(\mu_{scir})^r$on $[-1, 1]^r$, the test functions

$$S_{n_1+1,\ldots,n_r+1}(\theta_1,\ldots, \theta_r)$$

become the functions

$$U_{n_1, \ldots, n_r}(\theta\text{'s}) := \Pi_j U_{n_j}(u_j).$$

The measures

$$\mu(k, \psi, \chi\text{'s}) := (1/q)\Sigma_{t \text{ in } k}\, \delta_{\theta(\psi, \chi\text{'s})(t)}$$

on $[0, \pi]^r$ become the measures

$$\nu(k, \psi, \chi\text{'s}) := (1/q)\Sigma_{t \text{ in } k}\, \delta_{F(\psi, \chi\text{'s})(t)/2}$$

on $[-1, 1]^r$.

Theorem 3 bis Fix $r \ge 1$. Take any sequence of data $(k_i, \psi_i, \chi_i\text{'s})$ in which each k_i has characteristic at least 7, and in which $q_i := \mathrm{Card}(k_i)$ is strictly increasing. Then the r–tuples in $[-1, 1]^r$

$$\{F(\psi, \chi\text{'s})(t)/2\}_{t \text{ in } k}$$

are approximately equidistributed in $[-1, 1]^r$ with respect to $(\mu_{scir})^r$, in the sense that as $i \to \infty$, the measures $\nu(k_i, \psi_i, \chi_i\text{'s})$ tend weak * to to $(\mu_{scir})^r$. More precisely, for any nonzero r–tuple of nonnegative integers $(n_1, n_2, \ldots, n_r)$, we have the estimate

$$|\int_{[-1,1]^r} U_{n_1,n_2,\ldots,n_r} d\nu(k, \psi, \chi\text{'s})| = |(1/q)\Sigma_{t \text{ in } k} U_{n_1,n_2,\ldots, n_r}(F(\psi, \chi\text{'s})(t)/2)|$$
$$\le (\Pi_i (n_i + 1))/\mathrm{Sqrt}(q).$$

Here is a strengthening of Theorem 3, where we vary not just χ but the pair (ψ, χ). Given ψ and χ, we denote by $\bar{\psi}$ and $\bar{\chi}$ the complex conjugate characters

$$\bar{\psi}(x) := \psi(-x) = 1/\psi(x),$$
$$\bar{\chi}(x) := \chi(x^{-1}) = 1/\chi(x).$$

Fix an integer $r \ge 1$. Given a finite field k of odd characteristic, suppose we are given r pairs

$$\{(\psi_i, \chi_i)\}_{i=1 \text{ to } r},$$

each consisting of a non–trivial additive character ψ_i and a nontrivial multiplicative character χ_i. Suppose that for all $i \ne j$, we have

$$(\psi_i, \chi_i) \ne (\psi_j, \chi_j), \text{ and } (\psi_i, \chi_i) \ne (\bar{\psi}_j, \bar{\chi}_j).$$

[Equivalently, the (ψ_i, χ_i) and their complex conjugates form 2r distinct pairs.] We define a map from k to $[0, \pi]^r$ by

$$t \mapsto \theta(\psi's, \chi's)(t) := (\theta(\psi_1, \chi_1)(t), \theta(\psi_2, \chi_2)(t), ..., \theta(\psi_r, \chi_r)(t)).$$

We form the corresponding probability measure $\mu(k, \psi's, \chi's)$ on $[0, \pi]^r$, defined by

$$\mu(k, \psi, \chi's) := (1/q)\Sigma_{t \text{ in } k}\, \delta_{\theta(\psi's, \chi's)(t)}$$

i.e. for any $\mathbb{C}$–valued continuous function f on $[0, \pi]$, we have

$$\int_{[0, \pi]^r} f d\mu(k, \psi, \chi's) := (1/q)\Sigma_{t \text{ in } k}\, f(\theta(\psi's, \chi's)(t)).$$

Theorem 4 Fix $r \geq 1$. Take any sequence of data $(k_i, \psi_i's, \chi_i's)$ as above (i.e. we are given r distinct pairs $(\psi_{i_j}, \chi_{i_j})_{j=1 \text{ to } r}$ which together with their complex conjugates form 2r distinct pairs) in which each k_i has characteristic at least 7, and in which $q_i := \mathrm{Card}(k_i)$ is strictly increasing. Then the r–tuples of angles

$$\{\theta(\psi_i's, \chi_i's)(t)\}_{t \text{ in } k_i}$$

are approximately equidistributed in $[0, \pi]^r$ with respect to $(\mu_{ST})^r$, in the sense that as $i \to \infty$, the measures $\mu(k_i, \psi_i, \chi_i's)$ tend weak * to to $(\mu_{ST})^r$. More precisely, for any r tuple of strictly positive integers $(n_1, n_2, ..., n_r)$ with $\Sigma_j n_j \geq r+1$, and any datum $(k, \psi's, \chi's)$ with k of characteristic at least 7,we have the estimate

$$|\int_{[0, \pi]^r} S_{n_1, n_2, ..., n_r} d\mu(k, \psi's, \chi's)|$$
$$= |(1/q)\Sigma_{t \text{ in } k}\, S_{n_1, n_2, ..., n_r}(\theta(\psi's, \chi's)(t))| \leq (\Pi_i n_i)/\mathrm{Sqrt}(q).$$

Remark Theorem 3 is the special case of Theorem 4 in which all the ψ_i are equal to a single ψ [The point is that $\psi \neq \overline{\psi}$, because the characteristic p is odd.]

In terms of semicircle measure, the measures

$$\mu(k, \psi's, \chi's) := (1/q)\Sigma_{t \text{ in } k}\, \delta_{\theta(\psi, \chi's)(t)}$$

on $[0, \pi]^r$ become the measures

$$\nu(k, \psi's, \chi's) := (1/q)\Sigma_{t \text{ in } k}\, \delta_{F(\psi's, \chi's)(t)/2}$$

on $[-1, 1]^r$.

The statement of Theorem 4 becomes

Theorem 4 bis Fix $r \geq 1$. Take any sequence of data $(k_i, \psi_i's, \chi_i's)$ as above (i.e. we are given r distinct pairs $(\psi_{i_j}, \chi_{i_j})_{j=1 \text{ to } r}$ which together with their complex conjugates form 2r distinct pairs) in which each k_i has characteristic at least 7, and in which $q_i := \mathrm{Card}(k_i)$ is strictly increasing. Then the r–tuples in $[-1, 1]^r$,

$$\{F(\psi's, \chi's)(t)/2\}_{t \text{ in } k},$$

are approximately equidistributed in the r–fold product $[-1, 1]^r$ with respect to the r–fold product measure $(\mu_{scir})^r$ in the sense that as $i \to \infty$, the measures $\mu(k_i, \psi_i, \chi_i\text{'s})$ tend weak * to to $(\mu_{scir})^r$. More precisely, for any nonzero r–tuple of nonnegative integers $(n_1, n_2, ..., n_r)$, we have the estimate

$$|\int_{[-1,1]^r} U_{n_1,n_2,...,n_r} d\nu(k, \psi\text{'s}, \chi\text{'s})|$$
$$= |(1/q)\Sigma_{t \text{ in } k} U_{n_1,n_2,...., n_r}(F(\psi\text{'s}, \chi\text{'s})(t)/2)|$$
$$\le (\prod_i (n_i + 1))/\mathrm{Sqrt}(q)$$

Proofs of the theorems

Let us fix the finite field $k = \mathbb{F}_q$. For any pair (ψ, χ) consisting of a nontrivial additive character ψ and a nontrivial multiplicative character χ, both ψ and χ take values in the field $\mathbb{Q}(\xi_p, \xi_{q-1})$, viewed as a subfield of $\mathbb{C}$. The quantity $A(\psi, \chi)$ is an algebraic integer in $\mathbb{Q}(\xi_p, \xi_{q-1})$, which is a unit outside of p. If we adjoin to $\mathbb{Q}(\xi_p, \xi_{q-1})$ the square roots $B(\psi, \chi)$ of the $1/A(\psi, \chi)$ for all the finitely many such pairs (ψ, χ), we get a finite extension $F/\mathbb{Q}$ inside $\mathbb{C}$, in which the $B(\psi_i, \chi_i)$ are algebraic numbers, and units outside of p. The functions $H(\psi, \chi)$ and $F(\psi, \chi)$ take values in the number field K.

Now pick a prime number $\ell \neq p$, an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field $\mathbb{Q}_\ell$ of ℓ–adic numbers, and an embedding of the number field F into $\overline{\mathbb{Q}}_\ell$. Extend this embedding to a field isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$. By means of this isomorphism, we may and will view the characters ψ and χ, and the functions $H(\psi, \chi)$ and $F(\psi, \chi)$, as taking values in $\overline{\mathbb{Q}}_\ell$. The quantity $B(\psi, \chi)$ is an ℓ–adic unit in $\overline{\mathbb{Q}}_\ell$.

On the affine line $\mathbb{A}^1 \otimes k$, we have the Artin–Schreier sheaf $\mathcal{L}_\psi$ and the (extension by zero across 0 of) the Kummer sheaf $\mathcal{L}_\chi$. The $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{L}_{\psi(x^2)} \otimes \mathcal{L}_{\chi(x)}$ on $\mathbb{A}^1 \otimes k$ is lisse on $\mathbb{G}_m \otimes k$ of rank one, and it vanishes at x=0. Its naive Fourier Transform $\mathrm{NFT}_\psi(\mathcal{L}_{\psi(x^2)} \otimes \mathcal{L}_{\chi(x)})$, cf. [Ka–GKM, 8.2], will be denoted $\mathcal{H}(\psi, \chi)$:

$$\mathcal{H}(\psi, \chi) := \mathrm{NFT}_\psi(\mathcal{L}_{\psi(x^2)} \otimes \mathcal{L}_{\chi(x)}).$$

Because $\mathcal{L}_{\psi(x^2)} \otimes \mathcal{L}_{\chi(x)}$ is a geometrically irreducible middle extension on $\mathbb{A}^1$ which is pure of weight zero, lisse of rank one on $\mathbb{G}_m$, ramified but tame at 0 and with Swan conductor 2 at ∞, its naive Fourier Transform $\mathcal{H}(\psi, \chi)$ is a lisse $\overline{\mathbb{Q}}_\ell$–sheaf of rank 2 on $\mathbb{A}^1$, which is geometrically irreducible and pure of weight one. Its trace function is given as follows. For a finite extension E/k, denote by ψ_E (resp. χ_E) the nontrivial character of E obtained by composing ψ (resp. χ) with the relative trace $\mathrm{Trace}_{E/k}$ (resp. the relative norm $\mathrm{Norm}_{E/k}$). For any point t in $E = \mathbb{A}^1(E)$, we have

$$\mathrm{Trace}(\mathrm{Frob}_{t,E} \mid \mathcal{H}(\psi, \chi)) = -\Sigma_{x \text{ in } E}\, \psi_E(x^2 + tx)\chi_E(x).$$

In particular, for t in k, we have

$$\mathrm{Trace}(\mathrm{Frob}_{t,k} \mid \mathcal{H}(\psi, \chi)) = -H(\psi, \chi)(t).$$

Now define a second geometrically irreducible lisse $\overline{\mathbb{Q}}_\ell$–sheaf of rank 2 on $\mathbb{A}^1\otimes k$, $\mathcal{F}(\psi, \chi)$, now pure of weight zero, to be the following twist of $\mathcal{H}(\psi, \chi)$:

$$\mathcal{F}(\psi, \chi) := \mathcal{H}(\psi, \chi)\otimes\mathcal{L}_{\psi(t^2/8)}\otimes B(\psi, \chi)^{\deg}.$$

For any finite extension E/k, and any point t in E, we have

$$\mathrm{Trace}(\mathrm{Frob}_{t,E} \mid \mathcal{F}(\psi, \chi))$$
$$= (\mathrm{Trace}(\mathrm{Frob}_{t,k} \mid \mathcal{H}(\psi, \chi))\psi_E(t^2/8)B(\psi,\chi)^{\deg(E/k)}.$$

In particular, for t in k, we have

$$\mathrm{Trace}(\mathrm{Frob}_{t,k} \mid \mathcal{F}(\psi, \chi)) = F(\psi, \chi)(t).$$

Lemma 1 The lisse rank two sheaf $\mathcal{F}(\psi, \chi)$ on $\mathbb{A}^1\otimes k$ has trivial determinant. Equivalently, the determinant of the lisse rank two sheaf $\mathcal{H}(\psi, \chi)$ on $\mathbb{A}^1\otimes k$ is given by

$$\det(\mathcal{H}(\psi, \chi)) \cong \mathcal{L}_{\psi(-t^2/4)}\otimes A(\psi,\chi)^{\deg}.$$

proof By Chebotarov, it suffices to prove that for any finite extension E/k, and for any t in E, we have

$$\det(\mathrm{Frob}_{t,E} \mid \mathcal{H}(\psi, \chi)) = \mathcal{L}_{\psi_E(-t^2/4)}\otimes A(\psi,\chi)^{\deg(E/k)}.$$

By the Hasse–Davenport theorem, the Gauss sum $G(\psi_E, \chi_E)$ over E is related to the Gauss sum $G(\psi, \chi)$ over k by

$$(-G(\psi_E, \chi_E)) = (-G(\psi, \chi))^{\deg(E/k)}.$$

In view of the definition of $A(\psi, \chi)$, we have

$$A(\psi_E, \chi_E) = A(\psi,\chi)^{\deg(E/k)}.$$

So it is the same to prove

$$\det(\mathrm{Frob}_{t,E} \mid \mathcal{H}(\psi_E, \chi_E)) = \mathcal{L}_{\psi_E(-t^2/4)}\otimes A(\psi_E,\chi_E).$$

So we are reduced to proving universally that for any t in k, we have

$$\det(\mathrm{Frob}_{t,k} \mid \mathcal{H}(\psi, \chi)) = \mathcal{L}_{\psi(-t^2/4)}\otimes A(\psi,\chi).$$

For this we use tha classical Hasse–Davenport argument, cf. [Ka–MG, p. 53]. From the definition of $\mathcal{H}(\psi, \chi)$ as a naive Fourier Transform, we have

$$\det(1 - T\mathrm{Frob}_{t,k} \mid \mathcal{H}(\psi, \chi))$$
$$= \det(1 - T\mathrm{Frob}_k \mid H^1_c(\mathbb{G}_m\otimes\overline{k}, \mathcal{L}_{\psi(x^2 + tx)}\otimes\mathcal{L}_{\chi(x)}))$$

As this H^1_c is the only nonvanishing cohomology group, the Lefschetz Trace formula expresses the L–function on $\mathbb{G}_m\otimes k$ with coefficients in $\mathcal{L}_{\psi(x^2 + tx)}\otimes\mathcal{L}_{\chi(x)}$ as

$$L(\mathbb{G}_m\otimes k, \mathcal{L}_{\psi(x^2 + tx)}\otimes\mathcal{L}_{\chi(x)})(T)$$
$$= \det(1 - T\mathrm{Frob}_k \mid H^1_c(\mathbb{G}_m\otimes\overline{k}, \mathcal{L}_{\psi(x^2 + tx)}\otimes\mathcal{L}_{\chi(x)})).$$

Because this H^1_c has dimension 2, we obtain the determinant in question as the coefficient of T^2 in the power series expansion of the L–function:

$\det(\mathrm{Frob}_{t,k} \mid \mathcal{H}(\psi, \chi))$

$= \text{coef. of } T^2 \text{ in } L(\mathbb{G}_m \otimes k, \mathcal{L}_{\psi(x^2 + tx)} \otimes \mathcal{L}_{\chi(x)})(T).$

From the additive expression of this abelian L–function as a sum over all effective divisors on $\mathbb{G}_m \otimes k$, i.e. over all monic polynomials in k[X] with nonzero constant term, we see that for any integer $d \geq 1$, the coefficient of T^d is

$\sum_{\text{monic f of deg. d, } f(0)\neq 0} \psi(\sum_{\text{roots } \alpha \text{ of f}} (\alpha^2 + t\alpha))\chi(\prod_{\text{roots } \alpha \text{ of f}} (\alpha))$

Denote by $S_i(f)$ and by $N_i(f)$ the elementary and the Newton symmetric functions of the roots of f. Then the coefficient of T^d is

$$\sum_{\text{monic f of deg. d, } f(0)\neq 0} \psi(N_2(f) + tS_1(f))\chi(S_d(f)).$$

The expression of N_2 in terms of S_1 and S_2 is

$$N_2 = (S_1)^2 - 2S_2.$$

So all in all we find that the coefficient of T^d is

$$\sum_{\text{monic f of deg. d, } f(0)\neq 0} \psi(S_1(f))^2 + tS_1(f) - 2S_2(f))\chi(S_d(f)).$$

Now a monic f of degree d with $f(0) \neq 0$ is precisely given by its coefficients, which are the elementary symmetric functions of its roots:

$$f(X) = X^d - S_1(f)X^{d-1} + S_2(f)X^{d-2} + \ldots + (-1)^d S_d(f).$$

So we may write the coefficient of T^d as

$$\sum_{s_1, s_2, \ldots s_d \text{ in k, } s_d \neq 0} \psi(s_1{}^2 + ts_1 - 2s_2)\chi(s_d).$$

This expression shows that for $d > 2$ the coefficient of T^d vanishes (because the sum of $\chi(s_d)$ over nonzero s_d vanishes), as it must. The coefficient of T^2 is

$$\sum_{s_1, s_2 \text{ in k, } s_2 \neq 0} \psi((s_1{}^2 + ts_1 - 2s_2)\chi(s_2)$$

$$= (\sum_{s_1 \text{ in k}} \psi(s_1{}^2 + ts_1))(\sum_{s_2 \text{ in } k^\times} \psi(-2s_2)\chi(s_2)).$$

The second factor is $\chi(-1/2)G(\psi, \chi)$, and the first factor is

$$\sum_{s_1 \text{ in k}} \psi(s_1{}^2 + ts_1) = \sum_{s_1 \text{ in k}} \psi((s_1 + t/2)^2 - t^2/4)$$

$$= \psi(- t^2/4)\sum_{s \text{ in k}} \psi(s^2)$$

$$= \psi(- t^2/4)G(\psi, \chi_{\text{quad}}).$$

Putting this all together, we find that $\det(\mathrm{Frob}_{t,k} \mid \mathcal{H}(\psi, \chi))$, the coefficient of T^2 in the L function, is indeed equal to

$$\chi(-1/2)G(\psi, \chi)\psi(- t^2/4)G(\psi, \chi_{\text{quad}}) = \psi(-t^2/4)A(\psi, \chi),$$

as asserted. QED

Lemma 2 For $p \geq 7$, the lisse rank two sheaf $\mathcal{F}(\psi, \chi)$ on $\mathbb{A}^1 \otimes k$ has geometric monodromy group G_{geom} equal to SL(2), and under the ℓ–adic representation ρ of $\pi_1 := \pi_1(\mathbb{A}^1 \otimes k)$ corresponding to $\mathcal{F}(\psi, \chi)$, we have $\rho(\pi_1) \subset G_{geom}(\overline{\mathbb{Q}}_\ell)$.

proof We have already proven that $\mathcal{F}(\psi, \chi)$ has trivial determinant, so we trivially have the inclusions

$$\rho(\pi_1) \subset SL(2)(\overline{\mathbb{Q}}_\ell)$$

and

$$G_{geom} \subset SL(2).$$

So it remains only to prove that G_{geom} contains SL(2). As the sheaf $\mathcal{F}(\psi, \chi)$ is geometrically irreducible and starts life on $\mathbb{A}^1 \otimes k$, its G_{geom} is a semisimple subgroup of GL(2). So its identity component $(G_{geom})^0$, being a connected semisimple subgroup of GL(2), is either the group SL(2), or it is the trivial group. So either G_{geom} contains SL(2), or G_{geom} is a finite irreducible subgroup Γ of GL(2, $\overline{\mathbb{Q}}_\ell$). For $p \geq 7$, the second case cannot occur, thanks to the $n = 2$ case of the Feit–Thompson theorem [F–T]: for any $n \geq 2$, any finite subgroup Γ of GL(n, $\overline{\mathbb{Q}}_\ell$) and any prime $p > 2n+1$, any p–Sylow subgroup Γ_1 of Γ is both normal and abelian. Our Γ is a finite quotient of $\pi_1(\mathbb{A}^1 \otimes \overline{k})$, so it has no nontrivial quotients of order prime to p. The quotient Γ/Γ_1 is prime to p, hence trivial, and hence $\Gamma = \Gamma_1$. Then Γ is abelian, which is impossible since it is an irreducible subgroup of GL(2, $\overline{\mathbb{Q}}_\ell$). QED

Lemma 3 Let (ψ_i, χ_i) for i=1,2 be two pairs, each consisting of a nontrivial additive character ψ_i and a nontrivial multiplicative character χ_i. Put $\mathcal{F}_i := \mathcal{F}(\psi_i, \chi_i)$. Suppose that $(\psi_1, \chi_1) \neq (\psi_2, \chi_2)$ and that $(\psi_1, \chi_1) \neq (\overline{\psi}_2, \overline{\chi}_2)$ Then for any lisse rank one $\overline{\mathbb{Q}}_\ell$–sheaf L on $\mathbb{A}^1 \otimes \overline{k}$, the sheaves $L \otimes \mathcal{F}_1$ and $\mathcal{F}_2$ are not geometrically isomorphic (i.e., isomorphic as lisse $\overline{\mathbb{Q}}_\ell$–sheaves on $\mathbb{A}^1 \otimes \overline{k}$) and the sheaves $L \otimes \mathcal{F}_1$ and $(\mathcal{F}_2)^\vee$ are not geometrically isomorphic.

proof Since $\mathcal{F}_2$ has $G_{geom} = SL(2)$, $\mathcal{F}_2$ is geometrically self–dual, so it suffices to show that $L \otimes \mathcal{F}_1$ and $\mathcal{F}_2$ are not geometrically isomorphic. Since $\mathcal{F}_i$ is a twist of $\mathcal{H}_i := \mathcal{H}(\psi_i, \chi_i)$ by a lisse rank one sheaf, it suffices to show that for any lisse rank one $\overline{\mathbb{Q}}_\ell$–sheaf L on $\mathbb{A}^1 \otimes \overline{k}$, the sheaves $L \otimes \mathcal{H}_1$ and $\mathcal{H}_2$ are not geometrically isomorphic.

If L is tame at ∞, then L, being lisse on $\mathbb{A}^1 \otimes \overline{k}$, is trivial. So in this case we must show that $\mathcal{H}_1$ is not geometrically isomorphic to $\mathcal{H}_2$. We know that $\det(\mathcal{H}_i) = \mathcal{L}_{\psi_i(-t^2/4)}$. So if $\psi_1 \neq \psi_2$, then $\mathcal{H}_1$ and $\mathcal{H}_2$ have non–isomorphic determinants. Indeed, if if $\psi_1 \neq \psi_2$, then there exists an

$\alpha \neq 1$ in $k^\times$ for which $\psi_1(x) = \psi_2(\alpha x)$, and so

$$\det(\mathcal{H}_1)\otimes(\det(\mathcal{H}_2)^\vee \cong \mathcal{L}_{\psi_1((\alpha-1)t^2/4)}$$

is geometrically nontrivial, because it has Swan conductor two at ∞.

We next recover χ_i from $\mathcal{H}_i$. For this, we recall that

$$\mathcal{H}_i := \mathrm{NFT}(\mathcal{L}_{\psi_i(x^2)}\otimes\mathcal{L}_{\chi_i(x)}).$$

Laumon's stationary phase decomposition [Lau–TF] of $\mathcal{H}_i(\infty)$ (:= $\mathcal{H}_i$ as a representation of the inertia group $I(\infty)$) has the form

$$\mathcal{H}_i(\infty) = \mathcal{L}_{\overline{\chi}_i} \oplus \mathcal{M}_i$$

with $\mathcal{M}_i$ a one–dimensional representation of $I(\infty)$ of Swan conductor two. [In the notation of [Ka–ESDE, 7.4.1],

$$\mathcal{L}_{\overline{\chi}_i} = \mathrm{FTloc}(0, \infty)(\mathcal{L}_{\psi_i(x^2)}\otimes\mathcal{L}_{\chi_i(x)}),$$
$$\mathcal{M}_i = \mathrm{FTloc}(\infty, \infty)(\mathcal{L}_{\psi_i(x^2)}\otimes\mathcal{L}_{\chi_i(x)}).]$$

Looking at the determinant of $\mathcal{H}_i(\infty)$, we see that the above decomposition of $\mathcal{H}_i(\infty)$ is

$$\mathcal{H}_i(\infty) \cong \mathcal{L}_{\overline{\chi}_i} \oplus \mathcal{L}_{\chi_i}\otimes\det(\mathcal{H}_i)$$
$$\cong \mathcal{L}_{\overline{\chi}_i} \oplus \mathcal{L}_{\chi_i}\otimes\mathcal{L}_{\psi_i(-t^2/4)}.$$

Thus we recover χ_i from $\mathcal{H}_i$ from looking at the tame part of $\mathcal{H}_i(\infty)$.
So $\chi_1 \neq \chi_2$, then $\mathcal{H}_1$ cannot be geometrically isomorphic to $\mathcal{H}_2$.

Thus, if either $\psi_1 \neq \psi_2$, or if $\chi_1 \neq \chi_2$, then $\mathcal{H}_1$ is not geometrically isomorphic to $\mathcal{H}_2$.

Suppose now that L is not tame at ∞, but that $L\otimes\mathcal{H}_1 \cong \mathcal{H}_2$ geometrically. Looking at $I(\infty)$ representations, we have

$$L\otimes\mathcal{H}_1(\infty) \cong L\otimes\mathcal{L}_{\overline{\chi}_1} \oplus L\otimes\mathcal{L}_{\chi_1}\otimes\mathcal{L}_{\psi_1(-t^2/4)}$$

while

$$\mathcal{H}_2(\infty) \cong \mathcal{L}_{\overline{\chi}_2} \oplus \mathcal{L}_{\chi_2}\otimes\mathcal{L}_{\psi_2(-t^2/4)}.$$

There is at most one decomposition of a two–dimensional $I(\infty)$ representation as the sum of a tame character and of a nontame character. Since L is not tame at ∞, $L\otimes\mathcal{L}_{\overline{\chi}_1}$ is not tame at ∞. So in matching the terms, we must have

$$L\otimes\mathcal{L}_{\overline{\chi}_1} \cong \mathcal{L}_{\chi_2}\otimes\mathcal{L}_{\psi_2(-t^2/4)},$$

i.e.,

$$L\otimes\mathcal{L}_{\psi_2(t^2/4)} \cong \mathcal{L}_{\chi_2}\otimes\mathcal{L}_{\chi_1}$$

as $I(\infty)$–representation. Thus $L\otimes\mathcal{L}_{\psi_2(t^2/4)}$ is tame at ∞. As $L\otimes\mathcal{L}_{\psi_2(t^2/4)}$ is lisse on $\mathbb{A}^1\otimes\overline{k}$, it is trivial. Thus we get

$$L \cong \mathcal{L}_{\psi_2(-t^2/4)}, \text{ and } \chi_1 = \overline{\chi}_2.$$

Interchanging the indices and repeating the argument, we get

$$L^\vee \cong \mathcal{L}_{\psi_1(-t^2/4)}.$$

Thus we have $\mathcal{L}_{\psi_2(-t^2/4)} \cong \mathcal{L}_{\psi_1(t^2/4)}$. From this we conclude that $\psi_1 = \overline{\psi}_2$. So if L is not tame, the existence of a geometric isomorphism $L\otimes\mathcal{H}_1 \cong \mathcal{H}_2$ implies that $(\psi_1, \chi_1) = (\overline{\psi}_2, \overline{\chi}_2)$. QED

Lemma 4 For any pair (ψ, χ) consisting of a nontrivial additive character ψ and a nontrivial multiplicative character χ, the lisse sheaf $\mathcal{F} := \mathcal{F}(\psi, \chi)$ as $I(\infty)$–representation is the direct sum of two inverse characters, each of Swan conductor two:

$$\mathcal{F}(\infty) \cong \mathcal{L}_{\overline{\chi}}\otimes\mathcal{L}_{\psi(t^2/8)} \oplus \mathcal{L}_\chi\otimes\mathcal{L}_{\psi(-t^2/8)}.$$

proof Indeed, we have seen that $\mathcal{H} := \mathcal{H}(\psi, \chi)$ as $I(\infty)$–representation is given by

$$\mathcal{H}(\infty) \cong \mathcal{L}_{\overline{\chi}} \oplus \mathcal{L}_\chi\otimes\det(\mathcal{H})$$
$$\cong \mathcal{L}_{\overline{\chi}} \oplus \mathcal{L}_\chi\otimes\mathcal{L}_{\psi(-t^2/4)}.$$

Therefore $\mathcal{F}$, being an $\alpha^{\deg}$ twist of $\mathcal{F}\otimes\mathcal{L}_{\psi(t^2/8)}$, has $I(\infty)$–representation given by

$$\mathcal{F}(\infty) \cong \mathcal{L}_{\overline{\chi}}\otimes\mathcal{L}_{\psi(t^2/8)} \oplus \mathcal{L}_\chi\otimes\mathcal{L}_{\psi(-t^2/8)}. \text{ QED}$$

Lemma 5 Fix an integer $r \geq 1$. Suppose that the characteristic p of k is at least 7. Suppose we are given r pairs $\{(\psi_i, \chi_i)\}_{i=1 \text{ to } r}$, each consisting of a non–trivial additive character ψ_i and a nontrivial multiplicative character χ_i. Suppose that for all $i \neq j$, we have

$$(\psi_i, \chi_i) \neq (\psi_j, \chi_j), \text{ and } (\psi_i, \chi_i) \neq (\overline{\psi}_j, \overline{\chi}_j).$$

For each i =1 to r, put

$$\mathcal{F}_i := \mathcal{F}(\psi_i, \chi_i),$$

and denote by

$$\rho_i : \pi_1(\mathbb{A}^1\otimes k) \to SL(2, \overline{\mathbb{Q}}_\ell)$$

the ℓ–adic representation which $\mathcal{F}_i$ "is". Consider the lisse $\overline{\mathbb{Q}}_\ell$–sheaf $\mathcal{G}$ on $\mathbb{A}^1\otimes k$, defined as the direct sum

$$\mathcal{G} := \oplus_{i=1 \text{ to } r} \mathcal{F}_i.$$

Denote by

$$\rho := \oplus_{i=1 \text{ to } r} \rho_i. : \pi_1(\mathbb{A}^1\otimes k) \to \prod_{i=1 \text{ to } r} SL(2, \overline{\mathbb{Q}}_\ell)$$

the ℓ–adic representation which $\mathcal{G}$ "is". Then the group G_{geom} for $\mathcal{G}$ is the largest possible, namely $\prod_{i=1 \text{ to } r} SL(2)$.

proof By Lemma 2, the geometric monodromy group G_i of each $\mathcal{F}_i$ is SL(2). So the geometric monodromy group G for $\mathcal{G}$ is a closed subgroup of $\prod_{i=1 \text{ to } r} G_i$ which maps onto each factor. By the Goursat–Kolchin–Ribet theorem [Ka–ESDE, 1.8.2], it results from Lemma 3 that $G^{0,\mathrm{der}}$ is the full product $\prod_{i=1 \text{ to } r} G_i^{0,der}$. Since each G_i is SL(2), $G^{0,der}$ is the full product $\prod_{i=1 \text{ to } r}$ SL(2). Since G is in any case a subgroup of this product, we have $G = \prod_{i=1 \text{ to } r}$ SL(2). QED

Theorem 4 now result immediately from Deligne's Equidisribution Theorem [De–Weil II, 3.5], cf. [Ka–GKM, 3.6, 3.6.3], [Ka–Sar, 9.2.5, 9.2.6], applied to the sheaf $\mathcal{G}$ of Lemma 5. A maximal compact subgroup K of $\prod_{i=1 \text{ to } r} SL(2, \mathbb{C})$ is the product group $\prod_{i=1 \text{ to } r}$ SU(2). Its space of conjugacy classes is the product space $[0, \pi]^r$, with measure the r–fold self–product of the Sato–Tate measure μ_{ST}. Its irreducible representations are precisely the tensor products of irreducible representations of the factors. For the i'th factor SU(2), denote by std(i) its standard two–dimensional representation. There is one irreducible representation of the i'th factor of each dimension $n \geq 1$, given by $\mathrm{Symm}^{n-1}(\mathrm{std(i)})$. Its character is the function $S_n(\theta)$. So the **nontrivial** irreducible representations of the product group

$$\prod_{i=1 \text{ to } r} SU(2)$$

are the tensor products

$$\otimes_{i=1 \text{ to } r} \mathrm{Symm}^{n_i-1}(\mathrm{std(i)})$$

with all $n_i \geq 1$ and at least one $n_i > 1$. This representation has dimension $\prod_{i=1 \text{ to } r} n_i$. Its character is $\prod_{i=1 \text{ to } r} S_{n_i}(\theta_i)$.

The sheaf $\mathcal{G}$ of Lemma 5 has all its ∞–slopes equal to two. For each point t in k, the image $\rho(\mathrm{Frob}_{t,k})^{ss}$ in $\prod_{i=1 \text{ to } r} SL(2, \overline{\mathbb{Q}}_\ell)$, when viewed in $\prod_{i=1 \text{ to } r} SL(2, \mathbb{C})$ via the chosen field isomorphism of $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$, is conjugate in $\prod_{i=1 \text{ to } r} SL(2, \mathbb{C})$ to an element of $K = \prod_{i=1 \text{ to } r} SU(2)$, which is itself well–defined up to conjugacy in K. The resulting conjugacy class is none other that the r–tuple

$$(\theta(\psi_1, \chi_1)(t), \theta(\psi_2, \chi_2)(t), ..., \theta(\psi_r, \chi_r)(t)).$$

So Theorem 4 is just Deligne's Equidisribution Theorem applied to $\mathcal{G}$. The "more precisely" estimate results from [Ka–GKM, 3.6.3] and the fact that $\mathcal{G}$ is lisse on $\mathbb{A}^1$ and has highest ∞–slope two.

In more concrete terms, each $F(\psi_i, \chi_i)(t)$ is the trace of of a conjugacy class in SU(2), hence is real and lies in [–2, 2]. This gives Theorem 1. Theorem 2 is the special case r=1 of Theorem 4, and Theorem 3 is the special case "all ψ_i equal" of Theorem 4.

References

[De–Weil II] Deligne, P., La conjecture de Weil II, Pub. Math. I.H.E.S. 52 (1981), 313–428.

[F–T] Feit, W., and Thompson, J., On groups which have a faithful representation of degree less that (p–1)/2, Pacific Math. J. 4 (1961), 1257–1262.

[Ka−ESDE] Katz, N.,*Exponential sums and differential equations*, Annals of Math. Study 124, Princeton Univ. Press, 1990.

[Ka−GKM] Katz, N.,*Gauss sums, Kloosterman sums, and monodromy groups,* Annals of Math. Study 116, Princeton Univ. Press, 1988.

[Ka−MG] Katz, N., On the monodromy groups attached to certain families of exponential sums, Duke Math. J. 54 No. 1 (1987), 41−56.

[Ka−Sar] Katz, N., and Sarnak, P., *Random Matrices, Frobenius Eigenvalues, and Monodromy*, A.M.S. Colloquium Publications 45, 1999.

[Kol] Kolchin, E., Algebraic groups and algebraic independence, Amer. J. Math. 90 (1968), 1151−1164.

[Kur−Rud] Kurlberg, P., and Rudnick, Z., Value distribution for eigenfunctions of desymmetrized quantum maps, preprint, December, 2000.

[Lau−TF] Laumon, G., Transformation de Fourier, constantes d'équations fonctionelles et conjecture de Weil, Pub. Math. I.H.E.S. 65 (1987), 131−210.

[Ri] Ribet, K., Galois action on division points of abelian varieties with real multiplication, Amer. J. Math. 98 (1976), 751−805.