Remarks on polynomial count varieties

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1 Introduction

In this short note we prove a couple of facts about polynomial count varieties, answering natural questions that they raise. A *polynomial count* X variety is essentially one for which its number of points over finite fields $\mathbb{F}_q$ is given by a polynomial $\#X(\mathbb{F}_q) = C(q)$. Well-known examples include affine or projective space (or more generally the Grassmanian) and other standard varieties. For the general definition of "polynomial count", and the connection of $C(q)$ with the mixed Hodge structure of X see [12][Appendix, pp. 617-618]. There we began with $X/\mathbb{C}$ a separated scheme of finite type, and a polynomial $C[t] \in \mathbb{Z}[t]$. We said

that $X/\mathbb{C}$ was "polynomial count", with polynomial $C(t)$, if for some "spreading out" X_R/R of $X/\mathbb{C}$ to a subring $R \subset \mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra, we had:

For every finite field $\mathbb{F}_q$ and every ring homomorphism $\phi : R \to \mathbb{F}_q$,

$$\#X_R(\mathbb{F}_q) = C(q).$$

Notice that even if our $X/\mathbb{C}$ has a descent to $\mathbb{Z}$ and is polynomial count, it may not be polynomial count over $\mathbb{Z}$ in any naive sense. Simplest example: $x^2 + 1 = 0$ is polynomial count over $\mathbb{Z}[i, 1/2]$, with C the constant polynomial 2, but it is not polynomial count over $\mathbb{Z}$.

In this paper, we will be concerned with the following situation. We are given a separated scheme $X/\mathbb{Z}$ of finite type over $\mathbb{Z}$, and $C[t] \in \mathbb{Z}[t]$ a polynomial. Given an integer $D \geq 1$, We will say that $X/\mathbb{Z}$ is polynomial count "outside D" if, for every finite field $\mathbb{F}_q$ in which D is invertible, we have

$$\#X(\mathbb{F}_q) = C(q).$$

We will say $X/\mathbb{Z}$ is polynomial count if there exists such a D.

The two questions we address are the following

- 1) If $X/\mathbb{C}$ is smooth, polynomial count with $C(t) = t^n$ for some n, is X isomorphic to affine space $\mathbb{A}^n$?
- 2) If $X/\mathbb{C}$ is polynomial count, is it true that its Hodge numbers in a given graded piece of $H^i_c(X)$ of fixed weight $p + q$ satisfy $h^{p,q} = 0$ unless $p = q$?

We show that in both cases the answer is no.

2 Basic examples and open questions

Let $H \in \mathbb{C}[x_1, \ldots, x_N]$ be a polynomial with $N \geq 1$. To state the main result in this section we need the following combinatorial constants. For a subset $S \subseteq \{1, \ldots, N\}$ let Δ_S be the Newton polytope of the polynomial obtained by setting $x_i = 0$ in H for every $i \notin S$, if the specialization is non-zero. Otherwise, set $\Delta_S = \emptyset$ if the specialization is zero. Then define

$$f_{r,n} := \#\{S \mid \#S = r, \dim \Delta_S = n - 1\}, \tag{1}$$

with $\dim \Delta_S = -1$ if $\Delta_S = \emptyset$.

Let

$$\mathcal{F}_\Delta(x, y) := \sum_{r,n=0}^{N} f_{r,n} x^r y^n$$

be the placeholder polynomial for these constants.

For convenience we let $\sigma_n \subseteq \mathbb{Z}^n$ be the standard simplex of dimension $n - 1$, convex hull of $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. We will say two lattice polytopes in $\mathbb{Z}^n$ are *equivalent* if there is an invertible affine transformation over $\mathbb{Z}$ taking one to the other.

Theorem 2.1. *Let $X \subseteq \mathbb{A}^N$ be the zero locus over $\mathbb{Z}$ of a polynomial $H \in \mathbb{Z}[x_1, \dots, x_N]$ with $N \geq 1$ whose Newton polytope Δ is equivalent to σ_N. Let*

$$X' := X \cap T,$$

where

$$T := \mathit{Spec}(\mathbb{Z}[x_1, x_1^{-1}, \dots, x_N, x_N^{-1}])$$

is the torus.

Then the varieties X' and X are polynomial count outside D, for

$$D := \text{ the product of all nonzero coefficients of } H.$$

More precisely,

$$\#X'(\mathbb{F}_q) = c_N(q), \tag{2}$$

and

$$\#X(\mathbb{F}_q) = \sum_{r=0}^{N} \sum_{n=0}^{N} f_{r,n} c_n(q)(q-1)^{r-n}, \tag{3}$$

where the $f_{r,n}$ are defined in (1) *and, for $n \geq 0$,*

$$c_n(q) := \sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} q^{i-1} + (-1)^n = \frac{(q-1)^n + (-1)^n (q-1)}{q}.$$

Before we prove the theorem we need some fairly standard preliminaries; we include short proofs for the convenience of the reader.

Let Λ be an abelian group isomorphic to $\mathbb{Z}^r$ for some r. We call a non-empty set $\mathcal{M} = \{m_0, \dots, m_k\} \subseteq \Lambda$ *unimodular* if $v_1 := m_1 - m_0, \dots, v_k := m_k - m_0$ can be extended to a basis of Λ. Let $\Lambda_0 := \langle v_1, \dots, v_k \rangle$.

Lemma 2.2. *(i) The definition of unimodular does not depend on the choice of labeling of the vectors.*

(ii) Every nonempty subset of a unimodular set is unimodular.

(iii) A set $\mathcal{M} = \{m_0, \dots, m_k\} \subseteq \Lambda$ is unimodular if and only if $v_1, \dots, v_k$ are linearly independent and Λ/Λ_0 is torsion free.

(iv) If $\mathcal{M}$ is unimodular in Λ and $\Lambda' \subseteq \Lambda$ is a subgroup containing $\mathcal{M}$, then $\mathcal{M}$ is also unimodular in Λ'.

Proof. (i) Let $v_1, \dots, v_k, w_{k+1}, \dots, w_r$ be a basis of Λ. For $1 \leq j \leq k$ the map

$$v_j \mapsto -v_j, \qquad v_i \mapsto v_i - v_j, \qquad w_l \mapsto w_l, \qquad i \neq j,\ l = k+1, \dots, r,$$

is an involution of Λ and takes $v_1, \dots, v_k$ to $m_0 - m_j, \dots, \widehat{m_j - m_j}, \dots, m_k - m_j$. Hence these vectors, together with $w_{k+1}, \dots, w_r$, form a basis of Λ, proving (i).

(ii) Follows from (i).

(iii) If $v_1, \dots, v_k, w_{k+1}, \dots, w_r$ is a basis of Λ, then the v_i are linearly independent and Λ/Λ_0 is free on the images of the $w_{k+1}, \dots, w_r$. Conversely, Λ/Λ_0 is free, being torsion free and finitely generated. Lifting a basis of Λ/Λ_0 to $w_{k+1}, \dots, w_r \in \Lambda$ yields a basis of Λ, $v_1, \dots, v_k, w_{k+1}, \dots, w_r$.

(iv) Follows from (iii) as $\Lambda'/\Lambda_0 \subseteq \Lambda/\Lambda_0$ and again because a finitely generated torsion free abelian group is free. □

Fix a commutative ring R and consider schemes over R. For $k \geq 0$ we let $Y_k \subseteq \mathbb{G}_m^k$ be the hypersurface with equation $1 + y_1 + \cdots + y_k = 0$. Note that $Y_0 = \emptyset$.

Lemma 2.3. *Let $\{m_0, \dots, m_k\}$ be a unimodular set in $\mathbb{Z}^r$ and let $H = \sum_{j=0}^k a_j x^{m_j} \in R[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ with $a_0, \dots, a_k \in R^\times$. Then the zero locus of H in $\mathbb{G}_m^r$ is isomorphic to $Y_k \times_R \mathbb{G}_m^{r-k}$. In particular, if $R = \mathbb{F}_q$, then H has exactly $(q-1)^{r-k-1}\, c_{k+1}(q)$ zeros in $(\mathbb{F}_q^\times)^r$.*

Proof. With the above notation, let $v_1, \dots, v_k, w_{k+1}, \dots, w_r$ be a basis of $\Lambda := \mathbb{Z}^r$. Label these vectors $u_1, \dots, u_r$. The map $(x_1, \dots, x_r) \mapsto (x^{u_1}, \dots, x^{u_r})$ is an automorphism of $\mathbb{G}_m^r$ defined over $\mathbb{Z}$. It maps the zero locus of $H = x^{m_0}(a_0 + a_1 x^{u_1} + \cdots + a_k x^{u_k})$ isomorphically to that of $a_0 + a_1 y_1 + \cdots + a_k y_k$, and rescaling y_j by a_j/a_0 identifies this with $Y_k \times_R \mathbb{G}_m^{r-k}$. □

As a quick check we have for example:

- $k = 2$, $r = 3$, $H = a_0 + a_1 x_1 + a_2 x_2$. There are $c_3(q) = (q-1)(q-2)$ zeros, since the line $H = 0$ meets each axis once and x_3 is free.
- $k = 0$, arbitrary r, $H = a_0 x^{m_0}$. There are no zeros and $c_1(q) = 0$.

We are now ready to prove the theorem.

Proof. It is easy to verify that for $N \geq 1$

$$\#\{(x_1, \dots, x_N) \in (\mathbb{F}_q^\times)^N \mid x_1 + \cdots + x_N = 0\} = c_N(q)$$

by Möbius inversion as

$$\sum_{n=0}^{N} \binom{N}{n} c_n(q) = q^{N-1}, \qquad N \geq 1.$$

From now on we work over $\mathbb{Z}[1/D]$, and $\mathbb{F}_q$ denotes a finite field whose characteristic does not divide D. Since we assume that Δ is equivalent to σ_N, the set $\mathcal{M} := \{m_0, \dots, m_{N-1}\} \subseteq \mathbb{Z}^N$ of exponents of H is unimodular.

To prove (3) we proceed as follows. Specializing $H(x_1, \dots, x_N)$ by setting $x_i = 0$ for $i \notin S$ with $S \subseteq \{1, \dots, N\}$ results in a polynomial H_S whose Newton polytope Δ_S is a face of Δ (or empty). The exponents of the monomials in H_S lie in the lattice

$$\mathbb{Z}^S := \{m \in \mathbb{Z}^N \mid m_i = 0 \text{ for } i \notin S\}.$$

By Lemma 2.2(ii) and (iv), the set of vertices of Δ_S is also unimodular in $\mathbb{Z}^S$ or empty (if H_S vanishes completely). By Lemma 2.3, if Δ_S is empty we obtain $(q-1)^r$ solutions to $H_S = 0$, whereas if Δ_S is non-empty the number of solutions of $H_S = 0$ with $x_i \in \mathbb{F}_q^\times$, $i \in S$, is $c_n(q)(q-1)^{r-n}$, where $r := \#S$ and $n = \dim \Delta_S + 1$. This proves the claim. □

Example 2.4. *Let $n_1, \dots, n_r$ be a tuple of pairwise coprime positive integers and let*

$$X: \quad x_1^{n_1} + \cdots + x_r^{n_r} = 0.$$

Then $X/\mathbb{Z}$ is polynomial count with $D = 1$: more precisely,

$$\#X(\mathbb{F}_q) = q^{r-1}. \tag{4}$$

Proof. To see this note that the coprimality condition guarantees that the Newton polytope Δ is equivalent to σ_r. Indeed, translating by $-(n_1, 0, \dots, 0)$ the polytope Δ is equivalent to the simplex with vertices

$$(0, \dots, 0), (-n_1, n_2, 0, \dots, 0), \dots (-n_1, 0, \dots, 0, n_r).$$

Let

$$M := \begin{pmatrix} -n_1 & -n_1 & \cdots & & -n_1 \\ n_2 & 0 & \cdots & & 0 \\ 0 & n_3 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & & 0 & n_r \end{pmatrix}.$$

To prove the claim is enough to show that we may find a column vector $x := (x_1, \dots, x_r)$ with $x_i \in \mathbb{Z}$ such that the matrix $U := (x \,|\, M) \in \mathbb{Z}^{r \times r}$, adding x to M as its first column, is invertible. Given such a U we get

$$M = UM',$$

where

$$M' := \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$$

and since the simplex with vertices

$$(0, \dots, 0), (0, 1, 0, \dots, 0), \dots (0, \dots, 0, 1)$$

is equivalent to σ_r the same is true of Δ.

To find x we need to solve for integers x_i such that

$$\det(U) = -d_r x_1 + \cdots (-1)^r d_1 x_r = \pm 1, \qquad d_i := n_1 \cdots \widehat{n_i} \cdots n_r.$$

By the coprimality assumption the vector $(d_1, \dots, d_n)$ is primitive and hence such a solution exists. It is also clear that $\mathcal{F}_\Delta = \mathcal{F}_{\sigma_r}$. The claim now follows from Theorem 2.1.

For an alternate proof, see [19, page 502, lines 3-6]. In fact this Weil calculation shows the stronger result that merely having n_1 relatively prime to each $n_i, i \geq 2$, forces $\#X(\mathbb{F}_q) = q^{r-1}$.

More precisely, Weil shows such an X is polynomial count with $\#X(\mathbb{F}_q) = q^{r-1}$ if and only if there are no solutions to the equation

$$\prod_{i=1}^{r} \zeta_i = 1 \text{ with each } \zeta_i \in \mu_{n_i}(\mathbb{C}) \setminus \{1\}.$$

Notice that if all $n_i \geq 2$, such an $X_{\mathbb{C}}$ cannot be isomorphic to $\mathbb{A}^{r-1}$, simply because it fails to be smooth at the origin. What can we say about its mixed Hodge structure, or about its AK invariant [14]?

□

Example 2.5. *Consider the Russell threefold [14]*

$$X: \quad x^2y + x + z^2 + t^3 = 0.$$

Its Newton polytope is the 3*-dimensional simplex in* $\mathbb{Z}^4$ *with vertices the columns of the matrix*

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

which is in fact equivalent to σ_4*. We have*

$$\mathcal{F}_\Delta(x, y) = x^4y^4 + 3x^3y^3 + x^3y^2 + 4x^2y^2 + 2x^2y + 3xy + x + 1.$$

Then formula (3) *gives*

$$X(\mathbb{F}_q) = q^3.$$

(That the Russell threefold is polynomial count over $\mathbb{Z}$*, with* $D = 1$ *and* $C(t) = t^3$*, can also be verified directly, see the next example)*

It is known that $X(\mathbb{C})$ is diffeomorphic to $\mathbb{R}^6$ but X is not isomorphic to $\mathbb{A}^3$. (See the discussion in [14][§1],and the literature cited therein for more details on this type of phenomenon. See also [6, Corollary 3.2] and [11, Theorem A]. This answers our first question.

As another example, it is known that the fourfold

$$x + x^2y + z^2 + t^3 + u^5 = 0$$

is diffeomorphic to $\mathbb{R}^8$ but not isomorphic to $\mathbb{C}^4$, just as in Russell's case (see [14][Remark 11.1]).

Example 2.6. *In general, fix* $m \geq 2$, $r \geq 2$ *and let* $n_1, \ldots, n_r$ *be a tuple of pairwise coprime positive integers.Consider the affine variety*

$$X: \quad x^my + x + \sum_{i=1}^{r} x_i^{n_i} = 0$$

in $\mathbb{A}^{r+2}$ *with coordinates* $(x, y, x_1, \ldots, x_r)$. *Then* X *is smooth over* $\mathbb{Z}$ *and*

$$\#X(\mathbb{F}_q) = q^{r+1}.$$

If $X(\mathbb{C})$ *is contractible, then it is diffeomorphic to* $\mathbb{R}^{2r+2}$,

Proof. It is straightforward to check that X is smooth over $\mathbb{Z}$. It follows easily from the previous example that its Newton polytope is equivalent to σ_{r+2} (here again the coprimality assumption is crucial.) If $X(\mathbb{C})$ is contractible, it is (hence) diffeomorphic to $\mathbb{R}^{2r+2}$, see [5, Step 3 of proof of Theorem 5]. Presumably such an X is not isomorphic to $\mathbb{A}^{r+1}$ over $\mathbb{C}$, but this seems to be open in general.

It is not hard to verify directly that $\#X(\mathbb{F}_q) = q^{r+1}$ without appealing to Theorem 2.1. Indeed, for $x \neq 0$ we get $q^r(q-1)$ points by fixing $(x_1, \ldots, x_r)$ and solving for y. For $x = 0$ instead, we have q^{r-1} points from $\sum_{i=1}^{r} x_i^{n_i} = 0$ by (4) times q as y is arbitrary independent of $(x_1, \ldots, x_r)$.

In fact, there is a bit more we can say when $X(\mathbb{C})$ is contractible. Because $X(\mathbb{C})$ is diffeomorphic to $\mathbb{R}^{2r+2}$, we have

$$H^i_{c,\text{Betti}}(X(\mathbb{C})^{an}, \mathbb{Z}) = 0 \text{ for } i \neq 2r+2,$$

$$H^{2r+2}_{c,\text{Betti}}(X(\mathbb{C})^{an}, \mathbb{Z}) \text{ is free of rank one over } \mathbb{Z}.$$

Fix a prime ℓ. The comparison theorem gives that $H^i_{c,et}(X_{\mathbb{C}}, \mathbb{Q}_\ell)$ vanishes for $i \neq 2r+2$ and $\dim H^{2r+2}_{c,et}(X_{\mathbb{C}}, \mathbb{Q}_\ell) = 1$. By the constructiblity of $R^i f_! \mathbb{Q}_\ell$ for the structural morphism $f : X \to \text{Spec}(\mathbb{Z})$, and proper base change, it follows that for all but finitely many primes p, we have the same statements for the étale H^i_c of the various $X_{\overline{\mathbb{F}_p}}$. By the Lefschetz trace formula and the fact that our X is polynomial count, we see that Frob_p acts on $H^{2r+2}_{c,et}(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell)$ as p^{r+1} for almost all p. Thus $H^{2r+2}_{c,et}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is $\mathbb{Q}_\ell(-r-1)$. So for these X, there are no surprises (along the lines of the last section) about their mixed Hodge structures. □

Let us consider the following slightly more general situation. Suppose we are given, over $\mathbb{C}$, an affine X which is irreducible, of dimension $d \geq 1$. Suppose further that $X(\mathbb{C})$ is contractible. Then its rational Betti cohomology groups $H^i(X(\mathbb{C}), \mathbb{Q})$ are $\mathbb{Q}$ for $i = 0$ and vanish for $i > 0$. If in addition $X(\mathbb{C})$ is a rational homology manifold, then we have Poincare duality, hence $H^i_c(X(\mathbb{C}), \mathbb{Q})$ vanishes for $i \neq 2d$, and is one-dimensional for $i = 2d$. Then the comparison theorem, together with the fact that $H^{2d}_c(X_{\mathbb{C}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-d)$ for a geometrically irreducible X of dimension d, shows that such an X is polynomial count, with polynomial q^d.

In addition to smooth contractible X, there are other examples. Examples of such X include those Brieskorn varieties $\sum_{i=0}^{d} x_i^{a_i} = 0$, $d \geq 3$, whose exponents are each ≥ 2 and satisfy the following condition: there are at least two indices i such that a_i is relatively prime to all a_j with $j \neq i$. These are manifolds, cf. [3, footnote], [2], and [13, Corollary, page 21], not just rational homology manifolds.

A more exotic example is the space N of nilpotent matrices in $M_n(\mathbb{C})$, or equivalently the space of unipotent elements in $\text{SL}(n, \mathbb{C})$. Here the fact that this N is polynomial count with polynomial q^{n^2-n} goes back (at least) to [9] and [10], with a vast generalization by Steinberg

in [18, 15.3], though on [18, page 5], Steinberg says he doesn't know why this polynomial count holds. In fact, this N is geometrically irreducible of dimension $d = n^2 - n$, and is a rational homology manifold, as is proven in [1, 1.4 and 2.3], which is arguably the "right" conceptual reason it is polynomial count.

Are there examples of geometrically irreducible affine $X/\mathbb{C}$ of dimension $d \geq 1$ which are both contractible and polynomial count with polynomial q^d which are **not** rational homology manifolds? Simplest test case: X defined by $F = 0$ for a weighted homogeneous square-free F in $d+1$ variables (so automatically contractible) which is polynomial count with q^d. [Notice that such an X must be irreducible, otherwise F would factor into $r \geq 2$ irreducible factors, and the leading term in the $\mathbb{F}_q$ count would be rq^d over large enough $\mathbb{F}_q$.]

Consider the Brieskorn situations X where we fail to get a sphere but get a rational homology sphere, e.g. exponents $(2, 2, ..., 2)$ repeated an odd number $n \geq 3$ times, or $(2, 3, 3, 3, ..., 3)$, as examples of q^d cases. In general, the condition to get (at least) a rational homology sphere, in terms of the exponents $(a_1, .., a_n)$, is precisely [8, page 95,(E)] the Weil condition in [19, page 502, lines 3-6] discussed earlier that forces Weil's $M = 0$ for the Brieskorn hypersurface $\sum_i x_i^{a_i} = 0$ over any $\mathbb{F}_q$ with $q \equiv 1 \mod a_i$ for each a_i. Here the unique singularity is the point $x_0 := (0, \ldots, 0)$. The long exact rational homology sequence for $(X, X - x_0)$ shows that X is a rational homology manifold if and only if $X \cap$ (a small sphere around x_0) is a rational homology sphere, i.e., if and only if X is polynomial count with $q^{\dim(X)}$.

What about equations $X : x^r y + \sum_{i=1}^d t_i^{a_i} = 0$ with $r \geq 2$ and all $a_i \geq 2$? This is polynomial count with count q^{d+1} precisely when $\sum_{i=1}^d t_i^{a_i} = 0$ is polynomial count with count q^{d-1} (the Weil/Brieskorn situation). For these, the critical locus is the entire line $(0, y, 0, ..., 0)$. Is such an X a rational homology manifold? Same question for equations $XX : x^r y + u^s v + \sum_{i=1}^d t_i^{a_i} = 0$ with $r \geq 2$, $s \geq 2$, and all $a_i \geq 2$ with the a_i in the Weil/Brieskorn situation: now the polynomial count is q^{d+3}, and the critical locus is the plane $(0, y, 0, v, 0, ..., 0)$. Or an equation $XXX : x^r y + u^r v + z^r w + t^{r+1}$, to have a homogenous form with q^6 affine points, i.e., its projective locus has the same point count as $\mathbb{P}^5(\mathbb{F}_q)$. If we add x to the equation, we have a smooth affine hypersurface with the same q^6 point count, but we lose the homogeneity.

3 Browning-Munshi-Wang examples

In the paper [4], the authors consider the projective fourfold with equation

$$\sum_{i=1}^{3} x_i y_i^2 = 0.$$

This led us to wonder about the projective varieties $V_{n,d}$ of equation

$$V_{n,d} : \sum_{i=1}^{n} x_i y_i^d = 0,$$

for any $n \geq 1$ and any $d \geq 1$.

For $d = 1$, this is a smooth quadric in $\mathbb{P}^{2n-1}$, and over any $\mathbb{F}_q$, we have

$$\#V_{n,1}(\mathbb{F}_q) = \#\mathbb{P}^{2n-2} + q^{n-1}.$$

The reason for this is that the étale cohomology groups $H^i := H^i(V_{n,1} \otimes \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ vanish for i odd or $i > 4n-4$, and are given by

$$H^{2i} = \mathbb{Q}_\ell(-i) \text{ for } 2i \neq 2n-2,\ H^{2n-2} = \mathbb{Q}_\ell(1-n) \oplus \mathbb{Q}_\ell(1-n).$$

[As the referee pointed out to us, there is a more elementary way to arrive at this point count. It is equivalent to show that the affine point count of $\sum_{i=1}^n x_i y_i = 0$ in $\mathbb{A}^{2n}(\mathbb{F}_q)$ is $(q^n-1)q^{n-1} + q^n$. View the equation as $< v, w >= 0$ for pairs of vectors v, w in $\mathbb{A}^n(\mathbb{F}_q)$. For each $v \neq 0$, its orthogonal has dimension $n-1$, so q^{n-1} points. For $v = 0$, its orthogonal has full dimension n, so q^n points. So the affine point count is $(q^n-1)q^{n-1} + q^n$.]

Given the point count, and hence the zeta function, the calculation of the reversed characteristic polynomial of Frobenius on each cohomology group is then forced by the requirement (Deligne's theorem!) that each H^i is pure of weight i.

Lemma 3.1. *For any $d \geq 1$, the projective variety $V_{n,d}$ has the same point count as $V_{n,1}$ over any $\mathbb{F}_q$.*

Proof. It is equivalent to show that the affine cone $V_{n,d}^{aff}$ (equation $\sum_{i=1}^n x_i y_i^d = 0$ in $\mathbb{A}^{2n}$) has the same point count as $V_{n,1}^{\text{aff}}$. For each subset $S \subset \{1, 2, ..., n\}$, and each fixed choice of y's with $y_i = 0$ for $i \in S$, $y_j \neq 0$ for $j \notin S$, the number of such points on $V_{n,d}^{\text{aff}}$ is $q^{\#S}$ if S is the entire set (then each x_i is free), otherwise it is $q^{\#S} \times q^{n-1-\#S}$ (because then the remaining x_j are restricted by one nontrivial linear equation). This calculation, by matching exactly which y_i vanish, does not "know" what d is. □

The affine cone point count over $\mathbb{F}_q$ is $1 + (q-1)\times$(the projective point count). So for $V_{n,d}^{\text{aff}}$ it is

$$\#V_{n,d}^{\text{aff}}(\mathbb{F}_q) = q^{2n-1} + q^n - q^{n-1}.$$

The same "y by y" counting argument, together with the Weil result [19, page 502, lines 3-6] cited earlier, shows the following.

Lemma 3.2. *For any $n \geq 1$, any choices of exponents $b_i \geq 1$, and any choice of $a_1, \ldots, a_n$ with $\gcd(a_i, a_j) = 1$ for all $i \neq j$, the affine variety of equation*

$$\sum_{i=1}^n x_i^{a_i} y_i^{b_i} = 0$$

has the same $\mathbb{F}_q$ point count, namely $q^{2n-1} + q^n - q^{n-1}$, as any of the $V_{n,d}^{aff}$.

Let us return to the projective varieties $V_{n,d}$.

Theorem 3.3. *For $n \geq 1$ and $d \geq 1$, each cohomology group $H^i := H^i(V_{n,d} \otimes \overline{\mathbb{F}_q}, \mathbb{Q}_\ell)$ is pure of weight i, and hence $V_{n,d}$ has the same total cohomology (as graded $\mathbb{Q}_\ell$ space with Frobenius action) as $V_{n,1}$.*

We will prove this for a fixed d by induction on n, the case $n = 1$ being known: the projective variety $xy^d = 0$ in $\mathbb{P}^1$ consists of 2 points, here exploiting the fact that V and V^{red} (the associated reduced scheme) have the same étale cohomology.

Let us first consider a more general situation. Suppose we are given a hypersurface V of degree $d + 1 \geq 2$ in a projective space $\mathbb{P}^N$, defined by a homogeneous form

$$V : F(X_0, \ldots, X_N)$$

of degree $d + 1$. Consider the hypersurface V_2 in $\mathbb{P}^{N+2}$ of the same degree $d + 1$ of equation

$$V_2 : F(X_0, \ldots, X_N) + UZ^d.$$

The open set $V_2[1/Z]$ is the affine variety in $\mathbb{A}^{N+2}$ with equation

$$F(X_0, \ldots, X_N) + U = 0,$$

which is $\mathbb{A}^{N+1}$ (the $X_i, 0 \leq i \leq N$ are free, solve for U). The closed complement $V_2 \cap (Z = 0)$ is the hypersurface $V_1 \subset \mathbb{P}^{N+1}$, homogenous coordinates $X_i, 0 \leq i \leq N$ and U, with equation

$$V_1 : F(X_0, \ldots, X_N) = 0 \text{ in } \mathbb{P}^{N+1}.$$

The key point is that we start with $V \subset \mathbb{P}^N$, and use the **same** equation but view it in $\mathbb{P}^{N+1}$ with a “missing" variable, namely U.

Consider the excision sequence in compact cohomology,

$$\ldots H_c^i(\mathbb{A}^{N+1}) \to H^i(V_2) \to H^i(V_1) \to H_c^{i+1}(\mathbb{A}^{N+1}) \ \ldots.$$

From the fact that

$$H_c^{2N+2}(\mathbb{A}^{N+1}) = \mathbb{Q}_\ell(-1 - N), \text{ other } H_c^i(\mathbb{A}^{N+1}) = 0,$$

we see that

$$H^{2N+2}(V_2) = \mathbb{Q}_\ell(-1 - N),$$
$$H^{2N+1}(V_2) = 0,$$

(because V_1 has dimension $\leq N$) and, for $i \leq 2N$, we have

$$H^i(V_2) \cong H^i(V_1).$$

The key observation, due to Mark Goresky, is that V_1 is the Thom space attached to the hyperplane line bundle $\mathcal{O}_V(1)$ on V, and the Thom isomorphism gives what we need.

Here is the general picture, explained in down to earth terms.

Theorem 3.4. *Suppose we have a closed subscheme $V \subset \mathbb{P}^N$, defined by various homogeneous forms F_i in $X_0, \dots, X_N$, and we form the closed subscheme V_1, defined by these* **same** *equations, but viewed in the $\mathbb{P}^{N+1}$ with an extra homogenous coordinate U which does not appear. Then we have the following results.*

(1) For $i \geq 2$, $H^i(V_1) \cong H^{i-2}(V)(-1)$,

(2) For $i = 1$, $H^1(V_1) = 0$,

(3) For $i = 0$, $H^0(V_1) = \mathbb{Q}_\ell$.

Proof. Given a point $(x_0, \dots, x_N) \in V$, draw the line in V_1 from the point $(x_0, \dots, x_N, 0) \in V_1$ to the point $P := (0, \dots, 0, 1) \in V_1$: concretely, this line is the set of points $(sx_0, \dots, sx_N, t) \in V_1$ as (s, t) runs over $\mathbb{P}^1$. Now remove the point P from V_1, but keep each line minus its endpoint P. For each point $(x_0, \dots, x_N) \in V$, this affine line is the set of points $(x_0, \dots, x_N, t)$ as t runs over $\mathbb{A}^1$. A better way to think of this is to observe that the point $P := (0, \dots, 0, 1) \in V_1$ is the intersection of closed sets $V_1 \cap (X_i = 0)$ for $i \in \{0, \dots, N\}$. Thus $V_1 \setminus \{P\}$ is covered by the open sets $V_1[1/X_i], i \in \{0, \dots, N\}$. Each such open set is the affine variety

$$V_1[1/X_i] = V[1/X_i] \times \mathbb{A}^1,$$

in affine coordinates (the X_j/X_i, $j \neq i$, and $t = U/X_i$).

This construction shows with

$$P := (0, \dots, 0, 1) \in V_1,$$

that the projection

$$V_1 \setminus \{P\} \to V, (x_0, \dots, x_N, t) \mapsto (x_0, \dots, x_N)$$

makes $V_1 \setminus \{P\}$ into the total space E of the rank one hyperplane bundle $\mathcal{O}_V(1)$ on V.

Now consider the Leray spectral sequence in compact cohomology for the projection

$$\pi : E \to V.$$

Because the fibres are affine lines, we have

$$R^2\pi_!\mathbb{Q}_\ell \cong \text{ the constant sheaf } \mathbb{Q}_\ell(-1), \text{all other } R^i\pi_!\mathbb{Q}_\ell = 0.$$

So the E_2 terms are

$$E_2^{a,b} = H^a(V) \otimes H_c^2(\mathbb{A}^1), \text{if } b = 2, 0 \text{ otherwise.}$$

Thus we have

$$H_c^{i+2}(E) = H^i(V)(-1)$$

for $i \geq 0$, and

$$H_c^i(E) = 0 \text{ for } i = 0, 1.$$

We next compare $H^i_c(E)$ with $H^i(V_1)$. The excision sequence in compact cohomology for E as $V_1 \setminus \{P\}$ gives

$$0 \to H^0_c(E) \to H^0(V_1) \to H^0(\{P\}) = \mathbb{Q}_\ell \to H^1_c(E) \to H^1(V_1) \to 0$$

and

$$H^i_c(E) \cong H^i(V_1) \text{ for } i \geq 2.$$

This gives the asserted values for $H^i(V_1)$ for $i \geq 2$. Both $H^0_c(E), H^1_c(E)$ vanish, so the excision sequence gives the asserted values of $H^0(V_1), H^1(V_1)$. This concludes the proof of Theorem 3.4.

With this general Theorem 3.4 in hand, we apply it as follows. We put $V := V_{n,d}$, with equation

$$V_{n,d} : \sum_{i=1}^{n} x_i y_i^d = 0.$$

Then $V_{n+1,d}$ is the V_2 of the general discussion, the "added on" term $x_{n+1}y_{n+1}^d$ in the equation of $V_{n+1,d}$ is the "added on" term UZ^d of the general discussion. The V_1 of the general discussion is the equation $\sum_{i=1}^n x_i y_i^d = 0$ in the projective space with homogeneous coordinates $\{x_i, i \leq n,\ y_i, i \leq n,\ u\}$. The relation of the cohomology of V_2 to the cohomology of V, passing through the cohomology of V_1, is independent of d. So induction on n proves Theorem 3.3. □

4 Fake projective planes and fake $\mathbb{P}^4$s

Other interesting polynomial count varieties are the arithmetic fake projective planes. These are smooth connected projective complex surfaces X which have the same $\mathbb{Q}$-Betti numbers as $\mathbb{P}^2(\mathbb{C})^{\text{an}}$ but for which $X(C)^{\text{an}}$ is not homeomorphic to $\mathbb{P}^2(\mathbb{C})^{\text{an}}$. These are all obtained as quotients of the complex ball $\mathcal{B}^2$ by a cocompact, torsion free, discrete subgroup Π of $PU(2,1) = \text{Aut}(\mathcal{B}^2)$. One knows that Π is necessarily an arithmetic subgroup of $PU(2,1)$, see the introduction to [16] for a wonderful overview.

There are also arithmetic fake $\mathbb{P}^4$, cf. [16, Theorem 3]. These are smooth connected projective complex fourfolds X which have the same $\mathbb{Q}$-Betti numbers as $\mathbb{P}^4(\mathbb{C})^{\text{an}}$ but for which $X(\mathbb{C})^{\text{an}}$ is not homeomorphic to $\mathbb{P}^4(\mathbb{C})^{\text{an}}$. Here the adjective "arithmetic" means that these X are **defined** to be quotients of the complex four-ball $\mathcal{B}^4$ by a cocompact, torsion free, discrete subgroup Π of $PU(4,1) = \text{Aut}(\mathcal{B}^4)$ with Π arithmetic.

For an arithmetic fake $\mathbb{P}^2$ or an arithmetic fake $\mathbb{P}^4$, one knows (cf. [16, Theorem 4] and [17, A.11.Remark]) that $H_1(X(\mathbb{C})^{\text{an}}, \mathbb{Z})$ is nonzero, a fact which shows that $X(\mathbb{C})^{\text{an}}$ cannot be homeomorphic to a projective space. Here is a cohomological way of saying the same thing. By Poincaré duality on the compact oriented $X(\mathbb{C})^{\text{an}}$, we have

$$H_1(X(\mathbb{C})^{\text{an}}, \mathbb{Z}) \cong H^{2\dim(X)-1}(X(\mathbb{C})^{\text{an}}, \mathbb{Z}).$$

Therefore $H^{2\dim(X)-1}(X(\mathbb{C})^{\mathrm{an}},\mathbb{Z})$ is nonzero. But this abelian group is finite, simply because $H^{2\dim(X)-1}(X(\mathbb{C})^{\mathrm{an}},\mathbb{Q}) = 0$. The group $H^{2\dim(X)}(X(\mathbb{C})^{\mathrm{an}},\mathbb{Z})$ is torsion free (being $\mathbb{Z}$). So for a prime ℓ dividing $\#H^{2\dim(X)-1}(X(\mathbb{C})^{\mathrm{an}},\mathbb{Z})$, the universal coefficient sequence for

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z} \to 0$$

shows that

$$H^{2\dim(X)-1}(X(\mathbb{C})^{\mathrm{an}},\mathbb{Z})/\ell H^{2\dim(X)-1}(X(\mathbb{C})^{\mathrm{an}},\mathbb{Z}) \cong H^{2\dim(X)-1}(X(\mathbb{C})^{\mathrm{an}},\mathbb{Z}/\ell\mathbb{Z}),$$

and hence

$$H^{2\dim(X)-1}(X(\mathbb{C})^{\mathrm{an}},\mathbb{Z}/\ell\mathbb{Z}) \neq 0.$$

For any projective smooth $X/\mathbb{C}$, one can choose a "spreading out" of $X/\mathbb{C}$, namely a projective smooth $\mathbb{X}/A$ with A a subring of $\mathbb{C}$ which is finitely generated over $\mathbb{Z}$, such that after the extension of scalars $A \subset \mathbb{C}$, $\mathbb{X}_{\mathbb{C}}$ is the given $X/\mathbb{C}$. For any maximal ideal $\mathfrak{p}$ of A, the residue field $A/\mathfrak{p} := k(\mathfrak{p})$ is finite, with algebraic closure denoted $\overline{k(\mathfrak{p})}$. By proper and smooth base change, one knows that for any prime ℓ invertible in $k(\mathfrak{p})$, and R any of $\mathbb{Z}/\ell^n\mathbb{Z}, n \geq 1$ or $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, and any integer $i \geq 0$, there exist R-linear isomorphisms of cohomology groups

$$H^i_{\mathrm{top}}(X(\mathbb{C})^{\mathrm{an}},R) \cong H^i_{\mathrm{et}}(X,R) \cong H^i(\mathbb{X}_{\overline{k(\mathfrak{p})}},R).$$

Theorem 4.1. *Let $X_2/\mathbb{C}$ be an arithmetic fake $\mathbb{P}^2$, respectively let $X_4/\mathbb{C}$ be an arithmetic fake $\mathbb{P}^4$. Choose a prime ℓ_2 for which*

$$H^3(X_2(\mathbb{C})^{\mathrm{an}},\mathbb{Z}/\ell_2\mathbb{Z}) \neq 0,$$

respectively a prime ℓ_4 for which

$$H^7(X_4(\mathbb{C})^{\mathrm{an}},\mathbb{Z}/\ell_4\mathbb{Z}) \neq 0.$$

Then we have the following results.

(1) For any spreading out $\mathbb{X}_2/A_2$ of $X_2/\mathbb{C}$ (respectively any spreading out $\mathbb{X}_4/A_4$ of $X_4/\mathbb{C}$), and for any maximal ideal $\mathfrak{p}_2$ of $A_2[1/\ell_2]$, (respectively any maximal ideal $\mathfrak{p}_4$ of $A_4[1/\ell_4]$), the variety $\mathbb{X}_{2,k(\mathfrak{p}_2)}$, (respectively the variety $\mathbb{X}_{4,k(\mathfrak{p}_4)}$), is a projective smooth variety over $k(\mathfrak{p}_2)$, (respectively over $k(\mathfrak{p}_4)$) which, after extension of scalars to $\overline{k(\mathfrak{p}_2)}$ (respectively to $\overline{k(\mathfrak{p}_4)}$) has the same $\mathbb{Q}_{\ell_2}$-Betti numbers as $\mathbb{P}^2$, (respectively the same $\mathbb{Q}_{\ell_4}$-Betti numbers as $\mathbb{P}^4$).

(2) $\mathbb{X}_{2,\overline{k(\mathfrak{p}_2)}}$ is not isomorphic to $\mathbb{P}^2$, and $\mathbb{X}_{4,\overline{k(\mathfrak{p}_4)}}$ is not isomorphic to $\mathbb{P}^4$.

(3) For any finite extension field $\mathbb{F}_{q_2}$ of $k(\mathfrak{p}_2)$ (respectively any finite extension field $\mathbb{F}_{q_4}$ of $k(\mathfrak{p}_4)$), we have

$$\#X_{2,k(\mathfrak{p}_2)}(\mathbb{F}_{q_2}) = 1 + q_2 + q_2^2 \ \ (= \#P^2(\mathbb{F}_{q_2})),$$

$$\#X_{4,k(\mathfrak{p}_4)}(\mathbb{F}_{q_4}) = 1 + q_4 + q_4^2 + q_4^3 + q_4^4 \ \ (= \#P^4(\mathbb{F}_{q_4})).$$

Proof. Statement (1) is just a reminder of what proper smooth base change gives. Statement (2) holds by the nonvanishing of $H^{2\dim(X)-1}(X(\mathbb{C})^{\mathrm{an}}, \mathbb{Z}/\ell\mathbb{Z})$ for the chosen X either X_2 or X_4 with the prime ℓ_2 used with X_2 and ℓ_4 used with X_4, along, once again, with proper smooth base change.

For statement (3), we observe that for any projective, smooth $X/\mathbb{F}_q$ of dimension $d \geq 1$ with the same $\mathbb{Q}_\ell$-Betti numbers as $\mathbb{P}^d$, the one-dimensional $\mathbb{Q}_\ell$ space H^2 has as $\mathbb{Q}_\ell$ basis the class of a hyperplane section (as this class is always nonzero in H^2), and for each $1 \leq i \leq d$, the ith power of this class is a $\mathbb{Q}_\ell$ basis of H^{2i} (as the dth power of this class is nonzero in H^{2d}). Therefore $\mathrm{Frob}_q\,|H^{2i} = q^i$, as this holds for powers of the hyperplane class. □

Corollary 4.2. *An arithmetic fake $\mathbb{P}^2$ is polynomial count with polynomial $1 + t + t^2$, and an arithmetic fake $\mathbb{P}^4$ is polynomial count with polynomial $1 + t + t^2 + t^3 + t^4$.*

5 A mixed weight counterexample

Recall [7, 3.2.5] that Deligne creates, for any smooth, separated $X/\mathbb{C}$ of finite type, a mixed Hodge structure on each $H^i_c(X) := H^i_c(X^{an}, \mathbb{C})$. One defines

$$h^{p,q;i} := h^{p,q}(gr_W^{p+q}(H^i_c(X))).$$

Suppose X is polynomial count. Is it true that $h^{p,q;i} = 0$ for any given i unless $p = q$?

This is not the case in general. Here is the simplest example. Take an elliptic curve E, and denote by E_{aff} the affine curve which is the complement of the origin, e.g., E_{aff} is the curve in $\mathbb{A}^2$ of literal equation

$$E_{\mathrm{aff}}: \quad y^2 = f(x),$$

with f a cubic with distinct roots.

Now consider the abstract scheme X given by the disjoint union of the affine variety $Y := E_{\mathrm{aff}} \subseteq \mathbb{A}^2$, and the affine variety $Z := \mathbb{A}^2 \setminus E_{\mathrm{aff}} \subseteq \mathbb{A}^3$ defined by $z(y^2 - f(x)) = 1$.

By the excision sequence for Z

$$0 = H^1_c(\mathbb{A}^2) \to H^1_c(Y) \to H^2_c(Z) \to H^2_c(\mathbb{A}^2) = 0.$$

Hence

$$H^1_c(Y) \cong H^2_c(Z).$$

By the dual of Lefschetz affine, we have $H^1_c(Z) = 0$.

On the other hand, the excision sequence for Y as $E \setminus 0$ shows that $H^1_c(Y) \cong H^1(E)$ and $H^2_c(Y) \cong H^2(E)$. Thus we conclude that

$$gr^1_W(H^2_c(Z)) \cong H^1(E),$$

so Z has $h^{0,1;2} = h^{1,0;2} = 1$ and $h^{0,1;1} = h^{1,0;1} = 0$ (because its entire H^1_c vanishes). Meanwhile,

$$gr^1_W(H^1_c(Y)) = H^1(E),$$

so we conclude that Y has $h^{0,1;1} = h^{1,0;1} = 1$ and Y has $h^{0,1;2} = h^{1,0;2} = 0$ (because $H^2_c \cong H^2(E)$ is of type $(1,1)$). So for X we have $h^{0,1;1} = h^{1,0;1} = 1$ and $h^{0,1;2} = h^{1,0;2} = 1$. But this disjoint union is visibly polynomial count, it has q^2 points.

One may object that the example is artificial, it's not even connected. To "improve" the example, pick a point $\alpha \in \mathbb{A}^2 \setminus E_{\text{aff}}$, and a point $\beta \in \mathbb{A}^3$. Then in $\mathbb{A}^5 = \mathbb{A}^2 \times \mathbb{A}^3$, we have the disjoint closed sets $Y \times \beta$ and $\alpha \times Z$. In other words, we can see the above example X as being a closed set $W \subseteq \mathbb{A}^5$. Then again playing with the excision sequence for $M := \mathbb{A}^5 \setminus W$, we will get

$$H^i_c(W) \cong H^{i+1}_c(M), \qquad i \leq 8,$$

in particular for $i = 1$ and $i = 2$. Thus we find that M is smooth and irreducible and has $h^{0,1;2} = h^{1,0;2} = 1$ and $h^{0,1;3} = h^{1,0;3} = 1$, though it too is trivially polynomial count, it has $q^5 - q^2$ points.

References

[1] Borho, W.; MacPherson, R. Partial resolutions of nilpotent varieties *Astérisque*, **101-102** (1983), 23-74.

[2] Brieskorn, E. Beispiele zur Differentialtopologie von Singulariaten *Invent. Math.* **2** (1966), 1-14.

[3] Brieskorn, E. Examples of singular normal complex spaces which are topological manifolds. *Proc. Nat. Acad. Sci.* **55** (1966), 1395-1397.

[4] Browning, T.; Munshi,R; and Wang, V.Y. Beyond the square-root barrier: cubic forms of Perazzo type `https://arxiv.org/pdf/2604.19045`

[5] Choudary, A.D.R.; Dimca, A. Complex hypersurfaces diffeomorphic to affine spaces *Kodai Math J.* **17** (1994), 171-178.

[6] Crachiola, A. The hypersurface $x + x^2y + z^2 + t^3 = 0$ over a field of arbitrary characteristic *Proc. Amer. Math. Soc.* **134** (2006), no. 5, 1289-1298.

[7] Deligne, P. Théorie de Hodge:II *Pub. Math. IHES* **40** (1971), 5-57.

[8] Dimca, A. Singularities and Topology of Hypersurfaces, Springer (1992).

[9] Fine, N.J.; Herstein, I.N. The probability that a matrix be nilpotent *Illinois J. Math.* **2** (1958), 499-504.

[10] Gerstenhaber, M. On the number of nilpotent matrices with coefficients in a finite field *Illinois J. Math.* **5** (1961), 330-333.

[11] Gupta, N., On the family of affine threefolds $x^m y = F(x,z,t)$ *Compos. Math.* **150** (2014), no. 6, 979-998.

[12] Hausel, T.; Rodriguez-Villegas, F. Mixed Hodge polynomials of character varieties, *Inv. Math.* **174**, No. 3, (2008), 555–624, arXiv:math.AG/0612668.

[13] Hirzebruch, F. Singularities and exotic spheres *Séminaire Bourbaki* : années 1966/67 and 1967/68, exposé 314, (1968) no. 10, 13-32.

[14] Kaliman, Sh.; Makar-Limanov, L. AK-invariant of affine domains in *Affine algebraic geometry*, 231–255. Osaka University Press, Osaka, 2007 ISBN:978-4-87259-226-9

[15] Kraft, H.; Petrie, T.; Randall, J. Quotient Varieties *Advances in Mathematics* **74** (1989), 145-192.

[16] Prasad, G.; Yeung, S-K. Arithmetic Fake Projective Spaces and Arithmetic Fake Grassmannians *American Journal of Mathematics*, Apr., 2009, Vol. 131, No. 2, pp. 379-407.

[17] Prasad, G.; Yeung, S-K. Addendum to “Fake projective planes" *Invent. Math.*, 168, 321-370 (2007) in *Invent. Math.* **182** (2010), 213–227

[18] Steinberg, R. Endomorphisms of Linear Algebraic Groups *Memoirs of the A.M.S.* **80** (1968)

[19] Weil, A., Numbers of solutions of equations in finite fields *Bull. Amer. Math. Soc.* **55** (1949), 497-508.