WILD RAMIFICATION AND SOME PROBLEMS OF "INDEPENDENCE OF ℓ"

By NICHOLAS M. KATZ

Dedicated to André Weil on his 77th birthday

Introduction. Let us say that a variety X over a finite field $\mathbf{F}_q$ of characteristic p is "independent of ℓ" if, for each prime $\ell \neq p$, and each integer i, the characteristic polynomial of Frobenius

$$P_{i,\ell}(T) \stackrel{\text{dfn}}{=} \det(1 - TF \mid H^i_{\text{comp}}(X \underset{\mathbf{F}_q}{\otimes} \overline{\mathbf{F}}_q; \mathbf{Q}_\ell))$$

has coefficients in $\mathbf{Z}$ (rather than in $\mathbf{Q}_\ell$) and if, for each given i, the $\mathbf{Z}$-polynomial $P_{i,\ell}(T)$ is independent of the auxiliary choice of $\ell \neq p$.

That a proper and smooth $X/\mathbf{F}_q$ is "independent of ℓ" results instantly from Deligne's fundamental result [1]. In this paper we will combine Deligne's techniques with his trick [2] of systematically "twisting" by very wildly ramified characters to show that "independence of ℓ" also holds for proper varieties which are suitable *limits* of proper smooth varieties, and for certain related open varieties. Here is a precise statement:

THEOREM. *Let Y be a smooth geometrically connected curve over $\mathbf{F}_q$, X a smooth $\mathbf{F}_q$-scheme, and $f: X \to Y$ a proper morphism of $\mathbf{F}_q$-schemes. Then all of the fibres of f, as well as the total space X, are "independent of ℓ."*

We also give some "independence" results for the way the sheaves $R^i f_* \mathbf{Q}_\ell$ on Y degenerate, we give a "new" proof of the local invariant cycle theorem, we show that for each i, j, the characteristic polynomials

$$\det(1 - TF \mid H^i_{\text{comp}}(Y \underset{\mathbf{F}_q}{\otimes} \overline{\mathbf{F}}_q; R^j f_* \mathbf{Q}_\ell))$$

Manuscript received September 30, 1982.

*Research supported in part by NSF Grants MCS 79-06083 and MCS 78-01436.

have **Z**-coefficients independent of $\ell \neq p$, and we show that for each $\ell \neq p$ the Leray spectral sequence for f in compact $\mathbf{Q}_\ell$-cohomology

$$E_2^{i,j} = H^i_{\text{comp}}(Y \underset{\mathbf{F}_q}{\otimes} \overline{\mathbf{F}}_q, R^j f_* \mathbf{Q}_\ell)) \Rightarrow H^{i+j}_{\text{comp}}(X \underset{\mathbf{F}_q}{\otimes} \overline{\mathbf{F}}_q; \mathbf{Q}_\ell)$$

is degenerate at E_2.

In the text, we actually give a souped-up version of these results, where we allow *coefficients* on X in a compatible collection of "pure complexes" in the sense of [1, 6.2]. In this generality, we no longer need to assume X itself smooth over $\mathbf{F}_q$ (the smoothness assures that the *constant sheaf* is a pure complex), but for non-smooth X one is hard-pressed to construct compatible systems of pure complexes. For example. if $j : U \hookrightarrow X^{\text{red}}$ is the inclusion of a dense open set which is smooth over $\mathbf{F}_q$, then according to [0] for each $\ell \neq p$ the "middle extension" $j_{*!}(\mathbf{Q}_\ell)$ is a pure complex on X, but it is not yet proven that these middle extensions are compatible for variable ℓ.

In an appendix, we apply our methods to prove some "independence of ℓ" results for compatible families of pure lisse sheaves on open curves. These last results may also be obtained by combining [1], 1.9 with [2], 9.8, and seem to have been "well-known" to the specialists, though nowhere explicitly recorded.

The Notion of (E, Λ)-compatibility. Fix a prime number p, a number field E of finite degree over $\mathbf{Q}$, and a non-empty set Λ of non-archimedean places of E, none of which divides p.

Let X be a scheme (always separated) of finite type over a finite field $\mathbf{F}_q$ of characteristic p. Suppose that for each $\lambda \in \Lambda$ we are given a constructible E_λ-sheaf $\mathfrak{F}_\lambda$ on X. We say that the collection $\{\mathfrak{F}_\lambda\}$ is (E, Λ)-compatible on X if, for every closed point x of X, the local characteristic polynomial of geometric Frobenius at x,

$$\det(1 - TF_x | (\mathfrak{F}_\lambda)_{\bar{x}}),$$

has coefficients in E, independent of λ.

More generally, if for each $\lambda \in \Lambda$ we are given an object K_λ in $D^b_c(X, E_\lambda)$, we say that the collection $\{K_\lambda\}$ is (E, Λ) compatible on X if, for every integer i, the collection of i'th cohomology sheaves $\{\underline{\mathcal{H}}^i(K_\lambda)\}$ is (E, Λ)-compatible on X.

Here are some elementary sorites on (E, Λ)-compatibility.

Suppose that for each $\lambda \in \Lambda$ we are given a lisse E_λ-sheaf $\mathcal{L}_\lambda$ on X, and a complex $K_\lambda \in D_c^b(X, E_\lambda)$. If both of the collections $\{\mathcal{L}_\lambda\}$ and $\{K_\lambda\}$ are (E, Λ)-compatible on X, then so is the collection $\{K_\lambda \otimes \mathcal{L}_\lambda\}$.

If we fix an (E, Λ)-compatible collection of lisse, *rank one* $\mathcal{L}_\lambda$'s on X, then for any collection of complexes $K_\lambda \in D_c^b(X, E_\lambda)$, we have an equivalence

$$\{K_\lambda\}\text{'s are } (E, \Lambda)\text{-compatible on } X$$

$$\Leftrightarrow$$

$$\{K_\lambda \otimes \mathcal{L}_\lambda\}\text{'s are } (E, \Lambda)\text{-compatible on } X.$$

For any morphism of finite type $g: Z \to X$, the inverse image $\{g^*(K_\lambda)\}$ of an (E, Λ)-compatible family on X is an (E, Λ)-compatible family on Z.

Review of L-functions. The L-function of $X/\mathbf{F}_q$ with coefficients in a constructible E_λ-sheaf $\mathcal{F}_\lambda$ is the power series over E_λ in one variable T defined as the Euler product, extended to all closed points $x \in X$,

$$L(X/\mathbf{F}_q, \mathcal{F}_\lambda)(T) \stackrel{\text{dfn}}{=} \prod \det(1 - T^{\deg(x)} F_x \,|\, (\mathcal{F}_\lambda)_{\bar{x}})^{-1}.$$

Here "$\deg(x)$" means the degree over $\mathbf{F}_q$ of the residue field at the closed point x. The L-function of $X/\mathbf{F}_q$ with coefficients in an object K_λ in $D_c^b(X, E_\lambda)$ is defined to be

$$L(X/\mathbf{F}_q, K_\lambda) \stackrel{\text{dfn}}{=} \prod_i L(X/\mathbf{F}_q, \underline{\mathcal{H}}^i(K_\lambda))^{(-1)^i}.$$

We denote by $H^i_{\text{comp}}(X/\mathbf{F}_q, \mathcal{F}_\lambda)$, resp. $\mathbf{H}^i_{\text{comp}}(X/\mathbf{F}_q, K_\lambda)$, the compact cohomology (resp. hypercohomology) groups of $X \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q$ with coefficients in $\mathcal{F}_\lambda$ (resp. K_λ); these are finite dimensional E_λ-vector spaces on which $\text{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$ acts continuously. We denote by $F_q \in \text{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$ the geometric Frobenius (the *inverse* of the automorphism $\alpha \to \alpha^q$ of $\overline{\mathbf{F}}_q$). According to the Lefschetz trace formula, we have fundamental identities

$$L(X/\mathbf{F}_q, \mathcal{F}_\lambda)(T) = \prod_i \det(1 - TF_q \,|\, H^i_{\text{comp}}(X/\mathbf{F}_q, \mathcal{F}_\lambda))^{(-1)^{i+1}}$$

$$L(X/\mathbf{F}_q, K_\lambda)(T) = \prod_i \det(1 - TF_q \,|\, \mathbf{H}^i_{\text{comp}}(X/\mathbf{F}_q, K_\lambda))^{(-1)^{i+1}}$$

More generally, if

$$f: X \to Y$$

is a morphism of $\mathbf{F}_q$-schemes of finite type, then for $K_\lambda \in D_c^b(X, E_\lambda)$ we have

$$L(X, K_\lambda) = L(Y, Rf_! K_\lambda);$$

indeed for each closed point $y \in Y$, the stalk of $R^i f_! K_\lambda$ at $\bar{y}$, with its action of F_y, is none other than $\mathbf{H}^i_{\text{comp}}(f^{-1}(y)/\mathbf{F}_q(y); K_\lambda)$ with its action of $F_{\mathbf{F}_q(y)}$.

The Notion of Physical (E, Λ)-compatibility on curves. Let Y be a smooth geometrically connected curve over $\mathbf{F}_q$, and $\mathfrak{F}_\lambda$ a constructible E_λ-sheaf on Y. For any non-void open set $U \overset{j}{\hookrightarrow} Y$ over which $\mathfrak{F}_\lambda$ is lisse, we have a natural morphism of sheaves on Y

$$\mathfrak{F}_\lambda \to j_* j^* \mathfrak{F}_\lambda,$$

which is independent of the auxiliary choice of U. It is an isomorphism precisely over *the* "ouvert de lissité" of $\mathfrak{F}_\lambda$. Its kernel, image and cokernel are successively denoted (cf. [4], 4.3)

$$(\mathfrak{F}_\lambda)_{\text{pct}}, \qquad (\mathfrak{F}_\lambda)_{\text{npct}}, \qquad (\mathfrak{F}_\lambda)_{\text{quot}}.$$

They sit in tautological short exact sequences

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & (\mathfrak{F}_\lambda)_{\text{pct}} & \longrightarrow & \mathfrak{F}_\lambda & \longrightarrow & (\mathfrak{F}_\lambda)_{\text{npct}} & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & j_* j^* \mathfrak{F}_\lambda & & \\
 & & & & & & \downarrow & & \\
 & & & & & & (\mathfrak{F}_\lambda)_{\text{quot}} & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

We remark in passing that the formation of $(\mathfrak{F}_\lambda)_{\text{pct}}$, $(\mathfrak{F}_\lambda)_{\text{npct}}$, $(\mathfrak{F}_\lambda)_{\text{quot}}$, $j_*j^*\mathfrak{F}_\lambda$, and of the entire diagram in which they sit commutes with the operation of tensoring by any *lisse* E_λ-sheaf on Y.

The sheaves $(\mathfrak{F}_\lambda)_{\text{pct}}$ and $(\mathfrak{F}_\lambda)_{\text{quot}}$ are skyscraper sheaves, supported at the finite set of points of non-lissité of $\mathfrak{F}_\lambda$.

If Y is *open*, then the inclusion of $(\mathfrak{F}_\lambda)_{\text{pct}}$ into $\mathfrak{F}_\lambda$ induces an isomorphism

$$H^0_{\text{comp}}(Y/\mathbf{F}_q, (\mathfrak{F}_\lambda)_{\text{pct}}) \xrightarrow{\sim} H^0_{\text{comp}}(Y/\mathbf{F}_q, \mathfrak{F}_\lambda),$$

while the projection of $\mathfrak{F}_\lambda$ onto $(\mathfrak{F}_\lambda)_{\text{npct}}$ induces isomorphisms

$$H^i_{\text{comp}}(Y/\mathbf{F}_q, \mathfrak{F}_\lambda) \xrightarrow{\sim} H^i_{\text{comp}}(Y/\mathbf{F}_q, (\mathfrak{F}_\lambda)_{\text{npct}}),$$

for $i = 1, 2$ (both sides vanish for $i > 2$).

We say that $\mathfrak{F}_\lambda$ on Y satisfies the local invariant cycle theorem if $(\mathfrak{F}_\lambda)_{\text{quot}} = 0$, i.e., if we have a surjective map

$$\mathfrak{F}_\lambda \twoheadrightarrow j_*j^*\mathfrak{F}_\lambda.$$

Given, for each $\lambda \in \Lambda$, a constructible E_λ-sheaf $\mathfrak{F}_\lambda$ on Y, we say that the collection $\{\mathfrak{F}_\lambda\}$ is physically (E, Λ)-compatible if all of the collections

$$\{(\mathfrak{F}_\lambda)_{\text{pct}}\}, \quad \{(\mathfrak{F}_\lambda)\}, \quad \{(\mathfrak{F}_\lambda)_{\text{npct}}\}, \quad \{j_*j^*\mathfrak{F}_\lambda\}, \quad \{(\mathfrak{F}_\lambda)_{\text{quot}}\},$$

are (E, Λ)-compatible. (If we refer to these collections in the order they are given, it is enough to check 1, 3, 5, as is visible from the diagram of exact sequences they sit in.) The terminology "physical" for this sort of compatibility is motivated by the fact that if the $\mathfrak{F}_\lambda$'s are physically (E, Λ)-compatible, then they all have exactly the same ouvert de lissité.

Statement of the theorem. Let Y be a smooth geometrically connected curve over $\mathbf{F}_q$, X an $\mathbf{F}_q$-scheme of finite type,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & \operatorname{Spec}(\mathbf{F}_q) & \end{array}$$

a morphism of $\mathbf{F}_q$-schemes, and w an integer. For each $\lambda \in \Lambda$, let K_λ be an object of $D_c^b(X_\lambda, E_\lambda)$. Suppose that:

A. The K_λ's are each pure complexes of weight w.
B. The K_λ's are (E, Λ)-compatible on X.
C. For each λ, the natural map "forget supports" is an isomorphism

$$\mathbf{R}^i f_! K_\lambda \xrightarrow{\sim} \mathbf{R}^i f_* K_\lambda$$

for every i (a condition which is *automatically* satisfied if f is proper).

Then:

1. For each integer i, the sheaves $\mathbf{R}^i f_! K_\lambda$ on Y are physically (E, Λ)-compatible, and satisfy the local invariant cycle theorem.
2. For each integer i and each $\lambda \in \Lambda$ the punctual sheaf $(\mathbf{R}^i f_! K_\lambda)_{\text{pct}}$ on Y is punctually pure of weight $w + i$.
3. ("mise pour memoire") For each integer i and each $\lambda \in \Lambda$, the sheaf $\mathbf{R}^i f_! K_\lambda$ on Y, when restricted to any open set $U \underset{j}{\hookrightarrow} Y$ over which it is lisse, becomes punctually pure of weight $w + i$ on U.
4. For each i, j, the sheaves $R^i b_! \mathbf{R}^j f_! K_\lambda$ on $\operatorname{Spec}(\mathbf{F}_q)$ are (E, Λ)-compatible, i.e., the characteristic polynomial of F on $H^i_{\text{comp}}(Y/\mathbf{F}_q, \mathbf{R}^j f_! K_\lambda)$ has E-coefficients independent of λ. For $i \neq 1$ (and for all i if Y is complete) these polynomials are *pure*, of weight $i + j + w$.
5. The Leray spectral sequence of sheaves on $\operatorname{Spec}(\mathbf{F}_q)$

$$E_2^{i,j} = R^i b_! \mathbf{R}^j f_! K_\lambda \Rightarrow \mathbf{R}^{i+j} a_! K_\lambda$$

degenerates at E_2.
6. For each integer i, the sheaves $\mathbf{R}^i a_! K_\lambda$ on $\operatorname{Spec}(\mathbf{F}_q)$ are (E, Λ)-compatible, i.e., the characteristic polynomials of F on the $\mathbf{H}^i_{\text{comp}} \cdot (X/\mathbf{F}_q, K_\lambda)$ have E-coefficients independent of λ.

Before beginning the proof, let us explain how the theorem stated in the introduction is a special case of this one. Thus we assume X smooth over $\mathbf{F}_q$, and f proper. Because X is smooth, the constant sheaves $\{\mathbf{Q}_\ell\}_{\ell \neq p}$ on X are all pure complexes of weight $w = 0$, and they are obviously (E, Λ)-compatible with $E = \mathbf{Q}$, $\Lambda = \{\text{all primes } \ell \neq p\}$. Because f is proper, hypothesis C is automatic. Therefore the theorem applies, and gives that the coefficients in question lie in $\mathbf{Q}$, and are independent of

$\ell \neq p$. That they actually lie in $\mathbf{Z}$ follows from the general integrality theorems of [3].

Proof of 3. The hypothesis C, written as an isomorphism in $D_c^b(Y; E_\lambda)$

$$Rf_!K_\lambda \xrightarrow{\sim} Rf_*K_\lambda,$$

gives, upon applying Verdier duality, an isomorphism:

$$D(Rf_!K_\lambda) \xleftarrow{\sim} D(Rf_*K_\lambda) \xrightarrow{\sim} Rf_!(DK_\lambda),$$

whence $Rf_!K_\lambda$ is a pure complex on Y of weight w (cf. [1]). Therefore on any open set $U \hookrightarrow Y$ over which all the sheaves $R^if_!K_\lambda$ are lisse, they are punctually pure of weight $w + i$. Since in any case each $R^if_!K$ is *mixed* on Y, the fact that it is punctually pure of weight $w + i$ over some non-void open set of lissité implies that it is punctually pure of this weight over its entire ouvert de lissité. ([1], 1.8.12).

Deduction of 4, 5, 6 from 1, 2, 3. The "serious" conclusions of the theorem are 1, 2, and 3, in the sense that 4, 5, 6 result from these. Indeed 6 is an immediate consequence of 5 (the degeneration at E_2 of the Leray spectral sequence) and 4 (the (E, Λ)-compatibility of E_2 terms). Conclusion 5 is itself a direct consequence of 4, because, Y being a curve, we automatically have $E_2^{i,j} = 0$ for $i \neq 0, 1, 2$, so the only possibly non-zero differential in the entire spectral sequence is

$$d_2 : E_2^{0,j} \to E_2^{2,j-1}.$$

But this d_2 must vanish, because, again by 4, $E_2^{0,j}$ is pure of weight $j + w$ while $E_2^{2,j-1}$ is pure of a different weight $2 + j - 1 + w$.

We next deduce 4 from 1, 2, 3. Let us fix an integer j, and a place λ, and write temporarily

$$\mathfrak{F}_\lambda = \mathbf{R}^jf_*K_\lambda,$$

$$P_i(\mathfrak{F}_\lambda) = \det(1 - TF | H^i_{\text{comp}}(Y/\mathbf{F}_q, \mathfrak{F}_\lambda)).$$

Then by the local invariant cycle theorem we have a short exact sequence on Y ($j : U \hookrightarrow Y$ denoting any ouvert de lissité of $\mathfrak{F}_\lambda$)

$$0 \to \mathfrak{F}_{\lambda,\mathrm{pct}} \to \mathfrak{F}_\lambda \to j_* j^* \mathfrak{F}_\lambda \to 0.$$

Because $\mathfrak{F}_{\lambda,\mathrm{pct}}$ is punctual, it has no higher cohomology, and the long exact cohomology sequence gives

$$P_0(\mathfrak{F}_\lambda) = P_0(\mathfrak{F}_{\lambda,\mathrm{pct}}) P_0(j_* j^* \mathfrak{F}_\lambda)$$

$$P_1(\mathfrak{F}_\lambda) = P_1(j_* j^* \mathfrak{F}_\lambda)$$

$$P_2(\mathfrak{F}_\lambda) = P_2(j_* j^* \mathfrak{F}_\lambda).$$

The cohomological formula for the L-function of $\mathfrak{F}_{\lambda,\mathrm{pct}}$ gives the relation

$$L(Y/\mathbf{F}_q, \mathfrak{F}_{\lambda,\mathrm{pct}}) = \frac{1}{P_0(\mathfrak{F}_{\lambda,\mathrm{pct}})}.$$

Let us exploit this: by 1, the punctual sheaves $\mathfrak{F}_{\lambda,\mathrm{pct}}$ on Y are (E, Λ)-compatible, hence also their L-functions; because the $\mathfrak{F}_{\lambda,\mathrm{pct}}$ are both punctual and punctually pure of weight $j + w$ by 2, their L-functions are pure of weight $j + w$. Hence the polynomials $P_0(\mathfrak{F}_{\lambda,\mathrm{pct}})$ are (E, Λ)-compatible, and pure of weight $j + w$.

We now turn to the $P_i(j_* j^* \mathfrak{F}_\lambda)$. Suppose first that Y is complete. Because $\mathfrak{F}_\lambda$ is pure of weight $j + w$ on its ouvert de lissité by 3, Deligne's fundamental theorem assures us that $P_i(j_* j^* \mathfrak{F}_\lambda)$ is pure of weight $i + j + w$. Therefore there can be no cancellation in the cohomological expression for the L-function

$$L(Y/\mathbf{F}_q, j_* j^* \mathfrak{F}_\lambda) = \frac{P_1(j_* j^* \mathfrak{F}_\lambda)}{P_0(j_* j^* \mathfrak{F}) P_2(j_* j^* \mathfrak{F}_\lambda)}$$

By 1, the sheaves $j_* j^* \mathfrak{F}_\lambda$, and hence their L-functions, are (E, λ)-compatible. Therefore we recover $P_1(j_* j^* \mathfrak{F}_\lambda)$ intrinsically as *the numerator* of this L-function, and we recover $P_0 P_2$ as *the denominator.* Thus both P_1 and $P_0 P_2$ are in $E[T]$, independent of λ. Because P_0 and P_2 are *pure* of *different* weights, they must individually lie in $E[T]$, independent of λ, as the factors of *the denominator* of the L-function of weights $j + w$ and $2 + j + w$ respectively.

If Y is an open curve, then, as already remarked, we have

$$P_0(\mathfrak{F}_\lambda) = P_0(\mathfrak{F}_{\lambda,\text{pct}}), \qquad P_0(j_*j^*\mathfrak{F}_\lambda) = 1,$$

and the cohomological formula for L reduces to

$$L(Y/\mathbf{F}_q, j_*j^*\mathfrak{F}_\lambda) = \frac{P_1(j_*j^*\mathfrak{F}_\lambda)}{P_2(j_*j^*\mathfrak{F}_\lambda)}.$$

Here the $P_2(j_*j^*\mathfrak{F}_\lambda)$ is trivially pure of weight $2 + j + w$ (because the sheaf in question is pure of weight $j + w$ over its ouvert de lissité, and we are on a curve), while by Deligne we know that $P_1(j_*j^*\mathfrak{F}_\lambda)$ is mixed of weight $\leq 1 + j + w$. So again there can be no cancellation, and we recover the P_1 and P_2 intrinsically as the numerator and denominator of the L-functions, themselves (E, Λ)-compatible because, by 1, the sheaves $j_*j^*\mathfrak{F}_\lambda$ are (E, Λ)-compatible on Y.

Some elementary reduction steps.

I. (Field Extension). Let E' be a finite extension of E, and let Λ' be the set of all those places λ' of E' lying over some place $\lambda \in \Lambda$. Then it is enough to prove the theorem for (E', Λ') and the collection of $E'_{\lambda'}$-complexes $\{K_\lambda \otimes_{E_\lambda} E'_{\lambda'}\}$. Indeed this results from the following elementary lemma:

LEMMA. *Suppose given for each $\lambda \in \Lambda$ an element $e(\lambda) \in E_\lambda$. Suppose there exists an element $e' \in E'$ such that for every $\lambda \in \Lambda$, and every place λ' of E' lying over λ, we have*

$$e' = e(\lambda) \quad \textit{in} \quad E'_{\lambda'}.$$

Then there exists an element in E, such that

$$e = e(\lambda) \quad \textit{in} \quad E_\lambda$$

for every $\lambda \in \Lambda$. (Of course this e can be none other than e', i.e., e' lies in E).

Proof. Because the trace is the sum of local traces, we have for each λ the identity in E_λ

$$\begin{aligned}\operatorname{trace}_{E'/E}(e') &= \sum_{\lambda'|\lambda} \operatorname{trace}_{E'_{\lambda'}/E_\lambda}(e') \\ &= \sum_{\lambda'|\lambda} \operatorname{trace}_{E'_{\lambda'}/E_\lambda}(e(\lambda)) \\ &= \sum_{\lambda'|\lambda} \deg(E'_{\lambda'}/E_\lambda)\cdot e(\lambda) \\ &= \deg(E'/E)\cdot e(\lambda).\end{aligned}$$

Therefore for each $\lambda \in \Lambda$ we have

$$e(\lambda) = \frac{1}{\deg(E'/E)} \cdot \operatorname{trace}_{E'/E}(e'). \qquad \text{Q.E.D.}$$

Therefore we may and will henceforth assume that E contains the p'th roots of unity.

II. (Pairwise comparison). We may and will henceforth assume the set Λ consists of at most two places, say λ_1 and λ_2 (possibly $\lambda_1 = \lambda_2$).

III. (Local on Y). Statements 1 and 2 of the theorem (as well as its hypotheses), are clearly Zariski local on Y. Therefore it suffices to show that for every closed point $y \in Y$, there is some Zariski open neighborhood U of Y over which 1, 2 of the theorem are true. For a given closed point $y \in Y$, we may pick an affine open neighborhood U of y, such that over $U - y$ each of the finitely many E_{λ_i}-constructible sheaves $\mathbf{R}^j f_! K_{\lambda_i}$, $i = 1, 2$, is lisse (simply delete from $U - y$ any of the finitely many points of non-lissité of any of these finitely many sheaves).

Therefore we may and will henceforth suppose that Y is open, and that all the $\mathbf{R}^j f_! K_\lambda$, all j, all $\lambda \in \Lambda$, are lisse over $Y - y$. We must prove 1 and 2 of the theorem "at y."

IV. (Twisting). Suppose for each $\lambda \in \Lambda$ we are given a lisse, rank one E_λ-sheaf $\mathcal{L}_\lambda$ on Y, which is pure of weight zero, and suppose that the collection of $\mathcal{L}_\lambda$'s is (E, Λ)-compatible on Y. Then 1, 2 of the theorem hold for the K_λ's if and only if they hold for the $K_\lambda \otimes f^*(\mathcal{L}_\lambda)$'s. Indeed these complexes on X are still visibly pure of weight w and (E, Λ)-compatible. Because the $\mathcal{L}_\lambda$'s are lisse, the natural maps

$$\mathcal{L}_\lambda \otimes \mathbf{R}^i f_! K \to \mathbf{R}^i f_!(K_\lambda \otimes f^*(\mathcal{L}_\lambda))$$

$$\mathcal{L}_\lambda \otimes \mathbf{R}^i f_* K_\lambda \to \mathbf{R}^i f_*(K_\lambda \otimes f^*(\mathcal{L}_\lambda))$$

are isomorphisms. Therefore the hypotheses A, B, C hold for the K_λ's if and only if they hold for the $K_\lambda \otimes f^*(\mathcal{L}_\lambda)$'s. As already noted, we have natural isomorphisms of sheaves on Y.

$$\begin{array}{ccc} \mathcal{L}_\lambda \otimes (R^i f_! K_\lambda)_{\text{pct}} & \xrightarrow{\sim} & (\mathcal{L}_\lambda \otimes R^i f_! K_\lambda)_{\text{pct}} \\ & & \downarrow \wr \\ & & (R^i f_!(K_\lambda \otimes f^*(\mathcal{L}_\lambda)))_{\text{pct}}, \end{array}$$

and similarly for "npct" and "quot" and "$j_* j^*$." Therefore the conclusions 1, 2 of the theorem hold for the K_λ's if and only if they hold for the $K_\lambda \otimes f^*(\mathcal{L}_\lambda)$'s.

Twisting by wild characters. Let k be an algebraically closed field of characteristic p, C a proper smooth connected curve over k, $U \subset C$ a nonvoid affine open in C, and $D = C - U$ its finite complement. We denote by K the function field of the curve C, K^{sep} a separable closure of K, and by Gal the galois group of K^{sep}/K. For each closed point x of C, viewed as a place of K, we choose a place $\bar{x}$ of K^{sep} lying over it. We denote by $I(\bar{x})$ the corresponding inertia subgroup of Gal, and by $P(\bar{x}) \subset I(\bar{x})$ its p-Sylow subgroup, the wild inertia group, which is a closed normal subgroup sitting in a canonical short exact sequence

$$0 \to P(\bar{x}) \to I(\bar{x}) \to \prod_{\ell \neq p} \mathbf{Z}_\ell(1) \to 0.$$

The quotient $\Pi_{\ell \neq p} \mathbf{Z}_\ell(1)$ is the "tame inertia group" $I_{(\bar{x})}^{\text{tame}}$ at x.

The "upper numbering" filtration on $I(\bar{x})$ is a decreasing filtration of $I(\bar{x})$ by closed normal subgroups $I(\bar{x})^{(r)}$, indexed by real numbers $r > 0$; for $0 < r_1 < r_2$ we have

$$I \supset P \supset I^{(r_1)} \supset I^{(r_2)},$$

$$P = \text{the closure of } \bigcup_{r>0} I^{(r)}$$

$$\{1\} = \bigcap_{r>0} I^{(r)}.$$

Let V_λ be a continuous finite-dimensional representation of $P(\bar{x})$ in an E_λ-vector space (E_λ being a finite extension of $\mathbf{Q}_\ell$, $\ell \neq p$). Because P is pro-p, while an open subgroup of $\mathrm{Aut}_{E_\lambda}(V_\lambda)$ is pro-ℓ, the representation necessary factors through a finite quotient of P, hence is P-semisimple. Furthermore, the closed subgroup $I^{(r)}$ operates trivially on V_λ for $r \gg 0$. The Swan conductor of V_λ is a non-negative real number

$$\mathrm{Swan}_x(V_\lambda) \geq 0,$$

which is *additive* in V_λ and which for W_λ an *irreducible* representation of P is defined by

$$\frac{\mathrm{Swan}_x(W_\lambda)}{\dim(W_\lambda)} = g.\ell.b.\ \{r > 0 \text{ such that } I^{(r)} \text{ acts trivially on } W_\lambda\}.$$

Alternatively, we can define $\mathrm{Swan}_x(V_\lambda)$ as the total height of a certain "Swan polygon" which is analogous to the Newton polygon of an irregular singularity. Consider the integer-valued function ψ on $\mathbf{R}$ defined by

$$\psi(r) = \begin{cases} \lim_{\epsilon \to 0+} \dim(V_\lambda^{I^{(r+\epsilon)}}/V_\lambda^{I^{(r-\epsilon)}}) & \text{for} \quad r > 0; \\ \dim(V_\lambda^P) & \text{for} \quad r = 0; \\ 0 & \text{for} \quad r < 0. \end{cases}$$

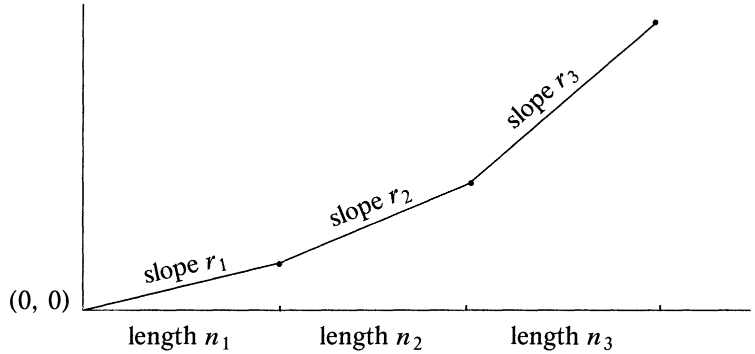
Clearly $\psi(r) = 0$ for all but finitely many r, and

$$\sum_r \psi(r) = \dim V_\lambda.$$

The points r where $\psi(r) \neq 0$ are called the *slopes* of V_λ, the integers $\psi(r)$ are called their *multiplicities.* By definition we have

$$\mathrm{Swan}_x(V_\lambda) = \sum_r r\psi(r).$$

The Swan polygon of V_λ is the polygon formed out of the slopes of V_λ, say $0 \leq r_1 < r_2 < \cdots < r_k$, and their multiplicities $n_1 = \psi(r_1), \ldots, n_k = \psi(r_k)$ in the usual way:



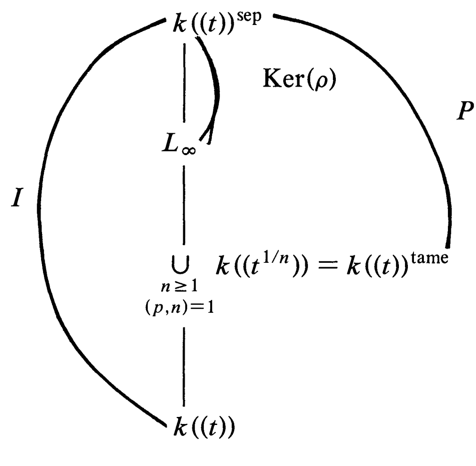
Its endpoint is $(\dim V_\lambda, \mathrm{Swan}(V_\lambda))$.

It follows from the Hasse-Arf theorem that if V_λ is the restriction to P of a continuous representation of the entire inertia group I, then the Swan polygon has integral break-points, i.e., each of the slopes r_i is a rational number, and the product $r_i n_i$ of a slope with its multiplicity is an integer. In particular, the Swan conductor $\Sigma n_i r_i$ of such a V_λ is an integer.

The following lemma and corollary clarify the possible non-integrality of $\mathrm{Swan}(V_\lambda)$ when V_λ doesn't extend to a representation of I.

LEMMA. *Any representation ρ of P which factors through a finite quotient H of P, in particular any V_λ of the type we are considering, extends to a representation ρ' of an open subgroup I' of I with $P \subset I'$, such that ρ' factors through an isomorphic finite quotient $H' \simeq H$ of I'.*

Proof. Fix a uniformizing parameter t at our place x of K. Then the x-adic completion of K is $k((t))$, and we have a diagram



The fixed field L_∞ of $\mathrm{Ker}(\rho)$ is a finite galois extension of $k((t))^{\mathrm{tame}}$, so is the "translation" to $k((t))^{\mathrm{tame}}$ of a finite galois extension $L_n/k((t^{1/n}))$ for some sufficiently divisible n prime to p, with isomorphic galois group. Then $I' = \mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t^{1/n})))$ "works." Q.E.D.

COROLLARY. *The slopes of V_λ are rational numbers, as is its Swan conductor.*

Proof. For some n prime to p, V_λ extends to $I_n = \mathrm{Gal}(k((t))^{\mathrm{sep}}/k((t^{1/n})))$. By the Hasse-Arf theorem, we know that the break-points of the Swan polygon of V_λ will be integral if we compute using the upper number filtration of I_n *rather* than that of I. But these filtrations are related by the simple change of scale

$$I^{(r)} = I_n^{(nr)} \quad \text{for} \quad r > 0.$$

Therefore the Newton polygon of V_λ as a representation of P (viewed as a subgroup of I) is obtained from the Newton polygon of V_λ (as a representation of the same P viewed as a subgroup of I_n), by applying the map $(x, y) \mapsto (x, y/n)$. Q.E.D.

We will make constant use of the following lemmas:

LEMMA. *Let V_λ be a continuous finite dimensional representation of $P(\bar{x})$ in an E_λ-vector space, E_λ a finite extension of $\mathbf{Q}_\ell$, $\ell \neq p$. Let $r > 0$ be a real number. Then we have the implications*

$$\begin{array}{ccc} r > \mathrm{Swan}_x(V_\lambda) & \Longrightarrow & r > \textit{largest slope of } V_\lambda \\ & & \Downarrow \\ & & I^{(r)} \text{ acts trivially on } V_\lambda \end{array}$$

Proof. Obvious from the definitions. Q.E.D.

LEMMA. *Let V_λ be as above. Let L_λ be a continuous one-dimensional E_λ-representation of P, and suppose that*

$$\mathrm{Swan}_x(L_\lambda) > \textit{the largest slope of } V_\lambda,$$

(*e.g., this is automatic if* $\mathrm{Swan}_x(L_\lambda) > \mathrm{Swan}_x(V_\lambda)$). *Then*

(1) *the tensor product representation $L_\lambda \otimes V_\lambda$ has only a single slope, namely* $\mathrm{Swan}_x(L_\lambda)$, *with multiplicity* $\dim(V_\lambda)$;

(2) *the representation* $L_\lambda \otimes V_\lambda$ *has no invariants* (*and no coinvariants, being semisimple*):

$$(L_\lambda \otimes V_\lambda)^P = 0 = (L_\lambda \otimes V_\lambda)_P.$$

Proof. For any real number $r > \mathrm{Swan}_x(L_\lambda)$, the subgroup $I^{(r)}$ operates trivially on both L_λ and V_λ, hence on $L_\lambda \otimes V_\lambda$. For r in the interval

$$\text{largest slope of } V_\lambda < r < \mathrm{Swan}_x(L_\lambda),$$

the group $I^{(r)}$ acts *trivially* on V_λ, but non-trivially on L_λ. Because L_λ is *one-dimensional*, we have $(L_\lambda)^{I^{(r)}} = 0$, whence we find

$$(L_\lambda \otimes V_\lambda)^{I^{(r)}} = (L_\lambda)^{I^{(r)}} \otimes V_\lambda = 0.$$

Therefore if $0 < r_1 < r$, we also have

$$(L_\lambda \otimes V_\lambda)^P \subset (L_\lambda \otimes V_\lambda)^{I^{(r_1)}} \subset (L_\lambda \otimes V_\lambda)^{I^{(r)}} = 0.$$

Thus the function $\psi(r)$ vanishes except at $r = \mathrm{Swan}_x(L_\lambda)$, where it takes the value $\dim(V_\lambda)$. Q.E.D.

We now return to our global situation of a proper smooth connected curve C over k, an affine open $U \subset C$ with finite complement $D = C - U$. Let $\mathfrak{F}_\lambda$ be a constructible E_λ-sheaf on U. Its geometric generic fibre $(\mathfrak{F}_\lambda)_{\bar\eta}$ is a continuous finite dimensional E_λ-representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$. For each closed point $x \in C$, we have $P(\bar x) \subset I(\bar x) \subset \mathrm{Gal}(K^{\mathrm{sep}}/K)$, which allows us to view $(\mathfrak{F}_\lambda)_{\bar\eta}$ as a representation of $I(\bar x)$, and of $P(\bar x)$. For each closed point x in C, we define $\mathrm{Swan}_x(\mathfrak{F}_\lambda)$ to be the Swan conductor of $(\mathfrak{F}_\lambda)_{\bar\eta}$ viewed as a representation of $P(\bar x)$. (Of course, $\mathrm{Swan}_x(\mathfrak{F}_\lambda) = 0$ for all but finitely many points x simply because $(\mathfrak{F}_\lambda)_{\bar\eta}$ is a trivial representation of $I(\bar x)$ for almost all x.)

If U' is an open subset of C with

$$U \underset{j}{\hookrightarrow} U'$$

then the sheaves $R^i j_* \mathfrak{F}_\lambda$ are described explicitly as follows: Over U, $j_*\mathfrak{F}_\lambda = \mathfrak{F}_\lambda$ and all $R^i j_* \mathfrak{F}_\lambda$, $i > 0$, vanish.

At a closed point $x \in U' - U$, the stalks of $R^i j_* \mathfrak{F}_\lambda$ at x are given by the formulas

$$\begin{array}{rl}(R^i j_* \mathcal{F}_\lambda)_{\bar{x}} = & H^i(I(\bar{x}), (\mathcal{F}_\lambda)_{\bar{\eta}}) \\ & \Big\| \\ & H^i(I^{\text{tame}}_{(\bar{x})}; ((\mathcal{F}_\lambda))_{\bar{\eta}})^{P(\bar{x})}),\end{array}$$

the second equality because P is pro-p, hence has no higher cohomology in λ-adic representations with $\lambda | \ell$, $\ell \neq p$.

The highest cohomology group $H^2_{\text{comp}}(U, \mathcal{F}_\lambda)$ is the group

$$((\mathcal{F}_\lambda)_{\bar{\eta}})_{\text{Gal}}(-1)$$

of (twisted) coinvariants of Gal acting on the geometric generic fibre. It is therefore a *quotient* of the (twisted) coinvariants under any subgroup of Gal. In particular, for any $x \in C$, we have a surjection

$$((\mathcal{F}_\lambda))_{\bar{\eta}})_{P(\bar{x})}(-1) \twoheadrightarrow H^2_{\text{comp}}(U, \mathcal{F}_\lambda).$$

Combining these remarks with the earlier lemma, we obtain the following extremely useful proposition.

Proposition. *Let C be a proper smooth geometrically connected curve over an algebraically closed field k of characteristic $p > 0$, U an affine open of C, U' a larger open, $j : U \hookrightarrow U'$ the inclusion. Let $\mathcal{F}_\lambda$ be a constructible E_λ-sheaf on U, and $\mathcal{L}_\lambda$ be a lisse E_λ-sheaf of rank one on U, with E_λ a finite extension of $\mathbf{Q}_\ell$, $\ell \neq p$. Then*

(1) *If at some point $x \in C$ we have*

$$\text{Swan}_x(\mathcal{L}_\lambda) > \text{Swan}_x(\mathcal{F}_\lambda),$$

then $H^2_{\text{comp}}(U, \mathcal{F}_\lambda \otimes \mathcal{L}_\lambda) = 0$.

(2) *If at every point x in $U' - U$ we have*

$$\text{Swan}_x(\mathcal{L}_\lambda) > \text{Swan}_x(\mathcal{F}_\lambda),$$

then the natural map on U' is an isomorphism

$$j_!(\mathcal{F}_\lambda \otimes \mathcal{L}_\lambda) \xrightarrow{\sim} Rj_*(\mathcal{F}_\lambda \otimes \mathcal{L}_\lambda),$$

(i.e., $j_! \xrightarrow{\sim} j_*$ and $R^i j_* = 0$ for $i > 0$).

Proof. The Swan inequalities at x guarantee that $(\mathfrak{F}_\lambda \otimes \mathcal{L}_\lambda)_{\bar{\eta}}$ has neither invariants nor coinvariants under $P(\bar{x})$. Q.E.D.

Construction of (E, Λ)-compatible systems of wild $\mathcal{L}_\lambda$'s. Let C be a proper smooth geometrically connected curve over $\mathbf{F}_q$, E a number field containing the p'th roots of unity, and Λ a set of prime-to-p places of E. Let $D \subset C$ be a non-empty finite set of closed points, $U = C - D$ the corresponding open curve over $\mathbf{F}_q$.

Construction. Given C, D, there exists an integer N (depending only the genus of C and the degree of D) such that for any integer $n \geq N$ which is prime to p, there exists an (E, Λ)-compatible collection of lisse rank one E_λ-sheaves $\mathcal{L}_\lambda$ on U with the following properties:

(1) each $\mathcal{L}_\lambda$ on U is pure of weight zero.

(2) at each closed point x of D, the Swan conductor of $\mathcal{L}_\lambda$ (at any closed point of $C \otimes \bar{\mathbf{F}}_q$ lying over x) is given by

$$\mathrm{Swan}_x(\mathfrak{F}_\lambda) = n \quad \text{for all} \quad x \in D.$$

Proof. By Riemann-Roch, there exists, for all n sufficiently large, a function f_n on C which is holomorphic on $C - D$ and which has a pole of exact order n at each point of D. If we further restrict n to be prime to p, then for any constructible E_λ-sheaf $\mathfrak{F}_\lambda$ on the affine line $\mathbf{A}^1$ over $\mathbf{F}_q$, we have, for any $x \in D$,

$$\mathrm{Swan}_x(f_n^*(\mathfrak{F}_\lambda)) = n \cdot \mathrm{Swan}_\infty(\mathfrak{F}_\lambda).$$

Therefore it suffices to construct on $\mathbf{A}^1$ an (E, Λ)-compatible family of lisse rank-one $\mathcal{L}_\lambda$'s which are all pure of weight zero and all of which have $\mathrm{Swan}_\infty(\mathcal{L}_\lambda) = 1$. This we do by standard Artin-Schreier theory. The Artin-Schreier covering $T^p - T = X$ of the X-line is finite etale galois with galois group $\mathbf{F}_p$. It defines a continuous homomorphism

$$\pi_1(\mathbf{A}^1_{\mathbf{F}_p}) \to \mathbf{F}_p.$$

If we compose with a non-trivial additive character of $\mathbf{F}_p$

$$\psi : \mathbf{F}_p \xrightarrow{\sim} \mu_p(E),$$

and then compose, for $\lambda \in \Lambda$, with the obvious inclusion

$$\mu_p(E) \subset E^{\times} \to E_{\lambda}^{\times},$$

we obtain an (E, Λ)-compatible system of continuous characters of weight zero

$$\psi_{\lambda}: \pi_1(\mathbf{A}^1_{\mathbf{F}_p}) \to E_{\lambda}^{\times},$$

whose corresponding $\mathcal{L}_{\lambda}$'s on $\mathbf{A}^1_{\mathbf{F}_p}$ are well-known to have $\mathrm{Swan}_{\infty}(\mathcal{L}_{\lambda}) = 1$.

The end of the proof. Let us recall the situation to which we have reduced ourselves. The number field E contains the p'th roots of unity, Λ consists of at most two places, Y is open, $y \in Y$ is a closed point, and all the $\mathbf{R}^i f_! K_{\lambda}$, all i and all λ, are lisse on $Y - y$. We must prove the theorem "at y," i.e., we must show that the stalks at y of the sheaves

$$\begin{aligned}&\mathfrak{F}^i_{\lambda} \overset{\text{notation}}{=\!=\!=} \mathbf{R}^i f_! K_{\lambda}\\&(\mathfrak{F}^i_{\lambda})_{\text{pct}}\\&(\mathfrak{F}^i_{\lambda})_{\text{npct}}\\&(\mathfrak{F}^i_{\lambda})_{\text{quot}}\end{aligned}$$

have (E, Λ)-compatible characteristic polynomials of F_y, and that the very last of these sheaves has stalk zero at y, this last vanishing being the local invariant cycle theorem "at y."

We denote by C the complete smooth geometrically connected curve over $\mathbf{F}_q$ in which Y is open;

$$Y - y \underset{k}{\hookrightarrow} Y \underset{j}{\hookrightarrow} C.$$

Let us define an auxiliary integer

$$N = \sup_{\lambda \in \Lambda} \sup_{i \in \mathbf{Z}} \sup_{x \in C - Y} \mathrm{Swan}_x(\mathfrak{F}^i_{\lambda}).$$

(Because $\#\Lambda$ is ≤ 2, and $\mathfrak{F}^i_{\lambda} = 0$ for all but finitely many values of i, and $C - Y$ is finite, the sup is well defined.)

Let $\{\mathcal{L}_\lambda\}$ be an (E, Λ)-compatible system of lisse rank-one E_λ-sheaves on Y, each pure of weight zero, such that at each point $x \in C - Y$ we have

$$\mathrm{Swan}_x(\mathcal{L}_\lambda) > N,$$

for each $\mathcal{L}_\lambda$. Such systems exist by our earlier Artin-Schreier construction.

As already explained, we may and will replace the K_λ's upstairs by the $K_\lambda \otimes f^*(\mathcal{L}_\lambda)$'s, which has the effect downstairs of replacing the $\mathcal{F}^i_\lambda$'s by the $\mathcal{F}^i_\lambda \otimes \mathcal{L}_\lambda$'s. In view of the proposition, this twisting reduces us to the case where in addition we have, for all i, λ

$$(*) \qquad j_!(\mathcal{F}^i_\lambda) \xrightarrow{\sim} Rj_*(\mathcal{F}^i_\lambda)$$

$$(**) \qquad H^2_{\mathrm{comp}}(U/\mathbf{F}_q, \mathcal{F}^i_\lambda) = 0 \quad \text{for } any \text{ open} \quad U \subset Y.$$

LEMMA 1. *For each $\lambda \in \Lambda$ and each pair of integers i, j, the natural map "forget supports" induces an isomorphism*

$$H^i_{\mathrm{comp}}(Y/F_q, \mathcal{F}^j_\lambda) \xrightarrow{\sim} H^i(Y/\mathbf{F}_q, \mathcal{F}^j_\lambda).$$

Proof. Follows from (*). Q.E.D.

LEMMA 2. *For each $\lambda \in \Lambda$ and each $n \in \mathbf{Z}$, the natural map "forget supports" induces an isomorphism*

$$\mathbf{H}^n_{\mathrm{comp}}(X/\mathbf{F}_q, K_\lambda) \xrightarrow{\sim} \mathbf{H}^n(X/\mathbf{F}_q, K_\lambda).$$

Proof. Compare the Leray spectral sequences for f in compact and in ordinary cohomology: there is a natural "forget supports" map between them

$$\begin{array}{ccc} E^{i,j}_{2,\mathrm{comp}} = H^i_{\mathrm{comp}}(Y/\mathbf{F}_q, \mathbf{R}^j f_! K_\lambda) & \Rightarrow & \mathbf{H}^{i+j}_{\mathrm{comp}}(X/\mathbf{F}_q, K_\lambda) \\ \downarrow & & \downarrow \\ E^{i,j}_2 = H^i(Y/\mathbf{F}_q, R^j f_* K_\lambda) & \Longrightarrow & \mathbf{H}^{i+j}(X/\mathbf{F}_q, K_\lambda) \end{array}$$

which in view of Lemma 1 above and hypothesis C induces an isomorphism at E_2, hence on the abutments. Q.E.D.

LEMMA 3. *For each $n \in \mathbf{Z}$, the characteristic polynomials of Frobenius on the E_λ-vector spaces $H^n_{\text{comp}}(X/\mathbf{F}_q, K_\lambda)$ are pure of weight $n + w$, and they are (E, Λ)-compatible.*

Proof. The purity follows formally from Lemma 2, and the fact that K_λ is by hypothesis a pure complex of weight w on X (cf. [1]). Because the complexes K_λ on X are (E, Λ)-compatible, their L-functions $L(X/\mathbf{F}_q, K_\lambda)$ are (E, Λ)-compatible. From their cohomological expression as alternating products of characteristic polynomials of Frobenius on the cohomology groups, the n'th of which is pure of weight $n + w$, it follows that for each λ and each n we can intrinsically recover the characteristic polynomial on $H^n(X/\mathbf{F}_q, K_\lambda)$ as *the* factor of *the* L-function which is pure of weight $n + 2$. Q.E.D.

LEMMA 4. *The Leray spectral sequence for f in compact cohomology*

$$E_2^{i,j} = H^i_{\text{comp}}(Y/\mathbf{F}_q, \mathfrak{F}^j_\lambda) \Rightarrow H^{i+j}_{\text{comp}}(X/\mathbf{F}_q, K_\lambda)$$

degenerates at E_2; indeed $E_2^{i,j} = 0$ unless $i = 0$ or $i = 1$.

Proof. The degeneration results from the asserted vanishing of $E_2^{i,j}$ for $i \neq 0, 1$, itself a consequence of (**) and the fact that Y is a curve. Q.E.D.

LEMMA 5. *For each i, j, the characteristic polynomial of Frobenius on $H^i_{\text{comp}}(Y/\mathbf{F}_q, \mathfrak{F}^j_\lambda)$ is pure of weight $i + j + w$.*

Proof. Immediate from Lemmas 3 and 4. Q.E.D.

LEMMA 6. *The characteristic polynomial of Frobenius on*

$$H^0_{\text{comp}}(Y/\mathbf{F}_q, (\mathfrak{F}^j_\lambda)_{\text{pct}})$$

is pure of weight $w + j$.

Proof. Because Y is open, the inclusion of $(\mathfrak{F}^j_\lambda)_{\text{pct}}$ into $\mathfrak{F}^j_\lambda$ induces an isomorphism on H^0_{comp}. The conclusion is now the special case $i = 0$ of Lemma 5. Q.E.D.

LEMMA 7. *The characteristic polynomial of F_y on the stalk at y of $(\mathfrak{F}^j_\lambda)_{\text{pct}}$ is pure of weight $w + j$. This polynomial is equal to the characteristic polynomial of Frobenius on $H^0_{\text{comp}}(Y/\mathbf{F}_q, \mathfrak{F}^j_\lambda)$.*

Proof. By hypothesis, the sheaf $(\mathfrak{F}^j_\lambda)_{\text{pct}}$ on Y is concentrated at y. Therefore it has no higher cohomology, and so the cohomological expression of its L-function (which consists of a *single* Euler factor) shows that the characteristic polynomial in question is equal to the one whose purity is given in Lemma 6. Because Y is open, the inclusion $(\mathfrak{F}^j_\lambda)_{\text{pct}} \hookrightarrow \mathfrak{F}^j_\lambda$ induces an isomorphism on H^0_{comp}, whence the second conclusion. Q.E.D.

LEMMA 8. *The characteristic polynomial of Frobenius on*

$$H^1_{\text{comp}}(Y/\mathbf{F}_q, (\mathfrak{F}^j_\lambda)_{\text{npct}})$$

is pure of weight $w + 1 + j$.

Proof. The projection of $\mathfrak{F}^j_\lambda$ onto $(\mathfrak{F}^j_\lambda)_{\text{npct}}$ induces an isomorphism on H^1_{comp}, so once again we have a special case ($i = 1$) of Lemma 5. Q.E.D.

LEMMA 9. *The sheaf* $(\mathfrak{F}^j_\lambda)_{\text{quot}}$ *vanishes* (*the local invariant cycle theorem*!).

Proof. Because $\mathfrak{F}^j_\lambda$ is lisse on the open set $Y - y \xrightarrow{k} Y$, we have a tautological short exact sequence on Y

$$0 \to (\mathfrak{F}^j_\lambda)_{\text{npct}} \to k_* k^* \mathfrak{F}^j_\lambda \to (\mathfrak{F}^j_\lambda)_{\text{quot}} \to 0.$$

Because Y is open, and $k^*\mathfrak{F}^j_\lambda$ is lisse on $Y - y$, the sheaf $k_* k^* \mathfrak{F}^j_\lambda$ on Y has vanishing H^0_{comp}. The cohomology sequence gives an inclusion

$$H^0_{\text{comp}}(Y/\mathbf{F}_q, (\mathfrak{F}^j_\lambda)_{\text{quot}}) \hookrightarrow H^1_{\text{comp}}(Y/\mathbf{F}_q, (\mathfrak{F}^j_\lambda)_{\text{npct}}).$$

Therefore, by Lemma 8, $H^0_{\text{comp}}(Y\mathbf{F}_q, (\mathfrak{F}^j_\lambda)_{\text{quot}})$ is pure of weight $w + 1 + j$. Because $(\mathfrak{F}^j_\lambda)_{\text{quot}}$ is punctual, indeed supported at y, this means that $(\mathfrak{F}^j_\lambda)_{\text{quot}}$ is punctually pure of weight $1 + j + w$. But $(\mathfrak{F}^j_\lambda)_{\text{quot}}$ is a quotient of $k_* k^* \mathfrak{F}^j_\lambda$, a sheaf which is of the form k_*(a lisse sheaf on $Y - y$, pure of weight $j + w$), and which by ([1], 1.8.9) is hence punctually mixed of weight $\leq j + w$. Q.E.D.

LEMMA 10. *For each* j, *the sheaves* $\mathfrak{F}^j_\lambda$ *on* $Y - y$ *are* (E, Λ)*-compatible on* $Y - y$.

Proof. As already proven (conclusion 3 of the theorem), these sheaves on $Y - y$ are punctually pure of weight $w + j$. For fixed λ and a fixed closed point z in $Y - y$, let $\mathbf{F}_q(z)$ denote the residue field at z, and let $X(z)/\mathbf{F}_q(z)$ be the fibre of $f: X \to Y$ over z. The cohomological expression for $L(X(z)/\mathbf{F}_q(z), K_\lambda)$ as the alternating product of the pure characteristic polynomials of F_z on the $(\mathfrak{F}^j_\lambda)_{\bar{z}}$, the j'th pure of weight $w + j$, allows us to intrinsically recover the j'th characteristic polynomial as the "factor purely of weight $w + j$" in the L-function. Because the K_λ's are (E, Λ)-compatible on X, they remain so when restricted to $X(z)$, and hence the L-functions in question are themselves (E, Λ)-compatible. Therefore so are their "factors purely of weight $w + j$" for each j. Q.E.D.

LEMMA 11. *For each j, the characteristic polynomials of Frobenius on the groups*

$$H^1_{\text{comp}}(Y - y/\mathbf{F}_q, \mathfrak{F}^j_\lambda)$$

are (E, Λ)-compatible.

Proof. Consider the L-function

$$L(Y - y/\mathbf{F}_q, \mathfrak{F}^j_\lambda).$$

By Lemma 10, these L-functions are (E, Λ)-compatible. Because $\mathfrak{F}^j_\lambda$ is lisse on $Y - y$, and Y − y is open, the cohomological expression of the L-function is simply

$$L(Y - y/\mathbf{F}_q, \mathfrak{F}^j_\lambda) = \frac{\det(1 - TF|H^1_{\text{comp}}(Y - y/\mathbf{F}_q; \mathfrak{F}^j_\lambda))}{\det(1 - TF|H^2_{\text{comp}}(Y - y/\mathbf{F}_q, \mathfrak{F}^j_\lambda))}.$$

Because $\mathfrak{F}^j_\lambda$ is lisse and pure of weight $j + w$ on $Y - y$, the denominator of the cohomological expression is pure of weight $w + j + 2$, while its numerator is mixed of weight $\leq w + j + 1$. Therefore there is no cancellation, and we recover the characteristic polynomials on H^1_{comp} as *the* numerators of *the* L-functions; hence these polynomials are themselves (E, Λ)-compatible. Q.E.D.

LEMMA 12. *For each j, the characteristic polynomials of F_y on the stalk at $\bar{y}$ of $(\mathfrak{F}^j_\lambda)_{\text{npct}}$ are (E, Λ)-compatible.*

Proof. Consider the sheaf $(\mathfrak{F}^j_\lambda)_{\text{npct}}$ on Y. Its restriction to $Y - y$ is the lisse sheaf $\mathfrak{F}^j_\lambda$ on $Y - y$. Consider the "excision" long exact sequence in compact cohomology for $(\mathfrak{F}^j_\lambda)_{\text{npct}}$ and the geometric situation

$$Y - y \overset{k}{\hookrightarrow} Y \overset{i}{\hookleftarrow} \{y\}$$

i.e., the long exact cohomology sequence associated to the short exact sequence on Y

$$0 \to k_! k^* \mathfrak{F}^j_\lambda \to (\mathfrak{F}^j_\lambda)_{\mathrm{npct}} \to i_* i^*((\mathfrak{F}^j)_{\mathrm{npct}}) \to 0.$$

The quotient is concentrated at y, and the first two have no punctual sections over Y, so the cohomology sequence gives

$$0 \to H^0_{\mathrm{comp}}(Y/\mathbf{F}_q, i_* i^*((\mathfrak{F}^j_\lambda)_{\mathrm{npct}})) \to H^1_{\mathrm{comp}}(Y - y/\mathbf{F}_q, \mathfrak{F}^j_\lambda)$$

$$\to H^1_{\mathrm{comp}}(Y/\mathbf{F}_q, (\mathfrak{F}^j_\lambda)_{\mathrm{npct}}) \to 0$$

Because $i_* i^*((\mathfrak{F}^j_\lambda)_{\mathrm{npct}}))$ is concentrated at y, it has no higher cohomology, and the cohomological formula for its L-function yields the identity

$$\det(1 - TF | H^0_{\mathrm{comp}}(Y/\mathbf{F}_q, i^* i_*((\mathfrak{F}^j_\lambda)_{\mathrm{npct}})) = \det(1 - TF_y | ((\mathfrak{F}^j_\lambda)_{\mathrm{npct}})_{\bar{y}}).$$

The exact sequence thus yields the identity

$$\det(1 - TF | H^1_{\mathrm{comp}}(Y - y/\mathbf{F}_q, \mathfrak{F}^j_\lambda))$$

$$= \det(1 - TF_y | ((\mathfrak{F}^j)_{\mathrm{npct}})_{\bar{y}}) \times \det(1 - TF | H^1_{\mathrm{comp}}(Y/\mathbf{F}_q, (\mathfrak{F}^j_\lambda)_{\mathrm{npct}}).$$

The left members, for variable λ are (E, Λ)-compatible, by Lemma 11. The first factor on the right is mixed of weight $\leq w + j$ (because $(\mathfrak{F}^j_\lambda)_{\mathrm{npct}}$ is a sub-sheaf of $k_* k^* \mathfrak{F}^j_\lambda$, which is punctually mixed of weight $\leq w + j$), while by Lemma 8, the second factor on the right is pure of weight $w + 1 + j$. Therefore these factors are individually describable as factors of the (E, Λ)-compatible left member defined by certain weight conditions. Q.E.D.

LEMMA 13. *For each j, the characteristic polynomials of Frobenius on the groups $H^1_{\mathrm{comp}}(Y/\mathbf{F}_q, \mathfrak{F}^j_\lambda)$ are (E, Λ)-compatible.*

Proof. The projection of $\mathfrak{F}^j_\lambda$ onto $(\mathfrak{F}^j_\lambda)_{\mathrm{npct}}$ induces an isomorphism on H^1_{comp}, and the (E, Λ)-compatibility for $H^1_{\mathrm{comp}}(Y | \mathbf{F}_q, (\mathfrak{F}^j_\lambda)_{\mathrm{npct}})$ is proven in the proof of Lemma 12 above. Q.E.D.

LEMMA 14. *For each j, the characteristic polynomials of F_y on the stalk of $\bar{y}$ of $(\mathfrak{F}^j_\lambda)_{\mathrm{pct}}$ are (E, Λ)-compatible.*

Proof. The degeneration of the Leray spectral sequence and the vanishing of $E_2^{i,j}$ for $i \neq 0, 1$ (Lemma 4) yields for each j a short exact sequence

$$0 \to H^1_{\text{comp}}(Y/\mathbf{F}_q, \mathfrak{F}^{j-1}_\lambda) \to H^j_{\text{comp}}(X/\mathbf{F}_q, K_\lambda) \to H^0_{\text{comp}}(Y/\mathbf{F}_q, \mathfrak{F}^j_\lambda) \to 0.$$

By Lemmas 13 and 3 respectively, the first two terms have (E, Λ)-compatible characteristic polynomials of Frobenius, hence also the last. By Lemma 7, these last characteristic polynomials are precisely those of F_y on the stalk at $\bar{y}$ of $(\mathfrak{F}^j_\lambda)$. Q.E.D.

Lemma 15. *For each j, the sheaves $\mathfrak{F}^j_\lambda$ on Y are physically (E, Λ)-compatible.*

Proof. By Lemmas 14, 12, and 9, for each j the collections of sheaves

$$\{(\mathfrak{F}^j_\lambda)_{\text{pct}}\}, \ \{(\mathfrak{F}^j_\lambda)_{\text{npct}}\}, \text{ and } \{(\mathfrak{F}^j_\lambda)_{\text{quot}} = 0\}$$

are all (E, Λ)-compatible. Q.E.D.

Lemma 16. *The theorem is proven.*

Proof. Concatenate Lemmas 15, 9, and 7. Q.E.D.

Appendix: Some independence results on curves. Throughout this section, we will deal with the following situation:

C is a smooth geometrically connected curve over a finite field $\mathbf{F}_q$ of characteristic p, $Y \subset C$ is a non-empty open in C, and $U \subset Y$ is an affine open in Y:

$$U \overset{k}{\hookrightarrow} Y \overset{j}{\hookrightarrow} C.$$

E is a number field of finite degree over $\mathbf{Q}$, and Λ is a non-empty set of prime-to-p nonarchimedean places of E.

w is an integer.

For each $\lambda \in \Lambda$, $\mathfrak{F}_\lambda$ is a lisse E_λ-sheaf on U which is pure of weight w, and the collection of $\{\mathfrak{F}_\lambda\}$'s is (E, Λ)-compatible on U.

Theorem. *In the above situation, we have*

1. *The sheaves $k_*\mathfrak{F}_\lambda$ on Y are (E, Λ)-compatible.*

2. *For every integer i, the characteristic polynomials of Frobenius on the cohomology groups $H^i_{\text{comp}}(Y/\mathbf{F}_q, k_*\mathfrak{F}_\lambda)$ are (E, Λ)-compatible.*

Proof. We first deduce the second conclusion from the first. By 1, the L-function $L(Y/\mathbf{F}_q, k_*\mathfrak{F}_\lambda)$ are (E, Λ)-compatible. Consider their cohomological expressions. If $Y = C$, then by [1] for each i the cohomology groups $H^i_{\text{comp}}(Y/\mathbf{F}_q, k_*\mathfrak{F}_\lambda)$ are pure of weight $w + i$, and hence their characteristic polynomials of Frobenius can be intrinsically recovered from the L-function as "the factor which is purely of weight $w + i$". If $Y \neq C$, then Y is open, so $H^0_{\text{comp}}(Y/\mathbf{F}_q, k_*\mathfrak{F}_\lambda) = 0$, the H^1_{comp} is mixed of weight $\leq w + 1$, and the H^2_{comp} is pure of weight $w + 2$. Therefore there is no cancellation in the cohomological expression of the L-function, and we recover the characteristic polynomial of Frobenius on H^2_{comp} (resp. on H^1_{comp}) as *the* denominator (resp. *the* numerator) of the L-function.

The statement 1 is Zariski local on Y, so we may assume that Y is affine and that $Y - U$ consists of a single closed point y. As already explained, we may also assume that the field E contains the p'th roots of unity, that Λ consists of at most two places. We are free to twist by any (E, Λ)-compatible family of lisse, rank-one E_λ-sheaves $\mathfrak{L}_\lambda$ on Y which are pure of weight zero (since $k_*(\mathfrak{F}_\lambda \otimes k^*\mathfrak{L}_\lambda) \xleftarrow{\sim} \mathfrak{L}_\lambda \otimes k_*\mathfrak{F}_\lambda$). Twisting by such a family which is sufficiently wild at all points of $C - Y$, we reduce to the case where the sheaves $k_*\mathfrak{F}_\lambda$ on Y satisfy

$$j_!(k_*\mathfrak{F}_\lambda) \xrightarrow{\sim} R_{j*}(k_*\mathfrak{F}_\lambda) \quad \text{on} \quad C,$$

so in particular satisfy

$$j_!(k_*\mathfrak{F}_\lambda) \xrightarrow{\sim} j_*k_*\mathfrak{F}_\lambda \quad \text{on} \quad C.$$

Passing to cohomology on C, we find, for all i:

$$H^i_{\text{comp}}(Y/\mathbf{F}_q, k_*\mathfrak{F}_\lambda) \xrightarrow{\sim} H^i(C/\mathbf{F}_q, j_*k_*\mathfrak{F}_\lambda),$$

so that for all i, $H^i_{\text{comp}}(Y/\mathbf{F}_q, k_*\mathfrak{F}_\lambda)$ is *pure* of weight $i + w$. Now consider the excision exact sequence for $k_*\mathfrak{F}_\lambda$ on Y and

$$U = Y - y \overset{k}{\hookrightarrow} Y \overset{i}{\hookleftarrow} \{y\}.$$

Because Y is open there is no H^0_{comp} on Y, and because $\{y\}$ is a point there is no H^1_{comp} on $\{y\}$, so the cohomology sequence gives

$$0 \to H^0_{\text{comp}}(Y/\mathbf{F}_q, i_* i^*(k_* \mathfrak{F}_\lambda)) \to H^1_{\text{comp}}(U/\mathbf{F}_q, \mathfrak{F}_\lambda)$$

$$\to H^1_{\text{comp}}(Y/\mathbf{F}_q, k_* \mathfrak{F}_\lambda) \to 0.$$

The characteristic polynomial of Frobenius on the H^0_{comp} term is just the characteristic polynomial of F_y on the stalk at $\bar{y}$ of $k_* \mathfrak{F}_\lambda$.

Because $\mathfrak{F}_\lambda$ is lisse on U and pure of weight w, the sheaf $k_* \mathfrak{F}_\lambda$ on Y is mixed of weight $\leq w$, and therefore this first characteristic polynomial is mixed of weight $\leq w$. The third characteristic polynomial, that on $H^1_{\text{comp}}(Y/\mathbf{F}_q, k_* \mathfrak{F}_\lambda)$ has already been proven pure of weight $w + 1$. Therefore it will suffice to prove that the middle characteristic polynomials on $H^1_{\text{comp}}(U/\mathbf{F}_q, \mathfrak{F}_\lambda)$, are (E, Λ)-compatible (for then the characteristic polynomial of F_y on $(k_* \mathfrak{F}_y)_{\bar{y}}$ are the "weight $\leq w$ factor" of these (E, Λ)-compatible polynomials, so are themselves (E, Λ)-compatible.

To see that the characteristic polynomials of Frobenius on $H^1_{\text{comp}} \cdot (U/\mathbf{F}_q, \mathfrak{F}_\lambda)$ are (E, Λ)-compatible, we notice that by hypothesis the $\mathfrak{F}_\lambda$'s are lisse and pure of weight w and (E, Λ)-compatible on U, so that once again we recover their H^1_{comp} as *the* numerator of the (E, Λ)-compatible L-functions $L(U/\mathbf{F}_q, \mathfrak{F}_\lambda)$. Q.E.D.

COROLLARY. *Suppose for each $\lambda \in \Lambda$ we are given a constructible E_λ-sheaf $\mathcal{G}_\lambda$ on C. Let $U_\lambda \subset C$ denote its exact "ouvert de lissité," j_λ the inclusion, and suppose that*

1. *For each λ, we have $\mathcal{G}_\lambda \xrightarrow{\sim} (j_\lambda)_*(j_\lambda)^*(\mathcal{G}_\lambda)$.*
2. *For each $\lambda, \lambda' \in \Lambda$, there exists a non-empty open set $U_{\lambda,\lambda'} \subset U_\lambda \cap U_{\lambda'}$ over which both $\mathcal{G}_\lambda$ and $\mathcal{G}_{\lambda'}$ are pure of some integer weight $w_{\lambda,\lambda'}$ and $(E, \{\lambda, \lambda'\})$-compatible.*

Then

1. *there is an integer w such that $w = w_{\lambda,\lambda'}$ for all λ, λ';*
2. *the sheaves $\mathcal{G}_\lambda$ on C are (E, Λ)-compatible on C;*
3. *the sheaves $\mathcal{G}_\lambda$ all have the same exact ouvert de lissité: $U_\lambda = U_{\lambda'}$ for all λ, λ'.*

Proof. Each $\mathcal{G}_\lambda$ is lisse on U_λ, hence on U_λ it is the direct image of its restriction to any non-empty open in U_λ. Applying this to the open

$U_{\lambda,\lambda'}$, we find that $\mathcal{G}_\lambda$ is lisse and mixed of weight $\leq w_{\lambda,\lambda'}$ on U_λ, and pure of weight $w_{\lambda,\lambda'}$ on $U_{\lambda,\lambda'}$. Therefore it is pure of weight $w_{\lambda,\lambda'}$ on U. Thus $w_{\lambda,\lambda'}$ is independent of λ', and by symmetry, it is also independent of λ. This proves 1. To prove 2 and 3, we may assume that λ consists of at most two places, say λ, λ'. Then both $\mathcal{G}_\lambda$ and $\mathcal{G}_{\lambda'}$ on C are the direct image of their restrictions to $U_{\lambda,\lambda'}$. Applying the previous theorem to the inclusion

$$U_{\lambda,\lambda'} \underset{k}{\hookrightarrow} Y = C.$$

and the sheaves $k_*\mathcal{G}_\lambda$, $k_*\mathcal{G}_{\lambda'}$, we obtain 2, the (E, Λ)-compatibility of $\mathcal{G}_\lambda$ and $\mathcal{G}_{\lambda'}$ on C. Finally, 3 follows from 2, because for a sheaf which is the direct image of its restriction to its ouvert de lissité, its ouvert de lissité consists exactly of the points where its stalk has maximal E_λ-dimension, and for a closed point y we recover this dimension as the *degree* of the characteristic polynomial of F_y on this stalk. Q.E.D.

REFERENCES

[0] Beilinson, Berstein, Deligne, Gabber, Faisceaux pervers, to appear in *Astérisque.*

[1] Deligne, P. La conjecture de Weil II, *Publ. Math. IHES* **52** (1981).

[2] Deligne, P. Les constantes des équations fonctionelles des fonctions L, in *Proceedings 1972 Antwerp Summer School on Modular Forms*, *Springer Lecture Notes*, **349** (1973).

[3] Deligne, P. Théorèmes d'intégralité, Appendix to Exposé XXI SGA VII, part II, *Springer Lecture Notes*, **340** (1973).

[4] Katz, N. Sommes exponentielles, redigé par G. Laumon, *Astérisque* **79** (1980).