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Introduction.

This talk is devoted to a part of Dwork's work on the variation of the zeta function of a variety over a finite field, as the variety moves through a family. Recall that for a single variety \( V/\mathbb{F}_q \), its zeta function is the formal series in \( t \)

\[
\text{Zeta}(V/\mathbb{F}_q; t) = \exp\left( \sum_{n \geq 1} \frac{t^n}{n} \right) \left( \# \text{ of points on } V \text{ rational over } \mathbb{F}_q^n \right).
\]

As a power series it has coefficients in \( \mathbb{Z} \), and in fact it is a rational function of \( t \) \([4]\). We shall generally view it as a rational function of a \( p \)-adic variable.

Suppose now we consider a one parameter family of varieties, i.e. a variety \( V/\mathbb{F}_p[\lambda] \). For each integer \( n \geq 1 \) and each point \( \lambda_0 \in \mathbb{F}_p^n \), the fibre \( V(\lambda_0)/\mathbb{F}_p^n \) has a zeta function \( \text{Zeta}(V(\lambda_0)/\mathbb{F}_p^n; t) \). We want to understand how this rational function of \( t \) varies when we vary \( \lambda_0 \) in the algebraic closure of \( \mathbb{F}_p \). Ideally, we might wish a "formula", of a \( p \)-adic sort, for, say, one of the reciprocal zeroes of \( \text{Zeta}(V(\lambda_0)/\mathbb{F}_p^n; t) \). A natural sort of "formula" would be a \( p \)-adic power series \( a(x) = \sum a_n x \) with coefficients \( a_n \in \mathbb{Z}_p \) tending to zero, with the property:

for every \( n \geq 1 \) and for every \( \lambda_0 \in \mathbb{F}_p^n \), let \( x_0 \in \mathbb{Q}_p \) be the unique quantity lying over \( \lambda_0 \) which satisfies

\( x_0 = x_0^p \).

Then

\[
a(x_0)a(x_0^p)\ldots a(x_0^{p^{n-1}})
\]

is a reciprocal zero of \( \text{Zeta}(V(\lambda_0)/\mathbb{F}_p^n; t) \), i.e., the numerator of \( \text{Zeta}(V(\lambda_0)/\mathbb{F}_p^n; t) \) is divisible by \( (1 - a(x_0)a(x_0^p)\ldots a(x_0^{p^{n-1}}))t \).
Now it is unreasonable to expect such a formula unless we can at least describe a priori which reciprocal zero it's a formula for! If, for example, we knew a priori that one and only one of the reciprocal zeroes were a p-adic unit, then we might reasonably hope for a formula for it. If, on the other hand, we knew a priori that precisely \( \nu \geq 2 \) of the reciprocal zeroes were p-adic units, we oughtn't hope to single one out; we could expect at best that we could describe the polynomial of degree \( \nu \) which has those \( \nu \) as its reciprocal zeroes. For instance, we might hope for a \( \nu \times \nu \) matrix \( A(X) \) with entries in \( \mathbb{Z}_p[[X]] \), their coefficients tending to zero, so that for each \( \lambda_o \in \mathbb{F}_p^n \), the characteristic polynomial

\[
\text{det}(I - t A(X_o)A(x)^{p^n-1})
\]

is the above polynomial.

In another optic, zeta functions come from cohomology, and to study their variation we should study the variation of cohomology. As Dwork discovered in 1961-63 in his study of families of hypersurfaces, their cohomology is quite rigid p-adically, forming a sort of structure on the base now called an F-crystal. Thanks to crystalline cohomology, we now know that this is a general phenomenon (cf. pt. 7 for a more precise statement). The relation with the "formula" viewpoint is this: a formula \( a(x) \) for one root is sub-F-crystal of rank 1, a formula \( A(x) \) for the \( \nu \) roots "at once" is a sub-F-crystal of rank \( \nu \).

So in fact this expose is about some of Dwork's recent work on variation of F-crystals, from the point of view of p-adic analysis. Due to space limitations, we have systematically suppressed the Monsky-Washnitzer "over-convergent" point of view in favor of the simpler but less rich "Krasner-analytic" or "rigid analytic" one (but cf. [16]). Among the casualties are Dwork's work on "excellent Liftings of Frobenius", and on the p-adic use of the Picard-Lefschetz formula, both of which are entirely omitted.
1. **F-crystals** ([1],[2]).

In down-to-earth terms, an F-crystal is a differential equation on which a "Frobenius" operates. Let us make this precise.

(1.0) Let \( k \) be a perfect field of characteristic \( p > 0 \), \( W(k) \) its Witt vectors, and \( S = \text{Spec}(A) \) a smooth affine \( W(k) \)-scheme. For each \( n \geq 0 \), we put \( S_n = \text{Spec}(A/p^{n+1}A) \), an affine smooth \( W(k) \)-scheme, and for \( n = \infty \) we put \( S^\infty = \text{the p-adic completion of } S = \text{Spec}(\lim A/p^n A) \). (Function theoretically, \( A^\infty = \lim A/p^n A \) is the ring of those rigid analytic functions of norm \( \leq 1 \) on the rigid analytic space underlying \( S \) which are defined over \( W(k) \)). For any affine \( W(k) \)-scheme \( T \) and any \( k \)-morphism \( f_o : T_o \rightarrow S_o \), there exists a compatible system of \( W(k) \)-morphisms \( f_n : T_n \rightarrow S_n \) with \( f_{n+1} \) lifting \( f_n \) (because \( T \) is affine and \( S \) smooth), or, equivalently, a \( W(k) \)-morphism \( f : T^\infty \rightarrow S^\infty \) lifting \( f_o \). Of course, there is in general no unicity in the lifting \( f \).

In particular, noting by \( \sigma \) the Frobenius automorphism of \( W(k) \), there exists a \( \sigma \)-linear endomorphism \( \varphi \) of \( S^\infty \) which lifts the \( p \)'th power endomorphism of \( S_o \). The interplay between \( S_o, S^\infty \) and \( \varphi \) is given by:

**Lemma 1.1.** (Tate-Monsky [24],[27]). Denote by \( \mathcal{C} \) the completion of the algebraic closure of the fraction field \( K \) of \( W(k) \), and by \( \mathcal{O}_\mathcal{C} \) its ring of integers.

1.1.1. The successive inclusions between the sets below are all bijections:

a) the \( \mathcal{C} \)-valued points of \( S \) (as \( W(k) \)-scheme)

b) the continuous \( W(k) \)-homomorphisms \( A^\infty \rightarrow \mathcal{O}_\mathcal{C} \)

c) " " \( A^\infty \rightarrow \mathcal{C} \)

d) the closed points of \( S^\infty \otimes \mathcal{C} \).
1.1.2. Every k-valued point $e_0$ of $S_0$ lifts uniquely to a $W(k)$-valued point $e$ of $S^\infty$ which verifies $\varphi \circ e = e \circ \sigma$. In fact, for any isometric extension $\tilde{\sigma}$ of $\sigma$ to $C$, $e$ is the unique $C$-valued point of $S^\infty$ which lifts $e_0$ and verifies $\varphi \circ e = e \circ \tilde{\sigma}$. The point $e$ is called the $\varphi$-Teichmüller representative of $e_0$. The Teichmüller points of $S^\infty$ ($C$-valued points $e$ satisfying $\varphi \circ e = e \circ \tilde{\sigma}$) are in bijective correspondence with the points of $S_0$ with values in the algebraic closure $\bar{k}$ of $k$, and all take values in $W(\bar{k})$.

(1.2) Let $H$ be a locally free $S^\infty$-module of finite rank, with an integrable connection $\nabla$ (for the continuous derivations of $S^\infty/W(k)$) which is nilpotent. This means that for any continuous derivation $D$ of $S^\infty/W(k)$ which is $p$-adically topologically nilpotent as additive endomorphism of $A^\infty$, the additive endomorphism $\nabla(D)$ of $H$ is also $p$-adically topologically nilpotent. For any affine $W(k)$-scheme $T$ which is $p$-adically complete, any pair of maps

$$f \quad g$$

which are congruent modulo a divided-power ideal of $T$ ($(p)$, for example), the connection $\nabla$ provides an isomorphism

$$\chi(f,g) : f^*H \sim g^*H.$$

This isomorphism satisfies

(i) $\chi(g,h) \chi(f,g) = \chi(f,h)$ if $T \xrightarrow{h} S^\infty$.

(ii) $\chi(fk, gk) = k^* \chi(f,g)$ if $R \xrightarrow{k} T \xrightarrow{f} S^\infty$.

(iii) $\chi(id, id) = id$.

The universal example of such a situation $T \xrightarrow{f} S^\infty$ is provided by
the "closed divided power neighborhood of the diagonal" $\text{P.D.-}\Delta(S^\infty)$, with its two projections to $S^\infty$. When, for examples, $S$ is etale over $\Delta^n_{W(k)}$, $\text{P.D.-}\Delta(S^\infty)$ is the spectrum of the ring of convergent divided power series over $A^\infty$ in $n$ indeterminates, the formal expressions

$$\sum a_{i_1,\ldots,i_n} t_1^{i_1} \cdots t_n^{i_n}$$

whose coefficients $a_{i_1,\ldots,i_n}$ are elements of $A^\infty$ which tend to zero (in the $p$-adic topology of $A^\infty$).

Any situation $T \xrightarrow{f} S^\infty$ of the type envisioned above can be factored uniquely

$$T \xrightarrow{f \times g} \text{P.D.-}\Delta(S^\infty) \xrightarrow{\text{pr}_2} S^\infty,$$

and we have

$$\chi(f,g) = (f \times g)^\ast \chi(\text{pr}_1,\text{pr}_2).$$

In fact, giving the isomorphism $\chi(\text{pr}_1,\text{pr}_2)$, subject to a cocycle condition, is equivalent to giving the nilpotent integrable connection $\nabla$.

(1.3) We may now define an $F$-crystal $H = (H,\nabla,F)$ as consisting of:

1. a "differential equation" $(H,\nabla)$ as above
2. for every lifting $\varphi : S^\infty \longrightarrow S^\infty$ of Frobenius, a horizontal morphism

$$F(\varphi) : \varphi^\ast H \longrightarrow H$$

which becomes an isomorphism upon tensoring with $Q$.

For different liftings $\varphi_1, \varphi_2$, we require the commutativity of the diagram below. (compare [11], section 5 and [12], section 2)
Given a $k$-valued point $e_0$ of $S_0$, let $\varphi_1$ and $\varphi_2$ be two liftings of Frobenius, and $e_1$ and $e_2$ the corresponding Teichmüller representatives. By inverse image, we obtain two $F$-crystals on $W(k)$, $(e_1^* H, e_1^* (F(\varphi_1)))$ and $(e_2^* H, e_2^* (F(\varphi_2)))$ which are explicitly isomorphic

\[ (e_1^* H, e_1^* (F(\varphi_1))) \xrightarrow{\sigma} (e_1^* H) \]
\[ (e_2^* H, e_2^* (F(\varphi_2))) \xrightarrow{\sigma} (e_2^* H) \]

We thus obtain an $F$-crystal on $W(k)$ (a free $W(k)$-module of finite rank together with a $\sigma$-linear endomorphism which is an isomorphism over $K$) which depends only on the point $e_0$ of $S_0$. In case $k$ is a finite field $\mathbb{F}_p^n$, then for every multiple, $m$, of $n$, the $m$-th iterate of the $\sigma$-linear endomorphism is linear over $W(\mathbb{F}_p^m)$. Its characteristic polynomial $\det(1 - t F^m)$ is denoted $F(H; e_0, \mathbb{F}_p^m, t)$.

2. $F$-crystals over $W(k)$ and their Newton polygons [19].

**Theorem 2.** (Manin–Dieudonné). Let $(H, F)$ be an $F$-crystal over $h/(k)$, and suppose $k$ algebraically closed.

2.1. $H$ admits an increasing finite filtration of $F$-stable sub-modules

\[ 0 \subset H_0 \subset H_1 \subset \ldots \]
whose associated graded is free, with the following property. There exists
a sequence of rational numbers in "lowest terms"

\[ 0 \leq \frac{a_0}{n_0} < \frac{a_1}{n_1} < \frac{a_2}{n_2} < \ldots \]

(if \( a_0 = 0 \), \( n_0 = 1 \); \( n_i \geq 1 \), \( a_i \geq 0 \), and \( (a_i, n_i) = 1 \) if \( a_i \neq 0 \))

such that

2.1.1. \( (H_i/H_{i-1}) \otimes K \) admits a K-base of vectors \( x \) which satisfy
\( F^{n_i}(x) = p^{a_i}x \), and its dimension is a multiple of \( n_i \).

2.1.2. If \( a_0/n_0 = 0 \), then \( H_i \) itself admits a \( W(k) \) base of elements
\( x \) satisfying \( Fx = x \), \( F \) is topologically nilpotent on \( H/H_o \), and the
rank of \( H_o \) is equal to the stable rank of the \( p \)-linear endomorphism of
the \( k \)-space \( H/pH \) induced by \( F \); \( H_o \) is then called the "unit root part" of
\( H \), or the "slope zero" part.

2.1.3. If (\( H,F \)) is deduced by extension of scalars from an \( F \)-crystal
\( (\mathfrak{M}, \mathfrak{F}) \) over \( W(k_0) \), \( k_o \) a perfect subfield of \( k \), then the filtration
descends to an \( \mathfrak{F} \)-stable filtration of \( \mathfrak{M} \). In case \( k_o \) is a finite field
\( \mathfrak{F}^n \), the eigenvalues of \( \mathfrak{F}^n \) on the i'th associated graded have \( p \)-adic
\( p \)-ordinal \( na_i/n_i \).

2.2. The rational numbers \( a_i/n_i \) are called the slopes of the \( F \)-crystal,
and the ranks of \( H_i/H_{i-1} \) are called the multiplicities of the slopes.
The slopes and their multiplicities characterize the \( F \)-crystal up to isogeny.
It is convenient to assemble the slopes and their multiplicities in the Newton polygon

When \((H, F)\) comes by extension of scalars from \((H, F)\) over \(W(F_n)\), this Newton polygon is the "usual" Newton polygon of the characteristic polynomial \(P(H; e, IF P^t)\), calculated with the ordinal function normalized by \(\text{ord}(p^n) = 1\).

3. Local Results; F-crystals on \(W(k)[[t_1, \ldots, t_n]]\).

(3.0) The completion of \(S^\infty\) along a \(k\)-valued point \(e_0\) of \(S_o\) is (non-canonically) isomorphic to the spectrum of \(W(k)[[t_1, \ldots, t_n]]\). In this optic, the set of \(W(k)\)-valued points of \(S^\infty\) lying over \(e_0\) becomes the n-fold product of \(pW(k)\), and the set of \(\mathfrak{c}_e\)-valued points of \(S^\infty\) lying over \(e_0\) becomes the n-fold product of the maximal ideal of \(\mathfrak{c}_e\) (namely, the values of \(t_1, \ldots, t_n\)).

By inverse image, any F-crystal on \(S^\infty\) gives an F-crystal on \(W(k)[[t_1, \ldots, t_n]]\).
Proposition 3.1. Let \((H, V, F)\) be an \(F\)-crystal over \(W(k)[[t_1, \ldots, t_n]]\).

3.1.1. Let \(W(k)\langle\langle t_1, \ldots, t_n \rangle\rangle\) denote the ring of convergent divided power series over \(W(k)\) (cf. 1.2). Then \(H \otimes W(k)\langle\langle t_1, \ldots, t_n \rangle\rangle\) admits a basis of horizontal (for \(V\)) sections.

3.1.2. Let \(K[[t_1, \ldots, t_n]]\) denote the ring of power series over \(K\) which are convergent in the open polydisc of radius one (i.e. series \(\sum a_{i_1 \ldots i_n} t_1^{i_1} \ldots t_n^{i_n}\) such that for every real number \(0 \leq r < 1\), \(|a_{i_1 \ldots i_n} r^{i_1 + \ldots + i_n}\) tends to zero). Then \(H \otimes K[[t_1, \ldots, t_n]]\) admits a basis of horizontal sections.

3.1.3. Every horizontal section of \(H \otimes W(k)\langle\langle t_1, \ldots, t_n \rangle\rangle\) fixed by \(F\) "extends" to a horizontal section of \(H\) (i.e. over all of \(W(k)[[t_1, \ldots, t_n]]\)).

Proof: 3.1.1. is completely formal: the two homomorphisms \(f, g : W(k)[[t_1, \ldots, t_n]] \rightarrow W(k)\langle\langle t_1, \ldots, t_n \rangle\rangle\) given by \(f = \text{natural inclusion}\), \(g = \text{evaluation at } (0, \ldots, 0)\), followed by the inclusion of \(W(k)\) in \(W(k)\langle\langle t_1, \ldots, t_n \rangle\rangle\), are congruent modulo the divided power ideal \((t_1, \ldots, t_n)\) of the \(\mathfrak{p}\)-adically complete ring \(W(k)\langle\langle t_1, \ldots, t_n \rangle\rangle\). Thus \(\chi(f, g)\) is an isomorphism between \(H \otimes W(k)\langle\langle t_1, \ldots, t_n \rangle\rangle\) with its induced connection and the "constant" module \(H(0, \ldots, 0) \otimes W(k)\langle\langle t_1, \ldots, t_n \rangle\rangle\) with connection \(1 \otimes d\).

3.1.2. is more subtle. Let's choose a particularly simple \(\varphi\) (as we may using 1.3.1), the one which sends \(t_i \mapsto t_i^p\), \(i=1, \ldots, n\), and is \(\sigma\)-linear. Choose a basis of the free \(W(k)[[t_1, \ldots, t_n]]\) module \(H\), and let \(A_\varphi\) denote the matrix of
Denote by $Y$ the matrix with entries in $W(k)[[t_1,\ldots,t_n]]$ whose columns are a basis of horizontal sections of $H \otimes W(k)[[t_1,\ldots,t_n]]$ (a "fundamental solution matrix"); in the notation of (2) above, it's the matrix of $\chi(g,f)$. Because $F$ is horizontal, we have the matricial relation

$$A_\varphi Y = Y \cdot A_\varphi (0,\ldots,0).$$

We must deduce that $Y$ converges in the open unit polydisc. We know this is true of $A_\varphi$, as it even has coefficients in $W(k)[[t_1,\ldots,t_n]]$. Since $A_\varphi (0,\ldots,0)$ is invertible over $K$ by definition of an $F$-crystal, we conclude that for any real number $0 \leq r < 1$, we have the implication

$$\varphi(Y) \text{ converges in the polydisc of radius } r \implies Y \text{ converges in the polydisc of radius } r.$$

On the other hand, writing $Y = \sum Y_{i_1} \ldots t_i^{i_1} \ldots t_n^{i_n}$, we have

$$\varphi(Y) = \sum \sigma(Y_{i_1} \ldots t_i^{i_1} \ldots t_n^{i_n}) t_i^{p_i^{i_1}} \ldots t_n^{p_i^{i_n}},$$

whence for any real $r > 0$, we have the implication

$$Y \text{ converges in the polydisc of radius } r \implies \varphi(Y) \text{ converges in the polydisc of radius } r^{1/p}.$$

Since $Y$ has entries in $W(k)[[t_1,\ldots,t_n]]$, it converges in the polydisc of radius $r_0 = |p|^{1/p-1}$, hence, iterating our two implications, in the polydisc of radius $r_o^{1/p^n}$ for every $n$; as $\lim (r_o^{1/p^n}) = 1$, we are done.

3.1.3. is similar to 3.1.2, only easier. If $y$ is a column vector with entries in $W(k)[[t_1,\ldots,t_n]]$ satisfying

$$A_\varphi \varphi(y) = y$$
then for every integer \( m \geq 1 \) we have

\[
A_\varphi \cdot \varphi(A_\varphi) \cdot \varphi^2(A_\varphi) \cdots \varphi^{m-1}(A_\varphi) \cdot \varphi^m(y) = y
\]

Since \( \varphi^m(y) \) is congruent to \( \varphi^m(y(0,\ldots,0)) \) modulo \( (t_1^{p^m}, \ldots, t_n^{p^m}) \), we have a \((t_1,\ldots,t_n)-adic\) limit formula for \( y \)

\[
y = \lim_{n \to \infty} A_\varphi \cdot \varphi(A_\varphi) \cdots \varphi^{n-1}(A_\varphi) \varphi^n(\varphi(0,\ldots,0))
\]

which shows that \( y \) has entries in \( W(k)[[t_1,\ldots,t_n]] \).

Q.E.D.

Remark 3.2. 3.1.2 shows that "most" differential equations do not admit any structure of F-crystal. For example, the differential equation for \( \exp(t^n) \) is nilpotent provided \( n \geq 1 \), but its local solutions around any point \( \alpha \in \mathbb{O} \) converge only in the disc of radius

\[
|p|^{1/p^n(p-1)}
\]

The meaning of 3.1.2 is this: for any two points \( e_1, e_2 \) of \( S \) with values in \( \mathbb{O} \) which are sufficiently near (congruent modulo \( p^{1/p^n(p-1)} \), the connection provides an explicit isomorphism of the two \( \mathbb{O} \)-modules \( e_1^\varphi(H) \) and \( e_2^\varphi(H) \). If the two points are further apart, but still congruent modulo the maximal ideal of \( \mathbb{O} \), 3.1.2 says the connection still gives an explicit isomorphism of the \( \mathbb{O} \)-vector spaces \( e_1^\varphi(H) \otimes \mathbb{C} \) and \( e_2^\varphi(H) \otimes \mathbb{C} \).
4. Global results: gluing together the "unit root" parts ([11], thm 4.1)

(4.0) Given an F-crystal $H = (H, V, F)$ and an integer $n \geq 0$, we denote by $H(-n)$ the F-crystal $(H, V, p^n F)$. An F-crystal of the form $H(-n)$ necessarily has all its slopes $\geq n$, though the converse need not be true.

Theorem 4.1. Suppose $k$ algebraically closed, and $H$ an F-crystal on $S^\infty$ such that at every $k$-valued point of $S_0$, its Newton polygon begins with a side of slope zero, always of the same length $\nu \geq 1$ (i.e., point by point, the unit root part has rank $\nu$). Suppose further that there exists a locally free submodule $\text{Fil} \subset H$ such that $H/\text{Fil}$ is locally free of rank $\nu$, and such that for every lifting $\phi$ of Frobenius, we have

$$F(\phi) (\phi^* \text{Fil}) \subset p H.$$

Then there exists a sub-crystal $U \subset H$, of rank $\nu$, whose underlying module $U$ is transversal to $\text{Fil}$ ($H = U \oplus \text{Fil}$) such that

4.1.1. $F$ is an isomorphism on $U$.
4.1.2. The connection $V$ on $U$ prolongs to a stratification.
4.1.3. The quotient F-crystal $H/U$ is of the form $V(-1)$.
4.1.4. The extension of F-crystals $0 \to U \to H \to H/U \to 0$ splits when pulled back to $W(k)$ along any $W(k)$-valued point of $S^\infty$.
4.1.5. If the situation $(H, \text{Fil})$ on $S^\infty/W(k)$ comes by extension of scalars from a situation $(H, \text{Fil})$ on $S^\infty/W(k_0)$, $k_0$ a perfect subfield of $k$, the F-crystal $U$ descends to an F-crystal $U$ on $S^\infty/W(k_0)$.
Proof. We may assume Fil, H and H/Fil are free, say of ranks \( r-v, r \) and \( v \). In terms of a basis of \( H \) adopted to the filtration \( Fil \subset H \), the matrix of \( F(\varphi) \) for some fixed choice of \( \varphi \) is of the form

\[
\begin{pmatrix}
    r-v & v \\
    \begin{pmatrix}
        pA & C \\
        pB & D
    \end{pmatrix}
\end{pmatrix}
\]

The hypothesis that there be \( v \) unit root point by point means \( D \) is invertible. Let's begin by finding for a free submodule \( U \subset H \) which is transversal to Fil and stable by \( F(\varphi) \cdot \varphi^* \). This means finding an \( r-v \times v \) matrix \( \eta \), such that the submodule of \( H \) spanned by the columns of

\[
\begin{pmatrix}
    \eta \\
    I
\end{pmatrix}
\]

(I denoting the \( v \times v \) identity matrix) is stable under \( F(\varphi) \cdot \varphi^* \).

But

\[
F(\varphi)\varphi^*(\eta) = \begin{pmatrix} pA & C \\ pB & D \end{pmatrix} \begin{pmatrix} \varphi^*(\eta) \\ I \end{pmatrix} = \begin{pmatrix} pA\varphi^*(\eta) + C \\ pB\varphi^*(\eta) + D \end{pmatrix},
\]

so that \( F \)-stability of \( \begin{pmatrix} \eta \\ I \end{pmatrix} \) is equivalent to having

\[
\begin{pmatrix}
    pA\varphi^*(\eta) + C \\
    pB\varphi^*(\eta) + D
\end{pmatrix}
= \begin{pmatrix}
    \eta(pB\varphi^*(\eta) + D) \\
    I(pB\varphi^*(\eta) + D)
\end{pmatrix},
\]

or equivalently (\( D \) being invertible) that \( \eta \) satisfy
Because the endomorphism of \( r \times r \) matrices given by
\[
\eta = (pA \varphi^*(\eta)) + C)(1 + pD^{-1}B_\varphi^*(\eta))^{-1}D^{-1}
\]
is a contraction mapping in the \( p \)-adic topology of \( A^\infty \), it has a unique fixed point.

In order to prove that \( U \) is horizontal, it suffices to do so over the completion of \( S^\infty \) along any closed point \( e_0 \) of \( S_0 \). Let \( e \) be the \( \varphi \)-Teichmüller point of \( S^\infty \) with values in \( W(k) \) lying over \( e_0 \). By hypothesis, \( e^*(H) \) contains \( \nu \) fixed points of \( e^*(\Phi(\varphi)) \) which span a direct factor of \( e^*(H) \), which is necessarily transverse to \( e^*(\text{Fil}) \). By 3.1.3, these fixed points extend to horizontal sections over \( H \otimes W(k)[[t_1,\ldots,t_n]] \), which span a direct factor of \( \hat{H}(e) \), still transversal to \( \text{Fil}(e) \). Write these sections as column vectors:

\[
\begin{pmatrix}
S_2 \\
S_1
\end{pmatrix}
\in M_{r-\nu}(W(k)[[t_1,\ldots,t_n]])
\]

By transversality we have \( S_1 \) invertible. The fixed-point property is

\[
\begin{pmatrix}
pA & C \\
pB & D
\end{pmatrix}
\begin{pmatrix}
\varphi^*(S_2) \\
\varphi^*(S_1)
\end{pmatrix} = \begin{pmatrix}
S_2 \\
S_1
\end{pmatrix}
\]

or equivalently
Let's put $\mu = S_2^{-1} S_1^{-1}$; we have

$$\begin{pmatrix} pA & C \\ pB & D \end{pmatrix} \begin{pmatrix} \varphi^g(S_2^{-1} S_1^{-1}) \\ \text{I} \end{pmatrix} = \begin{pmatrix} S_2^{-1} S_1 \varphi^g(S_1^{-1}) \\ S_1 \varphi^g(S_1^{-1}) \end{pmatrix}. $$

So $\mu$ satisfies $pA \varphi^g(\mu) + C = \mu S_1 \varphi^g(S_1^{-1})$ and $pB \varphi^g(\mu) + D = S_1 \varphi^g(S_1^{-1})$.

Since the endomorphism of $\mathbb{M}_{-\nu} \sqrt{W(k)[[t_1, \ldots, t_n]]}$ defined by 4.1.7 is still a contraction mapping in its $p$-adic topology, it follows that $\mu$ is its unique fixed point, and hence that $\mu$ is the power series expansion of our global fixed point $\eta$ near $e_0$. This proves that

4.1.8. the inverse image $\hat{U}(e)$ of $U$ over $W(k)[[t_1, \ldots, t_n]]$ is the module spanned by the horizontal fixed points of $F(\varphi) \cdot \varphi^g$ in $\hat{H}(e)$. Hence $\hat{U}(e)$ is horizontal, and stratified, which proves 4.1.2.

4.1.9. The matrices $\mu = S_2^{-1} S_1^{-1}$ and $S_1 \varphi^g(S_1^{-1})$ with entries in $W(k)[[t_1, \ldots, t_n]]$ are the local expansion of the global matrices $\eta$ and $pB \varphi^g(\eta) + D$ respectively. This is an example of analytic continuation par excellence.

To see that $U$ is $F$-stable, notice that once we know it's horizontal, it suffices for it to be $F(\varphi)$-stable for one choice of $\varphi$ (as it is), thanks to 1.3.1. In terms of the new base of $H$, adopted to $H = \text{Fil}^\oplus U$, the matrix of $F(\varphi)$ is
which proves 4.1.1 and 4.1.3. That 4.1.5 holds is clear from the "rational" way $\eta$ was determined.

It remains to prove 4.1.4. The matrix of $F$ in $M_r(W(k))$ looks like

$$
\begin{pmatrix}
pa & 0 \\
pb & d
\end{pmatrix}
\begin{pmatrix}
I \\
\sigma p(E)
\end{pmatrix} =
\begin{pmatrix}
pa \\
pb + pd\sigma(E)
\end{pmatrix},
$$

so $F$-stability is equivalent to the equation

$$
\begin{pmatrix}
pa \\
pb + pd\sigma(E)
\end{pmatrix} =
\begin{pmatrix}
pa \\
pE.pa
\end{pmatrix}.
$$

Thus $E$ must be a fixed point of $E \rightarrow \sigma^{-1}(-d^{-1}b + pd^{-1}Ea)$, which is again a contraction of $M_{\nu, r^{-\nu}}(W(k))$. Q.E.D.
5. Hodge F-crystals ([20])

5.0. A Hodge F-crystal is an F-crystal \((H, \nabla, F)\) together with a finite decreasing "Hodge filtration" \(H = \text{Fil}^0 \supset \text{Fil}^1 \supset \ldots\) by locally free sub-modules with locally free quotients, subject to the transversality condition

\[ \nabla \text{Fil}^i \subset \text{Fil}^{i-1} \otimes \Omega^1 \]

Its Hodge numbers are the integers \(h^i = \text{rank} (\text{Fil}^i / \text{Fil}^{i+1})\).

A Hodge F-crystal is called \textit{divisible} if for some lifting \(\varphi\) of Frobenius, we have

\[ F(\varphi) (\varphi^n(\text{Fil}^i)) \subset p^i H \quad \text{for } i = 0, 1, \ldots \]

It is rather striking that if \(p\) is sufficiently large that \(\text{Fil}^p = 0\), then 5.0.2 will hold for every choice of \(\varphi\) if it holds for one. [To see this, one uses the explicit formula (1.3.1) for the variation of \(F(\varphi)\) with \(\varphi\), transversality (5.0.1), and the fact that the function \(f(n) = \text{ord}(p^n/n!)\) satisfies \(f(n) \geq \inf(n, p^{-1})\) for \(n \geq 1\).]

The Hodge polygon associated to the Hodge numbers \(h^0, h^1, \ldots\) is the polygon which has slope \(\nabla\) with multiplicity \(h^\vee\).
By looking at the first slopes of all exterior powers, one sees:

**Lemma 5.1.** The Newton polygon of a divisible Hodge F-crystal is always above (in the (x, y) plane) its Hodge polygon.

5.2. A Hodge F-crystal is called autodual of weight $N$ if $H$ is given a horizontal autoduality $< , > : H \otimes H \longrightarrow \mathbb{Q}_\infty$ such that

5.2.1 the Hodge filtration is self-dual, meaning $\bigwedge (\text{Fil}^i) = \text{Fil}^{N+1-i}$.

5.2.2 $F$ is $p^N$-symplectic, meaning that for $x, y \in H$, and any lifting $\varphi$, we have $< F(\varphi)(\varphi^N x), F(\varphi)(\varphi^N y) > = p^N < x, y >$.

The Newton polygon of an autodual Hodge F-crystal of weight $N$ is symmetric, in the sense that its slopes are rational numbers in $[0, N]$ such that the slopes $\alpha$ and $N-\alpha$ occur with the same multiplicity.

As an immediate corollary of 4.1, we get

**Corollary 5.3.** Let $(H,\nabla, F, \text{Fil}, < , >)$ be an autodual divisible Hodge F-crystal, whose Newton polygon over every closed point of $S_0$ has slope zero with multiplicity $h^0$. Then $H$ admits a three-step
filtration

\[ \mathcal{U} \subset \mathcal{H} \]

with:

5.3.1. \( \mathcal{U} \) the "unit root" part of \( \mathcal{H} \), from 4.1.

5.3.2. \( \mathcal{H}/\mathcal{U} \) is of the form \( \mathcal{V}_N^{(-N)} \), where \( \mathcal{V}_N \) is a unit-root F-crystal (its \( \mathcal{F} \) is an isomorphism).

5.3.3. \( \mathcal{L}(\mathcal{U})/\mathcal{U} \) is of the form \( \mathcal{H}_1^{(-1)} \), where \( \mathcal{H}_1 \) is an autodual divisible Hodge F-crystal of weight \( N-2 \).

Similarly, we have

Corollary 5.4. Suppose \( (\mathcal{H}, \nabla, \mathcal{F}, \text{Fil}) \) is a Hodge F-crystal whose Newton polygon coincides with its Hodge polygon over every closed point of \( S_0 \). Then \( \mathcal{H} \) admits a finite increasing filtration

\[ 0 \subset \mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots \]

such that

5.4.1. \( \mathcal{U}_i/\mathcal{U}_{i+1} \) is of the form \( \mathcal{V}_i^{(-i)} \), with \( \mathcal{V}_i \) a unit-root F-crystal (\( \mathcal{F} \) an isomorphism).

5.4.2. the filtration is transverse to the Hodge filtration:

\[ \mathcal{H} = \text{Fil}^{\leq i} \oplus \mathcal{U}_{i-1} \]

5.4.3. if \( (\mathcal{H}, \nabla, \mathcal{F}, \text{Fil}) \) admits an autoduality of weight \( N \), the filtration by the \( \mathcal{U}_i \) is autodual: \( \mathcal{L}(\mathcal{U}_i) = \mathcal{U}_{N-1-i} \).
Remark 5.5. F-crystals and p-adic representations.

The category of "unit-root" F-crystals on $S^\infty$ (F an isomorphism), such as the $V_1$ occurring in 5.4, is equivalent to the category of continuous representations of the fundamental group $\pi_1(S_o)$ on free $\mathbb{Z}/p^n \mathbb{Z}$-modules of finite rank (i.e., to the category of "constant tordu" étale p-adic sheaves on $S_o$).

[Given $H$ and a choice of $\varphi$, one shows successively that for each $n \geq 0$, there exists a finite étale covering $T_\varphi$ of $S_n$ over which $H/p^{n+1} H$ admits a basis of fixed points of $F(\varphi, \varphi^*).$ The fixed points form a free $\mathbb{Z}/p^{n+1} \mathbb{Z}$ module of rank $= \text{rank} (H)$, on which $\text{Aut}(T_\varphi/S_n)$, hence $\pi_1(S_n) = \pi_1(S_o)$ acts. For $n$ variable, these representations fit together to give the desired p-adic representation of $\pi_1(S_o)$. This construction is inverse to the natural functor from constant tordu p-adic étale sheaves on $S_o$ to F-crystals on $S^\infty$ with $F$ invertible.]

6. A conjecture on the L-function of an F-crystal.

6.0. Suppose $H$ is an F-crystal on $S^\infty/W(\mathbb{F}_q)$. Denote by $\Delta_n$ the points of $S_o$ with values in $\mathbb{F}_q^n$ which are of degree precisely $n$ over $\mathbb{F}_q$. The L-function of $H$ is the formal power series in $1 + tW(\mathbb{F}_q)[[t]]$ defined by the infinite product (cf. [13], [26])

$$L(H; t) = \prod_{n \geq 1} \prod_{e_o \in \Delta_n} \left( \frac{1}{p(H; e_o, \mathbb{F}_q^n, t^n)} \right)^{-1/n}$$

When $H$ is a unit root F-crystal, its L-function is the L-function
associated to the corresponding étale p-adic sheaf (cf. [13], [26]).

**Conjecture 6.1.** (cf. [8], [13])

6.1.1. $L(H; t)$ is p-adically meromorphic.

6.1.2. if $H$ is a unit root $F$-crystal, denote by $M$ the corresponding
p-adic étale sheaf on $S_0$, and by $H^i_c(M)$ the étale cohomology groups
with compact supports of the geometric fibre $S_0 = S_0 \times \mathbb{F}_q$ with
coefficients in $M$. These are $\mathbb{Z}_p$-modules of finite rank, zero for
$i > \dim S_0$, on which $\text{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q)$ acts. Let $f \in \text{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q)$ denote
the inverse of the automorphism $x \mapsto x^q$. Then the function

$$L(H; t) \cdot \prod_{i=0}^{\dim S_0} \det (1-t f H^i_c(M)) (-1)^i$$

has neither zero nor pole on the circle $|t| = 1$.

**Remarks 6.1.1.** is (only) known in cases where the $F$-crystal $H$ on $S_\infty$
"extends" to the Washnitzer-Monsky weak completion $S^+$ of $S$ ([23]), in
which case it follows from the Dwork-Reich-Monsky fixed point formula
([4], [25], [24]). Unfortunately, such cases are as yet relatively rare
(but cf. [10] for a non-obvious example). It is known ([12a]) that when
$S_0 = \mathbb{A}^n$, then $L(H; t)$ is meromorphic in the closed disc $|t| \leq 1$.
The extension to general $S_0$ of this result should be possible by the
methods of ([25]); it would at least make the second part 6.1.2 of the
conjecture meaningful. As for 6.1.2 itself, it doesn't seem to be known
for any non-constant $M$. Even for $M = \mathbb{Z}_p$, when $L = \zeta$ of $S_0$,
6.1.2 has only been checked for curves and abelian varieties.
7. **F-crystals from geometry** ([1], [2])

Let \( f : X \longrightarrow S^\infty \) be a proper and smooth morphism, with geometrically connected fibres, whose de Rham cohomology is locally free (to avoid derived categories!). Crystalline cohomology tells us that for each integer \( i \geq 0 \), the de Rham cohomology \( H^i = R^if_*(\Omega^\ast_{X/S}) \) with its Gauss-Manin connection \( \nabla \) is the underlying differential equation of an F-crystal \( H^i \) on \( S^\infty \). When \( k \) is finite, say \( F_q \), then for every point \( e_0 \) of \( S_0 \) with values in \( F_{q^n} \), the inverse image \( X_{e_0} \) of \( X \) over \( e_0 \) is a variety over \( F_{q^n} \), and its zeta function is given by (cf. 1.4)

\[
\text{Zeta}(X_{e_0} / F_{q^n} ; t) = \prod_{i=0}^{2\dim X_{e_0}} p(H^i; e_0, F_{q^n}, t)^{-1+i+1}
\]

If in addition we suppose that the Hodge cohomology of \( X/S^\infty \) is locally free, and that \( X/S^\infty \) is projective, then according to Mazur [20], the Hodge F-crystal \( H^i \) is divisible, provided that \( p > i \).

For every \( p \) and \( i \) we have \( F(\varphi) \varphi^p(\text{Fil}^1) \subset p H^i \), and the \( p \)-linear endomorphism of \( H^i/pH^i + \text{Fil}^1 \cong R^if_*(\mathcal{O}_X)/pR^if_*(\mathcal{O}_X) = R^if_*(\mathcal{O}_{X_{e_0}}) \) \((f : X_0 \longrightarrow S_0 \) denoting the "reduction modulo \( p \)" of \( f : X \longrightarrow S^\infty \)) is the classical Hasse-Witt operation, deduced from the \( p \)'th power endomorphism of \( \mathcal{O}_{X_{e_0}} \). Thus if Hasse-Witt is invertible, we may apply 4.1 to the situation \( H^i, H^i \supset \text{Fil}^1 \).

When \( X/S^\infty \) is a smooth hypersurface in \( F_{q^n}^N \) of degree prime to \( p \) which satisfies a mild technical hypothesis of being "in general position", Dwork gives ([5], [7]) an a priori description of an
F-crystal on \( S \) whose underlying differential equation is (the primitive part of \( H^N_{\text{DR}}(X/S) \)) with its Gauss-Manin connection, and whose characteristic polynomial is the "interesting factor" in the zeta function ([14]).

The identification of Dwork's \( F \) with the crystalline \( F \) follows from [14] and (as yet unpublished) work of Berthelot and Meredith (c.f. the Introduction to [2]) relating the crystalline and Monsky-Washnitzer theories ([23], [24]). Dwork's F-crystal is isogenous to a divisible one for every prime \( p \) ([7], lemma 7.2).
8. **Local study of ordinary curves**: Dwork's period matrix $T$ ([11])

7.0. Let $f : X \to \text{Spec}(W(k)[[t_1, \ldots, t_n]])$ be a proper smooth curve of genus $g \geq 1$. It's crystalline $H^1$ is an autodual (cup-product) divisible Hodge $F$-crystal of weight 1. We assume that it is *ordinary*, in the sense that modulo $p$ its Hasse-Witt matrix is invertible, or equivalently that its **Newton** polygon is

![Newton polygon diagram](image)

(this means geometrically that the jacobian of the special fibre has $p^g$ points of order $p$). Let's also suppose $k$ algebraically closed, and denote by $e$ the homomorphism "evaluation at $(0, \ldots, 0)$": $W(k)[[t_1, \ldots, t_n]] \to W(k)$. By 2.1.2 and 4.14, $e^*(H^1)$ admits a symplectic base of $F$-eigenvectors

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$$

satisfying

$$e^*(F)(\alpha_i) = \alpha_i, \quad e^*(F)(\beta_i) = p\beta_i$$

7.0.1

$$\begin{cases}
<\alpha_i, \alpha_j> = <\beta_i, \beta_j> = 0, \\
<\alpha_i, \beta_j> = -<\beta_j, \alpha_i> = \delta_{ij}
\end{cases}$$

By 3.1.2, this base is the value at $(0, \ldots, 0)$ of a horizontal base of $H^1 \otimes K[[t_1, \ldots, t_n]]$, which we continue to note $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$. For each choice of lifting $\varphi$, we have
According to 3.1.3, the sections $\alpha_1, \ldots, \alpha_g$ extend to horizontal sections over "all" of $H^1$, where they span the submodule $U$ of $4.1$; in general the $\beta_i$ do not extend to all of $H^1$.

We now wish to express the position of the Hodge filtration $\text{Fil}^1 \subset H$ in terms of the horizontal "frame" provided by the $\alpha_i$ and $\beta_j$. Since $H^1 = U \oplus \text{Fil}^1$ is a decomposition of $H^1$ in submodules isotropic for $<,>$, there is a base $w_1, \ldots, w_g$ of $\text{Fil}^1$ dual to the base $\alpha_1, \ldots, \alpha_g$ of $U$.

$$<w_i, w_j> = 0, <\alpha_i, w_j> = \delta_{ij}.$$  

In $H \otimes K[[t_1, \ldots, t_n]]$, the differences $w_i - \beta_i$ are orthogonal to $U$, hence lie in $U$:

$$w_i - \beta_i = \sum_j \tau_{ji} \alpha_j; \quad \tau_{ji} = <w_i, \beta_j>.$$  

The matrix $T = (\tau_{ij})$ is Dwork's "period matrix"; it has entries in $W(k) \langle \langle t_1, \ldots, t_n \rangle \rangle \otimes_k K[[t_1, \ldots, t_n]]$. Differentiating 7.0.4 via the Gauss-Manin connection, we see:

**Lemma 7.1.** $T$ is an indefinite integral of the matrix of the mapping "cup-product with the Kodaira-Spencer class"; for every continuous $W(k)$-derivation $D$ of $W(k)[[t_1, \ldots, t_n]]$, $D(T)$ is the matrix of the composite
expressed in the dual bases $\omega_1, \ldots, \omega_g$ and $\alpha_1, \ldots, \alpha_g$.

Lemma 7.2. For any lifting $\varphi$ of Frobenius, we have the following congruences on the $\tau_{ij}$:

7.2.1 $\varphi^*(\tau_{ij}) - p\tau_{ij} \in \mathbb{p}W(k)[[t_1, \ldots, t_n]]$

7.2.2 $\tau_{ij}(0, \ldots, 0) \in \mathbb{p}W(k)$.

Proof. Applying $F(\varphi)\varphi^*$ to the defining equation (7.0.4), we get

$$F(\varphi)(\varphi^*(w_i)) - p\beta_i = \sum_j \varphi^*(\tau_{ij})\alpha_j.$$

Subtracting $p$ times (7.0.4), we are left with

$$F(\varphi)(\varphi^*(w_i)) - p\omega_i = \sum_j [\varphi^*(\tau_{ij}) - p\tau_{ij}]\alpha_j.$$

Since the left side lies in $\mathbb{p}H^1$, we get

$$\varphi^*(\tau_{ij}) - p\tau_{ij} = \langle F(\varphi)\varphi^*(w_i) - pw_i, w_j \rangle \in \mathbb{p}W(k)[[t_1, \ldots, t_n]].$$

To prove that $\tau_{ij}(0, \ldots, 0) \in \mathbb{p}W(k)$, choose a lifting $\varphi$ which preserves $(0, \ldots, 0)$, for instance, $\varphi(t_i) = t_i^p$ for $i = 1, \ldots, n$, and evaluate (7.2.1) at $(0, \ldots, 0)$:

$$\sigma(\tau_{ij}(0, \ldots, 0)) - p\tau_{ij}(0, \ldots, 0) \in \mathbb{p}W(k).$$

which implies $\tau_{ij}(0, \ldots, 0) \in \mathbb{p}W(k)$! QED.
7.3. According to a criterion of Dieudonné and Dwork ([3]), these congruences for $p \neq 2$ imply that the formal series

$$q_{ij} \overset{\text{defn}}{=} \exp(\tau_{ij})$$

lie in $W(k)[[t_1, \ldots, t_n]]$, and have constant terms in $1 + pW(k)$.

(When $p = 2$, we cannot define $q_{ij}$ unless $\tau_{ij}$ has constant term $= 0$ (4), in which case we would again have the $q_{ij}$ in $W(k)[[t_1, \ldots, t_n]]$).

It is expected that the $s^2$ principal units $q_{ij}$ in $W(k)[[t_1, \ldots, t_n]]$ are the Serre-Tate parameters of the particular lifting to $W(k)[[t_1, \ldots, t_n]]$ of the jacobian of the special fibre of $X$ given by the jacobian of $X/W(k)[[t_1, \ldots, t_n]]$ (cf. [18], [22]). This seems quite reasonable, because over the ring of ordinary divided power series $W(k)\langle t_1, \ldots, t_n \rangle$, $p \neq 2$, such liftings are known to be parameterized by the position of the Hodge filtration, ([21]), which is precisely what $(\tau_{ij})$ is.

Proposition 7.4. The following conditions are equivalent

7.4.1. The Gauss-Manin connection on $H^1$ extends to a stratification (i.e., horizontal section of $H^1 \otimes W(k)\langle t_1, \ldots, t_n \rangle$ extend to horizontal sections of $H^1$).

7.4.2. Every horizontal section of $\hom{W}{k}{k}{k}[[t_1, \ldots, t_n]]$ is bounded in the open unit polydisc (i.e., lies in $p^{-m}H^1$ for some $m$).

7.4.3. The $\tau_{ij}$ are all bounded in the open unit polydisc (i.e., lie in $p^{-m}W(k)[[t_1, \ldots, t_n]]$ for some $m$).
7.4.4. The $\tau_{ij}$ all lie in $W(k)[[t_1, \ldots, t_n]]$.

7.4.5. The $\tau_{ij}$ all lie in $pW(k)[[t_1, \ldots, t_n]]$.

Proof. Using the congruences 7.2, we get $7.4.3 \iff 7.4.4 \iff 7.4.5$, by choosing for $\varphi$ the lifting $\varphi(t_i) = t_i^p$ for $i = 1, \ldots, n$. By 7.0.4, $7.4.1 \iff 7.4.4$ and $7.4.2 \iff 7.4.3$.

QED.

Corollary 7.5. Suppose $X/W(k)[[t]]$ is an elliptic curve with ordinary special fibre, and that the induced curve over $k[t]/(t^2)$ is non-constant. Then every horizontal section of $H^1$ is a $W(k)$-multiple of $\alpha_1$, the horizontal fixed point of $F$ in $H^1$.

Proof. The non-constancy modulo $(p, t^2)$ means precisely that the Kodaira-Spencer class in $H^1(X_{\text{special}}, T)$ is non-zero, which for an elliptic curve is equivalent to the non-vanishing modulo $(p, t)$ of the composite mapping:

$$
\begin{align*}
\text{Fil}^1 &\hookrightarrow H^1 \xrightarrow{\nabla \left( \frac{d}{dt} \right)} H^1 \xrightarrow{\text{proj}} H/\text{Fil} \sim U,
\end{align*}
$$

whose matrix is $\frac{d\tau}{dt}$. Thus $\frac{d\tau}{dt} \notin (p, t)$, and hence by 7.4 there exists an unbounded horizontal section of $H^1(k[[t]])$. Writing it as $a \alpha_1 + b \beta_1$, we must have $b \neq 0$ because $\alpha_1$ is bounded. Hence $\beta_1$ is unbounded, hence any bounded horizontal section is a $K$-multiple of $\alpha_1$, and $H^1 \cap K\alpha_1 = W(k)\alpha_1$.

The interest of this corollary is that it describes the filtration $U \subset H^1$ purely in terms of the differential equation (i.e., without reference to $F$) as being the span of the horizontal sections of $H^1$ (the "bounded solutions" of the differential equation). (cf. [9], pt. 4 where this is worked out in great detail for...
Legendre's family of elliptic curves]. The general question of when the filtration by slopes can be described in terms of growth conditions to be imposed on the horizontal sections of $H^1 \otimes K[[t]]$ is not at all understood.
8. An example ([6], [10]). Let's see what all this means in a concrete case: the ordinary part of Legendre's family of elliptic curves. We take $p \neq 2$, $H(\lambda) \in \mathbb{Z}[\lambda]$ the polynomial $\Sigma (-1)^j \binom{p-1}{2j} \lambda^j$ of degree $p-1/2$, $S$ the smooth $\mathbb{Z}_p$-scheme $\text{Spec}(\mathbb{Z}_p[\lambda][1/\lambda(1-\lambda)H(\lambda)])$, and $X/S$ the Legendre curve whose affine equation is $y^2 = x(x-1)(x-\lambda)$ (*).

The De Rham $H^1$ is free of rank 2, on $w$ and $w'$, where

$$
\begin{cases}
  w & \text{is the class of the differential of the first kind } dx/y \\
  w' = \nabla \left( \frac{d}{d\lambda} \right)(w)
\end{cases}
$$

The Gauss-Manin connection is specified by the relation

$$
\lambda(1-\lambda) w'' + (1-2\lambda) w' = \frac{1}{4} w; \quad (w'' = \text{defn} (\nabla \left( \frac{d}{d\lambda} \right))^2(\omega))
$$

The Hodge filtration is $H^1_{\text{Fil}} \subset H^1 = \text{span of } w$. The cup-product is given by $\langle w, w' \rangle = \langle w', w' \rangle = 0$; $\langle w, w' \rangle = -2/\lambda(1-\lambda)$.

Horizontal sections are those of the form $\lambda(1-\lambda)f'w - \lambda(1-\lambda)f'w'$, where $f$ is a solution of the ordinary differential equation $\frac{df'}{df} = \frac{1}{4} f$.

8.0. For any point $\alpha \in \mathcal{W}(\mathbb{F}_q)$ for which $|H(\alpha).\alpha.(1-\alpha)| = 1$ we know by 7.5 and 4.1 that the $\mathcal{W}(\mathbb{F}_q)$-module of solutions in $\mathcal{W}(\mathbb{F}_q)[[\nu-\alpha]]$ of the differential equation 9.2 is free of rank one, and is generated by a solution whose constant term is 1. Denote this solution $f_\alpha$. According to 4.1.9, the ratio $f'/f_\alpha$ is the local expression of a "global" function $\eta \in \mathbb{P}$-adic completion of $\mathbb{Z}_p[\lambda][1/\lambda(1-\lambda)H(\lambda)]$. Now choose a lifting $\varphi$ of Frobenius, say the one with $\varphi^p(\lambda) = \lambda^p$. For each Teichmuller point $\alpha$, there exists a unit $C_\alpha$ in $\mathcal{W}(\mathbb{F}_q)$, such that the function $C_\alpha f_\alpha/\varphi^p(C_\alpha f_\alpha)$ is the local expression of the $1 \times 1$ matrix of $F(\varphi)$ on the rank one module $U$.

(*) $H(\lambda)$ modulo $p$ is the Hasse invariant = $1 \times 1$ Hasse-Witt matrix.
This is just the spelling out of 4.1.9, the constant $C_{\alpha}$ so chosen as to make $C_{\alpha} f_{\alpha}$ a fixed point of $F$. In terms of this matrix, call it $a(\lambda)$, we have a formula for $\zeta$:

For each $\alpha_o \in \mathbb{F}_q^n$ such that $y^2 = X(X-1)(X-\alpha_o)$ is the affine equation of an ordinary elliptic curve $E_{\alpha_o}$, denote by $\alpha \in \mathcal{W}(\mathbb{F}_p^n)$ its Teichmüller representative. The unit root of the numerator of $\zeta(E_{\alpha_o}/\mathbb{F}_p^n; t)$ is

$$u_n(\alpha) \overset{\text{def}}{=} a(\alpha)a(\alpha^p)\ldots a(\alpha^{p^{n-1}})$$

and hence

$$\zeta(E_{\alpha_o}/\mathbb{F}_p^n; t) = \frac{1 - u_n(\alpha) - (1 - (p^n/\alpha))t}{1 - t(1 - p^nt)}$$

This formula, known to Dwork by a completely different approach in 1957, ([6]) was the starting point of his application of $p$-adic analysis to $\zeta$!
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