p-Adic L-Functions for CM Fields.

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p-Adic L-Functions for CM Fields

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Introduction

As was known to Euler, the Riemann zeta function $\zeta(s)$ assumes rational values at negative integers:

$$\zeta(1-k) = -b_k/k \qquad \text{for } k \geqq 1$$

where the b_k are the Bernoulli numbers:

$$\sum_{k=0}^{\infty} b_k \cdot \frac{x^k}{k!} = \frac{x\,e^x}{e^x - 1}.$$

Furthermore, for any integer a, we have

$$a^{k+1}(1-a^{k+1})\,\zeta(-k) \in \mathbf{Z} \qquad \text{for } k \geqq 0.$$

(This was known to Euler at least for $a=2$; for general $a \in \mathbf{Z}$ it is the Sylvester-Lipschitz theorem, cf. [16]). Now fix a prime p, an integer a which is prime to p, and define

$$\zeta_{(a)}^{(p)}(-k) = (1-a^{k+1})(1-p^k)\,\zeta(-k) \qquad \text{for } k \geqq 0.$$

These numbers are p-integral, and satisfy the following congruences, which were essentially known to Kummer:

> For any polynomial $\sum c_k X^k \in \mathbf{Z}[X]$ which satisfies a congruence of the form
>
> $$\sum c_k x^k \equiv 0 \bmod p^N \qquad \text{for all } x \in \mathbf{Z}_p^\times,$$
>
> there is a corresponding congruence
>
> $$\sum c_k\, \zeta_{(a)}^{(p)}(-k) \equiv 0 \qquad \bmod p^N.$$

We can express the totality of these Kummer congruences by the assertion

> There exists a $\mathbf{Z}_p$-valued p-adic measure $\mu^{(a)}$ on the group $\mathbf{Z}_p^\times$ of p-adic units, such that for all integers $k \geqq 0$, we have
>
> $$\int_{\mathbf{Z}_p^\times} x^k\, d\mu^{(a)} = (1-a^{k+1})(1-p^k)\,\zeta(-k).$$

More generally, let ρ be a Dirichlet character modulo some power p^n of p. Because the "partial" zeta functions also take rational values at negative integers, the values at negative integers of the Dirichlet L-series $L(s,\rho)$ lie intrinsically in $\mathbf{Q}$(the values of ρ). If we pick a p-adic place of this field, and view ρ as a p-adic-valued locally constant character of $\mathbf{Z}_p^\times$, we find, for all integers $k \geqq 0$,

$$\int_{\mathbf{Z}_p^\times} x^k \rho(x)\, d\mu^{(a)} = (1-a^{k+1}\rho(a))(1-p^k \rho(p))\, L(-k,\rho).$$

(The equality takes place in the chosen p-adic completion of $\mathbf{Q}$(values of ρ); $\rho(p)$ is defined to be 1 if ρ is trivial, 0 if not.)

Now consider an arbitrary totally real field K of finite degree over $\mathbf{Q}$. We "replace" the group $\mathbf{Z}_p^\times$ by the group

$$\varprojlim_n \{\text{prime-to-}p \text{ fractional ideals of } K\}/\{(\alpha)/\alpha \in K^\times \text{ totally positive and } \equiv 1 \bmod^\times p^n\},$$

which by class-field theory is the galois group $\mathrm{Gal}(K(p^\infty)/K)$ of the maximal abelian unramified-outside-p extension of K. Let ρ be any $\mathbf{C}$-valued locally-constant character of $\mathrm{Gal}(K(p^\infty)/K)$, and consider the corresponding Dirichlet L-function

$$L(s,\rho) = \sum \rho(\mathfrak{a})\, \mathbf{N}\,\mathfrak{a}^{-s}$$

the sum extended to all prime-to-p integral ideals $\mathfrak{a}$ of K. According to Siegel and Klingen, the values of $L(s,\rho)$ at negative integers lie intrinsically in the field $\mathbf{Q}$(values of ρ). The generalization to these functions of the Kummer congruences is due to Coates-Sinnott in the case of real quadratic fields, and to Deligne-Ribet in the general case. Their result is

> Let $\mathfrak{a}$ be a prime-to-p ideal of K. Then there exists a $\mathbf{Z}_p$-valued p-adic measure $\mu^{(\mathfrak{a})}$ on $\mathrm{Gal}(K(p^\infty)/K$ such that for every integer $k \geqq 0$, and every locally constant character ρ, we have
>
> $$\int_{\mathrm{Gal}(K(p^\infty)/K)} \rho(\mathfrak{b})\, \mathbf{N}\,\mathfrak{b}^k\, d\mu^{(\mathfrak{a})} = (1-\rho(\mathfrak{a})\,\mathbf{N}\,\mathfrak{a}^{k+1})\, L(-k,\rho).$$

(As before, the equality is to be understood in a chosen p-adic completion of $\mathbf{Q}$(values of ρ); the function $\mathbf{N}\,\mathfrak{b}$ denotes the unique $\mathbf{Z}_p^\times$-valued character on $\mathrm{Gal}(K(p^\infty)/K)$ whose value on (the Artin symbol of) a prime-to-p ideal $\mathfrak{b}$ is its norm.).

What, if any, is the generalization of these results to other fields? An initial obstacle is the fact that if K is a finite extension of $\mathbf{Q}$ which is *not* totally real, then the values at negative integers of any Dirichlet L-series $L(s,\rho)$ are all *zero* (thanks to the Γ-factors in the functional equation). In compensation, we have the possibility of studying the values at $s=0$ of Hecke L-series $L(s,\chi)$ attached to grossencharacters of type A_0 in the sense of Weil [26]. Recall that such a χ, of conductor (dividing) a given integral ideal $\mathfrak{m}$ of K, is by definition a homomorphism

$$\{\text{prime-to-}m \text{ fractional ideals of } K\} \to \bar{\mathbf{Q}}^\times$$

such that there exists integers $n(\lambda)$, one for each embedding $\lambda: K \hookrightarrow \bar{\mathbf{Q}}$ such that, for $\alpha \in K^\times$ we have

$$\chi((\alpha)) = \prod_{\lambda} \lambda(\alpha)^{n(\lambda)} \quad \text{if } \alpha \equiv 1 \bmod^\times \mathfrak{m} \text{ and } \alpha \gg 0.$$

To define $L(s, \chi)$, we must choose a complex embedding of $\bar{\mathbf{Q}}$.

When K is totally real, the only grossencharacters of type A_0 are of the form

$$\chi(\mathfrak{a}) = \rho(\mathfrak{a}) \, \mathbf{N} \, \mathfrak{a}^n$$

for some Dirichlet character ρ and some integer n; furthermore, we have

$$L(0, \chi) = L(-n, \rho)$$

so for K totally real, we've been speaking prose all along.

For what fields are there more χ's of type A_0 than those of the form $\rho \cdot \mathbf{N}^n$? By Weil [27], they are exactly the fields which *contain* a CM field (meaning a quadratic, totally imaginary extension of a totally real field). Thus we *still* have no candidates for p-adic interpolation for fields which are *neither* totally real *nor* contain a CM-field.

Suppose now that L is a field which *does* contain a CM field. Then it contains a maximal one, say L_{CM}, whose real subfield K is precisely the maximal totally real subfield of L. Again by Weil, we know that every A_0-grossencharacter of L is of the form

$$\chi = \rho \cdot (\chi_0 \circ \mathbf{N}_{L/L_{\mathrm{CM}}})$$

with ρ a Dirichlet character of L, and χ_0 an A_0-grossencharacter of L_{CM}.

Now fix a "CM-type" for L_{CM}; i.e., a subset Σ of the set Λ of all embeddings of L_{CM} into $\bar{\mathbf{Q}}$, such that

$$\Lambda = \Sigma \cup \bar{\Sigma}, \qquad \Sigma \cap \bar{\Sigma} = \emptyset.$$

(Here the complex conjugate of an embedding σ is defined in terms of the intrinsic complex conjugation $a \to \bar{a}$ on L_{CM} by $\bar{\sigma}(a) = \sigma(\bar{a})$.) By definition, there exist integers k, $\{d(\sigma)\}_{\sigma \in \Sigma}$ such that χ_0 is given on suitably principal ideals by the formula

$$\chi_0((\alpha)) = \prod_{\sigma} \left(\frac{1}{\sigma(a)^k} \left(\frac{\sigma(\bar{a})}{\sigma(a)} \right)^{d(\sigma)} \right).$$

According to Shimura [24], we can attach complex constants $\Omega(\sigma) \in \mathbf{C}^\times$ to the field L_{CM}, the CM-type Σ, and the chosen embedding of $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$, such that the

Hecke L-series values $L(0,\chi_0)$ for the field L_{CM} satisfy the following algebraicity theorem:

the product $\prod_\sigma \left(\frac{\pi^{d(\sigma)}}{\Omega(\sigma)^{k+2d(\sigma)}}\right) \cdot L(0,\chi_0)$ is an algebraic number, provided $k \geqq 1$ and all $d(\sigma) \geqq 0$, *or* (functional equation) provided $k \leqq 1$ and all $k+d(\sigma)-1 \geqq 0$.

It would be tempting, but perhaps premature, to conjecture that for such a χ_0, the L-series values for $\rho \cdot (\chi_0 \circ \mathbf{N}_{L/L_{\mathrm{CM}}})$ attached to L satisfy:

the product $L(0, \rho \cdot (\chi_0 \circ \mathbf{N}_{L/L_{\mathrm{CM}}})) \cdot \left(\prod_\sigma \left(\frac{\pi^{d(\sigma)}}{\Omega(\sigma)^{k+d(\sigma)}}\right)\right)^{\deg(L\ L_{\mathrm{CM}})}$

is an algebraic number, provided χ_0 is as above.

This is trivially true if $\rho = 1$ and L is abelian over L_{CM}, but other cases remain obscure.

It is perhaps worth pointing out the meaning of the restrictions $k \geqq 1$, all $d(\sigma) \geqq 0$, or $k \leqq 1$, all $k+d(\sigma)-1 \geqq 0$. The functional equation of Hecke L-series with grossencharacter can be written

$$*(\chi)\, L(0,\chi) = *(\chi^{-1}\mathbf{N}^{-1})\, L(0,\chi^{-1}\mathbf{N}^{-1})$$

where $*(\chi)$ is the product of some "root numbers," which are always non-zero, some powers of π, and some values of the Γ-function. Let χ_0 be an A_0-grossencharacter of L_{CM} such that

$$\chi = \rho \cdot (\chi_0 \circ \mathbf{N}_{L/L_{\mathrm{CM}}}).$$

Then both $*(\chi)$ and $*(\chi^{-1}\mathbf{N}^{-1})$ are *finite* and *non-zero* if and only if there *exists* a CM-type Σ for L_{CM} with respect to which the integers $k, d(\sigma)$'s attached to χ_0 satisfy either $k \geqq 1$, all $d(\sigma) \geqq 0$ or $k \leqq 1$, all $k+d(\sigma)-1 \geqq 0$.

In this paper, we study the case when $L = L_{\mathrm{CM}}$ is itself a CM field, and attempt to construct a p-adic measure μ on the galois group $\mathrm{Gal}(L(p^\infty)/L)$ against which the integrals

$$\int_{\mathrm{Gal}(L(p^\infty)/L)} \chi\, d\mu$$

of p-power conductor A_0-grossencharacters whose $(k, d(\sigma)$'s) satisfy $k \geqq 1$, all $d(\sigma) \geqq 0$ or $k \leqq 1$, all $k+d(\sigma)-1 \geqq 0$ are closely related to the algebraic numbers

$$\prod_\sigma \left(\frac{\pi^{d(\sigma)}}{\Omega(\sigma)^{k+2d(\sigma)}}\right) \cdot L(0,\chi).$$

Of course to make this precise, we must

1) fix a CM-type Σ for L, to have the $\Omega(\sigma)$'s.
2) fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p (\mathbf{C}_p$ the completion of the algebraic closure of $\mathbf{Q}_p)$ to be able to view χ as a p-adic character of $\mathrm{Gal}(L(p^\infty)/L)$.

We *succeed* only when the fixed CM-type Σ is "*ordinary*" at the given embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$, in the following sense: whenever $\sigma \in \Sigma$ and $\bar{\tau} \in \bar{\Sigma}$,

$$\begin{tikzcd} L \arrow[r, hook, shift left, "\sigma"] \arrow[r, hook, shift right, "\bar{\tau}"'] & \overline{\mathbf{Q}} \arrow[r, hook, "\text{given}"] & \mathbf{C}_p \end{tikzcd}$$

the p-adic valuations of L to which they give rise are *distinct*. *Given* an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$, there exist such ordinary Σ's if and only if every place $\mathfrak{p}$ of K lying over p splits completely in the quadratic extension L/K:

$$\mathfrak{p} = \mathfrak{P}\overline{\mathfrak{P}} \quad \text{with } \mathfrak{P} \neq \overline{\mathfrak{P}}.$$

If this is so, the Σ's which are ordinary at the given embedding correspond exactly to the choice, for each $\mathfrak{p}$ of K, of a $\mathfrak{P}$ lying over it in L; thus if we put

$a =$ numbers of places $\mathfrak{p}$ of K lying over p

there are 2^a Σ's which are ordinary, out of a total of 2^g $(g = \deg(K/\mathbf{Q}))$.

So for a given CM field L, our theory only applies to those p for which all p-adic places of K split completely in L/K. This is a set of positive density (it contains the primes which split completely in L), but its complement also has positive density (if not, "all" primes of K would split in L). For example if L is abelian over $\mathbf{Q}$, and we denote by $c \in \mathrm{Gal}(L/\mathbf{Q})$ the intrinsic complex conjugation of L, then our theory applies to p if and only if the cyclic subgroup of $\mathrm{Gal}(L/\mathbf{Q})$ generated by F_p, the Frobenius at p, does *not* contain c. When $L/\mathbf{Q}$ is *cyclic*, this just means that F_p must have *odd* order, so the density of good p in that case is $1/2^b$, where 2^b is the exact power of 2 which divides the order of G. In the opposite direction, Ribet pointed out to me that if we consider the case of an abelian $L/\mathbf{Q}$ of exponent two (i.e., $\mathrm{Gal}(L/\mathbf{Q})$ is $\mathbf{Z}/2\mathbf{Z}$-vector space), then the good p are those for which $F_p \neq c$, and thus have density $1 - 1/(\#\mathrm{Gal})$.

Our construction is based on the well-known fact that for $k \geqq 1$, all $d(\sigma) \geqq 0$, the complex numbers $L(0, \chi)$ are finite sums of *values* of (not necessarily holomorphic) Eisenstein series with level on the Hilbert modular group attached to the real subfield K of L, at *points* corresponding to abelian varieties with complex multiplication by the field L. Such abelian varieties are known to be "defined" over $\overline{\mathbf{Q}}$, and to have everywhere good reduction. The p-adic embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ at which Σ is ordinary are precisely those for which the associated abelian variety has *ordinary* reduction at the corresponding place of $\overline{\mathbf{Q}}$. The non-holomorphic Eisenstein series in question are all obtained from holomorphic ones by applying certain C^∞ (but *non*-analytic) differential operators. These operators are themselves obtained from "universal" holomorphic differential operators on the symmetric algebra of H^1_{DR} (of the universal abelian scheme with suitable polarization and multiplication by K) which are themselves manufactured out of the Gauss-Manin connection. The non-analyticity is introduced by "projecting" our universal operators onto $\mathrm{Symm}(H^{1,0})$ by making use of the transcendental Hodge decomposition

$$H^1_{DR} = H^{1,0} \oplus H^{0,1}, \qquad H^{0,1} = \overline{H^{1,0}}.$$

Shimura's algebraicity theorem, in our context, is an easy consequence of the fact that the Hodge *decomposition* of the H^1_{DR} of our CM-abelian varieties is purely algebraic (i.e. the subspace $H^{0,1}$ is defined over $\bar{\mathbf{Q}}$), and is valid for any CM type .

The possibility of p-adic interpolation, when Σ is ordinary, of the resultingly algebraic numbers springs from the fact that, in general, we have a p-adic analogue of the Hodge decomposition for H^1_{DR} of p-adic abelian varieties with everywhere ordinary reduction; the analogue of $H^{0,1}$ is the "unit root subspace" introduced and systematically studied by Dwork, and more recently studied by Mazur, Bloch, and others. We obtain p-adic differential operators by projecting our universal operators onto $\mathrm{Symm}(H^{1,0})$ using the "unit root"-splitting. Applying these p-adic differential operators to holomorphic, and hence purely algebraic, Eisenstein series, we obtain p-adic Eisenstein series, whose values at *any* point have good interpolation properties. The fact that the values at our CM-point *coincide* with their "complex" analogues results from the fact that for *ordinary* CM abelian varieties, the unit-root subspace is also purely algebraic, and *coincides* with $H^{0,1}$.

In this way we construct a p-adic measure μ on $\mathrm{Gal}(L(p^\infty)/L)$ such that for χ a p-power conductor grossencharacter of type A_0, viewed as a p-adic character of Gal, the integral

$$\int_{\mathrm{Gal}} \chi \, d\mu$$

is closely related (cf. (5.3.6) for the explicit formula) to the complex number $L(0,\chi)$. We then establish a simple functional equation for our measure μ, which turns out, after a long computation (cf. 5.7), to be perfectly compatible with the *classical* functional equation for Hecke L-series with grossencharacter. Does this compatibility admit a proof by what Mazur calls "pure thought"? At any rate, this compatibility extends our p-adic-classical comparison to be domain, $k \leqq 1$, all $k+d(\sigma)-1 \geqq 0$.

In an earlier paper [10], we treated the case of a quadratic imaginary extension of $\mathbf{Q}$. In that case, the abelian varieties involved are simply elliptic curves, and we were able to prove the theorems by dipping into the wealth of classical material available for elliptic curves, e.g. the Halphen-Ficke operator

$$\eta_1 \frac{\partial}{\partial \omega_1} + \eta_2 \frac{\partial}{\partial \omega_2},$$

the non-analytic weight two Eisenstein series

$$\lim_{s \to 0} \sum \frac{1}{(n\omega_1 + m\omega_2)^2 |n\omega_1 + m\omega_2|^{2s}},$$

and all their marvelous interrelations. We were for a long time blinded by these riches to the sinple cohomological mechanism which in some sense underlies them.

Unfortunately, the simplicity of the underlying cohomological mechanism is itself obscured by the complications which arise in dealing with an arbitrary totally real number field K, rather than with $\mathbf{Q}$. For this reason, the reader should keep in mind the special case $K=\mathbf{Q}$, especially for the first three chapters.

The idea that Hilbert modular forms are relevant to the study of rationality properties of zeta and L functions of totally real fields occurs prominently in Siegel. That they could be used to develop a p-adic theory was seen clearly by Serre. In a December, 1973 letter of Deligne to Serre, Deligne explains how he could prove the "good" congruences on such L-functions (i.e., construct the measure $\mu^{(\mathfrak{a})}$ alluded to above) if enough were known about the moduli space over $\mathbf{Z}$ of Hilbert-Blumenthal abelian varieties. The general theory of such moduli spaces was then worked out by Rapoport in his thesis [17]. The *particular* irreducibility result upon which Deligne's program depended was supplied by Ribet, and is described in [18].

Another approach to the study of L-values is based on viewing them as *periods* of suitable cusp forms. This idea occurs in Shimura [25]. Recently, it has been vastly extended by Manin [12, 14] and Manin-Vishik [13]. It would be very interesting to understand the relation between their point of view and ours.

The debt that we owe to Deligne, Rapoport, and Ribet will be obvious to the reader. Less obvious, but no less real, is the influence of Weil, especially through [27] and [28]. It is a pleasure to acknowledge the support of the John Simon Guggenheim Memorial Foundation, and the hospitality of IHES.

Chapter I: Review of the Theory of Hilbert Modular Forms

Throughout this paper, we fix a totally real field K of finite degree g over $\mathbf{Q}$. We denote $\mathcal{O}$ its ring of integers, and by $\mathfrak{d}^{-1}$ its inverse different. We begin by recalling some notions concerning Hilbert-Blumenthal abelian varieties (HBAV) relative to the field K.

1.0. Over any ring R, an HBAV over R is a g-dimensional abelian scheme X over R, together with an action of $\mathcal{O}$ on it, such that, locally on R, the Lie algebra $\mathrm{Lie}(X/R)$ is a free $\mathcal{O}\underset{\mathbf{Z}}{\otimes} R$ module of rank one. We denote by $\underline{\omega}_{X/R}$ the $\mathcal{O}\underset{\mathbf{Z}}{\otimes} R$ module of invariant one-forms on X/R:

$$\underline{\omega}_{X/R}=H^0(X,\Omega^1_{X/R}).$$

As R-modules, $\mathrm{Lie}(X/R)$ and $\underline{\omega}_{X/R}$ are each locally free of rank g, and they are R-duals of each other. As invertible $\mathcal{O}\underset{\mathbf{Z}}{\otimes} R$ modules, they are related by an isomorphism

$$(1.0.1)\quad \mathrm{Lie}(X/R) \underset{\mathcal{O}\otimes R}{\otimes} \underline{\omega}_{X/R} \xrightarrow{\sim} \mathfrak{d}^{-1} \underset{\mathbf{Z}}{\otimes} R,$$

whose composition with the trace map

$$(1.0.2)\quad \mathfrak{d}^{-1} \underset{\mathbf{Z}}{\otimes} R \xrightarrow{\mathrm{trace}_{K/\mathbf{Q}}\otimes 1} \mathbf{Z} \underset{\mathbf{Z}}{\otimes} R = R$$

yields the R-pairing

(1.0.3) $\mathrm{Lie}(X/R)\underset{R}{\otimes}\underline{\omega}_{X/R}\twoheadrightarrow \mathrm{Lie}(X/R)\underset{\mathcal{O}\otimes R}{\otimes}\underline{\omega}_{X/R}\simeq \mathfrak{d}^{-1}\otimes R\xrightarrow{\mathrm{trace}\otimes 1} R$

which defines their R-duality.

A "nowhere vanishing differential" is an element $\omega\in\underline{\omega}_{X/R}$ which is an $\mathcal{O}\underset{\mathbf{Z}}{\otimes}R$ basis of $\underline{\omega}_{X/R}$. By "contraction" via (1.0.1), such an ω gives rise to (and is equivalent to) an $\mathcal{O}\underset{\mathbf{Z}}{\otimes}R$-isomorphism

(1.0.4) $\omega\colon \mathrm{Lie}(X/R)\xrightarrow{\sim}\mathfrak{d}^{-1}\underset{\mathbf{Z}}{\otimes}R.$

Let $\mathfrak{c}$ be a fractional ideal of K. Viewing an HBAV X over R as a functor (on Schemes/R) in $\mathcal{O}$-modules, we can define $X\underset{\mathcal{O}}{\otimes}\mathfrak{c}$ as a functor (on Schemes/R):

(1.0.5) $(X\underset{\mathcal{O}}{\otimes}\mathfrak{c})(S)\overset{\mathrm{dfn}}{=}X(S)\underset{\mathcal{O}}{\otimes}\mathfrak{c}.$

It is easily checked that $X\underset{\mathcal{O}}{\otimes}\mathfrak{c}$ is itself an HBAV over R, and that there is a natural injective $\mathcal{O}$-linear map

(1.0.6) $\mathfrak{c}\rightarrow\mathrm{Hom}_{\mathcal{O}\text{-lin}}(X,X\underset{\mathcal{O}}{\otimes}\mathfrak{c}).$

Let us denote by X^t the dual abelian scheme to X (or, more precisely, the functor $\underline{\mathrm{Pic}}^0_{X/R}$) with its induced $\mathcal{O}$-structure. A $\mathfrak{c}$-polarization is an isomorphism λ of HBAV's over R

(1.0.7) $\lambda\colon X^t\rightarrow X\underset{\mathcal{O}}{\otimes}\mathfrak{c}$

under which the symmetric elements of $\mathrm{Hom}_{\mathcal{O}\text{-lin}}(X,X^t)$ correspond precisely to the elements of $\mathfrak{c}$ in $\mathrm{Hom}_{\mathcal{O}\text{-lin}}(X,X\underset{\mathcal{O}}{\otimes}\mathfrak{c})$, and under which the totally positive elements of $\mathfrak{c}$ correspond precisely to those symmetric $\mathcal{O}$-homs coming from polarizations (in the sense of [29], 6.3) of X/R.

For any (super)-natural number N, we denote by $\boldsymbol{\mu}_N$ the (ind)-groupscheme of "Nth roots of unity." A $\Gamma_{00}(N)$-structure on an HBAV X over R is an $\mathcal{O}$-linear homomorphism which is a closed immersion of R-(ind) groupschemes

(1.0.8) $i\colon\mathfrak{d}^{-1}\underset{\mathbf{Z}}{\otimes}\boldsymbol{\mu}_N\hookrightarrow X.$

Later, we will be particularly concerned with the case $N=p^n$ with p prime, $0\leqq n\leqq\infty$.

For any integer $N\geqq 1$, and any fractional ideal $\mathfrak{c}$ of K, we denote by $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))$ the moduli stack of $\mathfrak{c}$-polarized HBAV's with $\Gamma_{00}(N)$-structure. We recall that

(1.0.9) for all $N\geqq 1$, $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))$ is an algebraic stack, smooth of relative dimension g over $\mathbf{Z}$.

(1.0.10) the natural forgetful map $\mathscr{M}(\mathfrak{c},\Gamma_{00}(NM))\to\mathscr{M}(\mathfrak{c},\Gamma_{00}(N))$ is formally étale.

(1.0.11) for $N\geqq 4$, $\mathscr{M}(\mathfrak{c},\Gamma_{00}(N))$ is *represented* by an algebraic space, itself denoted $\mathscr{M}(\mathfrak{c},\Gamma_{00}(N))$, which is smooth of relative dimension g over $\mathbf{Z}$.

Let us denote by

$$\begin{tikzcd}(X_{\mathrm{univ}},\lambda_{\mathrm{univ}},i_{\mathrm{univ}}) \arrow[d, "\pi"] \\ \mathscr{M}(\mathfrak{c},\Gamma_{00}(N))\end{tikzcd} \tag{1.0.12}$$

the universal $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(N)$ structure. We denote by $\underline{\omega}$ and $\underline{\mathrm{Lie}}$ the sheaves $\pi_*(\Omega^1_{X_{\mathrm{univ}}/\mathscr{M}})$ and $\underline{\mathrm{Lie}}(X_{\mathrm{univ}}/\mathscr{M})$ on $\mathscr{M}(\mathfrak{c},\Gamma_{00}(N))$ respectively. They are each invertible $\mathcal{O}\underset{\mathbf{Z}}{\otimes}\mathcal{O}_{\mathscr{M}(\mathfrak{c},\Gamma_{00}(N))}$-modules, linked by a canonical isomorphism

(1.0.13) $\underline{\mathrm{Lie}}\underset{\mathcal{O}\otimes\mathcal{O}_{\mathscr{M}}}{\otimes}\underline{\omega}\xrightarrow{\sim}\mathfrak{d}^{-1}\underset{\mathcal{O}}{\otimes}\mathcal{O}_{\mathscr{M}}.$

We denote by H^1_{DR} the sheaf $\mathbf{R}^1\pi_*(\Omega^{\bullet}_{X_{\mathrm{univ}}\mathscr{M}})$ on $\mathscr{M}(\mathfrak{c},\Gamma_{00}(N))$. It sits in the "Hodge filtration" short exact sequence

(1.0.14) $0\to\underline{\omega}\to H^1_{DR}\to\underline{\mathrm{Lie}}(\mathrm{X}^t/\mathscr{M})\to 0.$

Using the $\mathfrak{c}$-polarization λ: $X^t\xrightarrow{\sim}X\underset{\mathcal{O}}{\otimes}\mathfrak{c}$, we can rewrite this

(1.0.15) $0\to\underline{\omega}\to H^1_{DR}\to\underline{\mathit{Lie}}\otimes\mathfrak{c}\to 0.$

Under the Gauss-Manin connection

(1.0.16) ∇: $H^1_{DR}\to H^1_{DR}\underset{\mathcal{O}_{\mathscr{M}}}{\otimes}\Omega^1_{\mathscr{M}/\mathbf{Z}}$,

any local section $D\in T_{\mathscr{M}/\mathbf{Z}}\overset{\mathrm{dfn}}{=}\underline{\mathrm{Der}}(\mathcal{O}_{\mathscr{M}},\mathcal{O}_{\mathscr{M}})$ defines a derivation"

(1.0.17) $\nabla(D)$: $H^1_{DR}\to H^1_{DR}$

which induces an $\mathcal{O}\otimes\mathcal{O}_{\mathscr{M}}$-linear map $K{-}S(D)$

$$\begin{tikzcd}\underline{\omega} \arrow[r, hook] \arrow[rrr, bend right, "K-S(D)"'] & H^1_{DR} \arrow[r, "\nabla(D)"] & H^1_{DR} \arrow[r, two heads] & \underline{\mathrm{Lie}}\otimes\mathfrak{c}.\end{tikzcd} \tag{1.0.18}$$

The construction $D\mapsto K-S(D)$ defines the Kodaira-Spencer *isomorphism*

(1.0.19) $T_{\mathscr{M}/\mathbf{Z}}\xrightarrow{\sim}\underline{\mathrm{Hom}}_{\mathcal{O}\otimes\mathcal{O}_{\mathscr{M}}}(\underline{\omega},\underline{\mathrm{Lie}}\underset{\mathcal{O}}{\otimes}\mathfrak{c})$

which we may rewrite, using (1.0.1), as an isomorphism

(1.0.20) $T_{\mathscr{M}/\mathbf{Z}}\xrightarrow{\sim}\underline{\mathrm{Lie}}^{\otimes 2}\underset{\mathcal{O}}{\otimes}\mathfrak{d}\mathfrak{c}.$

Its $\mathcal{O}_{\mathscr{M}}$-dual is an isomorphism

(1.0.21) $\Omega^1_{\mathscr{M}/\mathbf{Z}}\xleftarrow{\sim}\underline{\omega}^{\otimes 2}\underset{\mathcal{O}}{\otimes}\mathfrak{c}^{-1}$

(in the formulas (1.0.20–21) above, $\underline{\mathrm{Lie}}^{\otimes 2}$ and $\underline{\omega}^{\otimes 2}$ mean "square as invertible $\mathcal{O}\otimes\mathcal{O}_{\mathcal{M}}$-module").

1.1. We next give a brief account of the HBAV-analogue of the Tate elliptic curve Tate(q) over $\mathbf{Z}((q))$, which we will need for the algebraic theory of q-expansion. Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ be fractional ideals of K, such that

(1.1.1) $\mathfrak{c}=\mathfrak{a}\,\mathfrak{b}^{-1}$.

We would *like* to work with the ring of all formal "series"

(1.1.2) $a_0+\sum_{\substack{\alpha\in\mathfrak{a}\mathfrak{b}\\ \alpha\gg 0}} a_\alpha q^\alpha$, coefficients $a_0, a_\alpha\in\mathbf{Z}$

indexed by the elements of $\mathfrak{a}\,\mathfrak{b}$ which are either zero or totally positive, with multiplication given by

(1.1.3) $q^\alpha\cdot q^\beta=q^{\alpha+\beta}$, $q^0=1$.

Unfortunately, when $g>1$, this ring is too pathological for the present state of the art, so we must resort to an artifice. Consider a set S consisting of g linearly independent $\mathbf{Q}$-linear forms

(1.1.4) $l_i\colon K\to\mathbf{Q}$, $i=1,\ldots,g$,

which each map the totally positive elements of K to positive rational numbers. We say that an element α of K is S-positive if it satisfies

(1.1.5) $l_i(\alpha)\geqq 0$ for all $l_i\in S$.

For any lattice (free $\mathbf{Z}$ submodule of rank g) $A\subset K$, we denote by $A_{S\text{-pos}}\subset A$ the submonoid consisting of its S-positive elements. Then $A_{S\text{-pos}}$ is a finitely generated monoid (unlike the submonoid $A_{\text{tot pos}}$ of all totally positive elements), and we can recover $A_{\text{tot pos}}\cup\{0\}$ as the intersection, over all S, of $A_{S\text{-pos}}$.

We denote by $\mathbf{Z}[[A;S]]$ the ring of all formal series

(1.1.6) $\sum_{\alpha\in A_{S\text{-pos}}} a_\alpha q^\alpha$ $a_\alpha\in\mathbf{Z}$

indexed by the S-positive elements of A, and by $\mathbf{Z}((A.S))$ the ring obtained by *inverting* any (or every!) element q^α with α *totally* positive. Thus $\mathbf{Z}((A,S))$ consists of all formal series

(1.1.7) $\sum_{\alpha\in A} a_\alpha q^\alpha$

such that for some integer $n\gg 0$, we have

(1.1.8) $a_\alpha=0$ unless $l_i(\alpha)\geqq -n$ for all $l_i\in S$.

We will now apply these constructions to the case $A=\mathfrak{a}\,\mathfrak{b}$.

Over the ring $\mathbf{Z}((\mathfrak{a}\,\mathfrak{b},S))$, we have the constant g-dimensional algebraic torus

$\mathbf{G}_m\otimes\mathfrak{d}^{-1}\mathfrak{a}^{-1}$, together with an $\mathcal{O}$-linear group homomorphism

(1.1.9) $\underline{q}\colon \mathfrak{b}\mapsto\mathbf{G}_m\otimes\mathfrak{d}^{-1}\mathfrak{a}^{-1}$

defined as follows. To give (1.1.9), it is equivalent to give an $\mathcal{O}$-linear homomorphism

(1.1.10) $\underline{q}\colon \mathfrak{a}\mathfrak{b}\to\mathbf{G}_m\otimes\mathfrak{d}^{-1}$.

To give (1.1.10), it is in turn equivalent to give the $\mathbf{Z}$-linear group homomorphism

(1.1.11) $q\colon \mathfrak{a}\mathfrak{b}\to\mathbf{G}_m$

obtained from (1.1.10) by composing with the trace map $\mathbf{G}_m\otimes\mathfrak{d}^{-1}\xrightarrow{1\otimes\text{trace}}\mathbf{G}_m\otimes\mathbf{Z}=\mathbf{G}_m$. We define the map (1.1.11) by the formula

(1.1.12) $\alpha\in\mathfrak{a}\mathfrak{b}\mapsto q^{\alpha}\in\mathbf{Z}((\mathfrak{a}\mathfrak{b},S))^{\times}=\mathbf{G}_m(\mathbf{Z}(\mathfrak{a}\mathfrak{b},S)))$.

The rigid analytic quotient (via (1.1.9))

(1.1.13) $\mathbf{G}_m\otimes\mathfrak{d}^{-1}\mathfrak{a}^{-1}/\underline{q}(\mathfrak{b})$

is algebrifiable to an HBAV denoted $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ over $\mathbf{Z}((\mathfrak{a}\mathfrak{b},S))$, which carries a canonical $\mathfrak{c}=\mathfrak{a}\mathfrak{b}^{-1}$ polarization

(1.1.14) $\lambda_{\text{can}}\colon \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)^{t}\xrightarrow{\sim}\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)\otimes\mathfrak{a}\mathfrak{b}^{-1}\simeq\mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}(q)$.

For every integer $N\geqq 1$, the scheme-theoretic kernel of multiplication by N sits in an exact sequence

(1.1.15) $0\to\mathfrak{d}^{-1}\mathfrak{a}^{-1}\otimes\boldsymbol{\mu}_N\to(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q))_N\to\dfrac{1}{N}\mathfrak{b}/\mathfrak{b}\to 0$.

In particular, every choice of an $\mathcal{O}$-isomorphism

(1.1.16) $\varepsilon\colon \mathcal{O}/N\mathcal{O}\to\mathfrak{a}^{-1}/N\mathfrak{a}^{-1}$

gives rise to a $\Gamma_{00}(N)$-structure $i(\varepsilon)$ on $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$. When the fractional ideal $\mathfrak{a}$ is prime to N, both $\mathcal{O}$ and $\mathfrak{a}^{-1}$ have the *same* N-adic completions (inside $\prod_{p|N}K\otimes\mathbf{Q}_p$), so there is, in that case, a canonical isomorphism ε. The corresponding $\Gamma_{00}(N)$-structure on $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is denoted simply i_{can}.

The Lie algebra of $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is given by a canonical isomorphism

(1.1.17) $\omega_{\mathfrak{a}}\colon \mathrm{Lie}(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q))\simeq\mathrm{Lie}(\mathbf{G}_m\otimes\mathfrak{d}^{-1}\mathfrak{a}^{-1})=\mathfrak{d}^{-1}\mathfrak{a}^{-1}\otimes\mathbf{Z}((\mathfrak{a}\mathfrak{b},S))$,

and $\omega_{\mathfrak{a}}$ *is* an $\mathcal{O}\otimes\mathbf{Z}((\mathfrak{a}\mathfrak{b},S))$-basis of $\mathfrak{a}^{-1}\otimes\underline{\omega}$. The Kodaira-Spencer mapping (cf., (1.0.19))

(1.1.18)
$$\begin{aligned}K\text{–}S\colon \underline{\mathrm{Der}}(\mathbf{Z}((\mathfrak{a}\mathfrak{b},S)),\mathbf{Z}((\mathfrak{a}\mathfrak{b},S)))\to\underline{\mathrm{Lie}}^{\otimes 2}\otimes\mathfrak{d}\mathfrak{c}=\mathfrak{d}^{-2}\mathfrak{a}^{-2}\mathfrak{d}\mathfrak{c}\otimes\mathbf{Z}((\mathfrak{a}\mathfrak{b},S))\\ \|\qquad\qquad\\ \mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}\otimes\mathbf{Z}((\mathfrak{a}\mathfrak{b},S))\end{aligned}$$

may be described as follows. Given an element $\gamma\in\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$, consider the derivation $D(\gamma)$ of $\mathbf{Z}((\mathfrak{a}\mathfrak{b},S))$ into itself defined by

(1.1.19) $D(\gamma)(\sum a_\alpha q^\alpha)=\sum_\alpha \operatorname{trace}(\alpha\gamma)\,a_\alpha q^\alpha.$

Then the image of $D(\gamma)$ under (1.1.18) is given by the simple formula

(1.1.20) $K-S(D(\gamma))=\gamma\otimes 1$ in $\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}\otimes\mathbf{Z}((\mathfrak{a}\mathfrak{b},S))$.

1.2. We are now in a position to recall the definitions and the principal foundational results concerning Hilbert modular forms. Fix a fractional ideal $\mathfrak{c}$ of K. Fix a ground-ring R_0, and let χ be an algebraic character of the g-dimensional torus $\coprod_{\mathcal{O}\,\mathbf{Z}}(\mathbf{G}_m)$, defined over R_0. In concrete terms, this means that for all R_0-algebras R, χ "is" a homomorphism

(1.2.1) $\chi\colon(\mathcal{O}\otimes R)^\times\to R^\times$

whose formation commutes with arbitrary extension of scalars $R\to R'$ of R_0-algebras.

A $\mathfrak{c}$-Hilbert modular form ($\mathfrak{c}$-HMF) of weight χ on $\Gamma_{00}(N)$, defined over R_0, is a rule f which assigns to every $\mathfrak{c}$-polarized HBAV (X,λ) over an R_0-algebra R, given with a nowhere vanishing differential ω and a $\Gamma_{00}(N)$-structure i, an element $f(X,\lambda,\omega,i)\in R$, subject to the following conditions (1.2.2–4).

(1.2.2) The value $f(X,\lambda,\omega,i)$ depends only on the R-isomorphism class of (X,λ,ω,i).

(1.2.3) Formation of the value $f(X,\lambda,\omega,i)\in R$ commutes with arbitrary extension of scalars $R\to R'$ of R_0-algebras.

(1.2.4) For any (X,λ,ω,i) over R, and any $a\in(\mathcal{O}\otimes R)^\times$, we have

$$f(X,\lambda,a^{-1}\omega,i)=\chi(a)\cdot f(X,\lambda,\omega,i).$$

(1.2.5) We denote by $M(\mathfrak{c},\Gamma_{00}(N),\chi;R_0)$

the R_0-module of all $\mathfrak{c}$-HMF's of weight χ on $\Gamma_{00}(N)$, defined over R.

Equivalently, we may define a $\mathfrak{c}$-HMF of weight χ on $\Gamma_{00}(N)$, defined over R_0, as a rule f which, to each $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(N)$ structure (X,λ,i) over an R_0-algebra R, assigns an element $f(x,\lambda,i)\in\underline{\omega}_{X/R}(\chi)$, where $\underline{\omega}_{X/R}(\chi)$ denotes the invertible R-module obtained from $\underline{\omega}_{X/R}$ by extension of the structural group by means of χ. This rule is subject to the following conditions (1.2.6–7):

(1.2.6) The value $f(X,\lambda,i)$ depends only upon the R-isomorphism class of (X,λ,i).

(1.2.7) Formation of the value $f(X,\lambda,i)$ commutes with arbitrary extension of scalars $R\to R'$ of R_0-algebras.

A form f in this latter sense gives rise to a form $\tilde{f}$ in the former sense by the rule

(1.2.8) $\tilde{f}(X, \lambda, \omega, i) \cdot \omega(\chi) = f(X, \lambda, i)$

where $\omega(\chi)$ is the R-basis of $\underline{\omega}(\chi)$ deduced from the $\mathcal{O} \otimes R$-basis ω of $\underline{\omega}$ by extension of the structural group.

Still another way to view $\mathfrak{c}$-HMF's is this, at least for $N \geqq 4$. Take the algebraic space $\mathcal{M}(\mathfrak{c}; \Gamma_{00}(N))_{R_0} \overset{\mathrm{dfn}}{=} \mathcal{M}(\mathfrak{c}; \Gamma_{00}(N)) \underset{\mathrm{Spec}(\mathbf{Z})}{\times} \mathrm{Spec}(R_0)$. Denote by $\underline{\omega}(\chi)$ the invertible sheaf on it obtained from $\underline{\omega}$ by extension of the structural group via χ. Then the elements of $H^0(\mathcal{M}(\mathfrak{c}; \Gamma_{00}(N))_{R_0}, \underline{\omega}(\chi))$ are precisely the $\mathfrak{c}$-HMF's of weight χ on $\Gamma_{00}(N)$ defined over R_0.

A $\mathfrak{c}$-HMF f of weight χ on $\Gamma_{00}(N)$, defined over R_0, gives rise to a plethora of q-expansions, as follows. Pick two fractional ideals $\mathfrak{a}$, $\mathfrak{b}$ such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$. Pick an $\mathcal{O}$-isomorphism

(1.2.9) $\varepsilon\colon \mathcal{O}/N\mathcal{O} \xrightarrow{\sim} \mathfrak{a}^{-1}/N\mathfrak{a}^{-1}$,

and an $\mathcal{O} \otimes R_0$-isomorphism

(1.2.10) $j\colon \mathfrak{a}^{-1} \underset{\mathbf{Z}}{\otimes} R_0 \to \mathcal{O} \underset{\mathbf{Z}}{\otimes} R_0$.

(Such a j need not exist globally on R_0, but does exist locally on R_0.) By extension of scalars from $\mathbf{Z}((\mathfrak{a}\mathfrak{b}; S))$ to $R_0 \otimes \mathbf{Z}((\mathfrak{a}\mathfrak{b}; S))$, we obtain $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, i(\varepsilon))$ over $R_0 \otimes \mathbf{Z}((\mathfrak{a}\mathfrak{b}; S))$, together with basis $\omega_{\mathfrak{a}}$ of $\mathfrak{a}^{-1} \otimes \underline{\omega}$. By means of j (1.2.10), $\omega_{\mathfrak{a}}$ gives rise to a basis $\omega_{\mathfrak{a}}(j)$ of $\underline{\omega}$. Thus we may form the value

(1.2.11)
$$\begin{tikzcd} f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, \omega_{\mathfrak{a}}(j), i(\varepsilon)) \in R_0 \otimes \mathbf{Z}((\mathfrak{a}\mathfrak{b}, S)) \arrow[d, hook] \\ R_0((\mathfrak{a}\mathfrak{b}, S)). \end{tikzcd}$$

The actual expression

(1.2.12) $f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathfrak{a}}(j), i(\varepsilon)) = \sum_{\alpha \in \mathfrak{a}\mathfrak{b}} a(f, \alpha; \mathfrak{a}, \mathfrak{b}, j, i(\varepsilon)) \cdot q^{\alpha}$

is called "the q-expansion of f at the cusp $(\mathfrak{a}; \mathfrak{b}, j, i(\varepsilon))$. The quantities $a(f, \alpha; \mathfrak{a}, \mathfrak{b}, j, i(\varepsilon)) \in R_0$ are called the q-expansion coefficients of f at that cusp. The q-expansion coefficients are independent of the auxiliary choice of S (cf., (1.1.4)). In the case $K \neq \mathbf{Q}$, one has the additional result that

(1.2.13) $a(f, \alpha; \mathfrak{a}, \mathfrak{b}, j, i(\varepsilon)) = 0 \quad \text{unless} \quad \alpha = 0 \quad \text{or } \alpha \gg 0.$

(1.2.14) According to a fundamental theorem of Ribet [18], the geometric fibres of $\mathcal{M}(\mathfrak{c}, \Gamma_{00}(N))$ over $\mathrm{Spec}(\mathbf{Z})$ are all geometrically irreducible, from which follows the q-expansion (cf. [17]), in the following strong form.

(*q-expansion principle*) Fix an integer $N \geqq 1$, a ring R_0, and a character χ of $(\coprod_{\mathcal{O}/\mathbf{Z}} \mathbf{G}_m)_{R_0}$. Let $(\mathfrak{a}, \mathfrak{b}, j, i(\varepsilon))$ be any cusp as in (1.2.9-12) above. Then we have:

(1.2.15) Let f be a $\mathfrak{c}$-HMF of weight χ on $\Gamma_{00}(N)$, defined over R_0. If the q-expansion of f at the cusp $(\mathfrak{a}, \mathfrak{b}, j, i(\varepsilon))$ vanishes identically, then $f=0$.

(1.2.16) Let $R_0 \subset R$, and let f be a $\mathfrak{c}$-HMF of weight χ on $\Gamma_{00}(N)$ defined over R. If the q-expansion of f at the cusp $(\mathfrak{a}, \mathfrak{b}, j, i(\varepsilon))$ has all its q-expansion coefficients in R_0, then there exists a unique $\mathfrak{c}$-HMF of weight χ on $\Gamma_{00}(N)$ defined over R_0, which gives rise to f by extension of scalars $R_0 \to R$.

1.3. We turn now to the case of a supernatural number N, say the inverse limit of the integers N_i, with $N_i | N_{i+1}$. The natural inclusions

(1.3.1) $\mathfrak{d}^{-1} \otimes \boldsymbol{\mu}_{N_i} \hookrightarrow \mathfrak{d}^{-1} \otimes \boldsymbol{\mu}_{N_{i+1}}$

lead to "forgetting" morphisms of moduli schemes

(1.3.2)
$$\begin{tikzcd}\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N_{i+1})) \arrow[d] \\ \mathscr{M}(\mathfrak{c}, \Gamma_{00}(N_i))\end{tikzcd}$$

Because these morphisms are *affine*, the inverse limit of these schemes exists.

(1.3.3) $\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N)) \overset{\text{dfn}}{=} \varprojlim \mathscr{M}(\mathfrak{c}, \Gamma_{00}(N_i))$.

This $\mathscr{M}(\Gamma_{00}(N))$ is affine over each $\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N_i))$, and it represents the functor "isomorphism classes of $\mathfrak{c}$-polarized HBAV's with $\Gamma_{00}(N)$ structure." Because the maps (1.3.2) are all étale, the scheme $\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N))$ is *pro-étale* over each $\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N_i))$, a fact which will be useful later.

To define a $\mathfrak{c}$-HMF of weight χ on $\Gamma_{00}(N)$ over a ring R_0, we can repeat either of the two definitions (1.2.2-4) or (1.2.6-7), or equivalently we can define it to be an element of

(1.3.4) $H^0(\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N)), \underline{\omega}(\chi))$

or, what is the same, an element of

(1.3.5) $\varinjlim_i H^0(\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N_i)), \underline{\omega}(\chi))$.

This last equivalence, expressed in the notation of (1.2.5), becomes

$$M(\mathfrak{c}, \Gamma_{00}(N), \chi; R_0) = \varinjlim_i M(\mathfrak{c}, \Gamma_{00}(N_i), \chi; R_0).$$

The q-expansion principle (1.2.15-16) remains valid.

1.4. We next give a transcendental description of the situation over $\mathbf{C}$. By a lattice $\mathscr{L} \subset K \otimes \mathbf{C}$, we mean an $\mathcal{O}$-submodule of $K \otimes \mathbf{C}$ which, as $\mathcal{O}$-module, is locally free of rank two, and whose real span $\mathscr{L} \otimes \mathbf{R}$ is all of $K \otimes \mathbf{C}$. Given an HBAV X over $\mathbf{C}$ together with a nowhere vanishing differential ω, we associate to it the lattice

$\mathscr{L} \subset K \otimes \mathbf{C}$ of all "periods" of ω, as follows. The complex torus X^{an} is the quotient of its universal covering, $\mathrm{Lie}(X^{\mathrm{an}})$, by its fundamental group $\pi_1(X^{\mathrm{an}}) \subset \mathrm{Lie}(X^{\mathrm{an}})$:

$$(1.4.1) \quad 0 \to \pi_1(X^{\mathrm{an}}) \to \mathrm{Lie}(X^{\mathrm{an}}) \to X^{\mathrm{an}} \to 0.$$

The nowhere-vanishing differential ω provides an $\mathcal{O} \otimes \mathbf{C}$ isomorphism

$$(1.4.2) \quad \omega: \mathrm{Lie}(X^{\mathrm{an}}) \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes \mathbf{C} = K \otimes \mathbf{C}$$

under which the image of $\pi_1(X^{\mathrm{an}})$ is the desired lattice $\mathscr{L} \subset K \otimes \mathbf{C}$. Conversely, given a lattice $\mathscr{L} \subset K \otimes \mathbf{C}$, one can show that the complex torus $K \otimes \mathbf{C}/\mathscr{L}$, whose Lie algebra is given as $\mathfrak{d}^{-1} \otimes \mathbf{C} = K \otimes \mathbf{C}$, is algebrifiable to a necessarily unique (X, ω). Thus we have a bijective correspondence

$$(1.4.3) \quad \{\text{pairs}(X, \omega) \text{ over } \mathbf{C}\} \leftrightarrow \{\text{lattices } \mathscr{L} \subset K \otimes \mathbf{C}\}.$$

Via this correspondence, a $\mathfrak{c}$-polarization λ on X corresponds exactly to an alternating $\mathcal{O}$-bilinear form

$$(1.4.4) \quad \langle\ ,\ \rangle: \Lambda^2_{\mathcal{O}} \mathscr{L} \xrightarrow{\sim} \mathfrak{d}^{-1} \mathfrak{c}^{-1}$$

which, for some (necessarily unique) totally positive element $A \in K \otimes \mathbf{R}$ is given by the formula

$$(1.4.5) \quad \langle u, v \rangle = \frac{\mathrm{Im}(\bar{u}\, v)}{A} \quad \text{for } u, v \in \mathscr{L}$$

– the complex conjugation takes place in $K \otimes \mathbf{C}$, and the equality holds in $\mathfrak{d}^{-1} \mathfrak{c}^{-1} \otimes \mathbf{R} = K \otimes \mathbf{R}$. The constant A is so denoted because in the case $K = \mathbf{Q}$, and $\langle\ ,\ \rangle$ corresponding to the canonical autoduality of an elliptic curve, it is nothing other than the *area* of a fundamental parallelogram for the period lattice. By (1.4.5) it is equivalent to know $\langle\ ,\ \rangle$ or to know A. We call $\langle\ ,\ \rangle$ a $\mathfrak{c}$-polarization of the lattice $\mathscr{L}$. Thus we have a bijective correspondence

$$(1.4.6) \quad \{\mathfrak{c}\text{-polarized } (X, \lambda, \omega)\text{'s over } \mathbf{C}\} \leftrightarrow \{\mathfrak{c}\text{-polarized lattices } (\mathscr{L}, \langle\ ,\ \rangle)\}.$$

A $\Gamma_{00}(N)$-structure, N an ordinary integer, on (X, ω) over $\mathbf{C}$, corresponds, via the exponential isomorphism

$$(1.4.7) \quad \frac{1}{N}\mathbf{Z}/\mathbf{Z} \xrightarrow[\sim]{\exp(2\pi i -)} \boldsymbol{\mu}_N,$$

to an *injective* $\mathcal{O}$-linear map

$$(1.4.8) \quad i: \frac{1}{N}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \hookrightarrow \frac{1}{N}\mathscr{L}/\mathscr{L},$$

or equivalently, as an injective $\mathcal{O}$-linear map

$$(1.4.8 \text{ bis}) \quad i: \mathfrak{d}^{-1} \otimes \mathbf{Z}/N\mathbf{Z} \hookrightarrow \mathscr{L} \otimes \mathbf{Z}/N\mathbf{Z}.$$

When N is supernatural, say $N=\varprojlim N_i$ with $N_i|N_{i+1}$, then a $\Gamma_{00}(N)$ structure corresponds precisely to an injective $\mathcal{O}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}$-linear map

(1.4.9) $i\colon \mathfrak{d}^{-1}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}\hookrightarrow\mathcal{L}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}$

whose cokernel is an invertible $\mathcal{O}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}$-module. (This "extra" condition on the cokernel i is automatic in the case of finite N.)

1.5. With this transcendental description at hand, we can write down some explicit families of complex HBAV's. Let $\tau\in K\otimes\mathbf{C}$ be any element with $\operatorname{Im}(\tau)$ totally positive. For any two fractional ideals $\mathfrak{a},\mathfrak{b}$ of K, we define a lattice $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)\subset K\otimes\mathbf{C}$ by

(1.5.1) $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)=2\pi i(\mathfrak{d}^{-1}\mathfrak{a}^{-1}+\mathfrak{b}\tau)$.

This lattice carries a canonical $\mathfrak{c}=\mathfrak{a}\,\mathfrak{b}^{-1}$ polarization $\langle\ ,\ \rangle_{\text{can}}$, given by

(1.5.2) $\langle 2\pi i(a+b\tau), 2\pi i(c+d\tau)\rangle_{\text{can}}=ad-bc$

for a,c in $\mathfrak{d}^{-1}\mathfrak{a}^{-1}, b,d$ in $\mathfrak{b}$. The totally positive $A(\tau)\in K\otimes\mathbf{R}$ for which

$$\langle u,v\rangle_{\text{can}}=\frac{\operatorname{Im}(\bar{u}\,v)}{A(\tau)}$$

is given by

(1.5.3) $A(\tau)=4\pi^2\operatorname{Im}(\tau)$.

(1.5.4) *Any* $\mathfrak{c}$-polarized lattice is siomorphic to an $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$ (for some τ with $\operatorname{Im}(\tau)\gg 0$).

Given a (possibly super-) natural number $N=\varprojlim N_i$, an $\mathcal{O}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}$ isomorphism

(1.5.5) $\varepsilon\colon\mathcal{O}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}\xrightarrow{\sim}\mathfrak{a}^{-1}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}$

determines a $\Gamma_{00}(N)$-structure $i(\varepsilon)$ on $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$, as the composite

(1.5.6) $i(\varepsilon)\colon\mathfrak{d}^{-1}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}\xrightarrow[\sim]{\varepsilon}\mathfrak{d}^{-1}\mathfrak{a}^{-1}\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}$

$\xhookrightarrow{\times 2\pi i}\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)\otimes\varprojlim\mathbf{Z}/N_i\mathbf{Z}$.

1.6. We now discuss $\mathfrak{c}$-HMF's from the complex analytic point of view. Let χ be a character of the torus $\coprod_{\mathcal{O}/\mathbf{Z}}(\mathbf{G}_m)$, defined over $\mathbf{C}$, and let f be a $\mathfrak{c}$-HMF of weight χ on $\Gamma_{00}(N)$, defined over $\mathbf{C}$. Then f gives rise to a holomorphic complex-valued f^{an} on the space of all $\mathfrak{c}$-polarized lattices with $\Gamma_{00}(N)$ structure $(\mathscr{L}, \langle\ ,\ \rangle, i)$, which transforms under the action of $a \in (K \otimes \mathbf{C})^{\times}$ by the formula

(1.6.1) $f^{\mathrm{an}}(a^{-1}\mathscr{L}, a\bar{a}\langle\ ,\ \rangle, a^{-1}i) = \chi(a)\, f^{\mathrm{an}}(\mathscr{L}, \langle\ ,\ \rangle, i)$.

Let us temporarily denote by

(1.6.2) $M(\mathfrak{c}, \Gamma_{00}(N), \chi)^{\mathrm{an}}$

the $\mathbf{C}$-vector space of all analytic functions on the space of $\mathfrak{c}$-polarized lattices with $\Gamma_{00}(N)$-structure which transform by (1.6.2).

We will make constant use of the following GAGA-style result:

(1.6.3) **Theorem.** *If the field K is different from $\mathbf{Q}$, the construction $f \mapsto f^{\mathrm{an}}$ defines an isomorphism*

(1.6.4) $M(\mathfrak{c}, \Gamma_{00}(N), \chi; \mathbf{C}) \to M(\mathfrak{c}, \Gamma_{00}(N), \chi)^{\mathrm{an}}$.

(In the case $K = \mathbf{Q}$, this is false, but may be "corrected" by imposing a suitable "meromorphic at ∞" condition on the analytic side. For $K \neq \mathbf{Q}$, this meromorphy is automatic. See [7] for a detailed discussion of the case $K = \mathbf{Q}$.)

1.7. We next recall the complex-analytic description of q-expansion. Let us denote by j_{can} the equality-isomorphism

(1.7.1) $j_{\mathrm{can}}\colon \mathfrak{a}^{-1} \otimes \mathbf{C} = \mathcal{O} \otimes \mathbf{C} \quad (\text{in } K \otimes \mathbf{C})$.

Then we can consider the q-expansion of f at the cusp $(\mathfrak{a}, \mathfrak{b}, j_{\mathrm{can}}, i(\varepsilon))$ as in (1.2.12):

(1.7.2) $f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, \omega_{\mathfrak{a}}(j), i(\varepsilon)) = \sum_{\alpha \in \mathfrak{a}} a(f, \alpha, \mathfrak{a}, \mathfrak{b}, j_{\mathrm{can}}, i(e)) \cdot q^{\alpha}$.

On the other hand, we can consider the holomorphic function $f_{\mathfrak{a},\mathfrak{b},\varepsilon}$ on

(1.7.3) $\{\tau \in K \otimes \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$

defined by

(1.7.4) $f_{\mathfrak{a},\mathfrak{b},\varepsilon}\colon \tau \mapsto f^{\mathrm{an}}(\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau), \langle\ ,\ \rangle_{\mathrm{can}}, i(\varepsilon))$.

Because the data $(\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau), \langle\ ,\ \rangle_{\mathrm{can}}, i(\varepsilon))$ remains unchanged if we additively translate τ by an element of $\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mathfrak{b}^{-1}$, it follows that the function $f_{\mathfrak{a},\mathfrak{b},\varepsilon}$ has a Fourier expansion of the form

(1.7.5) $f_{\mathfrak{a},\mathfrak{b},\varepsilon}(\tau) = \sum_{\alpha \in \mathfrak{a}\mathfrak{b}} b(f, \alpha, \mathfrak{a}, \mathfrak{b}, \varepsilon) \exp(2\pi i\, \mathrm{trace}(\alpha\tau))$.

The Fourier coefficients $b(f, \alpha, \mathfrak{a}, \mathfrak{b}, \varepsilon)$ occurring in (1.7.5) are none other than the algebraically defined q-expansion coefficients $a(f, \alpha, \mathfrak{a}, \mathfrak{b}, j_{\mathrm{can}}, i(\varepsilon))$ occurring in (1.7.2):

(1.7.6) $b(f, \alpha, \mathfrak{a}, \mathfrak{b}, \varepsilon) = a(f, \alpha, \mathfrak{a}, \mathfrak{b}, j_{\mathrm{can}}, i(\varepsilon))$.

1.8. We now recall the notion of a C^∞ (meaning C^∞ but not *necessarily* holomorphic) $\mathfrak{c}$-HMF. By a C^∞ $\mathfrak{c}$-HMF of weight χ on $\Gamma_{00}(N)$ we mean a C^∞ complex-valued function on the space of all $\mathfrak{c}$-polarized lattices with $\Gamma_{00}(N)$-structure which transforms as in (1.6.1).

If we denote by $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))^{\mathrm{an}}$ the complex manifold of all $\mathbf{C}$-valued points of $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))$, by $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))(C^\infty)$ the underlying C^∞-manifold, and by $\underline{\omega}(\chi)^{\mathrm{an}}$ and $\underline{\omega}(\chi)(C^\infty)$ the invertible sheaves on these spaces obtained by tensoring $\underline{\omega}(\chi)$ with the holomorphic and the C^∞-structural sheaves respectively, we have the following diagram of inclusions and identifications:

$$\begin{array}{lccc} & M(\mathfrak{c},\Gamma_{00}(N),\chi,\mathbf{C}) & = & H^0\cdot\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))_{\mathbf{C}},\underline{\omega}(\chi)) \\ & \cap & & \cap \\ (1.8.1) & M(\mathfrak{c},\Gamma_{00}(N),\chi)^{\mathrm{an}} & = & H^0(\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))^{\mathrm{an}},\underline{\omega}(\chi)^{\mathrm{an}}) \\ & \cap & & \cap \\ & M(\mathfrak{c},\Gamma_{00}(N),\chi)(C^\infty) & = & H^0(\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))(C^\infty),\underline{\omega}(\chi)(C^\infty)). \end{array}$$

For later use, we record here one of the principal merits of $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))(C^\infty)$: over it, the Hodge short exact sequence (1.0.15)

$$(1.8.2)\quad 0\to\underline{\omega}\to H^1_{DR}\to\underline{\mathrm{Lie}}\otimes\mathfrak{c}\to 0$$

is canonically split by the Hodge *decomposition:*

$$(1.8.3)\quad H^1_{DR}(C^\infty)=\underline{\omega}(C^\infty)\oplus\overline{\underline{\omega}(C^\infty)}$$

where the bar denotes complex conjugation. Over each point of $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))$, this splitting is simply the ordinary Hodge decomposition

$$(1.8.4)\quad H^1=H^{1,0}\oplus H^{0,1},H^{0,1}=\overline{H^{1,0}}.$$

The C^∞-submodule

$$(1.8.5)\quad \overline{\underline{\omega}(C^\infty)}\subset H^1_{DR}(C^\infty)$$

is *holomorphically horizontal* in the following sense:

For any local *holomorphic* section D of the holomorphic tangent bundle of $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))^{\mathrm{an}}$, we have

$$(1.8.6)\quad \nabla(D)(\overline{\underline{\omega}(C^\infty)})\subset\overline{\underline{\omega}(C^\infty)}.$$

In fact, if h is a holomorphic section of H^1_{DR}, and f is a C^∞ function on $\mathcal{M}(\mathfrak{c},\Gamma_{00}(N))(C^\infty)$, then we have the tautological formulas

$$(1.8.7)\quad \nabla(D)(f\cdot\bar{h})=D(f)\cdot\bar{h},\qquad \nabla(D)(\bar{h})=0.$$

Thus, the "complex conjugate" of *any* holomorphic subbundle of H^1_{DR}, in particular of $\underline{\omega}$, is holomorphically horizontal.

(1.9) We now turn to the notion of p-adic HMF's. Our object is to develop certain fruitful analogies between p-adic HMF's and C^∞ HMF's.

Fix a prime number p. We say that a ring R is "p-adic" if it is complete and separated in its p-adic topology, i.e., if

(1.9.1) $R \xrightarrow{\sim} \varprojlim R/p^n R$.

We will be concerned with $\mathfrak{c}$-polarized HBAV's with $\Gamma_{00}(p^\infty)$-structure, (X, λ, i), over p-adic rings R.

Let R_0 be a p-adic ring. A p-adic $\mathfrak{c}$-HMF defined over R_0 is a rule f which assigns to every $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(p^\infty)$-structure (X, λ, i) defined over a p-adic R_0-algebra R, an element $f(X, \lambda, i) \in R$, subject to the following conditions (1.9.2–3).

(1.9.2) The value $f(X, \lambda, i)$ depends only on the R-isomorphism class of (X, λ, i).

(1.9.3) Formation of the value $f(X, \lambda, i)$ commutes with arbitrary extension of scalars of p-adic R_0-algebras.

We denote by $V(\mathfrak{c}, R_0)$ the R_0-algebra of all p-adic $\mathfrak{c}$-HMF's defined over R_0. Notice that we have, tautologically,

(1.9.4) $V(\mathfrak{c}, R_0) \xrightarrow{\sim} \varprojlim V(\mathfrak{c}, R_0/p^n R_0)$,

and that $V(\mathfrak{c}, R_0)$ is itself a p-adic R_0-algebra. In case p is *nilpotent* in R_0, every R_0-algebra is automatically p-adic, so that we have

(1.9.5) *Tautology.* When p is nilpotent in R_0, the ring $V(\mathfrak{c}, R_0)$ is nothing other than the ring $M(\mathfrak{c}, \Gamma_{00}(p^\infty), \chi_{\text{trivial}}, R_0) = H^0(\mathcal{M}(\mathfrak{c}, \Gamma_{00}(p^\infty))_{R_0}, \mathcal{O}_{\mathcal{M}})$ of all $\mathfrak{c}$-HMF's of weight *zero* (i.e., transorming by the *trivial* character) on $\Gamma_{00}(p^\infty)$ defined over R_0.

Thus for *any* p-adic ring R_0, we may interpret $V(\mathfrak{c}, R_0)$ as the ring of global sections of the structural sheaf of the *formal scheme* $\mathcal{M}(\mathfrak{c})_{R_0}^{p\text{-adic}}$ over R_0 defined as

(1.9.6) $\mathcal{M}(\mathfrak{c})_{R_0}^{p\text{-adic}} = \{\mathcal{M}(\mathfrak{c}, \Gamma_{00}(p^\infty))\, R_0/p^n R_0\}$.

In order to define q-expansions, we proceed as follows. Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of K with $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$, and choose an $\mathcal{O} \otimes \mathbf{Z}_p$ isomorphism

(1.9.7) $\varepsilon\colon \mathcal{O} \otimes \mathbf{Z}_p \xrightarrow{\sim} \mathfrak{a}^{-1} \otimes \mathbf{Z}_p$.

By extension of scalars from $\mathbf{Z}((\mathfrak{a}\mathfrak{b}; S))$ to $\widehat{R_0((\mathfrak{a}\mathfrak{b}; S))}$ (the hat denoting p-adic completion), we obtain $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathfrak{c}\mathrm{n}}, i(\varepsilon))$ over the p-adic R_0-algebra $\widehat{R_0((\mathfrak{a}\,\mathfrak{b}; S))}$. Evaluating elements $f \in V(\mathfrak{c}, R_0)$ there gives a "q-expansion"-homomorphism

(1.9.8) $V(\mathfrak{c}, R_0) \to \widehat{R_0((\mathfrak{a}\mathfrak{b}; S))}$.

(1.9.9) **Theorem.** *The q-expansion homomorphism* (1.9.8) *is injective, and, when* R_0 *is flat over* $\mathbf{Z}_p$ (i.e., when R_0 has no p-torsion), *the cokernel*

(1.9.10) $\widehat{R_0((\mathfrak{a}\mathfrak{b}; S))}/V(\mathfrak{c}, R_0)$

is flat over $\mathbf{Z}_p$ (i.e., it has no p-torsion).

Proof. The injectivity is clear because (1.9.8) is the inverse limit of the q-expansion homomorphisms

$$(1.9.11) \quad V(\mathfrak{c}, R_0/p^n R_0) = M(\mathfrak{c}, \Gamma_{00}(p^\infty), \chi_{\mathrm{trivial}}, R_0/p^n R_0) \to (R_0/p^n R_0)((\mathfrak{a}\,\mathfrak{b}; S)),$$

each of which is injective by the usual q-expansion principle (1.2.15).

Suppose now that R_0 is flat over $\mathbf{Z}_p$, and suppose we are given a representative $f(q)$ of a p-torsion element in the cokernel (1.9.10). This means that there exists an element $g \in V(\mathfrak{c}, R_0)$ such that g has q-expansion $p \cdot f(q)$, and hence that the image of g in $V(\mathfrak{c}, R_0/pR_0)$ vanishes. Because $\mathcal{M}(\mathfrak{c}, \Gamma_{00}(p^\infty))$ is flat over $\mathbf{Z}$, and R_0 is flat over $\mathbf{Z}_p$, the formal scheme (1.9.6) is flat over R_0, and hence, denoting by $\mathcal{O}^{\wedge}_{\mathcal{M}}$ its structural sheaf, the sequence

$$(1.9.12) \quad 0 \to \mathcal{O}^{\wedge}_{\mathcal{M}} \xrightarrow{p} \mathcal{O}^{\wedge}_{\mathcal{M}} \to \mathcal{O}^{\wedge}_{\mathcal{M}}/p\mathcal{O}^{\wedge}_{\mathcal{M}} \to 0$$

is exact. Taking global sections, we get an exact sequence

$$(1.9.13) \quad 0 \to V(\mathfrak{c}, R_0) \xrightarrow{p} V(\mathfrak{c}, R_0) \to V(\mathfrak{c}, R_0/pR_0).$$

Thus our element g may be written pf for a unique $f \in V(\mathfrak{c}, R_0)$, and the q-expansion of f is our $f(q)$. QED

For p-adic R_0's which are flat over $\mathbf{Z}_p$, we *define*

$$(1.9.14) \quad V(\mathfrak{c}, R_0) \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p$$

to be the space of "p-adic $\mathfrak{c}$-HMF's defined over $R_0 \otimes \mathbf{Q}_p$." The q-expansion homomorphism (1.9.8) extends to a homomorphism

$$(1.9.15) \quad \begin{tikzcd} V(\mathfrak{c}, R_0) \otimes \mathbf{Q}_p \arrow[r] & (\widehat{R_0((\mathfrak{a}\,\mathfrak{b}; S))}) \otimes \mathbf{Q}_p \arrow[d, phantom, "\cap"] \\ & (R_0 \otimes \mathbf{Q}_p)((\mathfrak{a}\,\mathfrak{b})) \end{tikzcd}$$

where we denote by $(R_0 \otimes \mathbf{Q}_p)((\mathfrak{a}\,\mathfrak{b}))$ the $R_0 \otimes \mathbf{Q}_p$ module of *all* formal series

$$(1.9.16) \quad \sum_{\alpha \in \mathfrak{a}\,\mathfrak{b}} a_\alpha q^\alpha$$

with coefficients $a_\alpha \in R_0 \otimes \mathbf{Q}_p$.

It follows immediately from (1.9.9) that we have

(1.9.17) **Corollary.** *The q-expansion homomorphism* (1.9.15) *is injective, and the R_0-submodule $V(\mathfrak{c}, R_0)$ of $V(\mathfrak{c}, R_0) \otimes \mathbf{Q}_p$ consists precisely of those elements of $V(\mathfrak{c}, R_0) \otimes \mathbf{Q}_p$ whose images under the q-expansion homomorphism* (1.9.15) *have coefficients in R_0.*

Because the formal scheme (1.9.6) is itself flat over R_0, we can give a modular description of $V(\mathfrak{c}, R_0) \otimes \mathbf{Q}_p$ when R_0 is flat over $\mathbf{Z}_p$, as follows: an element of $V(\mathfrak{c}, R_0) \otimes \mathbf{Q}_p$ is precisely a rule f which assigns to every $\mathfrak{c}$-polarized HBAV with

$\Gamma_{00}(p^{\infty})$ structure (X, λ, i) over a p-adic R_0-algebra R which is flat over $\mathbf{Z}_p$, a quantity

(1.9.18) $f(X, \lambda, i) \in R \otimes \mathbf{Q}_p$

subject to the following rules:

(1.9.19) The value $f(X, \lambda, i) \otimes \mathbf{Q}_p$ depends only on the R-isomorphism class of (X, λ, i).

(1.9.20) Formation of the value $f(X, \lambda, i) \in R \otimes \mathbf{Q}_p$ commutes with arbitrary extension of scalars $R \to R'$ of p-adic algebras flat over $\mathbf{Z}_p$.

Alternately, we can view $V(\mathfrak{c}, R_0) \otimes \mathbf{Q}_p$ as the ring of global sections, over the formal scheme (1.9.6) $\mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0}$, of the sheaf (cf., (1.9.12)) $\mathcal{O}_{\hat{\mathcal{M}}} \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p$.

(1.10) We now come to one of the characteristic features of the p-adic theory, namely that over a p-adic R_0, any $\mathfrak{c}$-HMF on $\Gamma_{00}(p^{\infty})$ of *any* weight χ, defined over R_0, gives rise to a p-adic $\mathfrak{c}$-HMF over R_0 which has the *same* q-expansions. The construction is based on the following lemmas.

(1.10.1) **Lemma.** *Let X be an HBAV over a p-adic ring R. Then a $\Gamma_{00}(p^{\infty})$-structure over R*

(1.10.2) $i\colon \mathfrak{d}^{-1} \otimes \boldsymbol{\mu}_{p^{\infty}} \hookrightarrow X$

is equivalent to an isomorphism of formal groups, denoted $\hat{i}$,

(1.10.3) $\hat{i}\colon \mathfrak{d}^{-1} \otimes \hat{\mathbf{G}}_m \xrightarrow{\sim} \hat{X}$,

Proof. We can reduce immediately to the case when p is nilpotent. In this case, the *functors* $\boldsymbol{\mu}_{p^{\infty}}$ and $\hat{\mathbf{G}}_m$ coincide, so also the functors $\mathfrak{d}^{-1} \otimes \boldsymbol{\mu}_{p^{\infty}}$ and $\mathfrak{d}^{-1} \otimes \hat{\mathbf{G}}_m$. Since $\hat{X} \subset X$, an isomorphism (1.10.3) gives rise to an inclusion (1.10.2). Conversely, given an i as in (1.10.2), we can reinterpret it as an inclusion

(1.10.4) $i\colon \mathfrak{d}^{-1} \otimes \hat{\mathbf{G}}_m \hookrightarrow X$.

This map necessarily factors through $\hat{X}$, and so gives rise to an inclusion

(1.10.5) $\hat{i}\colon \mathfrak{d}^{-1} \otimes \hat{\mathbf{G}}_m \hookrightarrow \hat{X}$.

To see that it is an *isomorphism*, one reduces easily to checking the case when R is a *field* k of characteristic p. Both source and target are g-parameter formal Lie groups over K, and the injectivity of (1.10.5), applied to $k[\varepsilon]/(\varepsilon^2)$-valued points, shows that $\hat{i}$ is injective, and hence bijective, on tangent spaces. Hence $\hat{i}$ is an isomorphism.

(1.10.6) **Lemma.** *Let $\mathfrak{a}$ be a fractional ideal of K, and ε an isomorphism*

(1.10.7) $\varepsilon\colon \mathcal{O} \otimes \mathbf{Z}_p \xrightarrow{\sim} \mathfrak{a}^{-1} \otimes \mathbf{Z}_p$.

Consider $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), i(\varepsilon))$ *over* $\mathbf{Z}_p\widehat{((\mathfrak{a}\mathfrak{b}; S))}$. *Then the isomorphism*

(1.10.8) $i(\varepsilon)\colon \mathfrak{d}^{-1} \otimes \hat{\mathbf{G}}_m \xrightarrow{\sim} \widehat{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)}$

corresponding via (1.10.1) *to* $i(\varepsilon)$, *is none other than the composition of the isomorphism of formal groups*

(1.10.9) $\varepsilon: \mathfrak{d}^{-1} \otimes \hat{\mathbf{G}}_m \to \mathfrak{d}^{-1} \mathfrak{a}^{-1} \otimes \hat{\mathbf{G}}_m$

with the canonical isomorphism

(1.10.10) $\mathfrak{d}^{-1} \mathfrak{a}^{-1} \otimes \hat{\mathbf{G}}_m \xrightarrow{\sim} \widehat{\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)}$

deduced from the construction of $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ *as a quotient of* $\mathfrak{d}^{-1} \mathfrak{a}^{-1} \otimes \mathbf{G}_m$ *by a discrete subgroup.*

Proof. This is tautological, in view of the original definition of $i(\varepsilon)$ (cf., (1.1.15)). QED

(1.10.11) *Definition.* Given an HBAV with $\Gamma_{00}(p^{\alpha})$-structure (X, i) over a p-adic ring R, we denote by ω_{can} the isomorphism

(1.10.12) $\omega_{\mathrm{can}}: \mathrm{Lie}(\hat{X}) \xrightarrow{\sim} \mathfrak{d}^{-1} \otimes R$

which is *inverse* to the isomorphism

(1.10.13) $\mathrm{Lie}(\hat{i}): \mathfrak{d}^{-1} \otimes R \xrightarrow{\sim} \mathrm{Lie}(\hat{X}) = \mathrm{Lie}(X)$

deduced from $\hat{i}$ (1.10.3) by passage to Lie algebras.

(1.10.14) **Corollary.** *Given* ε *as in* (1.10.7), *let* j *denote its inverse. Then the isomorphism* ω_{can} *attached* $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), i(\varepsilon))$ *is none other than the one denoted* $\omega_{\mathfrak{a}}(j)$ *in* (1.2.11).

Proof. Again a tautology.

(1.10.15) **Theorem.** *Let* R_0 *be a p-adic ring. The construction* $(X, \lambda, i) \mapsto (X, \lambda, \omega_{\mathrm{can}}, i)$ *defines, by transposition, a ring homomorphism*

(1.10.16)
$$\begin{aligned} \bigoplus_{\chi} M(\mathfrak{c}, \Gamma_{00}(p^{\alpha}), \chi; R_0) &\to V(\mathfrak{c}, R_0) \\ f &\mapsto \tilde{f} \end{aligned}$$

which preserves q-expansions in the sense that for all fractional ideals $\mathfrak{a}, \mathfrak{b}$ *of* K *such that* $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$, *and for all isomorphisms* $\varepsilon: \mathcal{O} \otimes \mathbf{Z}_p \to \mathfrak{a}^{-1} \otimes \mathbf{Z}_p$, *we have*

(1.10.17) $\tilde{f}(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, i(\varepsilon)) = f(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, \omega_{\mathfrak{a}}(\varepsilon^{-1}), i(\varepsilon))$.

Proof. Simply define $f \mapsto \tilde{f}$ by

(1.10.18) $\tilde{f}(X, \lambda, i) \overset{\mathrm{dfn}}{=} f(X, \lambda, \omega_{\mathrm{can}}, i)$.

The assertion about q-expansions follows from (1.10.14). QED

When R_0 is flat over $\mathbf{Z}_p$, it is sometimes convenient to tensor the homomorphism (1.10.16) with $\mathbf{Q}_p$, thus obtaining a q-expansion-preserving homomorphism, still noted $f \mapsto \tilde{f}$,

(1.10.19) $\bigoplus_{\chi} M(\mathfrak{c}, \Gamma_{00}(p^{\alpha}), \chi; R_0 \otimes \mathbf{Q}_p) \to V(\mathfrak{c}, R_0) \otimes \mathbf{Q}_p$.

1.11. We now turn to the p-adic analogue of the C^{∞} splitting of the Hodge exact sequence (1.8.21) by a holomorphically horizontal C^{∞} subbundle of H^1_{DR}. This analogue is provided by the "unit root subspace" with respect to the action of the *Frobenius* (compare [6]).

Let X be an HBAV over a ring R in which p is nilpotent, and suppose that every geometric fibre of X/R is an ordinary abelian variety. (This is equivalent to supposing that for some $n \geqq 1$, X admits a $\Gamma_{00}(p^n)$-structure, at least after extending scalars from R to some finite étale over-ring.) Let $H_{\mathrm{can}} \subset X$ be the "canonical subgroup" (compare [7]), defined by

(1.11.1) $H_{\mathrm{can}} = \text{Kernel of multiplication by } p \text{ in } \hat{X}.$

In terms of *any* $\Gamma_{00}(p^n)$-structure

(1.11.2) $i: \mathfrak{d}^{-1} \otimes \boldsymbol{\mu}_{p^n} \hookrightarrow X$

we have

(1.11.3) $H_{\mathrm{can}} = i(\mathfrak{d}^{-1} \otimes \boldsymbol{\mu}_p).$

We denote by X' the quotient X/H_{can}, and by π the projection map

(1.11.4) $\pi: X \to X'.$

When $p=0$ in R, i.e., when R is an $\mathbf{F}_p$-algebra, then X' is just the scheme $X^{(p)}$ obtained from X/R by the extension of scalars $F_{\mathrm{abs}}: R \to R$ given by the absolute Frobenius ($F_{\mathrm{abs}}(x) = x^p$ for all $x \in R$), and the morphism π is the *relative* Frobenius morphism

(1.11.5) $F: X \to X^{(p)}.$

(1.11.6) **Lemma.** *Let (X, λ) be a $\mathfrak{c}$-polarized HBAV over a ring R in which p is nilpotent. Then there exists a unique $\mathfrak{c}$-polarization λ' on X' which reduces mod p to the polarization $\lambda^{(p)}$ on $X^{(p)}$.*

Proof. Unicity follows from standard deformation theory cf. [17]. To show existence, we argue as follows. By the unicity, it suffices to construct λ' after replacing R by a finite étale over-ring, over which X admits a $\Gamma_{00}(p)$-structure

(1.11.7) $i: \mathfrak{d}^{-1} \otimes \boldsymbol{\mu}_p \hookrightarrow X.$

Denoting by X_p the "kernel of p" in X, we have a short exact sequence

(1.11.8) $0 \to \mathfrak{d}^{-1} \otimes \boldsymbol{\mu}_p \xrightarrow{i} X_p \to \mathfrak{c}^{-1} \otimes \mathbf{Z}/p\mathbf{Z} \to 0.$

Therefore we can factor the morphism "multiplication by p" on X as

(1.11.9)
$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "p"'] & X' \arrow[r, "\pi'"] & X \end{tikzcd}$$

with the morphism π' sitting in a short exact sequence

(1.11.10) $0 \to \mathfrak{c}^{-1} \otimes \mathbf{Z}/p\mathbf{Z} \to X' \xrightarrow{\pi'} X \to 0.$

Passing to the *duals*, we get a short exact sequence

(1.11.11) $$0 \to \mathfrak{d}^{-1}\mathfrak{c}\otimes\boldsymbol{\mu}_p \to X^t \xrightarrow{(\pi')^t} (X')^t \to 0$$

so that $(\pi')^t$ provides an isomorphism

(1.11.12) $$(\pi')^t : (X^t)' \xrightarrow{\sim} (X')^t.$$

Combining the inverse of this isomorphism with the isomorphism given by λ

(1.11.13) $$(X^t)' \xrightarrow{\sim} (X \underset{\mathcal{O}}{\otimes} \mathfrak{c})' = X' \underset{\mathcal{O}}{\otimes} \mathfrak{c}$$

we obtain a $\mathfrak{c}$-polarization

(1.11.14) $$\lambda' : (X')^t \xrightarrow{\sim} X' \otimes \mathfrak{c}.$$

It is routine to check that, in case $p=0$ in R, we have simply constructed $\lambda^{(p)}$. QED

Given an HBAV with $\Gamma_{00}(p^\infty)$-structure (X, i) over a ring R in which p is nilpotent, we define a $\Gamma_{00}(p^\infty)$-structure i' on X' by requiring the commutativity of the following diagram:

(1.11.15) $$\begin{tikzcd} 0 \arrow[r] & \mathfrak{d}^{-1}\otimes\boldsymbol{\mu}_p \arrow[r] \arrow[d, "i", "\wr"'] & \mathfrak{d}^{-1}\otimes\boldsymbol{\mu}_{p^\infty} \arrow[r, "p"] \arrow[d, hook, "i"] & \mathfrak{d}^{-1}\otimes\boldsymbol{\mu}_{p^\infty} \arrow[r] \arrow[d, hook, "i'"] & 0 \\ 0 \arrow[r] & H_{\mathrm{can}} \arrow[r] & X \arrow[r] & X' \arrow[r] & 0. \end{tikzcd}$$

By passage to the limit, the construction

(1.11.16) $$(X, \lambda, i) \mapsto (X', \lambda', i')$$

continues to make sense in the case of a *p-adic* ground-ring R. Applying it to the universal object

(1.11.17) $$\begin{tikzcd} (X_{\mathrm{univ}}, \lambda_{\mathrm{univ}}, i_{\mathrm{univ}}) \arrow[d] \\ \mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0} \end{tikzcd}$$

we obtain

(1.11.18) $$\begin{tikzcd} (X'_{\mathrm{univ}}, \lambda'_{\mathrm{univ}}, i'_{\mathrm{univ}}) \arrow[d] \\ \mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0}. \end{tikzcd}$$

Let us denote by

(1.11.19) $F\colon \mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0} \to \mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0}$

the *endomorphism* of $\mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0}$ which "classifies" (1.11.18), i.e., such that

(1.11.20) $(X'_{\text{univ}}, \lambda'_{\text{univ}}, i'_{\text{univ}}) = (X^{(F)}_{\text{univ}}, \lambda^{(F)}_{\text{univ}}, i^{(F)}_{\text{univ}})$.

We also denote by F the induced endomorphism of $V(\mathfrak{c}, R_0)$, defined modularly by the rule

(1.11.21) $F(f)(X, \lambda, i) \overset{\text{dfn}}{=} f(X', \lambda', i')$.

(1.11.22) **Lemma.** *For any fixed q-expansion homomorphism* (1.9.8), *denoted simply $f \mapsto f(q)$, we have the formula*

(1.11.23) $(Ff)(q) = f(q^p)$

i.e., if $f(q) = \sum a_\alpha q^\alpha$, then $(Ff)(q) = \sum a_\alpha q^{p\alpha}$.

Proof. This follows from that fact, obvious by inspection, that $(\text{Tate}_{\mathfrak{a},\mathfrak{b}}(q)', \lambda'_{\text{can}}, i(\varepsilon)')$ is simply obtained from $(\text{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\text{can}}, i(\varepsilon))$ by the endomorphism $q \mapsto q^p$ of $\mathbf{Z}_p((\mathfrak{a}\,\mathfrak{b}; S))$. QED

We now turn to H^1_{DR} over $\mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0}$. The morphism

(1.11.24) $\pi\colon X_{\text{univ}} \to X'_{\text{univ}} = (X_{\text{univ}})^{(F)}$

induces an F-linear endomorphism Fr of $H^1_{DR} = H^1_{DR}(X_{\text{univ}}/\mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0})$:

(1.11.25) $Fr \overset{\text{dfn}}{=} \pi^*\colon (H^1_{DR})^{(F)} \to H^1_{DR}$

which respects the Hodge filtration, and whose 2×2 $\mathcal{O} \otimes \mathcal{O}^{\wedge}_{\mathcal{M}}$-"matrix", in blocks adopted to the Hodge filtration, is of the form

(1.11.26) $\begin{pmatrix} pA & C \\ 0 & D \end{pmatrix}$

where both A and D are invertible. As explained in ([6]), this gives

(1.11.27) **Theorem.** *Over $\mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0}$, there exists a unique Fr-stable splitting*

$$H^1_{DR} = \underline{\omega} \oplus U$$

with U an invertible $\mathcal{O} \otimes \mathcal{O}^{\wedge}_{\mathcal{M}}$ module, which is (necessarily) horizontal for the Gauss-Manin connection.

1.12. For our later arithmetic applications (cf. (2.6.13, 2.6.27)), we must study this p-adic splitting for Tate $_{\mathfrak{a},\mathfrak{b}}(q)$. To begin, we consider $\text{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ over $\mathbf{Z}((\mathfrak{a}\,\mathfrak{b}, S))$, and the Hodge filtration on its H^1_{DR}:

(1.12.1) $0 \to \underline{\omega} \to H^1_{DR} \to \underline{\text{Lie}} \otimes \mathfrak{a}\,\mathfrak{b}^{-1} \to 0$.

The canonical identifications (cf., (1.1.17))

(1.12.2) $\underline{\omega} \simeq \mathfrak{a} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}((\mathfrak{a}\,\mathfrak{b}; S))$

(1.12.3) $\underline{\mathrm{Lie}} \underset{\mathcal{O}}{\otimes} \mathfrak{a}\,\mathfrak{b}^{-1} \simeq \mathfrak{d}^{-1}\,\mathfrak{b}^{-1} \underset{\mathbf{Z}}{\otimes} \mathbf{Z}((\mathfrak{a}\,\mathfrak{b}; S))$

allow us to specify certain elements of $\underline{\omega}$ and of $\mathrm{Lie} \underset{\mathcal{O}}{\otimes} \mathfrak{a}\,\mathfrak{b}^{-1}$, as follows:

(1.12.4) for $\alpha \in \mathfrak{a}$, denote by $\omega(\alpha) \in \underline{\omega}$ the element corresponding to $\alpha \otimes 1$ under (1.12.2)

for $\beta \in \mathfrak{d}^{-1}\,\mathfrak{b}^{-1}$, we denote by $l(\beta) \in \underline{\mathrm{Lie}} \underset{\mathcal{O}}{\otimes} \mathfrak{a}\,\mathfrak{b}^{-1}$ the element corresponding to $\beta \otimes 1$ under (1.12.3).

For each element $\gamma \in \mathfrak{d}^{-1}\,\mathfrak{a}^{-1}\,\mathfrak{b}^{-1}$, we recall that $D(\gamma)$ is the derivation of $\mathbf{Z}((\mathfrak{a}\,\mathfrak{b}; S))$ defined by

(1.12.5) $D(\gamma) \sum\limits_{\alpha \in \mathfrak{a}} a_\alpha q^\alpha = \sum \mathrm{trace}(\alpha\,\gamma)\, a_\alpha q^\alpha.$

Under the Gauss-Manin connection, we have (cf., (1.1.20))

(1.12.6) $\nabla(D(\gamma))(\omega(\alpha)) \equiv l(\alpha\,\gamma) \bmod \underline{\omega}.$

(1.12.7) **Key Lemma.** *After extension of scalars to* $\mathbf{Z}_p((\mathfrak{a}\,\mathfrak{b}; S))$, *the "unit root" submodule* $U \subset H^1_{DR}$ *is spanned over* $\mathbf{Z}_p((\mathfrak{a}\,\mathfrak{b}; S))$ *by the elements* $\nabla(D(\gamma))(\omega(\alpha))$.

Proof. The quotient $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)'$ by the canonical subgroup H_{can} and the morphism

(1.12.8) $\pi\colon \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \to \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)'$

in fact live over $\mathbf{Z}((\mathfrak{a}\,\mathfrak{b}; S))$ (i.e., *without* passing to the p-adic completion). Indeed, as noted above (1.11.22) we have

(1.12.9) $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)' = \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q^p)$

where $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q^p)$ means the HBAV obtained from $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ by the extension of scalars $F\colon \mathbf{Z}((\mathfrak{a}\,\mathfrak{b}; S)) \to \mathbf{Z}((\mathfrak{a}\,\mathfrak{b}; S))$ given by $q \to q^p$. The morphism

(1.12.10) $\pi\colon \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \to \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q^p)$

is the map obtained from the map "multiplication by p" on $\mathbf{G}_m \otimes \mathfrak{a}^{-1}\mathfrak{d}^{-1}$ by passage to quotients:

(1.12.11)
$$\begin{tikzcd}
\mathbf{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1} \arrow[r, "p"] \arrow[d] & \mathbf{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1} \arrow[d] \\
\mathbf{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1}/q(\mathfrak{b}) \arrow[r, "\pi"] & \mathbf{G}_m \otimes \mathfrak{d}^{-1}\mathfrak{a}^{-1}/p \cdot q(\mathfrak{b}).
\end{tikzcd}$$

In order to prove (1.12.7), it is sufficient to show that

$$(1.12.12)\quad \pi^*(F^*(\nabla(D(\gamma))(\omega(\alpha))))=\nabla(D(\gamma))(\omega(\alpha))$$

i.e., that $Fr(=\pi^*)$, viewed as an F-linear *endomorphism* of H^1_{DR}, actually *fixes* each of the elements $\nabla(D(\gamma))(\omega(\alpha))$. To verify this, we may extend scalars from **Z** to **C**, and then check the corresponding complex analytic assertion.

Over **C**, the morphism π corresponds to the map of lattices

$$(1.12.13)\quad \mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau)=2\pi i(\mathfrak{d}^{-1}\mathfrak{a}^{-1}+\mathfrak{b}\tau)\xrightarrow{p}2\pi i(\mathfrak{d}^{-1}\mathfrak{a}^{-1}+\mathfrak{b}p\tau)=\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(p\tau),$$

the complex H^1_{DR} is given by

$$(1.12.14)\quad H^1_{DR}=\mathrm{Hom}_{\mathbf{Z}}(\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau),\mathbf{C}),$$

the element $\omega(\alpha)\in H^1_{DR}$ is the composite linear form

$$(1.12.15)\quad \mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau)\hookrightarrow K\otimes\mathbf{C}\xrightarrow{\times\alpha}K\otimes\mathbf{C}\xrightarrow{\text{trace}}\mathbf{C}$$

and the element $\nabla(D(\gamma))(\omega(\alpha))$ is the linear form given by

$$(1.12.16)\quad \begin{aligned}&2\pi i\alpha\in 2\pi i\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mapsto 0\\&2\pi i\beta\tau\in 2\pi i\mathfrak{b}\tau\mapsto 2\pi i\,\mathrm{trace}(\beta\alpha\gamma).\end{aligned}$$

Similarly, $F^*(H^1_{DR})$ is given by

$$(1.12.17)\quad F^*(H^1_{DR})=\mathrm{Hom}_{\mathbf{Z}}(\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(p\tau),\mathbf{C}),$$

and the "inverse image" $F^*(\nabla(D(\gamma)))$ is the linear form

$$(1.12.18)\quad \begin{aligned}&2\pi i\alpha\in 2\pi i\mathfrak{d}^{-1}\mathfrak{a}^{-1}\mapsto 0\\&2\pi i\beta p\tau\in 2\pi i\mathfrak{b}p\tau\mapsto 2\pi i\,\mathrm{trace}(\beta\alpha\gamma).\end{aligned}$$

The truth of (1.12.12) is now apparent, for by definition of π^*, we have

$$\begin{aligned}&(\pi^*(F^*(\nabla(D(\gamma))(\omega(\alpha)))))(2\pi i\alpha+2\pi i\beta\tau)\\&\quad=(F^*(\nabla(D(\gamma))(\omega(\alpha))))(2\pi ip\alpha+2\pi i\beta p\tau)\\&\quad=2\pi i\,\mathrm{trace}(\beta\alpha\gamma)\\&\quad=(\nabla(D(\gamma))(\omega(\alpha)))(2\pi i\alpha+2\pi i\beta\tau).\quad\text{QED}\end{aligned}$$

Chapter II. Differential Operators on C^∞ and p-Adic Hilbert Modular Forms; Algebraicity Theorems

We begin with some algebraic preliminaries.

2.0. Let us denote by $\overline{\mathbf{Q}}$ a fixed algebraic closure of **Q**, and by $\mathfrak{S}$ the set consisting of the g distinct field embeddings

$$(2.0.1)\quad \sigma\colon K\hookrightarrow\overline{\mathbf{Q}}.$$

Let $K^{\mathrm{gal}} \subset \overline{\mathbf{Q}}$ be the composition of all the fields $\sigma(K)$, and let $\mathcal{O}^{\mathrm{gal}}$ denote its ring of integers.

For any $\mathcal{O}^{\mathrm{gal}}$-algebra R, and any $\sigma \in \mathfrak{S}$, we have a ring homomorphism, still denoted σ:

$$(2.0.2)\qquad \begin{aligned} &\sigma\colon \mathcal{O} \underset{\mathbf{Z}}{\otimes} R \to R \\ &a \otimes r \mapsto \sigma(a)\cdot r. \end{aligned}$$

Taken simultaneously, these give a ring homomorphism

$$(2.0.3)\qquad \begin{aligned} &\mathcal{O} \otimes R \to R^{\mathfrak{S}} \\ &a \otimes r \mapsto (\ldots, \sigma(a)\, r, \ldots) \end{aligned}$$

which is an isomorphism if and only if the *discriminant* d_K of $K/\mathbf{Q}$ is invertible in R.

We can also view the σ's as being *characters* of the torus $\coprod_{\mathcal{O}/\mathbf{Z}} (\mathbf{G}_m)_R$. In this way, *any* element $\sum n(\sigma)\sigma$ in the free abelian group $\mathbf{Z}[\mathfrak{S}]$ generated by $\mathfrak{S}$ defines a *character* of this torus, by the formula

$$(2.0.4)\qquad \sum a_i \otimes r_i \in (\mathcal{O} \otimes R)^{\times} \to \prod_{\sigma} \Big(\sum \sigma(a_i)\, r_i\Big)^{n(\sigma)}.$$

Taken simultaneously, the characters σ define a homomorphism of R-tori

$$(2.0.5)\qquad \coprod_{\mathcal{O}/\mathbf{Z}} (\mathbf{G}_m)_R \to (\mathbf{G}_m)_R^{\mathfrak{S}}$$

which is an isomorphism if and only if d_K is invertible in R.

Given an $\mathcal{O} \otimes R$ module M, we denote by $M(\sigma)$ the largest *$\mathcal{O} \otimes R$-quotient* module of M on which $\mathcal{O}$ operates by the formula

$$(2.0.6)\qquad \mathrm{am}(\sigma) = \sigma(a)\, m(\sigma) \qquad \text{for } a \in \mathcal{O}, \qquad m(\sigma) \in M(\sigma)$$

(more precisely, $m(\sigma) \in M(\sigma)$ is to be annihilated by $a \otimes 1 - 1 \otimes \sigma(a) \in \mathcal{O} \otimes R$).

(2.0.7) Given an invertible $\mathcal{O} \otimes R$ module M, and a character χ of $\coprod_{\mathcal{O}/\mathbf{Z}} (\mathbf{G}_m)_R$, we denote by $M(\chi)$ the invertible R-module obtained from M by extension of the structural group by χ. In the case that χ is just a single σ, "$M(\sigma)$ as invertible R-module" is precisely the underlying R-module of "$M(\sigma)$ as $\mathcal{O} \otimes R$-quotient of M where $\mathcal{O}$ acts through σ," so that there is in fact no ambiguity in the notation.

(2.0.8) **Lemma.** *Let M be an invertible $\mathcal{O} \otimes R$ module, and suppose that R is an $\mathcal{O}^{\mathrm{gal}}$-algebra in which d_K is invertible. Then*

(2.0.9) *The canonical map $M \to \bigoplus_{\sigma} M(\sigma)$ is an $\mathcal{O} \otimes R$ isomorphism.*

(2.0.10) *For every integer $k \geqq 1$, denote by $M^{\otimes k}$ the k-th power of M as invertible $\mathcal{O} \otimes R$-module. Then we have a canonical R-isomorphism*

$$\mathrm{M}^{\otimes k} \xrightarrow{\sim} \bigoplus_{\sigma} M(k\sigma).$$

(2.0.11) *For every integer $k \geqq 1$, denote by* $\mathrm{Symm}_R^k(M)$ *the k-th symmetric power of M as R-module. Then we have a canonical R-isomorphism*

(2.0.12) $$\mathrm{Symm}_R^k(M) \xrightarrow{\sim} \bigoplus_{\chi \geqq 0, |\chi| = k} M(\chi),$$

the sum being extended to all characters χ of the form $\sum_\sigma n(\sigma) \cdot \sigma$, with integers $n(\sigma) \geqq 0$ satisfying $\sum n(\sigma) = k$.

Proof. The assertion (2.0.9) follows from the fact that (2.0.3) is an isomorphism when d_K is invertible in R. The last two assertions follow from (2.0.9).

2.1. This section is devoted to the construction and study of some purely algebraic differential operators. We fix an $\mathcal{O}^{\mathrm{gal}}$-algebra R_0 in which d_K is invertible, and a fractional ideal $\mathfrak{c}$ of K which is prime to R_0 in the sense that it is prime to every prime number p which is a non-unit in R_0. Then canonically we have

(2.1.1) $$\mathfrak{c}^{-1} \underset{\mathbf{Z}}{\otimes} R_0 = \mathcal{O} \underset{\mathbf{Z}}{\otimes} R_0$$

(equality inside $\mathcal{O}[\text{all } 1/p \text{ with } p \text{ invertible in } R_0] \underset{\mathbf{Z}}{\otimes} R_0$).
We denote simply by $\mathcal{M}$ any of the moduli schemes $\mathcal{M}(\mathfrak{c}, \Gamma_{00}(N))_{R_0}$. The Kodaira-Spencer isomorphism (1.0.21)

(2.1.2) $$\Omega^1_{\mathcal{M}/R_0} \simeq \underline{\omega}^{\otimes 2} \underset{\mathcal{O}}{\otimes} \mathfrak{c}^{-1}$$

may be rewritten, using (2.1.1), as an isomorphism

(2.1.3) $$\Omega^1_{\mathcal{M}/R_0} \simeq \underline{\omega}^{\otimes 2}.$$

The Gauss-Manin connection on H^1_{DR}

(2.1.4) $$\nabla\colon H^1_{DR} \to H^1_{DR} \underset{\mathcal{O}_{\mathcal{M}}}{\otimes} \Omega^1_{\mathcal{M}/R_0}$$

induces a connection, still denoted ∇, on each of the symmetric powers $\mathrm{Sym}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR})$, $k = 1, 2, \ldots$:

(2.1.5) $$\nabla\colon \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \to \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \underset{\mathcal{O}_{\mathcal{M}}}{\otimes} \Omega^1_{\mathcal{M}/R_0}.$$

Using the isomorphism (2.1.3), we can rewrite this as a map

(2.1.6) $$\nabla\colon \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \to \mathrm{Symm}^k(H^1_{DR}) \underset{\mathcal{O}_{\mathcal{M}}}{\otimes} \underline{\omega}^{\otimes 2}.$$

(2.1.7) $$\underline{\omega}^{\otimes 2} \xrightarrow{\sim} \bigoplus_\sigma \underline{\omega}(2\sigma),$$

so that (2.1.6) is the sum of maps $\nabla(\sigma)$:

(2.1.8) $$\nabla(\sigma)\colon \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \to \mathrm{Symm}^k(H^1_{DR}) \underset{\mathcal{O}_{\mathcal{M}}}{\otimes} \underline{\omega}(2\sigma).$$

Applying the decomposition (2.0.11) to $M = \underline{\omega}$, $k=2$, we get an inclusion

(2.1.9) $\underline{\omega}(2\sigma) \subset \mathrm{Symm}^2_{\mathcal{O}_{\mathcal{M}}}(\underline{\omega}) = \bigoplus\limits_{\sigma,\sigma'} \underline{\omega}(\sigma+\sigma')$

By means of the inclusion $\underline{\omega} \subset H^1_{DR}$, we have

(2.1.10) $\mathrm{Symm}^2_{\mathcal{O}_{\mathcal{M}}}(\underline{\omega}) \subset \mathrm{Symm}^2_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR})$.

Combining (2.1.19) and (2.1.10), we obtain an inclusion

(2.1.11) $\underline{\omega}(2\sigma) \subset \mathrm{Symm}^2_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR})$.

We can now define $\mathbf{D}(\sigma)$ as the composite

(2.1.12)
$$\begin{tikzcd}
\mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \arrow[r, dashed, "\nabla(\sigma)"] \arrow[rdd, "\mathbf{D}(\sigma)"'] & \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \underset{\mathcal{O}_{\mathcal{M}}}{\otimes} \underline{\omega}(2\sigma) \arrow[d, phantom, "\bigcap"] \\
 & \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \underset{\mathcal{O}_{\mathcal{M}}}{\otimes} \mathrm{Symm}^2_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \arrow[d, "\text{multiplication}"] \\
 & \mathrm{Symm}^{k+2}_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}).
\end{tikzcd}$$

It is clear from this definition that $\mathbf{D}(\sigma)$ is a first-order differential operator (being the composition of ∇ with several $\mathcal{O}_{\mathcal{M}}$-linear maps), and that it defines a derivation, homogeneous of degree two, of the symmetric algebra $\mathrm{Symm}_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR})$:

(2.1.13) $\mathbf{D}(\sigma)(xy) = \mathbf{D}(\sigma)(x) \cdot y + x \cdot \mathbf{D}(\sigma)(y)$ for $x, y \in \mathrm{Symm}$.

Less clear is the following

(2.1.14) **Lemma.** *The* $\mathbf{D}(\sigma)$*'s mutually commute.*

Proof. By reduction to the universal case ($R_0 = \mathcal{O}^{\mathrm{gal}}[1/d_K]$) and then by extension of scalars, we reduce to checking in the case $R_0 = \mathbf{C}$, in which case it is adequate to check complex-analytically.

Because all the objects involved (H^1_{DR}, its Gauss-Manin connection, ...) are the inverse images on $\mathcal{M}(\mathfrak{c}, \Gamma_{00}(N))$ of their analogues on the stack $\mathcal{M}(\mathfrak{c}, \Gamma_{00}(1))$ by the étale morphism "forget the $\Gamma_{00}(N)$-structure" (cf., (1.0.12)), it suffices to work on the stack of $\mathfrak{c}$-polarized lattices. Recall (cf., (1.5.4)) that given any two fractional ideals $\mathfrak{a}, \mathfrak{b}$ of K such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$, any $\mathfrak{c}$-polarized lattice $\mathcal{L}$ is isomorphic to one of the lattices $\mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$ for some $\tau \in K \otimes \mathbf{C}$ with $\mathrm{Im}(\tau) \gg 0$. Let us denote by $\mathfrak{H}(K)$ the space of all such τ. The morphism (of "complex analytic stacks")

$$\begin{tikzcd}
\mathfrak{H}(K) \arrow[d] & \tau \arrow[d, mapsto] \\
\mathcal{M}(\mathfrak{c}, \Gamma_{00}(1))^{\mathrm{an}} & \mathcal{L}_{\mathfrak{a},\mathfrak{b}}(\tau)
\end{tikzcd}$$

is étale and surjective, so it suffices to check the truth of (2.1.14) over $\mathfrak{H}(K)$. Over $\mathfrak{H}(K)$, we are given a horizontal isomorphism of H^1_{DR} with the "constant" sheaf

$$(2.1.15)\quad \mathrm{Hom}_{\mathbf{Z}}(2\pi i\mathfrak{d}^{-1}\mathfrak{a}^{-1}\oplus 2\pi i\mathfrak{b}, \mathbf{C})\underset{\mathbf{C}}{\otimes}\mathcal{O}_{\mathfrak{H}(K)}$$

endowed with the trivial connection $1\otimes d$. For each $\sigma\in\mathfrak{S}$, let $X(\sigma)$ and $Y(\sigma)$ be the global horizontal sections of H^1_{DR} corresponding to the linear forms on lattice $\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$ given by

$$(2.1.16)\quad \begin{cases} X(\sigma)(2\pi i\alpha+2\pi i\beta\tau)=2\pi i\sigma(\alpha) \\ Y(\sigma)(2\pi i+2\pi i\beta\tau)=-2\pi i\sigma(\beta). \quad \text{(sic!)} \end{cases}$$

Then $\mathrm{Symm}(H^1_{DR})$ becomes the sheaf of algebras

$$(2.1.17)\quad \mathbf{C}[X(\sigma_1),\dots,X(\sigma_g), Y(\sigma_1),\dots,Y(\sigma_g)]\underset{\mathbf{C}}{\otimes}\mathcal{O}_{\mathfrak{H}(K)}$$

with the trivial connection $1\otimes d$. The space $\mathfrak{H}(K)$ is simply the g-fold product $\mathfrak{H}^{\mathfrak{S}}$ of usual upper-half planes, via the isomorphism (cf., (2.0.3)):

$$(2.1.18)\quad \begin{cases} \mathfrak{H}(K)\xrightarrow{\sim}\mathfrak{H}^{\mathfrak{S}} \\ \tau\mapsto(\dots,\tau(\sigma),\dots). \end{cases}$$

The Hodge filtration $\underline{\omega}\subset H^1_{DR}$ is provided by the subsheaf

$$(2.1.19)\quad \underline{\omega}=\bigotimes_{\sigma}\underline{\omega}(\sigma)=\bigoplus_{\sigma}(X(\sigma)-\tau(\sigma)\,Y(\sigma))\cdot\mathcal{O}_{\mathfrak{H}(K)}.$$

The Kodaira-Spencer isomorphism (cf., 2.1.3)

$$(2.1.20)\quad \Omega^1_{\mathfrak{H}(K)/\mathbf{C}}\simeq\underline{\omega}^{\otimes 2}=\bigoplus_{\sigma}(\mathrm{X}(\sigma)-\tau(\sigma)\,X\,(\sigma))^2\cdot\mathcal{O}_{\mathfrak{H}(K)}$$

is given by

$$(2.1.21)\quad 2\pi i\,d\tau(\sigma)\mapsto(X(\sigma)-\tau(\sigma)\,Y(\sigma))^2.$$

From the Definition (2.1.12) of $\mathbf{D}(\sigma)$, we see easily that it is the derivation

$$(2.1.21)\quad \mathbf{D}(\sigma)=\frac{1}{2\pi i}(X(\sigma)-\tau(\sigma)\,Y(\sigma))^2\frac{\partial}{\partial\tau(\sigma)}$$

of $\mathbf{C}[X(\sigma)\text{'s}, Y(\sigma)\text{'s}]\otimes\mathcal{O}_{\mathfrak{H}(K)}$ (with $\mathbf{D}(\sigma)\,X(\sigma'))=\mathbf{D}(\sigma)(Y(\sigma'))=0$ for all σ,σ'). That the $\mathbf{D}(\sigma)$'s mutually commute is now obvious, since $\mathbf{D}(\sigma)$ "involves" only $X(\sigma)$, $Y(\sigma)$, and $\tau(\sigma)$. QED

2.2. In this section we will construct, "purely algebraically", some punctually supported distributions on modular forms out of the operators $\mathbf{D}(\sigma)$. We continue to work over an $\mathcal{O}^{\mathrm{gal}}[1/d_K]$-algebra R_0.

Suppose we are given, over an R_0-algebra R, the following data (2.2.1–3)

(2.2.1) an R-valued point $x=(X,\lambda,i)$ of our moduli scheme $\mathscr{M}=$some $\mathscr{M}(\mathfrak{c},\Gamma_{00}(N))_{R_0}$.

(2.2.2) a nowhere vanishing differential (cf., (1.0.4)) ω on X

(2.2.3) an $\mathcal{O} \underset{\mathbf{Z}}{\otimes} R$-splitting of the Hodge filtration on $H^1_{DR}(x) \overset{\text{dfn}}{=} H^1_{DR}(X/R)$, i.e., an invertible $\mathcal{O} \otimes R$ sub-module "Split (x)" $\subset H^1_{DR}(x)$ such that

(2.2.4) $\underline{\omega}(x) \oplus \mathrm{Split}(x) \xrightarrow{\sim} H^1_{DR}(x)$.

Now let χ be a character of the torus $\coprod_{\mathcal{O}/\mathbf{Z}} (\mathbf{G}_m)_{R_0}$, which is of the form

(2.2.5) $\chi = \sum_{\sigma \in \mathfrak{S}} n(\sigma)\sigma$ with integers $n(\sigma) \geqq 0$, $\sum n(\sigma) \overset{\text{dfn}}{=} k$.

For every element of $\mathbf{Z}[\mathfrak{S}]$ of the form

(2.2.6) $\sum_{\sigma \in \mathfrak{S}} d(\sigma)\sigma$, with all $d(\sigma) \geqq 0$

we will define an R_0-linear map

(2.2.7) $M(\mathfrak{c}, \Gamma_{00}(N), \chi; R_0) \xrightarrow{\vartheta(\sum d(\sigma)\sigma, x, \omega, \mathrm{Split}(x))} R,$

as follows. Given an element $f \in M(\mathfrak{c}, \Gamma_{00}(N), \chi; R_0)$, view it as a global section of $\underline{\omega}(\chi)$ over $\mathcal{M}$. Because χ is of the special form (2.2.5), we obtain inclusions

(2.2.8) $\underline{\omega}(\chi) \subset \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(\underline{\omega}) \subset \mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR})$.

Composing these with the differential operator

(2.2.9) $\prod_{\sigma} \mathbf{D}^{d(\sigma)}_{(\sigma)}$: $\mathrm{Symm}^k_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR}) \to \mathrm{Symm}^{k+2\Sigma d(\sigma)}_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR})$

we obtain a differential operator

(2.2.10) $\prod_{\sigma} \mathbf{D}^{d(\sigma)}_{(\sigma)}$: $\underline{\omega}(\chi) \to \mathrm{Symm}^{k+2\Sigma d(\sigma)}_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR})$,

under which the image of f is a global section of $\mathrm{Symm}^{k+2\Sigma d(\sigma)}(H^1_{DR})$. The inverse image of this section via the given R-valued point x: $\mathrm{Spec}(R) \to \mathcal{M}$ is an element

(2.2.10) $((\prod_{\sigma} \mathbf{D}^{d(\sigma)}_{(\sigma)})(f))(x) \in \mathrm{Symm}^{k+2\Sigma d(\sigma)}_R(H^1_{DR}(x))$.

The given splitting (2.2.4) of $H^1_{DR}(x)$ gives us a projection operator

(2.2.12) $H^1_{DR}(x) \to \underline{\omega}(x)$

which induces a projection

(2.2.13) $\mathrm{Symm}^{\bullet}_R(H^1_{DR}(x)) \to \mathrm{Symm}^{\bullet}_R(\underline{\omega}(x))$

whose kernel is precisely the ideal generated by Split(x). Applying (2.2.13) in degree $k+2\sum d(\sigma)$, we get a map

(2.2.14) $\quad \mathrm{Symm}_R^{k+2\Sigma d(\sigma)}(H_{DR}^1(x)) \to \mathrm{Symm}_R^{k+2\Sigma d(\sigma)}(\underline{\omega}(\chi)).$

By (2.0.12), we have a canonical decomposition

(2.2.15) $\quad \mathrm{Symm}_R^{k+2\Sigma d(\sigma)}(\underline{\omega}(x)) = \bigoplus_{\rho \geqq 0,\, |\rho| = k+2\Sigma d(\sigma)} \underline{\omega}(x)(\rho)$

and in particular a canonical projection

(2.2.16) $\quad \mathrm{Symm}_R^{k+2\Sigma d(\sigma)}(\underline{\omega}(x)) \to \underline{\omega}(x)(\chi+2\sum d(\sigma)\,\sigma).$

Finally, the given $\mathcal{O}\otimes R$ basis $\omega(x)$ gives an R-basis of each of the $\underline{\omega}(x)(\rho)$'s, whence, in particular,

(2.2.17) $\quad \underline{\omega}(x)(\chi+2\sum d(\sigma)\,\sigma) \xrightarrow{\sim} R.$

Combining (2.2.14), (2.2.16) and (2.2.17), we get an R-linear map

(2.2.18) $\quad \mathrm{Symm}_R^{k+2\Sigma d(\sigma)}(H_{DR}^1(x)) \to R.$

We *define* $\vartheta(\sum d(\sigma)\,\sigma, x, \omega, \mathrm{Split}(x)) f \in R$ to be the *image* of $((\prod_{\sigma} \mathbf{D}_{(\sigma)}^{d(\sigma)})(f))$ under the map (2.2.18).

(2.3) In this section, we will construct some C^∞ differential operators out of the $\mathbf{D}(\sigma)$'s, by using the antiholomorphic splitting (1.8.3) of the Hodge filtration.

To simplify notation, we will denote the sheaf $\overline{\underline{\omega}(C^\infty)}$ simply as

(2.3.1) $\quad \overline{\underline{\omega}(C^\infty)} \overset{\text{(notation)}}{=\!=} \mathrm{Split}(C^\infty).$

Given an element of $\mathbf{Z}[\mathfrak{S}]$ of the form

$\sum d(\sigma)\,\sigma \quad$ with all $d(\sigma) \geqq 0$

we will define a C^∞-differential operator

(2.3.2) $\quad \vartheta(\sum d(\sigma)\,\sigma, C^\infty)\colon \mathrm{Symm}^k(\underline{\omega}(C^\infty)) \to \mathrm{Symm}^{k+2\Sigma d(\sigma)}(\underline{\omega}(C^\infty))$

as follows. The inclusion

(2.3.3) $\quad \mathrm{Symm}^k(\underline{\omega}(C^\infty)) \subset \mathrm{Symm}^k(H_{DR}^1(C^\infty))$

together with the differential operator

(2.3.4) $\quad \prod_{\sigma} \mathbf{D}(\sigma)^{d(\sigma)}\colon \mathrm{Symm}^k(H_{DR}^1(C^\infty)) \to \mathrm{Symm}^{k+2\Sigma d(\sigma)}(H_{DR}^1(C^\infty))$

gives a differential operator

(2.3.5) $\quad \mathrm{Symm}^k(\underline{\omega}(C^\infty)) \to \mathrm{Symm}^{k+2\Sigma d(\sigma)}(H_{DR}^1(C^\infty)).$

The *projection*

(2.3.6) $H^1_{DR}(C^\infty) \to \underline{\omega}(C^\infty)$

deduced from the splitting (1.8.3) induces a projection

(2.3.7) $\mathrm{Symm}^\bullet(H^1_{DR}(C^\infty)) \to \mathrm{Symm}^\bullet(\underline{\omega}(C^\infty))$

whose kernel is the ideal generated by Split (C^∞). Composing (2.3.5) with the projection (2.3.7), we get the desired $\vartheta(\sum d(\sigma)\,\sigma, C^\infty)$.

(2.3.8) **Theorem.** *The differential operators* $\vartheta(\sum d(\sigma)\,\sigma, C^\infty)$ *enjoy the following properties*

(2.3.9) *They mutually commute.*

(2.3.10) *They are formed out of the operators* $\vartheta(\sigma, C^\infty)$ *by the rule*

$$\vartheta(\textstyle\sum d(\sigma)\,\sigma, C^\infty) = \displaystyle\prod_\sigma \vartheta(\sigma, C^\infty)^{d(\sigma)}.$$

(2.3.11) *Each* $\vartheta(\sigma, C^\infty)$ *acts as a derivation, homogeneous of degree two, of* $\mathrm{Symm}^\bullet(\underline{\omega}(C^\infty))$ *into itself.*

(2.3.12) *For any character* χ *of the form* (2.2.5), $\vartheta(\sum d(\sigma)\,\sigma, C^\infty)$ *maps* $\underline{\omega}(\chi)(C^\infty)$ *to* $\underline{\omega}(\chi + 2\sum d(\sigma)\,\sigma)(C^\infty)$.

Proof. Because the submodule Split $(C^\infty) \subset H^1_{DR}(C^\infty)$ is holomorphically horizontal, the ideal it generates in $\mathrm{Symm}^\bullet(H^1_{DR}(C^\infty))$ is holomorphically horizontal, (hence stable by the $\mathbf{D}(\sigma)$'s) and the quotient is $\mathrm{Symm}^\bullet(\underline{\omega}(C^\infty))$. The operator $\vartheta(\sum d(\sigma)\,\sigma, C^\infty)$ is, by definition, the one induced on this quotient by the operator $\prod_\sigma \mathbf{D}(\sigma)^{d(\sigma)}$ on $\mathrm{Symm}^\bullet(H^1_{DR}(C^\infty))$, whence (2.3.9–11) follow from (2.1.12–13). To prove (2.3.12), we use (2.3.10) to reduce to checking the operators $\vartheta(\sigma, C^\infty)$. Using the fact (2.3.11) that $\vartheta(\sigma, C^\infty)$ is a derivation, we are reduced to checking that $\vartheta(\sigma, C^\infty)$ maps $\mathcal{O}_{\mathcal{M}}(C^\infty)$ to $\underline{\omega}(2\sigma)(C^\infty)$, and that it maps $\underline{\omega}(\sigma')(C^\infty)$ to $\underline{\omega}(\sigma' + 2\sigma)(C^\infty)$. The first of these points is clear, for this map is just the map

(2.3.13)
$$\begin{tikzcd}
\mathcal{O}(C^\infty) \arrow[r, "d^{\text{holo}}"] \arrow[drr, "\vartheta(\sigma, C^\infty)"'] & \Omega^1_{\mathcal{M}}(C^\infty) \simeq \underline{\omega}^{\otimes 2}(C^\infty) \simeq & \bigoplus_\sigma \underline{\omega}(2\sigma)(C^\infty) \arrow[d, "\text{projection}"] \\
& & \underline{\omega}(2\sigma)(C^\infty).
\end{tikzcd}$$

The second map

(2.3.14) $\vartheta(\sigma, C^\infty)\colon \underline{\omega}(C^\infty) \to \mathrm{Symm}^3(\underline{\omega}(C^\infty))$

is the composite

(2.3.15)

$$\begin{tikzcd}
\underline{\omega}(C^\infty) \subset H^1_{DR}(C^\infty) \arrow[r, "\nabla"] \arrow[ddddr, "\vartheta(\sigma, C^\infty)"'] & H^1_{DR}(C^\infty) \otimes \Omega^1_{\mathscr{M}}(C^\infty) \arrow[d, "\wr\ \mathrm{id} \otimes K-S"] \\
& H^1_{DR}(C^\infty) \otimes \underline{\omega}^{\otimes 2}(C^\infty) \arrow[d, "\wr"'] \\
& H^1_{DR}(C^\infty) \otimes (\bigoplus_{\sigma} \underline{\omega}(2\sigma)(C^\infty)) \arrow[d, "\mathrm{id} \otimes \mathrm{projection}"] \\
& H^1_{DR}(C^\infty) \otimes \underline{\omega}(2\sigma)(C^\infty) \arrow[d, "\mathrm{proj} \otimes \mathrm{id}"] \\
& \underline{\omega}(C^\infty) \otimes \underline{\omega}(2\sigma)
\end{tikzcd}$$

In terms of any holomorphic local coordinates $t_1, \ldots, t_g$ on $\mathscr{M}$, and any local section ω of $\underline{\omega}(C^\infty)$, we have

(2.3.16) $\nabla(\omega) = \sum \nabla\left(\frac{d}{dt_i}\right)(\omega) \otimes dt_i$

whence

(2.3.17) $\vartheta(\sigma, C^\infty)(\omega) \equiv \sum \nabla\left(\frac{d}{dt_i}\right)(\omega) \cdot$ (the "2σ" component of $K-S(dt_i)$)

(the congruence modulo the ideal (Split(C^∞))). Thus to check that $\vartheta(\sigma, C^\infty)$ maps $\underline{\omega}(\sigma')(C^\infty)$ to $\underline{\omega}(\sigma' + 2\sigma)(C^\infty)$, it suffices to observe that the derivations

(2.3.18) $\nabla\left(\frac{d}{dt_i}\right): H^1_{DR} \to H^1_{DR}$

although not linear over $\mathcal{O}_{\mathscr{M}}$, *are* linear over $\mathcal{O}$, and hence by (2.0.6) map $H^1_{DR}(\sigma')$ to itself for any $\sigma' \in \mathfrak{S}$. QED

(2.3.19) For later applications, we give explicit formulas for the operators $\vartheta(\sigma, C^\infty)$. As in the proof of (2.1.14), fix two fractional ideals $\mathfrak{a}, \mathfrak{b}$ of K such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$, and an $\mathcal{O}$-isomorphism

$$\varepsilon: \mathcal{O}/N\mathcal{O} \xrightarrow{\sim} \mathfrak{a}^{-1}/N\mathfrak{a}^{-1}.$$

Then any $\mathfrak{c}$-polarized lattice with $\Gamma_{00}(N)$ structure is isomorphic to one of the form $(\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau), \langle\ ,\ \rangle_{\mathrm{can}}, i(\varepsilon))$ (cf., 1.5) for some $\tau = (\ldots, \tau(\sigma), \ldots) \in \mathfrak{H}(K) \simeq \mathfrak{H}^{\mathfrak{S}}$. The corresponding projection

(2.3.20)
$$\begin{tikzcd}
\mathfrak{H}(K) \arrow[d] & \tau \arrow[d, mapsto] \\
\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N))^{\mathrm{an}} & (\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau), \langle\ ,\ \rangle_{\mathrm{can}}, i(\varepsilon))
\end{tikzcd}$$

is étale and surjective. Indeed for $N \geqq 4$, it is precisely the *universal* covering of $\mathscr{M}(\mathfrak{c}, \Gamma_{00}(N))^{\mathrm{an}}$, whose fundamental group appears here as the group $\Gamma_{00}(N; \mathfrak{a}, \mathfrak{b})$ of all fractional linear transformations of τ of the form

(2.3.21) $$\tau \mapsto \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$$

with

(2.3.22) $$\begin{cases} \alpha, \delta \in \mathcal{O}, \gamma \in \mathfrak{d}\,\mathfrak{a}\,\mathfrak{b}, \ \beta \in \mathfrak{d}^{-1}\,\mathfrak{a}^{-1}\,\mathfrak{b}^{-1} \\ \alpha\delta - \beta\gamma = 1 \\ \alpha, \delta \equiv 1 \bmod N\mathcal{O} \\ \gamma \in N \cdot \mathfrak{d}\mathfrak{a}\mathfrak{b}. \end{cases}$$

From this point of view, a holomorphic (resp. C^∞) $\mathfrak{c}$-HMF on $\Gamma_{00}(N)$ of weight $\chi = \sum n(\sigma)\sigma$ of type (2.2.5), is a holomorphic (resp. C^∞) function $f(\tau)$ such that the global section

(2.3.23) $$f(\tau) \cdot \prod_\sigma (X(\sigma) - \tau(\sigma)\,Y(\sigma))^{n(\sigma)}$$

of $\mathrm{Symm}^\bullet(H^1_{DR})$ over $\mathfrak{H}(K)$ is *invariant* under the *left* action of $\Gamma_{00}(N; \mathfrak{a}, \mathfrak{b})$ given by

(2.3.24) $$\begin{cases} \tau \mapsto \dfrac{\alpha\tau+\beta}{\gamma\tau+\delta} \\ X \mapsto \alpha X + \beta Y \\ Y \mapsto \gamma X + \delta Y \end{cases}$$

i.e., f must satisfy

(2.3.25) $$f\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \left(\prod_\sigma (\sigma(c) \cdot \tau(\sigma) + \sigma(d))^{n(\sigma)}\right) f(\tau) \qquad \text{for all } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_{00}(N; \mathfrak{a}, \mathfrak{b}).$$

(2.3.26) **Formula.** *For any C^∞ function $f(\tau)$ on $\mathfrak{H}(K)$, we have for $\sigma_0 \in \mathfrak{S}$*

$$\begin{aligned} &\vartheta(\sigma_0, C^\infty)\left(f(\tau) \prod_\sigma (X(\sigma) - \tau(\sigma)\,Y(\sigma))^{n(\sigma)}\right) \\ &\quad = * \cdot (X(\sigma_0) - \tau(\sigma_0)\,Y(\sigma_0))^2 \cdot \prod_\sigma (X(\sigma) - \tau(\sigma)\,Y(\sigma))^{n(\sigma)} \end{aligned}$$

with

(2.3.27) $$* = \frac{1}{2\pi i}\,\frac{\partial}{\partial\tau(\sigma_0)} f(\tau) - \frac{n(\sigma_0)}{4\pi\,\mathrm{Im}(\tau(\sigma_0))} f(\tau)$$

Proof. By definition, $\vartheta(\sigma, C^\infty)$ is the reduction modulo $\mathrm{Split}(C^\infty)$ of $\mathbf{D}(\sigma)$. The explicit formula (cf., (2.1.21))

$$(2.3.28) \quad \mathbf{D}(\sigma)=(X(\sigma)-\tau(\sigma)\,Y(\sigma))^2\cdot\frac{1}{2\pi i}\,\frac{\partial}{\partial\tau(\sigma)}$$

reduces (2.3.27) to the assertion that (for $\sigma=\sigma_0$) we have

$$(2.3.29) \quad \frac{1}{2\pi i}\,\frac{\partial}{\partial\tau(\sigma)}(X(\sigma)-\tau(\sigma)\,Y(\sigma))$$
$$\equiv\frac{-1}{4\pi\,\mathrm{Im}(\tau(\sigma))}X(\sigma)-\tau(\sigma)\,Y(\sigma)) \qquad \text{modulo}\ \mathrm{Split}(C^\infty).$$

But Split (C^∞) is *spanned* by the elements

$$(2.3.30) \quad X(\sigma)-\overline{\tau(\sigma)}\,Y(\sigma)$$

so that

$$(2.3.31) \quad X(\sigma)\equiv\overline{\tau(\sigma)}\,Y(\sigma) \qquad \mathrm{mod}\,\mathrm{Split}(C^\infty).$$

Modulo this relation, (2.3.29) becomes the identity

$$-\frac{1}{2\pi i}Y(\sigma)=\frac{-1}{4\pi\,\mathrm{Im}(\tau(\sigma))}(\overline{\tau(\sigma)}\,\mathrm{Y}\,(\sigma)-\tau(\sigma)\,Y(\sigma)). \quad \text{QED}$$

(2.3.32) To include this section, we give the homogeneous version of (2.3.26). Let us denote by GL_K^+ the complex manifold

$$(2.3.33) \quad GL_K^+=\{(\omega_1,\omega_2)|\omega_1,\omega_2\in K\otimes\mathbf{C},\ \mathrm{Im}(\bar\omega_1\,\omega_2)\ \text{is totally positive in}\ K\otimes\mathbf{R}\}.$$

To each point $(\omega_1,\omega_2)\in GL_K^+$, we associate

$$(2.3.34) \quad \begin{cases} \text{the lattice}\ \mathfrak{d}^{-1}\mathfrak{a}^{-1}\,\omega_1+\mathfrak{b}\omega_2 \\ \text{the}\ \mathfrak{c}\text{-polarization}\ \langle u,v\rangle_{\mathrm{can}}\overset{\mathrm{dfn}}{=}\dfrac{\mathrm{Im}(\bar u\,v)}{\mathrm{Im}(\bar\omega_1\,\omega_2)} \\ \text{the}\ \Gamma_{00}(N)\text{-structure}\ i(\varepsilon)\text{:}\ \mathfrak{d}^{-1}\otimes\mathbf{Z}/N\mathbf{Z}\xrightarrow{\sim}\mathfrak{d}^{-1}\mathfrak{a}^{-1}\,\omega_1\otimes\mathbf{Z}/N\mathbf{Z}. \end{cases}$$

From this point of view, a holomorphic (resp. C^∞) $\mathfrak{c}$-HMF f on $\Gamma_{00}(N)$ of weight $\chi=\sum n(\sigma)\sigma$ is a holomorphic (resp. C^∞) function $f(\omega_1,\omega_2)$ on GL^+_K, which is *invariant* under the *right* action of $\Gamma_{00}(N;\mathfrak{a},\mathfrak{b})$ on GL_K^+,

$$(2.3.35) \quad (\omega_1,\omega_2)\mapsto(\omega_1,\omega_2)\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$$

and which transforms under $a\in(K\otimes\mathbf{C})^\times$ by the rule

$$(2.3.36) \quad f(a^{-1}\,\omega_1,a^{-1}\,\omega_2)=\chi(a)f(\omega_1,\omega_2).$$

Let us denote by $\omega_i(\sigma)$, $i=1,2$, $\sigma\in\mathfrak{S}$, the holomorphic coordinates on GL_K^+ we obtain by writing, for $i=1,2$

$$(2.3.37) \quad \omega_i=(\ldots,\omega_i(\sigma),\ldots)\in K\otimes\mathbf{C}\simeq\mathbf{C}^{\mathfrak{S}}.$$

(2.3.38) **Formula.** *The operator* $\vartheta(\sigma, C^{\propto})$ *on* C^{∞} *functions* $f(\omega_1, \omega_2)$ *on* GL_K^+ *satisfying* (2.3.35–36) *coincides with the operator* $W(\sigma)$ *on* GL_K^+ *given by*

$$(2.3.39)\quad W(\sigma) \overset{\mathrm{dfn}}{=} \frac{-\pi}{\operatorname{Im}(\overline{\omega_1(\sigma)}\,\omega_2(\sigma))}\left(\overline{\omega_1(\sigma)}\frac{\partial}{\partial\omega_1(\sigma)} + \overline{\omega_2(\sigma)}\frac{\partial}{\partial\omega_2(\sigma)}\right)$$

Proof: Given $F(\tau)$ on $\mathfrak{H}(K)$ satisfying (2.3.25), the corresponding function $f(\omega_1, \omega_2)$ on GL_K^+ satisfying (2.3.35–36) is given by

$$(2.3.40)\quad f(\omega_1, \omega_2) = \chi^{-1}(\omega_1/2\pi i)\cdot F(\omega_2/\omega_1)$$
$$\text{i.e., } F(\tau) = f(2\pi i, 2\pi i \tau).$$

We must show that

$$(2.3.41)\quad \chi(\omega_1/2\pi i)\cdot(\omega_1(\sigma)/2\pi i)^2\, W(\sigma) f(\omega_1, \omega_2)$$
$$= \frac{1}{2\pi i}\frac{\partial F}{\partial\tau(\sigma)}(\omega_2/\omega_1) - \frac{n(\sigma)\,F(\omega_2/\omega_1)}{4\pi \operatorname{Im}(\omega_2(\sigma)/\omega_1(\sigma))}.$$

Substituting for f via (2.3.40), the left-hand member becomes

$$(2.3.42)\quad (\omega_1(\sigma)/2\pi i)^2\cdot W(\sigma)(F(\omega_2/\omega_1))$$
$$+ F(\omega_2/\omega_1)\cdot(\omega_1(\sigma)/2\pi i)^2 \chi(\omega_1/2\pi i) \times W(\sigma)(\chi^{-1}(\omega_1/2\pi i))$$
$$= (\omega_1(\sigma)/2\pi i)^2 \frac{\partial F}{\partial\tau(\sigma)}(\omega_2/\omega_1)\cdot W(\sigma)(\omega_2(\sigma)/\omega_1(\sigma))$$
$$+ F(\omega_2/\omega_1)\cdot(\omega_1(\sigma)/2\pi i)^2 \times \left(\frac{-n(\sigma)\,W(\sigma)(\omega_1(\sigma)/2\pi i)}{(\omega_1(\sigma)/2\pi i)}\right).$$

Comparing this with (2.3.41), we see that we need only check that

$$(2.3.43)\quad \begin{cases} (\omega_1(\sigma)/2\pi i)^2\cdot W(\sigma)(\omega_2(\sigma)/\omega_1(\sigma)) = \dfrac{1}{2\pi i} \\[2ex] (\omega_1(\sigma)/2\pi i)\cdot W(\sigma)(\omega_1(\sigma)/2\pi i) = \dfrac{1}{4\pi \operatorname{Im}(\omega_2(\sigma)/\omega_1(\sigma))}, \end{cases}$$

a task we leave to the reader. QED

(2.3.43) *Elaboration.* For fixed $(\alpha, \beta) \in \mathfrak{d}^{-1}\mathfrak{a}^{-1} \times \mathfrak{b}$, let us denote by l the $K \otimes \mathbf{C}$-valued function

$$(2.3.44)\quad l = \alpha\,\omega_1 + \beta\,\omega_2$$

on GL_K^+, and by $l(\sigma)$, $\sigma \in \mathfrak{S}$, its $\mathbf{C}$-valued component functions.

$$(2.3.45)\quad l(\sigma) = \sigma(\alpha)\,\omega_1(\sigma) + \sigma(\beta)\,\omega_2(\sigma).$$

Let us denote by a the $K \otimes \mathbf{R}$-valued "area" function

$$(2.3.46)\quad a = \operatorname{Im}(\bar{\omega}_1\,\omega_2)$$

and by $a(\sigma)$, $\sigma \in \mathfrak{S}$ its $\mathbf{R}$-valued component functions

(2.3.47) $a(\sigma) = \operatorname{Im}(\overline{\omega_1(\sigma)}\,\omega_2(\sigma))$.

The operators $W(\sigma)$ may be characterized as the *unique* C^∞ derivations of GL_K^+ which satisfy the following identies:

$$(2.3.48) \quad \begin{cases} W(\sigma)(l(\sigma)) = -\dfrac{\pi}{a(\sigma)} \cdot \overline{l(\sigma)}, \\ W(\sigma)(\overline{l(\sigma)}) = 0 \\ W(\sigma)(a(\sigma)) = 0 \end{cases}$$

$$(2.3.49) \quad W(\sigma)(l(\sigma')) = W(\sigma)(\overline{l(\sigma')}) = W(\sigma)(a(\sigma')) = 0 \qquad \text{if } \sigma \neq \sigma'.$$

(2.4) In this section, we give a fundamental algebraicity theorem for the operators $\vartheta(\sum d(\sigma)\sigma, C^\infty)$. This theorem is essentially equivalent to Shimura's generalization ([24]) of Damerell's theorem ([2]), but the proof is quite different from Shimura's.

Let R be an $\mathcal{O}^{\text{gal}}[1/d_K]$-algebra given with an inclusion

(2.4.1) incl: $R \hookrightarrow \mathbf{C}$,

and let $\mathfrak{c}$ be a fractional ideal of K which is prime to R (cf., (2.1.1)). Let $x = (X, \lambda, i)$ be an R-valued point of the moduli scheme $\mathcal{M} = \mathcal{M}(\mathfrak{c}, \Gamma_{00}(N))_R$, given with a nowhere vanishing differential ω on X. Let $\operatorname{Split}(x) \subset H^1_{DR}(x)$ be an $\mathcal{O} \otimes R$ splitting of the Hodge filtration which satisfies the following very strong condition:

(2.4.2) after the given extension of scalars $R \xhookrightarrow{\text{incl.}} \mathbf{C}$, the submodule $\operatorname{Split}(x) \underset{R}{\otimes} \mathbf{C} \subset H^1_{DR}(X) \underset{R}{\otimes} \mathbf{C} \xrightarrow{\sim} H^1(X^{\text{an}}_{\mathbf{C}}, \mathbf{C})$ is the antiholomorphic subspace $H^{0,1} \subset H^1(X^{\text{an}}_{\mathbf{C}}, \mathbf{C})$.

Let χ be any character of $\coprod_{\mathcal{O}/\mathbf{Z}} (\mathbf{G}_m)_R$ of the form

(2.4.3) $\chi = \sum n(\sigma)\sigma$ with integers $n(\sigma) \geqq 0$, $\sum n(\sigma) \overset{\text{dfn}}{=} k$

and let f be an element of $M(\mathfrak{c}, \Gamma_{00}(N), \chi; R)$. After the given extension of scalars $R \hookrightarrow \mathbf{C}$, we can view f as a (holomorphic) global section of $\underline{\omega}(\chi)(C^\infty)$ on $\mathcal{M}(\mathfrak{c}, \Gamma_{00}(N))(C^\infty)$. For any element

(2.4.4) $\sum d(\sigma)\sigma \in \mathbf{Z}[\mathfrak{S}]$, all $d(\sigma) \geqq 0$,

we can apply $\vartheta(\sum d(\sigma)\sigma, C^\infty)$ to f, and obtain a C^∞ global section of $\underline{\omega}(\chi + 2\sum d(\sigma)\sigma)(C^\infty)$, which we can then view as a C^∞ $\mathfrak{c}$-HMF on $\Gamma_{00}(N)$ of weight $\chi + 2\sum d(\sigma)\sigma$.

(2.4.5) **Theorem.** *Hypotheses and notations as in* (2.4.1–4) *above, the complex number*

$$(\vartheta(\textstyle\sum d(\sigma)\sigma, C^\infty) f)(X, \lambda, \omega, i)_{\mathbf{C}} \in \mathbf{C}$$

actually "lies in R": it is given, in the notation (2.2.7), *by the formula*

(2.4.6) $(\vartheta(\sum d(\sigma)\sigma, C^\infty)f)(X,\lambda,\omega,i)_{\mathbf{C}} = \text{incl.}\ (\vartheta(\sum d(\sigma)\sigma, x, \omega, \mathrm{Split}(x))f)$.

Proof. The first assertion is a consequence of the explicit formula (2.4.6). To prove (2.4.6), we are immediately reduced to the case $R=\mathbf{C}$. Then $(\vartheta(\sum d(\sigma)\sigma, C^\infty)f)(X,\lambda,\omega,i)$ (resp. $\vartheta(\sum d(\sigma)\sigma, x, \omega, \mathrm{Split}(x))f$) is obtained from

(2.4.7) $\left(\prod_\sigma \mathbf{D}(\sigma)^{d(\sigma)}\right)f$, a global section of $\mathrm{Symm}^{k+2\Sigma d(\sigma)}_{\mathcal{O}_{\mathcal{M}}}(H^1_{DR})$

by going around the left (resp. right) side of the diagram

(2.4.8)

$$\begin{tikzcd}
& \mathrm{Symm}^{k+2\Sigma d(\sigma)}(H^1_{DR}) \arrow[dl, "\substack{\text{(reduce modulo the ideal}\\ \text{gen. by Split } (C^\infty))}"'] \arrow[dr, "\text{(take fibre at } x)"] & \\
\mathrm{Symm}^{k+2\Sigma d(\sigma)}(\underline{\omega}(C^\infty)) \arrow[dr, "\text{(take fibre at } x)"'] & & \mathrm{Symm}^{k+2\Sigma d(\sigma)}(H^1_{DR}(x)) \arrow[dl, "\substack{\text{(reduce modulo the ideal}\\ \text{gen. by Split } (x))}"] \\
& \mathrm{Symm}^{k+2\Sigma d(\sigma)}(\underline{\omega}(x)) \arrow[d, "\substack{\text{(project. by (2.0.12), onto the}\\ \chi+2\sum d(\sigma)\sigma \text{ comp't.)}}"] & \\
& \underline{\omega}(x) \arrow[d, "\wr"', "\text{(use the given } \omega)"] & \\
& \mathbf{C} &
\end{tikzcd}$$

That this diagram is commutative results from the hypothesis (2.4.2) that, over $\mathbf{C}$, we have

(2.4.9) $\mathrm{Split}(x) = (\mathrm{Split}(C^\infty))(x)$. QED

(2.5) In this section, we develop the p-adic analogues of operators $\vartheta(\sum d(\sigma)\sigma, C^\infty)$. We fix a p-adic $\mathcal{O}^{\text{gal}}$-algebra R_0 which satisfies the following condition:

(2.5.1) the discriminant d_K of $K/\mathbf{Q}$ is not a zero divisor in R_0.

When p is *unramified* in K, this condition is automatically fulfilled, since d_K is then a unit in R_0. When p is *ramified* in K, this condition is equivalent to R_0's being flat over $\mathbf{Z}_p$.

We will denote simply by $\mathcal{M}(p\text{-adic})$ the formal scheme (cf., (1.9.6))

(2.5.2) $\mathcal{M}(p\text{-adic}) \overset{\text{dfn}}{=} \mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0} = \{\mathcal{M}(\mathfrak{c}, \Gamma_{00}(N))_{R_0/p^n R_0}\}$

and by $\mathcal{O}_{\mathcal{M}}(p\text{-adic})$ its structural sheaf. To unify notation, and to emphasize the analogy with the C^∞ case, we will denote by

(2.5.3) $\underline{\omega}(p\text{-adic})$, $H^1_{DR}(p\text{-adic})$, $\operatorname{Split}(p\text{-adic})$

the sheaves $\underline{\omega}$, H^1_{DR}, U (cf., (1.11.27)) on $\mathscr{M}(p\text{-adic})$.

Let us temporarily view the formal scheme $\mathscr{M}(p\text{-adic})$ as being the topological space $\mathscr{M}(\mathfrak{c}, \Gamma_{00}(p^\infty))_{R_0/pR_0}$, together with the p-adically complete and separated structural sheaf of $\mathcal{O}\otimes R_0$-algebras $\mathcal{O}_{\mathscr{M}}(p\text{-adic})$. Because this sheaf is *flat* over R_0, the hypothesis (2.5.1) on R_0 assures the natural map

(2.5.4) $$\mathcal{O}_{\mathscr{M}}(p\text{-adic}) \to \mathcal{O}_{\mathscr{M}}(p\text{-adic}) \underset{\mathbf{Z}_p}{\otimes} \mathbf{Z}_p[1/d_K] \overset{\text{dfn}}{=} \mathcal{O}_{\mathscr{M}}(p\text{-adic})[1/d_K]$$

is injective. For any locally free $\mathcal{O}_{\mathscr{M}}(p\text{-adic})$ module $\mathscr{F}$, the corresponding map

(2.5.5) $$\mathscr{F} \to \mathscr{F}[1/d_K] \overset{\text{dfn}}{=} \mathscr{F} \underset{\mathbf{Z}_p}{\otimes} \mathbf{Z}_p[1/d_K]$$

is also injective.

We now define, for every element of $\mathbf{Z}[\mathfrak{S}]$

(2.5.6) $\sum d(\sigma)\sigma$ with all $d(\sigma) \geqq 0$

a differential operator

(2.5.7) $$\vartheta(\textstyle\sum d(\sigma)\sigma, p\text{-adic}): \quad \operatorname{Symm}^k(\underline{\omega}(p\text{-adic}))[1/d_k]$$
$$\to \operatorname{Symm}^{k+2\Sigma d(\sigma)}(\underline{\omega}(p\text{-adic}))[1/d_K]$$

by repeating the Definition (2.3.3–7) of $\vartheta(\sum d(\sigma)\sigma, C^\infty)$, with the holomorphically horizontal splitting $\operatorname{Split}(C^\infty)$ of the Hodge filtration on $H^1_{DR}(C^\infty)$ replaced by the horizontal splitting $\operatorname{Split}(p\text{-adic})[1/d_K]$ of the Hodge filtration on $H^1_{DR}(p\text{-adic})[1/d_K]$.

(2.5.8) **Theorem.** (*p-adic analogue of* (2.3.8)). *The differential operators* $\vartheta(\sum d\sigma, p\text{-adic})$ *enjoy the following properties:*

(2.5.9) *They mutually commute*

(2.5.10) *They are formed out of the operators* $\vartheta(\sigma, p\text{-adic})$ *by the rule*

$$\vartheta(\textstyle\sum d(\sigma))\sigma, p\text{-adic}) = \displaystyle\prod_\sigma \vartheta(\sigma, p\text{-adic})^{d(\sigma)}$$

(2.5.11) *Each* $\vartheta(\sigma, p\text{-adic})$ *acts as a derivation, homogeneous of degree two, of* $\operatorname{Symm}^{\cdot}(\underline{\omega}(p\text{-adic}))[1/d_K]$ *into itself.*

(2.5.12) *For any character* χ *of* the *form* (2.2.5), $\vartheta(\sum d(\sigma)\sigma, p\text{-adic})$ *maps* $\underline{\omega}(\chi)(p\text{-adic})[1/d_K]$ *to* $\underline{\omega}(\chi + 2\sum d(\sigma)\sigma)(p\text{-adic})[1/d_K]$.

Proof. Just repeat the proof of (2.3.8). QED

(2.6) In this section, we give the p-adic analogue of the algebraicity theorem (2.4.5), and define some differential operators $\theta(\sigma)$ on p-adic $\mathfrak{c}$-HMF's. We fix an $\mathcal{O}^{\text{gal}}$-algebra R_1 which is separated for the p-adic topology and a fractional ideal $\mathfrak{c}$ of K which is prime to R_1. We denote by R_0 the p-adic completion of R_1:

(2.6.1) $R_1 \subset R_0 \overset{\text{dfn}}{=} \varprojlim R_1/p^n R_1$.

We make the hypothesis

(2.6.2) the discriminant d_K is not a zero divisor in R_0.

Let $x=(X,\lambda,i)$ be an R_1-valued point of the moduli scheme $\mathcal{M}=\mathcal{M}(\mathfrak{c},\Gamma_{00}(p^\infty))_{R_1}$, given with a nowhere vanishing differential ω on X. Let $\operatorname{Split}(x)\subset H^1_{DR}(x)$ be an $\mathcal{O}\otimes R_1$-splitting of the Hodge filtration, which satisfies the following very strong condition:

(2.6.3) after the extension of scalars $R_1\hookrightarrow R_0$, the submodule

$\operatorname{Split}(x)\underset{R_1}{\otimes}R_0\subset H^1_{DR}(x)\underset{R_0}{\otimes}R_1=H^1_{DR}(p\text{-adic})(x)$ is the "unit root" subspace $\underline{U}(x)(=\operatorname{Split}(p\text{-adic})(x)$ in the notation 2.5.3).

Let χ be a character of $\coprod_{\mathcal{O}/\mathbf{Z}}(\mathbf{G}_m)_{R_1}$ of the form

(2.6.4) $\chi=\sum n(\sigma)\sigma$ with integers $n(\sigma)\geqq 0$, $\sum n(\sigma)\overset{\mathrm{dfn}}{=}k$,

and let f be an element of $M(\mathfrak{c},\Gamma_{00}(p^\infty),\chi,R_1)$. After the extension of scalars $R_1\hookrightarrow R_0$, we can view f as a global section $f(p\text{-adic})$ of $\underline{\omega}(\chi)(p\text{-adic})$ over $\mathcal{M}(\mathfrak{c})^{p\text{-adic}}_{R_0}=\mathcal{M}(p\text{-adic})$. For any element

(2.6.5) $\sum d(\sigma)\sigma\in\mathbf{Z}[\mathfrak{S}]$, all $d(\sigma)\geqq 0$

we can apply $\vartheta(\sum d(\sigma)\sigma, p\text{-adic})$ to $f(p\text{-adic})$, and obtain a global section of $\underline{\omega}(\chi+2\sum d(\sigma)\sigma)(p\text{-adic})[1/d_K]$, which has a well-defined value

(2.6.6) $(\vartheta(\sum d(\sigma)\sigma, p\text{-adic})f(p\text{-adic}))((X,\lambda,\omega,i)_{R_0})\in R_0[1/d_K]$

(2.6.7) **Theorem** (p-adic analogue of (2.4.5)). *Hypotheses and notations as in* (2.6.1-6) *above, the "p-adic" number*

$(\vartheta(\sum d(\sigma)\sigma, p\text{-adic})f(p\text{-adic}))((X,\lambda,\omega,i)_{R_0})\in R_0[1/d_K]$

actually lies in $R_1(1/d_K]$: *it is given, in the notation* (2.2.7), *by the formula*

(2.6.8) $(\vartheta(\sum d(\sigma)\sigma, p\text{-adic})f(p\text{-adic}))(X,\lambda,\omega,i)=\vartheta(\sum d(\sigma)\sigma,x,\omega,\operatorname{Split}(x))f.$

Proof. Just repeat the proof of (2.4.5). QED

For "numerical" applications, it is important to reinterpret this theorem in terms of p-adic $\mathfrak{c}$-HMF's and the construction (1.10.16). The sheaf $\underline{\omega}(p\text{-adic})$ on $\mathcal{M}(p\text{-adic})$ is canonically trivialized

(2.6.9) $\omega_{\mathrm{can}}\colon \mathcal{O}\underset{\mathbf{Z}}{\otimes}\mathcal{O}_{\mathcal{M}}(p\text{-adic})\xrightarrow{\sim}\underline{\omega}(p\text{-adic})$

by the construction (1.10.11), applied in the "universal case." For any character χ of $\coprod_{\mathcal{O}/\mathbf{Z}}(\mathbf{G}_m)_{R_0}$, this trivialization induces a canonical trivialization

(2.6.10) $\omega(\chi)_{\mathrm{can}}\colon \mathcal{O}_{\mathcal{M}}(p\text{-adic})\xrightarrow{\sim}\underline{\omega}(\chi)(p\text{-adic})$

by "extension of the structural group." In terms of these trivializations, the construction (1.10.16) sits in a commutative diagram

(2.6.11)
$$\begin{tikzcd}
f \arrow[rr] \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & & \tilde{f} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
M(\mathfrak{c}, \Gamma_{00}(p^{\infty}), \chi; R_0) \arrow[rr, "(1.10.16)"] \arrow[d, equal] & & V(\mathfrak{c}, R_0) \arrow[d, equal] \\
f \in H^0(\mathscr{M}(\mathfrak{c}, \Gamma_{00}(p^n))_{R_0}, \underline{\omega}(\chi)) \arrow[drr, dashed] \arrow[drr, dashed, shift left] & & H^0(\mathscr{M}(p\text{-adic}), \mathcal{O}_{\mathscr{M}}(p\text{-adic})) \arrow[d, "\omega(\chi)_{\mathrm{can}}", "\wr"'] \\
& & f(p\text{-adic}) \in H^0(\mathscr{M}(p\text{-adic}), \underline{\omega}(\chi)(p\text{-adic}))
\end{tikzcd}$$

in which the diagonal dotted arrow is just the map "pass to the associated formal scheme," i.e.,

(2.6.12) $f(p\text{-adic}) = \tilde{f} \cdot \omega(\chi)_{\mathrm{can}}.$

(2.6.13) **Theorem.** *For* χ *of type* (2.2.5), *and any* σ, *we have*

(2.6.14) $\vartheta(\sigma, p\text{-adic})(\omega(\chi)_{\mathrm{can}}) = 0$ *in* $\mathrm{Symm}^{\bullet}(\underline{\omega}(p\text{-adic}))[1/d_K]$.

Proof. When χ is the trivial character, $\omega(\chi)_{\mathrm{can}}$ is the constant function "1", and (2.6.14) holds. When χ is of the form $\sum n(\sigma)\sigma$ with all $n(\sigma) \geqq 0$, some $n(\sigma) \geqq 1$, then

(2.6.15) $\omega(\chi)_{\mathrm{can}} = \prod_{\sigma \text{ with } n(\sigma) \geqq 1} (\omega(\chi)_{\mathrm{can}})^{n(\sigma)}$ in $\mathrm{Symm}^{\bullet}(\underline{\omega}(p\text{-adic})[1/d_K]$.

Because $\vartheta(\sigma, p\text{-adic})$ is a *derivation* (2.5.11), it suffices to show that

(2.6.16) $\vartheta(\sigma, p\text{-adic})(\omega(\sigma')_{\mathrm{can}}) = 0$ in $\underline{\omega}(\sigma' + 2\sigma)(p\text{-adic})[1/d_K]$.

Because the $\omega(\chi)_{\mathrm{can}}$ give global trivializations, we can write

$$\vartheta(\sigma, p\text{-adic})(\omega(\sigma')_{\mathrm{can}}) = * \, \omega(\sigma' + 2\sigma)_{\mathrm{can}}$$
with $* \in V(\mathfrak{c}, R_0)[1/d_K]$.

In order to prove that $* = 0$, it suffices to show that at least one of its q-expansions vanishes. This, in turn, is equivalent to establishing (2.6.16) over some $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, i(\varepsilon))_{R_0}$.

(2.6.17) For simplicity, choose both $\mathfrak{a}$ and $\mathfrak{b}$ prime to p (i.e., prime to R_0; this is possible because we have assumed that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$ is prime to R_1, and hence to p) and take for ε the *equality* (cf., (2.1.1))

(2.6.18) $\mathcal{O} \otimes \mathbf{Z}_p = \mathfrak{a}^{-1} \otimes \mathbf{Z}_p.$

For each $\sigma \in \mathfrak{S}$, we denote by $D(\sigma)$ the derivation of $R_0((\mathfrak{a}\mathfrak{b}, S))$ given by

(2.6.19) $D(\sigma)(\sum_{\alpha \in \mathfrak{a}} a_\alpha q^\alpha) = \sum \sigma(\alpha) a_\alpha q^\alpha$

Let $\omega(\sigma)$ be the element of $\underline{\omega}[1/d_K]$ corresponding, via the isomorphism (cf., (1.12.2)) furnished by ω_{can},

$$(2.6.20) \quad \underline{\omega}[1/d_K] = \simeq (\mathfrak{a}[1/d_K] \underset{\mathbf{Z}}{\otimes} R_0) \otimes_{R_0} R_0((\mathfrak{a}\,\mathfrak{b}, S))$$

to the element "σ"$\otimes 1$, where "σ" is the unique element of $\mathfrak{a}[1/d_K] \underset{\mathbf{Z}}{\otimes} R_0$ such that

$$(2.6.21) \quad \mathrm{trace}(\text{“}\sigma\text{”} \cdot \alpha) = \sigma(\alpha) \quad \text{for all } \alpha \in \mathcal{O} \underset{\mathbf{Z}}{\otimes} R_0.$$

It is thus a tautology that on $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$, we have

$$(2.6.22) \quad \omega(\sigma) = \omega(\sigma)_{\mathrm{can}}.$$

By (2.1.21), the derivation $\mathbf{D}(\sigma)$ of $\mathrm{Symm}^{\bullet}(H^1_{DR})[1/d_K]$ is given, over $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$, by the formula

$$(2.6.23) \quad \mathbf{D}(\sigma) = \omega(\sigma)^2 \cdot V(D(\sigma)).$$

We can now verify (2.6.16) over $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ by a simple computation:

$$\begin{aligned}(2.6.24) \quad \vartheta(\sigma, p\text{-adic})(\omega(\sigma')_{\mathrm{can}}) &= \vartheta(\sigma, p\text{-adic})(\omega(\sigma')) \quad (\text{by } (2.6.22)) \\ &\equiv \mathbf{D}(\sigma)(\omega(\sigma')) \quad \mathrm{mod\,Split}(p\text{-adic}) \\ &\equiv (V(D(\sigma))(\omega(\sigma'))) \cdot \omega(\sigma)^2 \quad \mathrm{mod\,Split}(p\text{-adic}) \\ &= 0 \quad \text{by Key Lemma (1.12.7). QED}\end{aligned}$$

(2.6.25) **Corollary.** *For each σ, there exists a derivation $\theta(\sigma)$ of $\mathcal{O}_{\mathcal{M}}$ (p-adic)$[1/d_K]$ such that for any χ of type* (2.2.5), *the diagram*

$$(2.6.26) \quad \begin{tikzcd} \mathcal{O}_{\mathcal{M}}(p\text{-adic})[1/d_K] \arrow[r, "\theta(\sigma)"] \arrow[d, "\wr"', "\omega(\chi)_{\mathrm{can}}"] & \mathcal{O}_{\mathcal{M}}(p\text{-adic})[1/d_K] \arrow[d, "\wr"', "\omega(\chi+2\sigma)_{\mathrm{can}}"] \\ \underline{\omega}(\chi)(p\text{-adic})[1/d_K] \arrow[r, "\vartheta(\sigma,\, p\text{-adic})"] & \underline{\omega}(\chi+2\sigma)(p\text{-adic})[1/d_K] \end{tikzcd}$$

commutes. The q-expansion of $\theta(\sigma)$ at a cusp of type (2.6.17–18) *is given by the following formula:*

(2.6.27) *if $g \in V(\mathfrak{c}, R_0)[1/d_K]$ has q-expansion $\sum_{\alpha \in \mathfrak{a}\mathfrak{b}} a_\alpha q^\alpha$, its image $\theta(\sigma) g$ in $V(\mathfrak{c}, R_0)[1/d_K]$ has q-expansion*

$$(\theta(\sigma)\, g)(q) = \sum_{\alpha \in \mathfrak{a}\mathfrak{b}} \sigma(\alpha) \cdot a_\alpha q^\alpha$$

Proof. The first assertion follows immediately from (2.6.14). The second follows from (2.6.14), (2.6.23), and the definition of $\vartheta(\sigma, p\text{-adic})$ in terms of Split (p-adic) and $\mathbf{D}(\sigma)$. QED

(2.6.28) *Remark.* These operators $\theta(\sigma)$ should be viewed as the Hilbert modular version of Ramanujan's operator $\theta = q\frac{d}{dq}$ in the theory of modular forms in one variable.

We now return to the situation (2.6.1–6). Recall that $\tilde{f}$ denotes the image of f under the map (1.10.16):

(2.6.29) $M(\mathfrak{c}, \Gamma_{00}(p^{\infty}), \chi; R_0) \to V(\mathfrak{c}, R_0)$.

Let us denote by

(2.6.30) $c \in (\mathcal{O} \otimes R_0)^{\times}$

the "ratio" between the *given* nowhere vanishing differential ω on $x = (X, \lambda, i)$ and ω_{can}:

(2.6.31) $\omega = c \cdot \omega_{\mathrm{can}} \quad \text{on } X_{R_0}$.

(2.6.32) **Lemma.** *Hypotheses as in* (2.6.1–6), *the values*

$$(\vartheta(\textstyle\sum d(\sigma)\,\sigma, p\text{-adic})\, f(p\text{-adic}))(X, \lambda, \omega, i)_{R_0} \in R_0[1/d_K]$$

and

$$((\textstyle\prod \theta(\sigma)^{d(\sigma)})\, \tilde{f})((X, \lambda, i)_{R_0}) \in R_0[1/d_K]$$

are related by the formula

(2.6.33) $$(\vartheta(\textstyle\sum d(\sigma)\,\sigma, p\text{-adic})\, f(p\text{-adic}))(X, \lambda, \omega, i) = \frac{((\prod \theta(\sigma)^{d(\sigma)})\, \tilde{f})(X, \lambda, i)}{\chi(c) \cdot \prod\limits_{\sigma} \sigma(c)^{2d(\sigma)}}.$$

Proof. By (2.6.12), we have

(2.6.34) $f(p\text{-adic}) = \tilde{f} \cdot \omega(\chi)_{\mathrm{can}}$

whence by (2.6.26) and (2.5.10–12), we have

(2.6.35) $$\vartheta(\textstyle\sum d(\sigma)\,\sigma, p\text{-adic})\, f(p\text{-adic}) = (\prod \theta(\sigma)^{d(\sigma)})\, \tilde{f} \cdot \omega(\chi + 2\sum d(\sigma)\,\sigma)_{\mathrm{can}}.$$

By definition, the value at $(X, \lambda, \omega, i)_{R_0}$ of the left-hand member is the *ratio* of its value at $x = (X, \lambda, i)$ in $\underline{\omega}(x)(\chi + 2\sum d(\sigma)\,\sigma)$ to the *given* basis $\omega(\chi + 2\sum d(\sigma)\,\sigma)$ of $\underline{\omega}(x)(\chi + 2\sum d(\sigma)\,\sigma)$. The formula (2.6.33) now follows from (2.6.35) and (2.6.31). QED

Concatenating (2.6.8) and (2.6.33), we get

(2.6.36) **Theorem.** *Hypothesis as in* (2.6.1–6) *and notations as above, the p-adic ratio*

$$\frac{((\prod \theta(\sigma)^{d(\sigma)})\, \tilde{f})(X, \lambda, i)}{\chi(c) \prod\limits_{\sigma} \sigma(c)^{2d(\sigma)}} \in R_0[1/d_K]$$

actually lies in $R_1[1/d_K]$: *it is given, in the notation* (2.2.7), *by the formula*

$$\frac{((\prod \theta(\sigma)^{d(\sigma)})\tilde{f})(X,\lambda,i)}{\chi(c)\prod\limits_{\sigma}\sigma(c)^{2d(\sigma)}}=\vartheta(\sum d(\sigma)\,\sigma,x,\omega,\ \mathrm{Split}(x))\,f.$$

Chapter III. Eisenstein Series

We begin with some convenient conventions and notations concerning q-expansions.

(3.0) For the rest of this paper, we fix a prime number p. We will permanently adopt the following conventions (3.0.1–2).

(3.0.1) We will only discuss the moduli scheme $\mathscr{M}(\mathfrak{c},\Gamma_{00}(p^{\infty}))$, or the associated modular forms, over a ring R when the fractional ideal $\mathfrak{c}$ is "prime to R" (in the sense of (2.1.1)), and is prime to p.

(3.0.2) Whenever we take q-expansions of $\mathfrak{c}$-HMF's over R by picking fractional ideals $\mathfrak{a},\mathfrak{b}$ of K with $\mathfrak{c}=\mathfrak{a}\mathfrak{b}^{-1}$, we will always choose $\mathfrak{a}$ to be "prime to R" and prime to p. We will endow $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ with the $\Gamma_{00}(p^{\infty})$-structure $i(\varepsilon_{\mathrm{can}})$, where $\varepsilon_{\mathrm{can}}$ is the *equality*

$$\varepsilon_{\mathrm{can}}\colon \mathfrak{a}^{-1}\otimes\mathbf{Z}_p=\mathcal{O}\otimes\mathbf{Z}_p \qquad \text{inside } K\otimes\mathbf{Z}_p.$$

We will denote this $\Gamma_{00}(p^{\infty})$ structure simply "i_{can}." We will endow Tate $_{\mathfrak{a},\mathfrak{b}}(q)$ with the nowhere vanishing differential $\omega_{\mathfrak{a}}(j_{\mathrm{can}})$, (cf., (1.2.11)), where j_{can} is the *equality*

$$j_{\mathrm{can}}\colon \mathfrak{a}^{-1}\otimes R=\mathcal{O}\otimes R \qquad \text{inside } \mathcal{O}[\text{all } 1/l \text{ with } l \text{ invertible in } R]\otimes R.$$

We will denote $\omega_{\mathfrak{a}}(j_{\mathrm{can}})$ simply "ω_{can}." (This should lead to no confusion, because when R is p-adic, this "ω_{can}" *is* the ω_{can} associated to i_{can} (cf., (1.10.14)).) We will use the catch-phrase "q-expansion at $(\mathfrak{a},\mathfrak{b})$" to mean "evaluation at $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q),(q),\ \lambda_{\mathrm{can}},\ \omega_{\mathrm{can}},\ i_{\mathrm{can}})$," (or to mean "evaluation at $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q),\ \lambda_{\mathrm{can}},\ i_{\mathrm{can}})$" in the p-adic case).

(3.0.3) *Remark.* In practice, the restriction that $\mathfrak{c}$ be prime to R and to p is not severe, for the following reason. Given any totally positive $\alpha\in K^{\times}$, and any (super)-natural number N, the moduli problems $\mathscr{M}(\mathfrak{c},\Gamma_{00}(N))$ and $\mathscr{M}((\alpha)\,\mathfrak{c},\Gamma_{00}(N))$ are isomorphic, via the map

$$(X,\lambda,i)\mapsto(X,\alpha\lambda,i).$$

So we are "free" to move $\mathfrak{c}$ within its strict ideal class, and thus to "make" it prime to any *given* finite set of primes. The restriction on $\mathfrak{a}$ is even less severe, since we are *always* permitted the choice $\mathfrak{a}=\mathcal{O}$.

(3.1) In this section, we develop some notions of "partial fourier transform," in the context of a $\mathfrak{c}$-polarized lattice with $\Gamma_{00}(p^{\infty})$-structure $(\mathscr{L},\langle\ ,\ \rangle,i)$. Let F be a locally constant function

$$(3.1.1)\quad F\colon (\mathcal{O}\otimes\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)\to\mathbf{C}.$$

Its "partial fourier transform" PF is the compactly supported function

$$(3.1.2)\quad PF\colon (\mathfrak{d}^{-1}\otimes\mathbf{Q}_p/\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)\to\mathbf{C}$$

defined by "fourier transform in the first variable" as follows: for any integer n such that F is constant on cosets modulo p^n, PF is *supported* in

$$(3.1.3)\quad \left(\mathfrak{d}^{-1}\otimes\frac{1}{p^n}\mathbf{Z}_p/\mathbf{Z}_p\right)\times(\mathcal{O}\otimes\mathbf{Z}_p),$$

where it is given by the formula

$$(3.1.4)\quad PF(x,y)=\frac{1}{p^{ng}}\sum_{a\in\mathcal{O}/p^n\mathcal{O}} F(a,y)\exp(2\pi i\,\mathrm{trace}(2x))$$

$$\text{for } x\in\mathfrak{d}^{-1}\otimes\frac{1}{p^n}\mathbf{Z}_p/\mathbf{Z}_p,\qquad y\in\mathcal{O}\otimes\mathbf{Z}_p.$$

One easily checks that this definition of PF is independent of the auxiliary n which enters into it.

Now consider a $\mathfrak{c}$-polarized lattice with $\Gamma_{00}(p^\infty)$-structure $(\mathscr{L},\langle\ ,\ \rangle,i)$. The polarization form

$$(3.1.5)\quad \Lambda^2_{\mathcal{O}}(\mathscr{L})\xrightarrow{\sim}\mathfrak{d}^{-1}\mathfrak{c}^{-1},$$

when tensored with $\mathbf{Z}_p$, gives an isomorphism

$$(3.1.6)\quad \Lambda^2_{\mathcal{O}\otimes\mathbf{Z}_p}(\mathscr{L}\otimes\mathbf{Z}_p)\xrightarrow{\sim}\partial^{-1}\mathfrak{c}^{-1}\otimes\mathbf{Z}_p=\mathfrak{d}^{-1}\otimes\mathbf{Z}_p$$

(the last equality, inside $K\otimes\mathbf{Z}_p$, holds because $\mathfrak{c}$ is prime to p). The $\Gamma_{00}(p^\infty)$-structure i on $\mathscr{L}$ gives a short exact sequence of free $\mathcal{O}\otimes\mathbf{Z}_p$-modules

$$(3.1.7)\quad 0\to\mathfrak{d}^{-1}\otimes\mathbf{Z}_p\xrightarrow{i}\mathscr{L}\otimes\mathbf{Z}_p\to ?\to 0,$$

whence an isomorphism

$$(3.1.8)\quad \Lambda^2(\mathscr{L}\otimes\mathbf{Z}_p)\xrightarrow{\sim}(\mathfrak{d}^{-1}\otimes\mathbf{Z}_p)\underset{\mathcal{O}\otimes\mathbf{Z}_p}{\otimes} ?.$$

Comparing this with (3.1.6), we find an isomorphism

$$(3.1.9)\quad ?\simeq\mathcal{O}\otimes\mathbf{Z}_p,$$

and so we can rewrite (3.1.7) as

$$(3.1.10)\quad 0\to\mathfrak{d}^{-1}\otimes\mathbf{Z}_p\xrightarrow{i}\mathscr{L}\otimes\mathbf{Z}_p\to\mathcal{O}\otimes\mathbf{Z}_p\to 0.$$

Tensoring with $\mathbf{Q}_p$, we get

$$(3.1.11)\quad 0\to\mathfrak{d}^{-1}\otimes\mathbf{Q}_p\xrightarrow{i}\mathscr{L}\otimes\mathbf{Q}_p\to\mathcal{O}\otimes\mathbf{Q}_p\to 0.$$

(3.1.12) *Definition.* We denote by $PV_p(\mathscr{L})$ the $\mathcal{O}\otimes\mathbf{Z}_p$-submodule of $\mathscr{L}\otimes\mathbf{Q}_p$ defined by

(3.1.12.1) $PV_p(\mathscr{L})=\mathscr{L}\otimes\mathbf{Z}_p+i(\mathfrak{d}^{-1}\otimes\mathbf{Q}_p).$

Alternately, we can say that $PV_p(\mathscr{L})$ is the complete inverse image of $\mathcal{O}\otimes\mathbf{Z}_p$ in $\mathscr{L}\otimes\mathbf{Q}_p$ by (3.1.11). (The notation "PV_p" is for "partial V_p" from the Tate-module notation which puts $T_p(\mathscr{L})=\mathscr{L}\otimes\mathbf{Z}_p$, $V_p(\mathscr{L})=\mathscr{L}\otimes\mathbf{Q}_p$).

Thus PV_p sits in two short exact sequences

(3.1.13) $0\rightarrow\mathscr{L}\otimes\mathbf{Z}_p\rightarrow PV_p(\mathscr{L})\rightarrow\mathfrak{d}^{-1}\otimes\mathbf{Q}_p/\mathbf{Z}_p\rightarrow 0,$

(3.1.14) $0\rightarrow\mathfrak{d}^{-1}\otimes\mathbf{Q}_p\rightarrow PV_p(\mathscr{L})\rightarrow\mathcal{O}\otimes\mathbf{Z}_p\rightarrow 0.$

We denote by pr_1, pr_2 the corresponding surjective maps

(3.1.15)
$$\begin{tikzcd}
 & \mathfrak{d}^{-1}\otimes\mathbf{Q}_p/\mathbf{Z}_p \\
PV_p(\mathscr{L}) \arrow[ur, two heads, "\mathrm{pr}_1"] \arrow[dr, two heads, "\mathrm{pr}_2"'] & \\
 & \mathcal{O}\otimes\mathbf{Z}_p.
\end{tikzcd}$$

Given a locally constant function F as in (3.1.1), its partial fourier transform PF gives rise to a locally constant, compactly supported function, still noted PF, on $PV_p(\mathscr{L})$, by

(3.1.16) $PF(l)\overset{\mathrm{dfn}}{=}PF(pr_1(l),\mathrm{pr}_2(l))\quad$ for $l\in PV_p(\mathscr{L})$.

(3.2) In this section, we will construct Eisenstein series. We denote by E the group $\mathcal{O}^\times$ of units of K, and by $\mathbf{N}$ the norm mapping $\mathcal{O}\rightarrow\mathbf{Z}$. For any ring R, the norm mapping provides a map

(3.2.1) $\mathcal{O}\otimes R\rightarrow R$

whose restriction to $(\mathcal{O}\otimes R)^\times$ is a group homomorphism

(3.2.2) $(\mathcal{O}\otimes R)^\times\rightarrow R^\times.$

Thus $\mathbf{N}$ may be viewed as a character of the torus $\coprod_{\mathcal{O}/\mathbf{Z}}(\mathbf{G}_m)$, whose expression over $\mathcal{O}^{\mathrm{gal}}$ is just $\sum\sigma$ (i.e., all $n(\sigma)=1$). When speaking of $\mathfrak{c}$-HMF's, we will say "weight k" instead of "weight $\mathbf{N}^k$."

(3.2.3) **Theorem** (Hecke). *Let $k\geqq 1$ be an integer, R an arbitrary ring, and let*

(3.2.4) $F\colon(\mathcal{O}\otimes\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)\rightarrow R$

be a locally constant R-valued function, which is supported in $(\mathcal{O}\otimes\mathbf{Z}_p)^\times\times(\mathcal{O}\otimes\mathbf{Z}_p)^\times$, and which satisfies

(3.2.5) $F(e^{-1}x,ey)=\mathbf{N}(e)^k F(x,y)\quad$ *for all* $e\in E$.

Then there exists a $\mathfrak{c}$*-HMF of weight* k *on* $\Gamma_{00}(p^\infty)$, *defined over* R, *denoted* $G_{k,F}$, *whose* q*-expansion at* $(\mathfrak{a},\mathfrak{b})$ *is given by the formula*

$$(3.2.6)\quad G_{k,F}(q;\mathfrak{a},\mathfrak{b})=\mathbf{N}\mathfrak{a}\sum_{\substack{\alpha\in\mathfrak{a}\mathfrak{b}\\ \alpha\gg 0}} q^\alpha \sum_{\substack{\text{factorizations } \alpha=ab,\\ a\in\mathfrak{a},\, b\in\mathfrak{b},\\ \text{modulo } E}} \operatorname{Sgn}(\mathbf{N}(a))\cdot\mathbf{N}(a)^{k-1}\cdot F(a,b).$$

When $R=\mathbf{C}$, *the transcendental expression of* $G_{k,F}$ *is given by the formula*

(3.2.7) $G_{k,F}(\mathscr{L},\langle\ ,\ \rangle,i)=$ *the value at* $s=0$ *of the entire function of* s *whose expression for* $\mathrm{Re}(s)>1-\frac{k}{2}$ *is the absolutely convergent series*

$$(3.2.8)\quad \frac{(-1)^{kg}\,\Gamma(k+s)^g}{\sqrt{d_K}} \sum_{\substack{l\in\mathscr{L}[1/p]\cap PV_p(\mathscr{L})\\ \text{modulo } E}} \frac{PF(l)}{\mathbf{N}(l)^k\cdot|\mathbf{N}(l)|^{2s}}.$$

Proof. We begin by explaining how to reduce to the case $R=\mathbf{Z}$. We claim that any locally constant F supported in $(\mathscr{O}\otimes\mathbf{Z}_p)^\times\times(\mathscr{O}\otimes\mathbf{Z}_p)^\times$ and satisfying (3.2.5), is an R-linear combination of such $\mathbf{Z}$-valued functions. To see this, let $E^+\subset E$ be the subgroup, of index 1 or 2, consisting of all units of norm one, and consider the profinite groups

$$(3.2.9)\quad \begin{aligned} G^+&=(\mathscr{O}\otimes\mathbf{Z}_p)^\times\times(\mathscr{O}\otimes\mathbf{Z}_p)^\times/\text{closure of } E^+\\ G&=\ldots\ldots\ldots\ldots\qquad /\textit{closure of } E. \end{aligned}$$

Suppose first either k even or $E=E^+$. Then our F is just an R-valued locally constant function on G, so an R-linear combination of characteristic functions of compact open sets of G. If k is odd and $E^+\neq E$, let e_0 be any unit of norm -1. Then e_0 projects to an element $\bar{e}_0$ of G^+ which has exact order two, and our F is precisely a locally constant R-valued function on G^+ satisfying

$$(3.2.10)\quad F(\bar{e}_0 g)=-F(g)\quad \text{for } g\in G^+.$$

Let $H\subset G^+$ be a compact open subgroup of G^+ modulo which F is constant, and which doesn't contain $\bar{e}_0$. Then $\bar{e}_0$ projects to an element $\bar{\bar{e}}_0$ of exact order two in G^+/H, and F "is" an R-valued function on G^+/H satisfying

$$(3.2.11)\quad F(\bar{\bar{e}}_0 g)=-F(g)\quad \text{for } g\in G^+/H.$$

Thus if we choose any set of coset representatives $\{g_i\}$ in G^+/H for the cosets modulo $\{1,\bar{\bar{e}}_0\}$, F is uniquely determined by (3.2.11) by the values $F(g_i)$, and conversely these values may be assigned arbitrarily.

The case $R=\mathbf{Z}$ follows from the case $R=\mathbf{C}$ by the q-expansion principle. By GAGA, we are then "reduced" to showing that the transcendental expression (3.2.8) is an entire function of s, whose value at $s=0$ is a *holomorphic* function of variable $(\mathscr{L},\langle\ ,\ \rangle,i)$, of weight k, whose fourier series is indeed given by (3.2.6). These analytic facts are due to Hecke (cf., especially p. 394 of his *Werke* [4]).

For the convenience of the reader, we recall the main steps. First, we remark that the transformation law (3.2.5) of F yields, for PF, the transformation law

$$(3.2.12)\quad PF(ex, ey) = \mathbf{N}(e)^k\, PF(x, y),$$

so that, in (3.2.8), the summation "modulo E" makes sense. Let F be constant on cosets modulo p^n, so that PF is supported in $\left(\mathfrak{d}^{-1} \otimes \frac{1}{p^n}\mathbf{Z}/\mathbf{Z}\right) \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times$. Then the value of (3.2.8) on the Tate lattice $\mathscr{L}_{\mathfrak{a},\mathfrak{b}}(\tau)$ with its canonical $\Gamma_{00}(p^\infty)$-structure is

$$(3.2.13)\quad \frac{(-1)^{kg}\,\Gamma(k+s)^g}{\sqrt{d_K}} \sum_{\substack{(a,b)\in\mathfrak{d}^{-1}\mathfrak{a}^{-1}\times\mathfrak{b} \\ \bmod E}} \frac{PF\left(\frac{a}{p^n}, b\right)}{\mathbf{N}\left(2\pi i\left(\frac{a}{p^n}+b\tau\right)\right)^k \left|\mathbf{N}\left(2\pi i\left(\frac{a}{p^n}+b\tau\right)\right)\right|^{2s}}.$$

In the summation, we can restrict to terms (a, b) with $b \neq 0$, because $PF(x, 0) = 0$ by the support hypothesis on F. Thus we may rewrite (3.2.13) as

$$(3.2.14)\quad \frac{(-1)^{kg}\,\Gamma(k+s)^g}{\sqrt{d_K}(2\pi i)^{kg}(2\pi)^{2gs}} \cdot \sum_{\substack{b\in\mathfrak{b}-\{0\} \\ \bmod E}} \sum_{a\in\mathfrak{d}^{-1}\mathfrak{a}^{-1}} \frac{PF\left(\frac{a}{p^n}, b\right)}{\mathbf{N}\left(\frac{a}{p^n}+b\tau\right)^k \left|\mathbf{N}\left(\frac{a}{p^n}+b\tau\right)\right|^{2s}}.$$

The *inner* sum can itself be rewritten as

(3.2.15)

$$\sum_{a_0\in\mathfrak{d}^{-1}\mathfrak{a}^{-1}/p^n\mathfrak{d}^{-1}\mathfrak{a}^{-1}} PF\left(\frac{a_0}{p^n}, b\right) \cdot \sum_{a\in\mathfrak{d}^{-1}\mathfrak{a}^{-1}} \frac{1}{\mathbf{N}\left(a+\frac{a_0}{p^n}+b\tau\right)^k \left|\mathbf{N}\left(a+\frac{a_0}{p^n}+b\tau\right)\right|^{2s}}.$$

Let us agree that for *any* $z\in K\otimes\mathbf{C}$ whose imaginary part $\mathrm{Im}(z)\in K\otimes\mathbf{R}$ is totally *non-zero*, we will denote by $S_k(z; \mathfrak{d}^{-1}\mathfrak{a}^{-1}; s)$ the sum

$$(3.2.16)\quad S_k(z\colon \mathfrak{d}^{-1}\mathfrak{a}^{-1}; s) \overset{\text{dfn}}{=} \sum_{a\in\mathfrak{d}^{-1}\mathfrak{a}^{-1}} \frac{1}{\mathbf{N}(a+z)^k\,|\mathbf{N}(a+z)|^{2s}}.$$

Then the quantity

$$(3.2.17)\quad \frac{a_0}{p^n}+b\tau$$

occurring in (3.2.15) is such a z, simply because $b \neq 0$ in $\mathfrak{b}$. Thus we can rewrite (3.2.14) in the form

(3.2.18)

$$\frac{(-1)^{kg}\,\Gamma(k+s)^g}{\sqrt{d^K}(2\pi i)^{kg}(2\pi)^{2gs}} \cdot \sum_{\substack{b\in\mathfrak{b}-\{0\} \\ \bmod E}} \sum_{a_0\in\mathfrak{d}^{\,1}\mathfrak{a}^{-1}/p^n\mathfrak{d}^{-1}\mathfrak{a}^{\,1}} PF\left(\frac{a_0}{p^n}, b\right) S_k\left(\frac{a_0}{p^n}+b\tau; \mathfrak{d}^{-1}\mathfrak{a}^{-1}; s\right).$$

The S_k's are now dealt with directly.

For each fixed $z \in K \otimes \mathbf{C}$ with $\operatorname{Im}(z)$ totally non-zero, we can apply the Poisson summation formula to the function

$$(3.2.19) \quad t \mapsto \frac{1}{\mathbf{N}(t+z)^k \, |\mathbf{N}(t+z)|^{2s}}$$

on $K \otimes \mathbf{R}$, summed over $\mathfrak{d}^{-1}\mathfrak{a}^{-1}$, and obtain

$$(3.2.20) \quad S_k(z; \mathfrak{d}^{-1}\mathfrak{a}^{-1}, s) = \mathbf{N}\mathfrak{a} \cdot \sqrt{d_K} \sum_{\lambda \in \mathfrak{a}} C_k(\lambda, z, s)$$

where $C_k(\lambda, z, s)$ is the Fourier transform of the function (3.2.19):

$$(3.2.21) \quad C_k(\lambda, z, s) = \int_{K \otimes \mathbf{R}} \frac{\exp(-2\pi i \operatorname{trace}(\lambda t))\, dt}{\mathbf{N}(t+z)^k \, |\mathbf{N}(t+z)|^{2s}}$$

and dt means ordinary Lebesgue measure $dt(\sigma_1) \ldots dt(\sigma_g)$ on $K \otimes \mathbf{R} \simeq \mathbf{R}^{\mathfrak{S}}$. Substituting (3.2.20) into (3.2.18), we get an expression for (3.2.14):

$$(3.2.22) \quad \frac{(-1)^{kg}\, \Gamma(k+s)^g \, \mathbf{N}\mathfrak{a}}{(2\pi i)^k (2\pi)^{2gs}} \cdot \sum_{\substack{b \in \mathfrak{b} - \{0\} \\ \operatorname{mod} E}} \; \sum_{a_0 \in \mathfrak{d}^{-1}\mathfrak{a}^{-1}/p^n \mathfrak{d}^{-1}\mathfrak{a}^{-1}} PF\left(\frac{a_0}{p^n}\right) \sum_{\lambda \in \mathfrak{a}} C_k\left(\lambda, \frac{a_0}{p^n} + b\tau, s\right).$$

Using the obvious relation

$$(3.2.23) \quad C_k\left(\lambda, \frac{a_0}{p^n} + b\tau, s\right) = \exp(2\pi i \operatorname{trace}(\lambda a_0/p^n))\, C_k(\lambda, b\tau, s)$$

and interchanging the order of summation, (3.2.22) becomes

$$(3.2.24) \quad \frac{(-1)^{kg}\, \Gamma(k+s)^g \, \mathbf{N}\mathfrak{a}}{(2\pi i)^k (2\pi)^{2gs}} \cdot \sum_{\substack{b \in \mathfrak{b} - \{0\} \\ \operatorname{mod} E}} \sum_{\lambda \in \mathfrak{a}} C_k(\lambda, b\tau, s) \sum_{a_0 \in \mathfrak{d}^{-1}\mathfrak{a}^{-1}/p^n \mathfrak{d}^{-1}\mathfrak{a}^{-1}} PF\left(\frac{a_0}{p^n}, b\right) \exp(2\pi i \operatorname{trace}(a_0/p^n)).$$

The innermost sum is just $F(\lambda, b)$ (partial fourier inversion), so (3.2.24) becomes

$$(3.2.25) \quad \frac{(-1)^{kg}\, \Gamma(k+s)^g \, \mathbf{N}\mathfrak{a}}{(2\pi i)^{kg} (2\pi)^{2gs}} \sum_{\substack{b \in \mathfrak{b} - \{0\} \\ \operatorname{mod} E}} \sum_{\lambda \in \mathfrak{a}} F(\lambda, b)\, C_k(\lambda, b\tau, s).$$

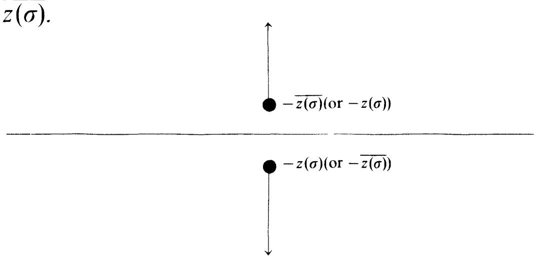
Because $F(0, b) = 0$, by hypothesis we can rewrite this

$$(3.2.26) \quad \frac{(-1)^{kg}\, \Gamma(k+s)^g \, \mathbf{N}\mathfrak{a}}{(2\pi i)^k (2\pi)^{2gs}} \sum_{\substack{b \in \mathfrak{b} - \{0\} \\ \operatorname{mod} E}} \; \sum_{\lambda \in \mathfrak{a} - \{0\}} F(\lambda, b)\, C_k(\lambda, b\tau, s).$$

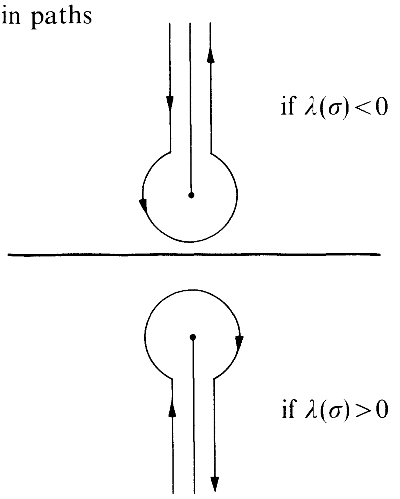
Of course, the above calculations are only meaningful for $\operatorname{Re}(s) \gg 0$. Hecke now provides an explicit analytic continuation of each integral $C_k(\lambda, z, s)$, λ totally non-zero in $K\otimes\mathbf{R}$, $\mathrm{Im}(z)$ totally non-zero in $K\otimes\mathbf{R}$, as follows. He writes C_k as the product

$$(3.2.2.7)\quad C_k(\lambda,z,s)=\prod_{\sigma}\int_{-\infty}^{\infty}\frac{\exp(-2\pi i\,\lambda(\sigma)\,t(\sigma))\,dt(\sigma)}{(t(\sigma)+z(\sigma))^{k+s}(t(\sigma)+\overline{z(\sigma)})^{s}}.$$

In each integral the complex powers extend to well-defined holomorphic functions of t in the complex plan, minus two vertical slits emanating from the points $-z(\sigma)$, $-\overline{z(\sigma)}$.



For $\mathrm{Re}(s)\gg 0$ and $\lambda(\sigma)>0$ (resp. $\lambda(\sigma)<0$), we can deform the path of integration into one of the mandolin paths



with neck-width $\varepsilon\to 0$, and radius around $-z(\sigma)$ or $-\overline{z(\sigma)}$ equal to $\frac{1}{2}|\mathrm{Im}(z(\sigma))|$. The integrals over *these* paths converge for *all* s, and define entire functions of s. Further, for s in a fixed compact set, we have an estimate for these integrals (Hecke, p. 393, (10)) which insures that the series (3.2.26) without Γ's)

$$\sum\sum F(\lambda,b)(C_k(\lambda,b\tau,s)$$

converges for all s to an entire function of s.

The evaluation of these $C_k(\lambda,z,s)$ at $s=0$ is then a simple residue calculation. When $s=0$, each of the integrands

$$(3.2.28)\quad \frac{\exp(-2\pi i\,\lambda(\sigma)\,t(\sigma))\,dt(\sigma)}{(t(\sigma)+z(\sigma)^{k+s}(t(\sigma)+\overline{z(\sigma)})^{s}}$$

becomes a single-valued function of t, and we get

(3.2.29) $C_k(\lambda, z, 0) = \prod_\sigma -2\pi i \,\mathrm{sgn}(\lambda(\sigma))$ (residue of (3.2.28) $(s=0)$ at whichever of $-z(\sigma)$, $-\overline{z(\sigma)}$ has imaginary part of the opposite sign as $\lambda(\sigma)$).

Since the integrands (3.2.28), at $s=0$, are holomorphic except at $-z(\sigma)$ we get

(3.2.30) $C_k(\lambda, z, 0) =$ unless $\mathrm{Im}(\lambda z)$ is totally positive and, if $\mathrm{Im}(\lambda z)$ is totally positive,

(3.2.31)
$$\begin{aligned} C_k(\lambda, z, 0) &= (2\pi i)^g \cdot (-1)^g \,\mathrm{sgn}(\mathbf{N}\lambda) \prod_\sigma (\text{residue of (3.2.28) } (s=0) \text{ at } -z(\sigma)) \\ &= (2\pi i)^g (-1)^g \,\mathrm{sgn}(\mathbf{N}\lambda) \prod_\sigma \left(\frac{1}{\Gamma(k)} \left(\frac{d}{dt(\sigma)} \right)^{k-1} (\exp(-2\pi i \lambda(\sigma) t(\sigma))_{|-z} \right) \\ &= (2\pi i)^g (-1)^g \,\mathrm{sgn}(\mathbf{N}\lambda) \prod_\sigma \left(\frac{1}{\Gamma(k)} (2\pi i \lambda(\sigma))^{k-1} \exp(2\pi i \lambda(\sigma) z(\sigma) \right) \\ &= \frac{(2\pi i)^{kg}(-1)^{kg}}{\Gamma(k)^g} \mathrm{sgn}(\mathbf{N}(\lambda)) \cdot \mathbf{N}(\lambda)^{k-1} \exp(2\pi i \,\mathrm{trace}(\lambda z)). \end{aligned}$$

Thus for $\lambda \neq 0$ in $\mathfrak{a}$ we have

$$(3.2.32) \quad C_k(\lambda, b\tau, 0) = \begin{cases} 0 & \text{unless } \lambda b \gg 0 \\ \dfrac{(2\pi i)^{kg}(-1)^{kg}}{\Gamma(k)^g} \mathrm{sgn}(\mathbf{N}(\lambda))\, \mathbf{N}(\lambda)^{k-1} q^{\lambda b} & \text{if } \lambda b \gg 0. \end{cases}$$

Substituting this into (3.2.26) then gives the asserted (3.2.6) (with $a = \lambda$, $\alpha = \lambda b$). It remains to explain why (3.2.26) is entire, i.e., to explain why the entire function

$$\sum \sum F(\lambda, b)\, C_k(\lambda, b\tau, s)$$

has a g-fold zero at each point $s = -n-k$, $n = 0, 1, 2, \ldots$. In fact, this is true of the individual terms $C_k(\lambda, b\tau, s)$, because for these values of s, each of the g integrands (3.2.28) becomes singlevalued and everywhere holomorphic in $t(\sigma)$, hence gives residue zero a la (3.2.29). QED

(3.3) In this section, we give a "functional equation" for the Eisenstein series $G_{1,F}$, in terms of a duality of HBAV's.

Given a $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(p^\infty)$-structure (X, λ, i) over a ring R, we will define its "dual" (X^t, λ^t, i^t), which will be a $\mathfrak{c}^{-1}$-polarized HBAV with $\Gamma_{00}(p^\infty)$-structure, as follows. The underlying HBAV is simply the dual, X^t, of X. The $\mathfrak{c}^{-1}$-polarization λ^t is the isomorphism defined by the commutative diagram

(3.3.1)
$$\begin{tikzcd} (X^t)^t \arrow[r, "\lambda^t"] & (X^t) \otimes_{\mathcal{O}} \mathfrak{c}^{-1} \arrow[d, "\lambda\otimes\mathrm{id}", "\sim"'] \\ X \arrow[u, "\text{biduality } \wr"] \arrow[r, phantom, "="] & X \otimes_{\mathcal{O}} \mathfrak{c} \otimes_{\mathcal{O}} \mathfrak{c}^{-1}. \end{tikzcd}$$

The $\Gamma_{00}(p^\infty)$-structure i^t is defined by the commutative diagram

$$\begin{tikzcd}
\mathfrak{d}^{-1}\otimes\boldsymbol{\mu}_{p^\infty} \arrow[r, hook, "i^t"] \arrow[d, "*\wr"'] & X^t \arrow[d, "\wr\, \lambda"] \\
(\mathfrak{d}^{-1}\underset{\mathbf{Z}}{\otimes}\boldsymbol{\mu}_{p^\infty})\underset{\mathcal{O}}{\otimes}\mathfrak{c} \arrow[r, hook, "i\otimes\mathrm{id}"] & X\underset{\mathcal{O}}{\otimes}\mathfrak{c}.
\end{tikzcd} \tag{3.3.2}$$

(The isomorphism $*$ is constructed out of the fact that $\mathfrak{c}$ is assumed prime to p, by the diagram

$$\begin{tikzcd}
\mathfrak{d}^{-1}\underset{\mathbf{Z}}{\otimes}\boldsymbol{\mu}_{p^\infty} \arrow[r, equal] \arrow[dd, "*\wr"'] & (\mathfrak{d}^{-1}\otimes\mathbf{Z}_p)\underset{\mathbf{Z}_p}{\otimes}\boldsymbol{\mu}_{p^\infty} \arrow[d, equal, "\text{(because } \mathfrak{c} \text{ is prime to } p)"] \\
 & ((\mathfrak{d}^{-1}\mathfrak{c})\otimes\mathbf{Z}_p)\underset{\mathbf{Z}_p}{\otimes}\boldsymbol{\mu}_{p^\infty} \arrow[d, equal] \\
(\mathfrak{d}^{-1}\otimes\boldsymbol{\mu}_{p^\infty})\otimes\mathfrak{c} \arrow[r, equal] & \mathfrak{d}^{-1}\mathfrak{c}\underset{\mathbf{Z}}{\otimes}\boldsymbol{\mu}_{p^\infty}.
\end{tikzcd} \tag{3.3.3}$$

The construction $(X,\lambda,i)\to(X^t,\lambda^t,i^t)$ defines an anti-equivalence

$$\mathscr{M}(\mathfrak{c},\Gamma_{00}(p^\infty)) \xrightarrow{t} \mathscr{M}(\mathfrak{c}^{-1},\Gamma_{00}(p^\infty)). \tag{3.3.4}$$

Given a nowhere vanishing differential ω on an (X,λ,i) over a ring R to which $\mathfrak{c}$ is prime, we can define a nowhere vanishing differential ω^t on (X^t,λ^t,i^t) by the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Lie}(X^t) \arrow[rr, "\omega^t", "\sim"'] \arrow[d, dash, "\wr\,\lambda"] & & \mathfrak{d}^{-1}\otimes R \arrow[dd, equal, "\text{(because } \mathfrak{c} \text{ is prime to } R)"] \\
\mathrm{Lie}(X\underset{\mathcal{O}}{\otimes}\mathfrak{c}) \arrow[d, equal] & & \\
\mathrm{Lie}(X)\otimes\mathfrak{c} \arrow[rr, "\omega\otimes\mathrm{id}", "\sim"'] & & \mathfrak{d}^{-1}\mathfrak{c}\otimes R.
\end{tikzcd} \tag{3.3.5}$$

This construction $(X,\lambda,\omega,i)\mapsto(X^t,\lambda^t,\omega^t,i^t)$ carries

$$(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q),\lambda_{\mathrm{can}},\omega_{\mathrm{can}},i_{\mathrm{can}})\to(\mathrm{Tate}_{\mathfrak{b},\mathfrak{a}}(q),\lambda_{\mathrm{can}},\omega_{\mathrm{can}},i_{\mathrm{can}}). \tag{3.3.6}$$

By transposition, it defines an isomorphism of spaces of $\mathfrak{c}$-HMF's

$$\begin{gathered} M(\mathfrak{c}^{-1},\Gamma_{00}(p^\infty),\chi;R) \xrightarrow{\sim} M(\mathfrak{c},\Gamma_{00}(p^\infty),\chi;R) \\ f\mapsto f^t \end{gathered} \tag{3.3.7}$$

by

$$f^t(X,\lambda,\omega,i) \overset{\mathrm{dfn}}{=} f(X^t,\lambda^t,\omega^t,i^t). \tag{3.3.8}$$

By (3.3.6) we have

(3.3.9) the q-expansion of f at $(\mathfrak{b},\mathfrak{a})=$ the q-expansion of f^t at $(\mathfrak{a},\mathfrak{b})$.

When R is a p-adic ring, the construction $(X,\lambda,i)\mapsto(X^t,\lambda^i,i^t)$ defines, by transposition, an R-algebra isomorphism

$$(3.3.10)\quad \begin{aligned} V(\mathfrak{c}^{-1},R) &\xrightarrow{\sim} V(\mathfrak{c},R) \\ f &\mapsto f^t \end{aligned}$$

by

$$(3.3.11)\quad f^t(X,\lambda,i)=f(X^t,\lambda^t,i^t).$$

which again (compare (3.3.9)) satisfies

(3.3.12) the q-expansion of f at $(\mathfrak{b},\mathfrak{a})=$ the q-expansion of f^t at $(\mathfrak{a},\mathfrak{b})$.

(3.3.13) **Theorem.** *Let F be a locally constant R-valued function on $\mathcal{O}\otimes\mathbf{Z}_p\times\mathcal{O}\otimes\mathbf{Z}_p$, which is supported in $(\mathcal{O}\otimes\mathbf{Z}_p)^\times\times(\mathcal{O}\otimes\mathbf{Z}_p)^\times$ and which satisfies* (3.2.5) *with $k=1$. Denote by F^t the function*

$$(3.3.14)\quad F^t(x,y)=F(y,x).$$

For any fractional ideal $\mathfrak{c}$ of K prime to both p and to R, let $G_{1,F;\mathfrak{c}}$ temporarily denote the $\mathfrak{c}$-HMF of "$G_{1,F}$" of weight one on $\Gamma_{00}(p^\infty)$ constructed in (3.2.3). *Then we have the functional equation*

$$(3.3.15)\quad (G_{1,F;\mathfrak{c}^{-1}})^t=\mathbf{N}\,\mathfrak{c}^{-1}G_{1,F^t;\mathfrak{c}}.$$

i.e. for any $\mathfrak{c}$-polarized (X,λ,ω,i), we have

$$(3.3.16)\quad G_{1,F;\mathfrak{c}^{-1}}(X^t,\lambda^t,\omega^t,i^t)=\mathbf{N}\,\mathfrak{c}^{-1}G_{1,F^t;\mathfrak{c}}(X,\lambda,\omega,i).$$

Proof. It suffices to check that both $(G_{1,F;\mathfrak{c}^{-1}})^t$ and $\mathbf{N}\,\mathfrak{c}^{-1}G_{1,F^t;\mathfrak{c}}$ have the same q-expansion at some cusp$(\mathfrak{a},\mathfrak{b})$. By (3.3.9), this comes down to the assertion

$$(3.3.17)\quad G_{1,F}(q;\mathfrak{b},\mathfrak{a})=\mathbf{N}\,\mathfrak{c}^{-1}\cdot G_{1,F^t}(q;\mathfrak{a},\mathfrak{b})$$

which follows immediately from the explicit formulas (3.2.6). QED

3.4. In this section, we construct *p-adic* Eisenstein series $G_{k,F}$.

(3.4.1) **Theorem.** *Fix a p-adic ring R, a fractional ideal $\mathfrak{c}$ of K which is prime to p, and an integer $k\geqq 1$. Let F be any R-valued continuous (but not necessarily locally constant) function on $\mathcal{O}\otimes\mathbf{Z}_p\times\mathcal{O}\otimes\mathbf{Z}_p$ which is supported in $(\mathcal{O}\otimes\mathbf{Z}_p)^\times\times(\mathcal{O}\otimes\mathbf{Z}_p)^\times$, and which satisfies*

$$(3.4.2)\quad F(e^{-1}x,ey)=\mathbf{N}(e)^kF(x,y)\qquad \textit{for all } e\in E.$$

Then there exists an element $G_{k,F}$ in $V(\mathfrak{c},R)$ whose q-expansion at $(\mathfrak{a},\mathfrak{b})$ is given by *the formula*

$$(3.4.3)\quad G_{k,F}(q;\mathfrak{a},\mathfrak{b})=\mathbf{N}\,\mathfrak{a}\sum_{\substack{\alpha\in\mathfrak{a}\mathfrak{b}\\ \alpha\gg 0}} q^{\alpha}\sum_{\substack{\text{factorizations } \alpha=ab\\ a\in\mathfrak{a},\, b\in\mathfrak{b},\\ \text{modulo } E}} \operatorname{sgn}(\mathbf{N}(a))\cdot\mathbf{N}(a)^{k-1}F(a,b).$$

When the function F is locally constant, this $G_{k,F}\in V(\mathfrak{c},R)$ is just the image of the earlier-constructed $G_{k,F}\in M(\mathfrak{c},\Gamma_{00}(p^{\infty}),k;R)$ by the canonical map

$$(3.4.4)\quad \bigoplus_{\chi} M(\mathfrak{c},\Gamma_{00}(p^{\infty}),\chi;R)\to V(\mathfrak{c},R).$$

Proof. For all $n\geqq 1$, the $R/p^n R$-valued function "$F \bmod p^n$" is continuous, hence locally constant, so gives rise, by (3.2.3), to a well-defined element

$$(3.4.5)\quad G_{k,\text{“}F \bmod p^n\text{”}}\in M(\mathfrak{c},\Gamma_{00}(p^{\infty}),k,R/^{pn}R)$$

whose image under (3.4.4) is an element

$$(3.4.6)\quad G_{k,F \bmod p^n}\in V(\mathfrak{c},R/p^n R),$$

whose q-expansion over $R/p^n R$ is the reduction $\bmod p^n$ of the right-hand side of (3.4.3). By the q-expansion principle, these $G_{k,F \bmod p^n}$, taken together, give an element

$$(3.4.7)\quad G_{k,F}=\{G_{k,F \bmod p^n}\}_n\in\varprojlim_n V(\mathfrak{c},R/p^n R)=V(\mathfrak{c},R),$$

whose q-expansion is necessarily given by (3.4.3). The final compatibility follows from the q-expansion principle. QED

(3.4.8) **Corollary.** *Hypotheses and notations as in* (3.4.1), *we have the formula*

$$(3.4.9)\quad G_{k,F}=G_{1,\mathbf{N}(x)^{k-1}F(x,y)}\quad \text{in } V(\mathfrak{c},R),$$

where $\mathbf{N}(x)^{k-1}F(x,y)$ means the function on $(\mathcal{O}\otimes\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)$ given by

$$(3.4.10)\quad (x,y)\to\mathbf{N}(x)^{k-1}F(x,y).$$

Proof. Obvious from the q-expansion principle.

(3.4.11) **Corollary.** *Hypotheses as in* (3.4.1), *under the ring isomorphism* (3.3.10)

$$\begin{aligned} V(\mathfrak{c}^{-1},R) &\xrightarrow{\sim} V(\mathfrak{c},R)\\ f &\to f^{t}, \end{aligned}$$

we have

$$(3.4.12)\quad (G_{1,F})^{t}=\mathbf{N}\,\mathfrak{c}^{-1}\,G_{1,F^{t}}\quad \textit{in } V(\mathfrak{c},R).$$

(In the over-precise notation of (3.3.15), this should read

$$(3.4.13)\quad (G_{1,F;\mathfrak{c}^{-1}})^{t}=\mathbf{N}\,\mathfrak{c}^{-1}\cdot G_{1,F^{t};\mathfrak{c}}.$$

Proof. Just as in (3.2.13) by the q-expansion principle. QED

(3.4.14) **Corollary.** *Hypotheses and notations as in* (3.4.1), *suppose in addition that R is an $\mathcal{O}^{\mathrm{gal}}$-algebra. Then the behavior of the $G_{k,F}$ under the derivation $\theta(\sigma)$* (cf. (2.6.25)) *is given by the formula*

$$(3.4.15)\quad \theta(\sigma)\,G_{k,F}=G_{k,\sigma(x)\sigma(y)F(x,y)}.$$

Proof. Combine (2.6.27) with the q-expansion principle.

3.5. In this section, we apply the algebraicity theorems of Chapter II to Eisenstein series.

Let us denote by $\mathbf{C}_p$ the completion of a fixed algebraic closure of $\mathbf{Q}$. Choose field embeddings

$$(3.5.1)\quad \begin{tikzcd} & \mathbf{C} \\ \overline{\mathbf{Q}} \arrow[ur, hook, "\mathrm{incl}\,(\infty)"] \arrow[dr, hook, "\mathrm{incl}\,(p)"'] & \\ & \mathbf{C}_p. \end{tikzcd}$$

(3.5.2) **Theorem.** *Let A be a subring of $\overline{\mathbf{Q}}$ which contains $\mathcal{O}^{\mathrm{gal}}$, and suppose given over A a $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(p^\infty)$-structure $x=(X,\lambda,i)$, a nowhere vanishing differential ω on X, and an $\mathcal{O}\otimes A$ splitting* $\mathrm{Split}(x)\subset H^1_{DR}(x)=H^1_{DR}(X/A)$ *of the Hodge filtration on $H^1_{DR}(X/A)$. Let $k\geqq 1$ be an integer, and F a locally constant A-valued function on $\mathcal{O}\otimes\mathbf{Z}_p\times\mathcal{O}\otimes\mathbf{Z}_p$ which is supported in $(\mathcal{O}\otimes\mathbf{Z}_p)^\times\times(\mathcal{O}\otimes\mathbf{Z}_p)^\times$ and which satisfies*

$$(3.5.3)\quad F(e^{-1}x,e\,y)=\mathbf{N}(e)^k(F(x,y))\qquad \text{for all } e\in E.$$

Suppose that after extension of scalars

$$(3.5.4)\quad A\subset\overline{\mathbf{Q}}\xhookrightarrow{\mathrm{incl}(\infty)}\mathbf{C}$$

the splitting $\mathrm{Split}(x)$ *becomes the antiholomorphic splitting, i.e., that* $\mathrm{Split}(x)\otimes\mathbf{C}=H^{0,1}$ *in* $H^1(X_{\mathbf{C}},\mathbf{C})=H^1_{DR}(x)\otimes\mathbf{C}$. *Let $(\mathcal{L},\langle\,,\rangle,i)$ be the $\mathfrak{c}$-polarized lattice with $\Gamma_{00}(p^\infty)$-structure in $K\otimes\mathbf{C}$, corresponding to (X,λ,ω), with polarization constant $a(\mathcal{L},\langle\,,\rangle)\in K\otimes\mathbf{R}$;*

$$(3.5.5)\quad \langle u,v\rangle=\frac{\mathrm{Im}(\bar{u}\,v)}{a(\mathcal{L},\langle\;\rangle)}.$$

Let the $a(\sigma)(\mathcal{L},\langle\,,\rangle)\in\mathbf{R}$ denote the components of $a(\mathcal{L},\langle\,,\rangle)$.

Then for any element $\sum d(\sigma)\,\sigma$ in $\mathbf{Z}[\mathfrak{S}]$ with all $d(\sigma)\geqq 0$, the complex number

"*the value at $s=0$ of the entire function of s whose expression for* $\mathrm{Re}(s)>1-\dfrac{k}{2}$ *is the absolutely convergent series*

$$(3.5.6)\quad \left(\frac{1}{\sqrt{d_K}}\right)(-1)^{kg}\prod_\sigma\left(\frac{\pi^{d(\sigma)}\Gamma(k+d(\sigma)+s)}{a(\sigma)(\mathcal{L},\langle\,,\rangle)^{d(\sigma)}}\right)\cdot\sum_{\substack{l\in\mathcal{L}\left[\frac{1}{p}\right]\cap PV_p(\mathcal{L})\\ \text{modulo } E}}\frac{PF(l)\cdot\prod_\sigma(\bar{l}(\sigma))^{d(\sigma)}}{\mathbf{N}(l)^k\prod_\sigma(l(\sigma))^{d(\sigma)}\cdot|\mathbf{N}(l)|^{2s}}\text{"}$$

lies in $A[1/d_K]$; *in fact, it is the image by* incl(∞) *of the quantity*

(3.5.7) $\vartheta(\sum d(\sigma)\sigma, x, \omega, \mathrm{Split}(x))(G_{k,F}) \in A[1/d_K]$.

Proof. This is just (2.4.5) in the case $R = A[1/d_K], f = G_{k,F}$, once we use (2.3.38) and (2.3.48–49) to compute that $\vartheta(\sum d(\sigma)\sigma, C^{\infty})G_{k,F}$ is indeed given by the formula (3.5.6).

(3.5.8) **Theorem.** *Hypotheses and notations as in* (3.5.2), *suppose that under the given p-adic embedding* incl(p): $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$, *the subring* $A \subset \bar{\mathbf{Q}}$ *lands in the ring of integers of* $\mathbf{C}_p$. *Let* $\hat{A} \subset \mathbf{C}_p$ *be the completion of* A *for the absolute value on* $\mathbf{C}_p$. *Let*

(3.5.9) $c \in (\mathcal{O} \otimes \hat{A})^{\times}$

be the ratio between the given ω *as* $x = (X, \lambda, i)$ *and the canonical* ω_{can} *on* (X, λ, i) *considered over* $\hat{A}$:

(3.5.10) $\omega = c \cdot \omega_{\mathrm{can}}$ on $X_{\hat{A}}$.

Then the p-adic number,

(3.5.11) $$\frac{1}{\mathbf{N}(c)^k \prod_{\sigma} \sigma(c)^{2d(\sigma)}} (G_{k, \prod_{\sigma} (\sigma(x)\sigma(y))^{d(\sigma)} F(x,y)})((X, \lambda, i)_{\hat{A}})$$

a priori *in* $\hat{A}$, *actually lies in* $A[1/d_K]$. *In fact, it is the image by* incl(p) *of the quantity*

$$\vartheta(\sum d(\sigma)\sigma, x, \omega, \mathrm{Split}(x))\, G_{k,F} \in A[1/d_K].$$

Proof. This is (2.6.33) for $G_{k,F}$, combined with (3.4.15). QED

Chapter IV. Review of Measures and their Mellin Transforms; the Eisenstein Measure

4.0. In this section, we review abstract p-adic measures. Let Y be a profinite (i.e., compact and totally disconnected) topological space, and R a p-adic ring. Denote by Contin (Y, R) the p-adic R-algebra of all continuous R-valued functions on Y. An R-valued p-adic measure μ on Y is, at choice, one of the following (4.0.1–3):

(4.0.1) a $\mathbf{Z}_p$-linear map μ: Contin$(Y, \mathbf{Z}_p) \to R$

written symbolically $f \mapsto \int_Y f\, d\mu$

or sometimes $f \mapsto \int_Y f(y)\, d\mu(y)$

(4.0.2) an R-linear map μ: Contin$(Y, R) \to R$

written symbolically $f \mapsto \int_Y f\, d\mu$

or sometimes $f \mapsto \int_Y f(y)\, d\mu(y)$

The two notions are equivalent, via the canonical isomorphism

$$\mathrm{Contin}(Y, \mathbf{Z}_p) \underset{\mathbf{Z}_p}{\hat{\otimes}} R \xrightarrow{\sim} \mathrm{Contin}(Y, R).$$

If perchance R is an R_0-algebra with R_0 itself a p-adic ring, these notions are equivalent to that of

(4.0.3) an R_0-linear map μ: $\mathrm{Contin}(Y, R_0) \to R$

written symbolically $\quad f \mapsto \int_Y f\, d\mu$

or sometimes $\quad f \mapsto \int_Y f(y)\, d\mu(y)$

Given an R-valued p-adic measure μ on Y, and a function $g \in \mathrm{Contin}(Y, R)$, the product $g\mu$ is the R-valued measure defined by

(4.0.4) $\int_Y f\, d(g\mu) \overset{\mathrm{dfn}}{=} \int_Y fg\, d\mu.$

Given an R-valued p-adic measure μ on Y, and a homomorphism $\varphi: R \to R'$ of p-adic rings, we get an R'-valued measure $\mu(\varphi)$ on Y, defined by

(4.0.5) $\int_Y f\, d\mu(\varphi) = \varphi(\int_Y f\, d\mu)$ for $f \in \mathrm{Contin}(Y, R)$.

Suppose that R is flat over $\mathbf{Z}_p$. Let μ be an R-valued measure on Y. Because Y is compact, we have

$$\mathrm{Contin}(Y, R)[1/p] = \mathrm{Contin}(Y, R[1/p]),$$

so that μ extends by linearity to give an $R[1/p]$ linear map

$$\mathrm{Contin}(Y, R[1/p]) \to R[1/p]$$

still denoted

$$f \mapsto \int_Y f\, d\mu.$$

(4.0.6) **Proposition** (abstract Kummer congruences). *Suppose R flat over $\mathbf{Z}_p$, and let $\{f_i\}_{i\in I}$ be a collection of elements of* Contin(Y, R), *whose $R[1/p]$-span is uniformly dense in* Contin$(Y, R[1/p])$. *Let $\{a_i\}_{i\in I}$ be a family of elements of R (with the same indexing set I). Then there exists an R-valued p-adic measure μ on Y such that*

(4.0.7) $\int f_i\, d\mu = a_i$ *for all* $i \in I$

if and only if the a_i satisfy the following "Kummer congruences":

(4.0.8) *for every collection $\{b_i\}_{i\in I}$ of elements of $R\left[\frac{1}{p}\right]$ which are zero for all but* *finitely many i, and every integer n such that*

$$\sum_i b_i f_i(y) \in p^n R \qquad \textit{for all } \ y \in Y,$$

we have

$$\sum b_i a_i \in p^n R.$$

Proof. The necessity is clear, because

$$(4.0.9) \quad \begin{aligned} \sum b_i a_i &= \int (\text{a } p^n R\text{-valued function})\, d\mu \\ &= p^n \int (\text{an } R\text{-valued function})\, d\mu \\ &\in p^n R. \end{aligned}$$

To prove the sufficiency, we will construct μ out of the a_i. Given $f \in \mathrm{Contin}(Y, R)$, and an integer $n \geqq 1$, there exist $b_i \in R\left[\frac{1}{p}\right]$, almost all zero, such that

$$(4.0.10) \quad f - \sum b_i f_i \in p^n \,\mathrm{Contin}(Y, R).$$

By hypothesis, the quantity

$$(4.0.11) \quad \sum b_i a_i$$

lies in R, and is well-defined (i.e., independent of the choice of the b_i) modulo $p^n R$. If we call it "$\int f\, d\mu \bmod p^n$," then we get the desired element

$$(4.0.12) \quad \int f\, d\mu \overset{\mathrm{dfn}}{=} \{\text{``}\textstyle\int f\, d\mu \bmod p^n\text{''}\}_n \in \varprojlim R/p^n R = R. \quad \text{QED}$$

4.1. In this section, we recall the "Mellin transform" (L-function) of a p-adic measure on a profinite abelian group. Let μ be an R-valued p-adic measure on a profinite abelian group G. Let $G^\vee(R)$ denote the group of all continuous group homomorphisms

$$(4.1.0) \quad \chi: G \to R^\times,$$

with the uniform topology.

The Mellin transform of μ is the continuous R-valued function L_μ on $G^\vee(R\}$ defined by

$$(4.1.1) \quad L_\mu(\chi) \overset{\mathrm{dfn}}{=} \int_G \chi\, d\mu.$$

(4.1.2) **Proposition.** *Suppose that R is flat over* $\mathbf{Z}_p$, *and that for every divisor n of the supernatural order of G, R contains a primitive n'th root of unity* (*meaning a root of the n'th cyclotonic polynomial*). *Let* μ *be an R-valued p-adic measure on G. Fix a character* $\chi_0 \in G^\vee(R)$, *and consider the values*

$$\int_G \chi_0 \chi\, d\mu = L_\mu(\chi_0 \chi) \qquad \textit{for all locally constant } \ \chi \in G^\vee(R).$$

Then μ is uniquely determined by these values.

Conversely, if we fix $\chi_0 \in G^\vee(R)$, then a family $\{a_{\chi_0\chi}\}$ of elements of R, indexed by all characters of the form $\chi_0\chi$ with χ locally constant, arises from a (necessarily unique) measure μ on G by the rule

$$L_\mu(\chi_0\chi) = a_{\chi_0\chi}$$

if and only if the $\{a_{\chi_0\chi}\}$ satisfy the Kummer congruences (4.0.8) (*with respect to the family of functions $\{\chi_0\chi\}$ on G*).

Proof. Under the hypothesis on R, any locally constant $R[1/p]$-valued locally constant function on G is a finite $R[1/p]$-linear combination of locally constant χ's. Hence, the $R[1/p]$-span of the $\{\chi_0\chi\}$'s is uniformly dense in $\mathrm{Contin}(G, R[1/p])$, as it contains all functions of the form

$$\chi_0\chi(\text{any locally constant function})$$

and χ_0 is *invertible* in $\mathrm{Contin}(G, R)$. The result now follows from (4.0.6).

4.2. We now give a non-trivial example of a p-adic measure, the "Eisenstein measure."

(4.2.0) **Lemma.** *Over any p-adic ring R, the inverse constructions*

$$(4.2.1)\quad \begin{aligned} H(x,y) &= \frac{1}{\mathbf{N}(x)} \cdot F\left(\frac{1}{x}, y\right) \\ F(x,y) &= \frac{1}{\mathbf{N}(x)} \cdot H\left(\frac{1}{x}, y\right) \end{aligned}$$

define an R-linear bijection between

(4.2.2) *continuous R-valued functions F on $(\mathcal{O} \otimes \mathbf{Z}_p)^\times \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times$, which satisfy $F(e^{-1}x, ey) = \mathbf{N}(e) F(x,y)$ for all $e \in E$*

and

(4.2.3) *continuous R-valued functions H on $(\mathcal{O} \otimes \mathbf{Z}_p)^\times \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times$, which satisfy $H(ex, ey) = H(x,y)$ for all $e \in E$.*

Proof. Compute. QED

If we define G to be the group

(4.2.4) $G = (\mathcal{O} \otimes \mathbf{Z}_p)^\times \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times$/the closure of E, viewed diagonally as the subgroup of (e,e)'s.

then the functions H of (4.2.3) are precisely the continuous R-valued functions on G.

(4.2.5) *Definition.* Let $\mathfrak{c}$ be a fractional ideal of K, prime to p. The *Eisenstein measure $\mu_\mathfrak{c}$ is the $V(\mathfrak{c}, \mathbf{Z}_p)$*-valued measure on G defined by

$$(4.2.6)\quad \int_G H\, d\mu_\mathfrak{c} = G_{1,F} \quad \text{in } V(\mathfrak{c}, \mathbf{Z}_p)$$

for $H \in \mathrm{Contin}(G, \mathbf{Z}_p)$, and F the corresponding function

$$(4.2.7)\quad F(x,y)=\frac{1}{\mathbf{N}(x)}H\left(\frac{1}{x},y\right)$$

of type (4.2.2) (and extended by zero to all of $\mathcal{O}\otimes\mathbf{Z}_p\times\mathcal{O}\otimes\mathbf{Z}_p$). The *q-expansion* of this measure at $(\mathfrak{a},\mathfrak{b})$ is given by

$$(4.2.8)\quad \Big(\int_G H\cdot d\mu_{\mathfrak{c}}\Big)(q;\mathfrak{a},\mathfrak{b})=\mathbf{N}\,\mathfrak{a}\sum_{\substack{\alpha\in\mathfrak{a}\mathfrak{b}\\ \alpha\gg 0}} q^{\alpha}\sum_{\substack{\text{factorizations } \alpha=ab\\ a\in\mathfrak{a},\, b\in\mathfrak{b},\\ \text{modulo units}}}\frac{\mathrm{sgn}(\mathbf{N}(a))}{\mathbf{N}(a)}H\left(\frac{1}{a},b\right)$$

By extension of scalars, we get a $V(\mathfrak{c},R)$-valued measure $\mu_{\mathfrak{c}}$ on G for any p-adic ring R, defined by

$$(4.2.9)\quad \begin{cases}\int_G H\,d\mu_{\mathfrak{c}}=G_{1,F} \quad \text{in } V(\mathfrak{c},R)\\ \text{for } H\in\mathrm{Contin}(C,R),\quad F(x,y)=\dfrac{1}{\mathbf{N}(x)}H\left(\dfrac{1}{x},y\right).\end{cases}$$

If we are given a $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(p^{\infty})$-structure (X,λ,i) over a p-adic ring R, then we can define an R-valued measure $\mu_{\mathfrak{c}}(X,\lambda,i)$ on G by "evaluation":

$$(4.2.10)\quad \begin{cases}\int_G H\,d\mu_{\mathfrak{c}}(X,\lambda,i)\overset{\mathrm{dfn}}{=}G_{1,F}(X,\lambda,i)\\ \text{for } H\in\mathrm{Contin}(G,R),\quad F(x,y)=\dfrac{1}{\mathbf{N}x}H\left(\dfrac{1}{x},y\right)\end{cases}$$

i.e.,

$$(4.2.11)\quad \int_G H\,d\mu_{\mathfrak{c}}(X,\lambda,i)=\Big(\int_G H\,d\mu_{\mathfrak{c}}\Big)(X,\lambda,i).$$

Given a function $H=H(x,y)$ on G, we define $\check{H}$ to be the function on G defined by

$$(4.2.12)\quad \check{H}(x,y)=\frac{1}{\mathbf{N}x\,\mathbf{N}y}\cdot H\left(\frac{1}{y},\frac{1}{x}\right).$$

This definition is so rigged that under the correspondence (4.2.1), the involution $H\mapsto\check{H}$ corresponds to $F\mapsto F^t$. In view of (3.4.11), we have

(4.2.13) **Lemma.** *Let (X,λ,i) be a $\mathfrak{c}$-polarized HBAV with $\Gamma_{00}(p^{\infty})$-structure over a p-adic ring R, and denote by (X^t,λ^t,i^t) its $\mathfrak{c}^{-1}$-polarized dual. Then for any continuous function H on G, we have the "functional equation"*

$$(4.2.14)\quad \int_G \check{H}\,d\mu_{\mathfrak{c}^{-1}}(X^t,\lambda^t,i^t)=\mathbf{N}\,\mathfrak{c}^{-1}\cdot\int_G H\,d\mu_{\mathfrak{c}}(X,\lambda,i).$$

Chapter V. p-Adic L-Functions for CM-Fields

5.0. In this section we review some basic notions concerning grossencharacters of type A_0. Let L be a finite extension of $\mathbf{Q}$. Λ the set of all field embeddings

(5.0.1) $\lambda: L \hookrightarrow \overline{\mathbf{Q}}$,

and $\mathfrak{m}$ an integral ideal of L. Let $I(\mathfrak{m})$ denote the group of all fractional ideals of L which are prime to $\mathfrak{m}$, and let $\operatorname{Prin}(\mathfrak{m}) \subset I(\mathfrak{m})$ denote the subgroup

(5.0.2) $\operatorname{Prin}(\mathfrak{m}) = \{(\alpha) | \alpha \in L^{\times}, \alpha \equiv 1 \bmod^{\times} \mathfrak{m}, \alpha \gg 0\}$.

(as customary, "$\alpha \gg 0$" means "α positive at all real places"). The factor group

(5.0.3) $H(\mathfrak{m}) = I(\mathfrak{m}) / \operatorname{Prin}(\mathfrak{m})$

is the "strict ideal class group of conductor $\mathfrak{m}$." Let $L(\mathfrak{m})$ denote the maximal abelian extension of L whose conductor divides $\mathfrak{m}$ (and is possibly "ramified" at some real places). The Artin symbol defines an isomorphism

(5.0.4) $H(\mathfrak{m}) \xrightarrow{\sim} \operatorname{Gal}(L(\mathfrak{m})/L)$

$\mathfrak{A} \mapsto (\mathfrak{A}, L(\mathfrak{m})/L)$

Now let $\sum n(\lambda)\lambda$ be an element of $\mathbf{Z}[\Lambda]$. A homomorphism

(5.0.5) $\chi: I(\mathfrak{m}) \to \overline{\mathbf{Q}}^{\times}$

is said to be a "grossencharacter of type A_0" of "∞-type" $\sum n(\lambda)\lambda$ and conductor dividing $\mathfrak{m}$ if its restriction to $\operatorname{Prin}(\mathfrak{m})$ is given by the formula

(5.0.6) $\chi((\alpha)) = \prod_{\lambda} \lambda(\alpha)^{n(\lambda)}$ for $(\alpha) \in \operatorname{Prin}(\mathfrak{m})$.

For fixed $\mathfrak{m}$, the A_0-grossencharacters of conductor dividing $\mathfrak{m}$ form a group A_0-Grossen $(\mathfrak{m})$, which sits in an exact sequence

(5.0.7) $0 \to \operatorname{Hom}(H(\mathfrak{m}), \overline{\mathbf{Q}}^{\times}) \to A_0\text{-Grossen}(\mathfrak{m}) \xrightarrow{\infty\text{-type}} \mathbf{Z}[\Lambda]$.

Let us now fix field embeddings

(5.0.8)
$$\begin{tikzcd}
 & \mathbf{C} \\
\overline{\mathbf{Q}} \arrow[ur, hook, "\operatorname{incl}(\infty)"] \arrow[dr, hook, "\operatorname{incl}(p)"'] & \\
 & \mathbf{C}_p
\end{tikzcd}$$

Associated to $\chi_{\infty} \overset{\text{dfn}}{=} \operatorname{incl}(\infty) \cdot \chi$ is the Hecke L-series

(5.0.9) $L(s, \chi_{\infty}) = \sum_{\substack{\mathfrak{A} \text{ integral ideal} \\ \text{prime to } \mathfrak{m}}} \chi_{\infty}(\mathfrak{A}) \mathbf{N}\mathfrak{A}^{-s}$.

This series is convergent only for $\mathrm{Re}(s) \gg 0$, but it has a meromorphic continuation to the whole s-plane, which is known to be holomorphic at $s=0$ provided that $\chi \neq \mathbf{N}^{-1}$. We define

(5.0.10) $L_{\infty}(\chi) \overset{\text{dfn}}{=}$ the value at $s=0$ of $L(s, \chi_{\infty})$.

On the p-adic side, let $D_p \subset \mathbf{C}_p$ denote the ring of integers. Then $\chi_p \overset{\text{dfn}}{=} \mathrm{incl}(p) \cdot \chi$ maps $I(p\mathfrak{m})$ to $D_p^{\times}$, as follows immediately from (5.0.6) and the finiteness of $H(p\mathfrak{m})/\mathrm{Prin}(p\mathfrak{m})$, and χ_p maps Prin $(p^n\mathfrak{m})$ to $1+p^n D_p$, as follows from (5.0.6).

Passing to the inverse limit over n, we get a continuous homomorphism

(5.0.11) $\chi_p\colon \varprojlim I(p\mathfrak{m})/\mathrm{Prin}(p^n\mathfrak{m}) \to \varprojlim D_p^{\times}/(1+p^n D_p) = D_p^{\times}$.

Now let $L(p^{\infty}\mathfrak{m})$ be the union of all the fields $L(p^n\mathfrak{m})$. Then the Artin symbol defines an isomorphism

(5.0.12)
$$\begin{tikzcd}
\varprojlim_{n} I(p\mathfrak{m})/\mathrm{Prin}(p^n\mathfrak{m}) \arrow[r, "\sim"] & \varprojlim \mathrm{Gal}(L(p^n\mathfrak{m})/L) \arrow[d, equal] \\
 & \mathrm{Gal}(L(p^{\infty}\mathfrak{m})/L
\end{tikzcd}$$

$$\mathfrak{A} \mapsto (\mathfrak{A}, L(p^{\infty}\mathfrak{m})/L).$$

This allows us to view χ_p as a continuous caracter

(5.0.13) $\chi_p\colon \mathrm{Gal}(L(p^{\infty}\mathfrak{m}))/L \to D_p^{\times}$

whose relation to the "original" χ on $I(p\mathfrak{m})$ is given by

(5.0.14) $\chi_p((\mathfrak{A}, L(p^{\infty}\mathfrak{m})/L)) = \mathrm{incl}(p)(\chi(\mathfrak{A}))$.

(5.1) We now recall the HBAV's attached to a CM-field. Thus we suppose that L is a quadratic, totally imaginary extension of our fixed totally real field K, necessarily of the form $K(\sqrt{-\alpha})$ with $\alpha \in K$ totally positive. We denote by $x \mapsto \bar{x}$ the non-trivial automorphism of L/K. For any $\lambda \in \Lambda = \mathrm{Hom}(L, \overline{\mathbf{Q}})$, we denote by $\bar{\lambda} \in \Lambda$ the embedding defined by

(5.1.0) $\bar{\lambda}(x) = \lambda(\bar{x})$.

Let us *fix* a CM-type for L, i.e., a subset $\Sigma \subset \Lambda$ such that

(5.1.1) $\Sigma \cap \bar{\Sigma} = \emptyset, \quad \Sigma \cup \bar{\Sigma} = \Lambda$.

Given an element $\sigma \in \mathfrak{S} = \mathrm{Hom}(K, \overline{\mathbf{Q}})$, there is a unique element of Σ which prolongs it. This unique element of Σ we also call σ, when no confusion can arise.

In terms of Σ, we can label the ∞-types which arise from A_0-

grossencharacters of L; they are of the form

(5.1.2) $$-k \sum_{\sigma\in\Sigma} \sigma - \sum_{\sigma\in\Sigma} d(\sigma)(\sigma-\bar{\sigma})$$

with arbitrary integers k and $d(\sigma)$'s (cf. [21, 27]).

We now use the field embeddings chosen above (5.0.8):

(5.1.3) $$\begin{tikzcd} & \mathbf{C} \\ \bar{\mathbf{Q}} \arrow[ur, hook, "\mathrm{incl}(\infty)"] \arrow[dr, hook, "\mathrm{incl}(p)"'] & \\ & \mathbf{C}_p. \end{tikzcd}$$

Using incl(∞), we can view each $\sigma\in\Sigma$ as a complex embedding of L, thus obtaining an isomorphism

(5.1.4) $$\begin{gathered} L\otimes\mathbf{R} \xrightarrow{\sim} K\otimes\mathbf{C} \simeq \mathbf{C}^{\mathfrak{S}} \\ L \rightarrow (\dots, \sigma(l), \dots) \end{gathered}$$

By means of this isomorphism (5.1.4), each fractional ideal $\mathfrak{A}$ of L becomes a *lattice* in $K\otimes\mathbf{C}$. In order to polarize these lattices, we fix an element $\delta\in L$ which satisfies

(5.1.5) δ is purely imaginary, i.e., $\bar{\delta}=-\delta$, and its image in $K\otimes\mathbf{C}$ by (5.1.4) has $\mathrm{Im}(\delta)$ totally positive in $K\otimes\mathbf{R}$.

(5.1.6) the alternating $\mathcal{O}$-bilinear form

$$\langle u, v\rangle = \frac{\bar{u}\,v - u\,\bar{v}}{2\delta}$$

defines an *isomorphism* of invertible $\mathcal{O}$-modules

$$\Lambda^2_{\mathcal{O}}(\underline{\mathcal{O}}_L) \xrightarrow{\sim} \mathfrak{d}^{-1}\,\mathfrak{c}^{-1}$$

with $\mathfrak{c}$ *prime to* p.

(The strict ideal class of $\mathfrak{c}$ is a well-defined invariant of L/K. The possibility of "correcting" a δ which satisfies only (5.1.5) to one satisfying (5.1.5–6) results from the existence of a prime-to-p representative of the class of $\mathfrak{c}$.)

(5.1.7) For any fractional ideal $\mathfrak{A}$ of L, the pairing (5.1.6) defines a $\mathfrak{c}\,\mathbf{N}_{L/K}(\mathfrak{A})^{-1}$-polarization of $\mathfrak{A}$, whose *area* $a(\mathfrak{A}, \langle\,,\rangle)\in K\otimes\mathbf{R}$ is none other than $\mathrm{Im}(\delta)$.

We next wish to endow the prime-to-p fractional ideals $\mathfrak{A}$ of L with $\Gamma_{00}(p^\infty)$-structure. In order to do this in a way "adopted to incl(p)," we must assume that the CM-type Σ is "ordinary at incl(p)," in the following sense:

(5.1.8) whenever $\sigma\in\Sigma$ and $\lambda\in\bar{\Sigma}$, the p-adic valuations on L induced from the p-adic embeddings incl$(p)\circ\sigma$ and incl$(p)\circ\lambda$ are inequivalent.

The *existence* of a CM-type which is ordinary at incl(p) is equivalent to the following condition on p and L/K:

(5.1.9) every prime $\mathfrak{p}$ of K which lies over p splits completely in the quadratic extension L/K.

If this condition is satisfied, then there is a bijection

$$(5.1.10)\quad \left\{\begin{matrix}\text{CM-types } \Sigma \text{ which are} \\ \text{ordinary at } \mathrm{incl}(p)\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{the choice, for each } \mathfrak{p} \text{ of } K \text{ over } p, \\ \text{of an overlying } \mathfrak{P} \text{ of } L\end{matrix}\right\}$$

$$\Sigma \to \left\{\begin{matrix}\mathfrak{P} \text{ induced by the } p\text{-adic embeddings} \\ \mathrm{incl}(p)\circ\sigma, \quad \sigma\in\Sigma\end{matrix}\right\}$$

$$\Sigma = \left\{\lambda\in\Lambda \,\middle|\, \begin{matrix}\mathrm{incl}(p)\circ\lambda \text{ induces} \\ \text{a } \mathfrak{P}, \text{ not a } \overline{\mathfrak{P}}\end{matrix}\right\} \leftarrow \{\text{chosen } \mathfrak{P}\text{'s}\}.$$

Now suppose that Σ is ordinary at incl(p), and let $\mathfrak{P}$'s denote the chosen primes of L, and $\overline{\mathfrak{P}}$'s their conjugates. Then we have a canonical ring isomorphism

$$(5.1.11)\quad \begin{aligned}&\mathcal{O}_L\otimes\mathbf{Z}_p \xrightarrow{\sim} \mathcal{O}\otimes\mathbf{Z}_p\times\mathcal{O}\otimes\mathbf{Z}_p \\ &x\mapsto(\varphi(x),\varphi(\bar{x}))\end{aligned}$$

where φ is the map

$$(5.1.12)\quad \begin{tikzcd} \mathcal{O}_L\otimes\mathbf{Z}_p \arrow[r] & \prod(\mathfrak{P}\text{-adic completion of } \mathcal{O}_L) \arrow[d, dash, "\wr"'] \\ & \prod(\mathfrak{p}\text{-adic completion of } \mathcal{O}) \arrow[d, equal] \\ & \mathcal{O}\otimes\mathbf{Z}_p. \end{tikzcd}$$

Given a fractional ideal $\mathfrak{A}$ of L which is prime to p, we have isomorphisms

$$(5.1.13)\quad \begin{tikzcd} \mathfrak{A}\otimes\mathbf{Z}_p \arrow[r, phantom, "="] \arrow[dr, "\wr"'] & \mathcal{O}_L\otimes\mathbf{Z}_p \quad (\text{inside } L\otimes\mathbf{Z}_p) \arrow[d, dash, "\wr"', "(5.1.11)"] \\ & \mathcal{O}\otimes\mathbf{Z}_p\times\mathcal{O}\otimes\mathbf{Z}_p \end{tikzcd}$$

whence an isomorphism

$$(5.1.14)\quad \Lambda^2_{\mathcal{O}\otimes\mathbf{Z}_p}(\mathfrak{A}\otimes\mathbf{Z}_p)\simeq\mathcal{O}\otimes\mathbf{Z}_p.$$

The polarization (5.1.6), p-adically completed, gives an isomorphism

$$(5.1.15)\quad \Lambda^2_{\mathcal{O}\otimes\mathbf{Z}_p}(\mathfrak{A}\otimes\mathbf{Z}_p)\simeq(\mathfrak{d}^{-1}\mathfrak{c}^{-1}\mathbf{N}_{L/K}(\mathfrak{A}))\otimes\mathbf{Z}_p=\mathfrak{d}^{-1}\otimes\mathbf{Z}_p$$

which, combined with (5.1.14) gives

$$(5.1.16)\quad \mathcal{O}\otimes\mathbf{Z}_p\simeq\mathfrak{d}^{-1}\mathbf{Z}_p.$$

Combining (5.1.16) with the diagonal arrow of (5.1.13), we get an isomorphism

(5.1.17) $\mathfrak{A}\otimes\mathbf{Z}_p \simeq (\mathfrak{d}^{-1}\otimes\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)$.

We now define the $\Gamma_{00}(p^\infty)$-structure on $\mathfrak{A}$.

(5.1.18) $i(\mathfrak{A})\colon \mathfrak{d}^{-1}\otimes\mathbf{Z}_p \hookrightarrow \mathfrak{A}\otimes\mathbf{Z}_p$

to be "the inclusion of the first factor" in (5.1.17).

According to the theory of complex multiplication, we have the following algebraicity and good reduction results:

(5.1.19) for any CM-type Σ, and any fractional ideal $\mathfrak{A}$ of L, the $\mathfrak{c}\,\mathbf{N}_{L/K}(\mathfrak{A})^{-1}$-polarized complex HBAV $(\mathfrak{A},\langle\,,\rangle)$ comes from a $\mathfrak{c}\,\mathbf{N}_{L/K}(\mathfrak{A})^{-1}$-polarized HBAV $(X(\mathfrak{A}),\lambda(\delta))$ over $\overline{\mathbf{Q}}$ by the extension of scalars $\mathrm{incl}(\infty)\colon \overline{\mathbf{Q}}\hookrightarrow\mathbf{C}$, and $(X(\mathfrak{A}),\lambda(\delta))$ has good reduction everywhere.

(5.1.20) Suppose further that the CM-type Σ is ordinary at incl(p), and that $\mathfrak{A}$ is prime to p. Then $(X(\mathfrak{A}),\lambda(\delta),\ i(\mathfrak{A}))$ over $\overline{\mathbf{Q}}$ has good reduction at all places of residue characteristic prime to p, and at the p-adic place given by incl(p).

We now consider in more detail the situation of a CM-type Σ which is ordinary at incl(p), and an $\mathfrak{A}$ which is prime to p. Let us denote by $A\subset\overline{\mathbf{Q}}$ the valuation ring of incl(p), i.e.,

(5.1.21) $A=\{a\in\overline{\mathbf{Q}}\,|\,\mathrm{incl}(p)(a)\in D_p\}$.

Then we have a canonical ring isomorphism

(5.1.22)
$$\begin{aligned}\mathcal{O}_L\otimes A &\xrightarrow{\sim} \mathcal{O}\otimes A\times\mathcal{O}\otimes A\\ x\otimes a&\mapsto(\varphi_\Sigma(x\otimes a),\ \varphi_\Sigma(\bar{x}\otimes a))\end{aligned}$$

where φ_Σ is the ring homomorphism which sits in the commutative diagram

(5.1.23)
$$\begin{tikzcd}
x\otimes a \arrow[d] & \in & \mathcal{O}_L\otimes A \arrow[r, "\varphi_\Sigma"] \arrow[d] & \mathcal{O}\otimes A \arrow[d, hook] & \ni & y\otimes a \arrow[d] \\
(\dots,\sigma(x)\,a,\dots) & \in & A^\Sigma \arrow[r, equal] & A^{\mathfrak{S}} & \ni & (\dots,\sigma(y)\,a,\dots).
\end{tikzcd}$$

Given any $\underline{\mathcal{O}}_L\otimes A$-module M, it has a corresponding $\mathcal{O}\otimes A$ decomposition as the direct sum of two $\mathcal{O}\otimes A$ modules

(5.1.24) $M\simeq M(\Sigma)\oplus M(\bar{\Sigma})$

where

(5.1.25)
$$\begin{cases} M(\Sigma)=\{m\in M\,|\ \text{for all } x\in\underline{\mathcal{O}}_L,\ x\cdot m=\varphi_\Sigma(x)\cdot m\}\\ M(\bar{\Sigma})=\{m\in M\,|\ \text{for all } x\in\mathcal{O}_L,\ x\cdot m=\varphi_\Sigma(\bar{x})\cdot m\}.\end{cases}$$

If M is invertible as $\mathcal{O}_L\otimes A$-module, then $M(\Sigma)$ and $M(\bar{\Sigma})$ are invertible as $\mathcal{O}\otimes A$-modules.

In particular, we can apply this decomposition to

(5.1.26) $H^1_{DR}(X(\mathfrak{A})/A)$,

when we view $(X(\mathfrak{A}), \lambda(\delta), i(\mathfrak{A}))$ as a $\mathfrak{c}\,\mathbf{N}_{L/K}(\mathfrak{A})^{-1}$-polarized HBAV with $\Gamma_{00}(p^\infty)$-structure over A (cf., (5.1.20)). The key point for our applications is the following apparently trivial lemma.

(5.1.27) **Key Lemma.** *Suppose that Σ is ordinary at* incl(p), *and that $\mathfrak{A}$ is prime to p. Then the $\mathcal{O}\otimes A$-submodule $(H^1_{DR})(\bar{\Sigma})\subset H^1_{DR}(X(\mathfrak{A})/A)$ splits the Hodge filtration $\underline{\omega}\subset H^1_{DR}$; in fact, the supspace $\underline{\omega}\subset H^1_{DR}$ is none other than $(H^1_{DR})(\Sigma)$. After the extension of scalars*

incl(∞): $A\to\mathbf{C}$,

the subspace $(H^1_{DR})(\bar{\Sigma})$ becomes the antiholomorphic subspace $H^{0,1}$, and after the extension of scalars

incl(p): $A\to D_p$ = the ring of integers in $\mathbf{C}_p$,

the subspace $(H^1_{DR})(\bar{\Sigma})$ becomes the unit root subspace $U\subset H^1_{DR}$.

Proof. By construction, the action of $\mathcal{O}_L$ on Lie$(X(\mathfrak{A}))$ is through the homomorphism φ_Σ: $\mathcal{O}_L\to\mathcal{O}\otimes A$, while its action by transposition on

$$\mathrm{Lie}(X(\mathfrak{A})^t)\simeq\mathrm{Lie}(X(\mathfrak{A}))\underset{\mathcal{O}}{\otimes}\mathfrak{c}\,\mathbf{N}_{L/K}(\mathfrak{A})^{-1}$$

is by the homomorphism $x\in\mathcal{O}_L\mapsto\varphi_\Sigma(\bar{x})\in\mathcal{O}\otimes A$. Now consider the Hodge exact sequence for $(X(\mathfrak{A})$:

(5.1.28) $0\to\underline{\omega}\to H^1_{DR}\to\mathrm{Lie}(X(\mathfrak{A})^t)\to 0$

Because this is a sequence of $\mathcal{O}_L\otimes A$-modules with H^1_{DR} invertible, we have

(5.1.29) $H^1_{DR}=(H^1_{DR})(\Sigma)\oplus(H^1_{DR})(\bar{\Sigma})$

with $(H^1_{DR})(\Sigma)$ and $(H^1_{DR})(\bar{\Sigma})$ invertible $\mathcal{O}\otimes A$-modules. Because $\mathcal{O}_L$ acts on Lie$(X(\mathfrak{A})^t)$ by $x\mapsto\varphi_\Sigma(\bar{x})$, we must have

(5.1.30) $(H^1_{DR})(\Sigma)$ projects to zero in Lie$(X(\mathfrak{A})^t)$

whence

(5.1.31) $(H^1_{DR})(\Sigma)\subset\underline{\omega}$.

Because both the quotients $H^1_{DR}/(H^1_{DR})(\Sigma)$ and $H^1_{DR}/\underline{\omega}$ are *invertible* $\mathcal{O}\otimes A$-modules, we necessarily have

(5.1.32) $(H^1_{DR})(\Sigma)=\underline{\omega}$.

Thus $(H^1_{DR})(\bar{\Sigma})$ does provide a splitting of the Hodge filtration. To see that this splitting becomes the Hodge decomposition after extension of scalars to $\mathbf{C}$, we simply remark that

(5.1.33) $H^1_{DR}\otimes\mathbf{C}=(\underline{\omega}\otimes\mathbf{C})\oplus H^{0,1}$

is a $\mathcal{O}_L$-stable decomposition into two invertible $\mathcal{O}\otimes\mathbf{C}$ modules, on the first of which $\mathcal{O}_L$ operates by φ_Σ. If we take the Σ-components in (5.1.33), we get

(5.1.34) $(H^1_{DR}\otimes\mathbf{C})(\Sigma)=(\underline{\omega}\otimes\mathbf{C})(\Sigma)\oplus(H^{0,1})(\Sigma)$

In view of (5.1.32), we conclude that

(5.1.35) $(H^{0,1})(\Sigma)=0,$

and hence, by (5.1.24), that

(5.1.36) $H^{0,1}=(H^{0,1})(\bar{\Sigma}).$

Taking the $\bar{\Sigma}$-components of (5.1.33), we get

(5.1.37) $(H^1_{DR}\otimes\mathbf{C})(\bar{\Sigma})=(\underline{\omega}\otimes\mathbf{C})(\bar{\Sigma})\oplus(H^{0,1})(\bar{\Sigma}).$

Combining (5.1.32) and (5.1.36), we find

(5.1.38) $(H^1_{DR}\otimes\mathbf{C})(\bar{\Sigma})=(H^{0,1})(\bar{\Sigma})=H^{0,1}.$

The proof that $(H^1_{DR}\otimes D_p)(\bar{\Sigma})$ becomes the unit root subspace U follows in a similar formal fashion from the functoriality of the splitting

(5.1.39) $H^1_{DR}\otimes D_p=(\underline{\omega}\otimes D_p)\oplus U.$ QED

To conclude this section, we will explain how to simultaneously pick a nowhere vanishing differential on all the $X(\mathfrak{A})/A$, once we have done so on a single one. For variable $\mathfrak{A}$'s which are prime to p, we have

(5.1.40) $\mathrm{Lie}(X(\mathfrak{A}))=\mathrm{Lie}(X(\mathcal{O}_L))\otimes_{\mathcal{O}_L}\mathfrak{A}.$

Because $\mathfrak{A}$ is prime to p, it is prime to the ring A, and hence

(5.1.41) $\mathrm{Lie}(X(\mathcal{O}_L))\underset{\mathcal{O}_L}{\otimes}\mathfrak{A}=\mathrm{Lie}(X(\mathcal{O}_L))$

(equality in $\mathrm{Lie}\underset{\mathcal{O}_L}{\otimes}L$). Combining (5.1.40) and (5.1.41), we obtain a canonical identification of $\mathcal{O}_L\otimes A$-modules

(5.1.42) $\mathrm{Lie}(X(\mathfrak{A}))=\mathrm{Lie}(X(\mathcal{O}_L))$ for $\mathfrak{A}$ prime to p.

Fix a nowhere vanishing differential on $X(\mathcal{O}_L)$ over A:

(5.1.43) $\omega\colon\mathrm{Lie}(X(\mathcal{O}_L))\xrightarrow{\sim}\mathfrak{d}^{-1}\otimes A.$

For each $\mathfrak{A}$ prime to p, denote by $\omega(\mathfrak{A})$ the nowhere-vanishing differential on $X(\mathfrak{A})$ over A obtained as the composite

(5.1.44) $\omega(\mathfrak{A})\colon\mathrm{Lie}(X(\mathfrak{A}))\overset{(5.1.42)}{=}\mathrm{Lie}(X(\mathcal{O}_L))\xrightarrow[\sim]{\omega}\mathfrak{d}^{-1}\otimes A.$

After extension of scalars by $\operatorname{incl}(\infty)\colon A\to\mathbf{C}$, we obtain *another* nowhere vanishing differential $\omega_{\text{trans}}(\mathfrak{A})$ on $X(\mathfrak{A})_{\mathbf{C}}$, such that the pair $(X(\mathfrak{A})_{\mathbf{C}}, \omega_{\text{trans}}(\mathfrak{A}))$ corresponds to the lattice $\mathfrak{A}\subset K\otimes\mathbf{C}$ by the correspondence (1.4.3).

(5.1.45) **Lemma.** *There exists a unit* $\Omega=(\ldots,\Omega(\sigma),\ldots)$ *in* $(K\otimes\mathbf{C})^\times\simeq(\mathbf{C}^\times)^{\Sigma}$ *such that for every prime-to-p ideal* $\mathfrak{A}$ *of L, the differentials* $\omega(\mathfrak{A})$ *and* $\omega_{\text{trans}}(\mathfrak{A})$ *are related by*

(5.1.46) $\omega(\mathfrak{A})=\Omega\cdot\omega_{\text{trans}}(\mathfrak{A})$.

Proof. Obvious, in view of the definition of $\omega(\mathfrak{A})$'s in terms of $\omega(\mathcal{O}_L)$.

After extension of scalars by $\operatorname{incl}(p)\colon A\hookrightarrow D_p$, we obtain the canonical nowhere vanishing differential $\omega_{\text{can}}(\mathfrak{A})$ on $X(\mathfrak{A})$, attached to the $\Gamma_{00}(p^\infty)$-structure $i(\mathfrak{A})$.

(5.1.47) **Lemma.** *There exists a unit* $c=(\ldots,c(\sigma),\ldots)$ *in* $(\mathcal{C}\otimes D_p)^\times\subset(D_p^\times)^{\Sigma}$, *such that for every prime-to-p ideal* $\mathfrak{A}$ *of L, the differentials* $\omega(\mathfrak{A})$ *and* $\omega_{\text{can}}(\mathfrak{A})$ *are related by*

(5.1.48) $\omega(\mathfrak{A})=c\cdot\omega_{\text{can}}(\mathfrak{A})$.

Proof. Over A, the p-divisible groups $X(\mathfrak{A})(p^\infty)$ of the various $X(\mathfrak{A})$, $\mathfrak{A}$ prime to p, are canonically isomorphic, via

(5.1.49) $$X(\mathfrak{A})(p^\infty)=X(\mathcal{O}_L)(p^\infty)\underset{\mathcal{O}_L\otimes\mathbf{Z}_p}{\otimes}(\mathfrak{A}\otimes\mathbf{Z}_p)=X(\mathcal{O}_L)(p^\infty)$$

the last equality because $\mathcal{O}_L\otimes\mathbf{Z}_p=\mathfrak{A}\otimes\mathbf{Z}_p$ inside $L\otimes\mathbf{Z}_p$. The $\Gamma_{00}(p^\infty)$-structure $i(\mathfrak{A})$ sit in the commutative diagram.

(5.1.50)
$$\begin{tikzcd}
\mathfrak{d}^{-1}\otimes\boldsymbol{\mu}_{p^\infty} \arrow[r, hook, "i(\mathfrak{A})"] \arrow[dr, hook, "i(\mathcal{O}_L)"'] & X(\mathfrak{A})(p^\infty) \arrow[d, equal, "(5.1.49)"] \\
 & X(\mathcal{O}_L)(p^\infty)
\end{tikzcd}$$

Over D_p, we can pass to the associated formal groups

(5.1.51)
$$\begin{tikzcd}
\mathfrak{d}^{-1}\otimes\hat{\mathbf{G}}_m \arrow[r, "\overset{i(\mathfrak{A})}{\sim}"] \arrow[dr, dashed, "\sim", "i(\mathcal{O}_L)"'] & \widehat{X(\mathfrak{A})} \arrow[d, equal, "(5.1.49)"] \\
 & \widehat{X(\mathcal{O}_L)}
\end{tikzcd}$$

Passing to Lie algebras, we get

(5.1.52)
$$\begin{tikzcd}
\mathfrak{d}^{-1}\otimes D_p \arrow[r, "\overset{\operatorname{Lie}(i(\mathfrak{A}))}{\sim}"] \arrow[dr, "\operatorname{Lie}(i(\mathcal{O}_L))"'] & \operatorname{Lie}(X(\mathfrak{A})) \arrow[d, equal, "\operatorname{Lie}(5.1.49)"] \\
 & \operatorname{Lie}(X(\mathcal{O}_L))
\end{tikzcd}$$

and now the vertical equality "Lie(5.1.49)" is none other then the equality (5.1.42) used to define $\omega(\mathfrak{A})$ in terms of $\omega(\mathcal{O}_L)$. QED

5.2. In this section, we give some conventions concerning conductors of A_0-grossencharacters of L, and define a "local term" for those whose conductor divides p^∞.

As in 5.1, we fix the following data (5.2.1–3):

(5.2.1) field embeddings

$$\begin{tikzcd} & \mathbf{C} \\ \overline{\mathbf{Q}} \arrow[ur, hook, "\text{incl}\,(\infty)"] \arrow[dr, hook, "\text{incl}\,(p)"'] & \\ & \mathbf{C}_p \end{tikzcd}$$

(5.2.2) a quadratic totally imaginary extension L of K, and a CM-type Σ for L which is ordinary at incl(p).

(5.2.3) an element $\delta \in L$ satisfying (5.1.5) and (5.1.6).

We adopt the following convention concerning conductors. Given an A_0-grossencharacter χ of L, we denote by cond(χ) its *exact* conductor, and we view χ as defined on all fractional ideals of L which are prime to cond(χ). We extend χ to all *integral* ideals $\mathfrak{A}$ of L by the convention

(5.2.4) $\chi(\mathfrak{A}) = 0 \quad$ if $\mathfrak{A}$ is not prime to $\operatorname{cond}(\chi)$.

We also adopt the convention that $L(s,\chi)$ means

(5.2.5) $\displaystyle\sum_{\mathfrak{A}\,\text{integral}} \chi(\mathfrak{A})\, \mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A})^{-s}$,

the L-series for χ viewed as having its *exact* conductor.

Given an A_0-grossencharacter χ of L, we denote by $\check{\chi}$ the A_0-grossencharacter defined by

(5.2.6) $\chi(\mathfrak{A})\, \check{\chi}(\overline{\mathfrak{A}}) = (\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A}))^{-1}$;

thus its exact conductor is given by the formula

(5.2.7) $\operatorname{cond}(\check{\chi}) = \overline{\operatorname{cond}(\chi)}$.

Given an A_0-grossencharacter χ of L whose conductor divides p^∞, we can attach to it a "local term"

(5.2.8) $\operatorname{Local}(\chi;\Sigma,\delta) \in \operatorname{incl}(\infty)\overline{\mathbf{Q}} \subset \mathbf{C}$

as follows. Write the exact conductor of χ in terms of the chosen (by Σ and incl(p)) primes $\mathfrak{P}_i$ of L as

(5.2.9) $\operatorname{cond}(\chi) = \prod \mathfrak{P}_i^{a_i}\, \overline{\mathfrak{P}}_i^{b_i}$, and write the ideal $\prod \mathfrak{P}_i^{a_i}$ as a product

$$\prod \mathfrak{P}_i^{a_i} = (a)\cdot \mathfrak{B}, \qquad a\in L^\times, \quad \mathfrak{B} \text{ prime to } p. \tag{5.2.10}$$

Write the ∞-type of χ in the form

$$-k\,\Sigma\sigma - \Sigma\, d(\sigma)(\sigma-\bar{\sigma}) \tag{5.2.11}$$

and denote by χ_{finite} the unique locally constant character

$$\chi_{\text{finite}}\colon (\underline{\mathcal{O}}_L\otimes \mathbf{Z}_p)^\times \to \overline{\mathbf{Q}}^\times \tag{5.2.12}$$

such that for any $b\in L^\times$ which is prime to p, we have

$$\chi((b)) = \chi_{\text{finite}}(b)\cdot \prod_{\sigma\in\Sigma}\left(\left(\frac{1}{\sigma(b)}\right)^k \left(\frac{\sigma(\bar{b})}{\sigma(b)}\right)^{d(\sigma)}\right). \tag{5.2.13}$$

Via the canonical isomorphism (cf., 5.1.11)

$$(\mathcal{O}_L\otimes \mathbf{Z}_p)^\times \xrightarrow{\sim} (\mathcal{O}\otimes \mathbf{Z}_p)^\times \times (\mathcal{O}\otimes \mathbf{Z}_p)^\times \tag{5.2.14}$$

we view χ_{finite} as a locally constant character of $(\mathcal{O}\otimes \mathbf{Z}_p)^\times \times (\mathcal{O}\otimes \mathbf{Z}_p)^\times$. We define another locally constant character $\tilde{F}(x,y)$ of this group by the formula

$$\tilde{F}(x,y) \overset{\text{dfn}}{=} \chi_{\text{finite}}\left(\frac{1}{x}, y\right). \tag{5.2.15}$$

Using the canonical isomorphism

$$\mathcal{O}\otimes \mathbf{Z}_p \simeq \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}} \qquad \mathcal{O}_{\mathfrak{p}} = \mathfrak{p}\text{-adic completion of } \mathcal{O} \tag{5.2.16}$$

we can view $\tilde{F}(x,y)$ as a product function on $\prod\limits_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times \times \prod\limits_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}^\times$:

$$\tilde{F}(x,y) = \prod \tilde{F}_{\mathfrak{p},1}(x_{\mathfrak{p}})\, \tilde{F}_{\mathfrak{p},2}(y_{\mathfrak{p}}). \tag{5.2.17}$$

Now let $F_{\mathfrak{p},1}$ be the function on $\mathcal{O}_{\mathfrak{p}}$ defined by

$$F_{\mathfrak{p},1} = \begin{cases} \tilde{F}_{\mathfrak{p},1} & \text{on } \mathcal{O}_{\mathfrak{p}}^\times, \text{ extended by zero if } \tilde{F}_{\mathfrak{p},1} \text{ is nontrivial} \\ 1 & \text{if } \tilde{F}_{\mathfrak{p},1} \text{ is trivial.} \end{cases} \tag{5.2.18}$$

Let $F_{\mathfrak{p},2}$ be the function on $\mathcal{O}_{\mathfrak{p}}$

$$F_{\mathfrak{p},2} = \tilde{F}_{\mathfrak{p},2} \quad \text{on } \mathcal{O}_{\mathfrak{p}}^\times, \text{ extended by zero,} \tag{5.2.19}$$

and let $F(x,y)$ be the function on $(\mathcal{O}\otimes \mathbf{Z}_p)\times(\mathcal{O}\otimes \mathbf{Z}_p)$ defined by

$$F(x,y) = \prod_{\mathfrak{p}} F_{\mathfrak{p},1}(x_{\mathfrak{p}})\, F_{\mathfrak{p},2}(y_{\mathfrak{p}}). \tag{5.2.20}$$

Taking its partial Fourier transform in the first variable, we obtain

$$PF\colon (\mathfrak{d}^{-1}\otimes \mathbf{Q}_p/\mathbf{Z}_p)\times(\mathcal{O}\otimes \mathbf{Z}_p) \to \mathbf{C}. \tag{5.2.21}$$

Let us denote by δ_0 the $\mathcal{O}\otimes\mathbf{Z}_p$ basis of $\mathfrak{d}^{-1}\otimes\mathbf{Z}_p$ given by (5.1.16), and denote by $P_{\delta_0}F$ the function on $(\mathcal{O}\otimes\mathbf{Q}_p/\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)$ defined by

$$(5.2.22)\quad P_{\delta_0}F(x,y)\overset{\mathrm{dfn}}{=}PF(x\delta_0,y).$$

Via the isomorphism (5.1.11)

$$(5.2.23)\quad \mathcal{O}_L\otimes\mathbf{Z}_p\xrightarrow{\sim}\mathcal{O}\otimes\mathbf{Z}_p\times\mathcal{O}\otimes\mathbf{Z}_p$$

we have

$$(5.2.24)\quad \Big(\bigcup_{n_i\geqq 0}\prod\mathfrak{P}_i^{-n_i}\otimes\mathbf{Z}_p\Big)\Big/\mathcal{O}_L\otimes\mathbf{Z}_p\xrightarrow{\sim}(\mathcal{O}\otimes\mathbf{Q}_p/\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p),$$

by means of which we can view $P_{\delta_0}F$ as a function on

$$(5.2.25)\quad \bigcup_{n_i\geqq 0}\Big(\prod\mathfrak{P}_i^{-n_i}\Big)\otimes\mathbf{Z}_p\Big/\mathcal{O}_L\otimes\mathbf{Z}_p.$$

Returning to (5.2.10), we can now define

$$(5.2.26)\quad \mathrm{Local}(\chi;\textstyle\sum,\delta)\overset{\mathrm{dfn}}{=}\dfrac{P_{\delta_0}F(a^{-1})}{\chi(\mathfrak{B})\prod\limits_{\sigma}\left(\dfrac{1}{\sigma(a)}\right)^{k}\left(\dfrac{\sigma(\bar{a})}{\sigma(a)}\right)^{d(\sigma)}}.$$

That the right-hand member is independent of the particular factorization of $\prod\mathfrak{P}_i^{a_i}$ used in (5.2.10) follows from the fact that the function $P_{\delta_0}F$ transforms under $(\mathcal{O}_L\otimes\mathbf{Z}_p)^{\times}$ by χ_{finite} $\Big($this last fact is clear from the definition of $P_{\delta_0}F$ as the partial Fourier transform in the first variable of $\chi_{\mathrm{finite}}\left(\dfrac{1}{x},y\right)\Big)$.

(5.2.27) **Lemma.** *If* χ *unramified at all the chosen primes* $\mathfrak{P}_i$, *then*

$$(5.2.28)\quad \mathrm{Local}(\chi,\Sigma,\delta)=1.$$

Proof. We can take $a=1$, $\mathfrak{B}$ trivial. We need $P_{\delta_0}F(1)=1$. But our F on $\mathcal{O}\otimes\mathbf{Z}_p\times\mathcal{O}\otimes\mathbf{Z}_p$ is $F(x,y)=F_2(y)$, where F_2 is a character of $(\mathcal{O}\otimes\mathbf{Z}_p)^{\times}$, extended by 0. Thus PF is the product function on $(\mathfrak{d}^{-1}\otimes\mathbf{Q}_p/\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)$ given by (char. fct. of $\mathfrak{d}^{-1}\otimes\mathbf{Z}_p)\times F_2$, whence P_{δ_0} becomes the function (char. fct. of $\mathcal{O}\otimes\mathbf{Z}_p)\times F_2$ on $(\mathcal{O}\otimes\mathbf{Q}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)$, and this last function does indeed take the value 1 at the point (1.1). QED

(5.2.29) **Lemma.** *For any* χ, *we have* $\mathrm{Local}(\chi,\Sigma,\delta)\neq 0$.

Proof. In view of the definition (cf., (5.2.26)), it suffices to show that

$$P_{\delta_0}F(a^{-1})\neq 0.$$

In fact, we have the more precise

$$(5.2.30)\quad |P_{\delta_0}F(a^{-1})|=\frac{1}{\sqrt{\mathbf{N}_{L/\mathbf{Q}}(\prod\mathfrak{P}_i^{a_i})}}$$

simply because $P_{\delta_0}F(a^{-1})$ is essentially a product of gauss sums. To be more explicit, use the decomposition

$$(5.2.31) \quad K\otimes\mathbf{Z}_p \simeq \prod_{\mathfrak{p}} K_{\mathfrak{p}}, \qquad \mathcal{O}\otimes\mathbf{Z}_p \simeq \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$$

to write δ_0 as a vector $(\dots,\delta_0(\mathfrak{p}),\dots)\in\prod K_{\mathfrak{p}}$. Thus $\delta_0(\mathfrak{p})$ generates the inverse different of $K_{\mathfrak{p}}$ relative to $\mathbf{Q}_p$. The function P_{δ_0} on $(\mathcal{O}\otimes\mathbf{Q}_p/\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)$ becomes the product function on

$$(5.2.32) \quad \Bigl(\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}\otimes\mathbf{Q}_p/\mathbf{Z}_p\Bigr)\times\Bigl(\prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}\Bigr)$$

given by

$$(5.2.33) \quad \prod_{\mathfrak{p}} \hat{F}_{\mathfrak{p},1}\times\prod_{\mathfrak{p}} F_{\mathfrak{p},2},$$

where $\hat{F}_{\mathfrak{p},1}$ is the Fourier transform of $F_{\mathfrak{p},1}$ defined by

$$(5.2.34) \quad \hat{F}_{\mathfrak{p},1}(t_{\mathfrak{p}})=\frac{1}{\#(\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n)}\sum_{\substack{a_{\mathfrak{p}}\in\mathcal{O}_{\mathfrak{p}}\\ \bmod \mathfrak{p}^n}} F_{\mathfrak{p},1}(a_{\mathfrak{p}})\exp(-2\pi i\,\mathrm{trace}_{\mathfrak{p}}(a_{\mathfrak{p}}\,t_{\mathfrak{p}}\,\delta_0(\mathfrak{p})))$$

when $t_{\mathfrak{p}}\in\mathfrak{p}^{-n}\mathcal{O}$, and when $F_{\mathfrak{p},1}$ is constant on cosets modulo $\mathfrak{p}^n$.

Via (5.1.11), the element $a^{-1}\in L$ becomes an element

$$(5.2.35) \quad (\dots,\alpha_{\mathfrak{p}},\dots)\times(\dots,\beta_{\mathfrak{p}},\dots)\in\prod K_{\mathfrak{p}}\times\prod\mathcal{O}_{\mathfrak{p}}$$

with

$$(5.2.36) \quad \alpha_{\mathfrak{p}_i}\mathcal{O}_{\mathfrak{p}_i}=\mathfrak{p}_i^{-a_i}, \qquad \beta_{\mathfrak{p}}\in\mathcal{O}_{\mathfrak{p}}^{\times},$$

and we get

$$(5.2.37) \quad P_{\delta_0}F(a^{-1})=\prod_{\mathfrak{p}} \hat{F}_{\mathfrak{p},1}(\alpha_{\mathfrak{p}})\,F_{\mathfrak{p},2}(\beta_{\mathfrak{p}}).$$

In this expression, each term $F_{\mathfrak{p},2}(\beta_{\mathfrak{p}})$ is a root of unity. What about the $\hat{F}_{\mathfrak{p}_i,1}(\alpha_{\mathfrak{p}_i})$? When $a_i=0$, $F_{\mathfrak{p}_i,1}$ is the constant function 1, $\alpha_{\mathfrak{p}_i}$ is a unit, and $\hat{F}_{\mathfrak{p}_i,1}(\alpha_{\mathfrak{p}_i})=1$. When $a_i\neq 0$, then $F_{\mathfrak{p}_i,1}$ is a character of $\mathcal{O}_{\mathfrak{p}_i}^{\times}$ of exact conductor $\mathfrak{p}_i^{a_i}$, extended by zero, $\alpha_{\mathfrak{p}_i}$ lies in $\mathfrak{p}_i^{-a_i}\mathcal{O}_{\mathfrak{p}_i}^{\times}$, $\delta_0(\mathfrak{p}_i)$ generates $\mathfrak{d}_{\mathfrak{p}_i}^{-1}$, and $\hat{F}_{\mathfrak{p}_i,1}(\alpha_{\mathfrak{p}_i})$ is a standard gauss sum for a primitive character mod $\mathfrak{p}_i^{a_i}$ and hence has absolute value

$$(5.2.38) \quad |\hat{F}_{\mathfrak{p}_i,1}(\alpha_{\mathfrak{p}_i})|^2=\frac{1}{\#(\mathcal{O}_{\mathfrak{p}_i}/\mathfrak{p}_i^{a_i})}$$

The asserted formula (5.2.30) now follows, if we remember that $\mathfrak{p}_i=\mathbf{N}_{L/K}\mathfrak{P}_i$). QED

5.3. We can now state the main theorem.

(5.3.0) **Theorem.** *Suppose we are given*

(5.3.1) *field embeddings*

$$\begin{tikzcd} & \mathbf{C} \\ \overline{\mathbf{Q}} \arrow[ur, hook, "\mathrm{incl}(\infty)"] \arrow[dr, hook, "\mathrm{incl}(p)"] & \\ & \mathbf{C}_p \end{tikzcd}$$

(5.3.2) *a quadratic totally imaginary extension L of K, and a* CM-*type Σ for L which is ordinary at* $\mathrm{incl}(p)$.

(5.3.3) *an element $\delta \in L$ which is purely imaginary, has $\mathrm{Im}(\sigma(\delta)) > 0$ for all $\sigma \in \Sigma$, and such that*

$$\langle u, v \rangle = \frac{\bar{u}\, v - \bar{v}\, u}{2\delta}$$

defines an isomorphism $\Lambda^2 \mathcal{O}_L \xrightarrow{\sim} \mathfrak{d}^{-1} \mathfrak{c}^{-1}$, with $\mathfrak{c}$ prime to p.

Then there exists a D_p-valued p-adic measure μ on the galois group $\mathrm{Gal}(L(p^\infty)/L)$ of the maximal abelian unramified-outside-p extension of L, whose Mellin transform $L_\mu(\chi_p)$ provides a p-adic interpolation of the complex Hecke L-function $L_\infty(\chi)$ in the following sense. Choose a nowhere vanishing differential $\omega(\mathcal{O}_L)$ on $X(\mathcal{O}_L)$ over A, ($A =$ *the valuation ring of* $\mathrm{incl}(p)$ in $\overline{\mathbf{Q}}$), and denote by $\Omega \in (K \otimes \mathbf{C})^\times$ and $c \in (\mathcal{O} \otimes D_p)^\times$ *the constants to which it gives rise by* (5.1.46) *and* (5.1.48). *Denote by $\Omega(\sigma) \in \mathbf{C}^\times$* and *$c(\sigma) \in D_p^\times$ the σ-components of these constants. Then*

(5.3.4) *For any A_0-grossencharacter χ* of *L whose conductor divides p^∞, and whose infinity-type*

$$-k \sum_{\sigma \in \Sigma} \sigma - \sum_{\sigma \in \Sigma} d(\sigma)(\sigma - \bar{\sigma})$$

satisfies

$k \geqq 1$, all $d(\sigma) \geqq 0$,

the p-adic number (*cf.* (5.0.13))

$$Ⓐ = \frac{L_\mu(\chi_p)}{\prod\limits_{\sigma} c(\sigma)^{k+2d(\sigma)}} \quad \text{lies in } \mathrm{incl}(p)\, \overline{\mathbf{Q}}$$

(5.3.5) *For χ as in* (5.3.4), *the complex number*

$$Ⓑ = \mathit{Local}(\chi, \Sigma, \delta)\, [\text{units of } L\text{: units of } K] \cdot \prod_{\mathfrak{P}} ((1 - \check{\chi}(\overline{\mathfrak{P}}))(1 - \chi(\overline{\mathfrak{P}})))$$
$$\times \frac{(-1)^{kg} (\pi)^{\Sigma d(\sigma)}}{\sqrt{d_K} \prod\limits_{\sigma} \mathrm{Im}(\sigma(\delta))^{d(\sigma)}} \cdot \frac{\prod\limits_{\sigma} \Gamma(\mathrm{k} + \mathrm{d}(\sigma))}{\prod\limits_{\sigma} (\Omega(\sigma)^{k+2d(\sigma)}} \cdot L_\infty(\chi)$$

lies in $\mathrm{incl}(\infty)\, \overline{\mathbf{Q}}$.

(5.3.6) *For χ as in* (5.3.4–5), *we have the equality* (in $\bar{\mathbf{Q}}$)

$$\mathrm{incl}(p)^{-1}(\textcircled{A}) = \mathrm{incl}(\infty)^{-1}(\textcircled{B}).$$

(5.3.7) *Moreover, the p-adic function L_μ has the following functional equation: for any $\rho \in \mathrm{Hom}_{\mathrm{contin}}(\mathrm{Gal}(L(p^\infty)/L), D_p^\times)$, denote by $\check{\rho}$ the character defined by*

$$\check{\rho}(\mathfrak{A}) = \frac{1}{\rho(\bar{\mathfrak{A}})\,\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A})};$$

then

$$L_\mu(\check{\rho}) = \mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c}) \cdot \rho(\mathfrak{c})\, L_\mu(\rho).$$

The proof will be given in the following sections. The role of the auxiliary choice of $\omega(\mathcal{O}_L)$ is simply to provide a common ground for comparing the p-adic ω_{can} to the complex ω_{trans}: if we replace $\omega(\mathcal{O}_L)$ by $\lambda\,\omega(\mathcal{O}_L)$ with $\lambda \in (\mathcal{O} \otimes A)^\times$, we simultaneously change the constants Ω and c to $\lambda\Omega$ and λc respectively, and do not alter the validity of (5.3.5).

5.4. In this section, we will define the required p-adic measure μ on $\mathrm{Gal}(L(p^\infty)/L)$, in terms of the Eisenstein measure of 4.2. As in 4.0, we denote by $L(p^n)$ the maximal abelian extension of L of conductor dividing p^n, and by $H(p^n)$ the "ideal class group with conductor p^n." Thus $L(1)$ is the Hilbert class-field of L, and $H(1)$ is the usual ideal class group of L. Let us denote

$$(5.4.1)\quad \begin{cases} E(L) = \text{units of } L = (\mathcal{O}_L)^\times \\ E(K) = \text{units of } K = \mathcal{O}^\times \end{cases}$$

Consider the exact sequence of galois groups

$$(5.4.2)\quad 0 \to \mathrm{Gal}(L(p^\infty)/L(1)) \to \mathrm{Gal}(L(p^\infty)/L) \to \mathrm{Gal}(L(1)/L) \to 0.$$

By means of the Artin symbol, we have isomorphisms

$$(5.4.3)\quad H(1) \xrightarrow{\sim} \mathrm{Gal}(L(1)/L),$$

$$(5.4.4)\quad \varprojlim H(p^n) \xrightarrow{\sim} \mathrm{Gal}(L(p^\infty)/L),$$

$$(5.4.5)\quad \begin{tikzcd} \varprojlim (\mathrm{Ker}\colon H(p^n) \to H(1)) \arrow[r, "\sim"] \arrow[d, dash, "\wr"'] & \mathrm{Gal}(L(p^\infty)/L(1)). \\ \varprojlim_n (\mathcal{O}_L \otimes \mathbf{Z}_p)^\times / E(L)(1 + p^n\, \mathcal{O}_L \otimes \mathbf{Z}_p) \arrow[d, dash, "\wr"'] & \\ (\mathcal{O}_L \otimes \mathbf{Z}_p)^\times/\text{closure of } E(L). & \end{tikzcd}$$

We denote this last group by G_0:

$$(5.4.6)\quad G_0 = (\mathcal{O}_L \otimes \mathbf{Z}_p)^\times/\text{closure of } E(L).$$

Thus we can rewrite (5.4.2) as

(5.4.7) $0 \to G_0 \to \mathrm{Gal}(L(p^\infty)/L) \to H(1) \to 0.$

For any prime-to-p fractional ideal $\mathfrak{A}$ of L, we denote by

(5.4.8) $\mathfrak{A} \in \mathrm{Gal}(L(p^\infty)/L)$

the *image* of $\mathfrak{A}$ by the Artin symbol. Thus if we choose any $h = \#(H(1))$ prime-to-p fractional ideals $\mathfrak{A}_i^{-1}$ which are *representatives* of the ideal classes of L, then we have an explicit coset decomposition:

(5.4.9) $\mathrm{Gal}(L(p^\infty)/L) = \coprod \mathfrak{A}_i^{-1} \cdot G_0.$

Thus Gal is the disjoint union of h pieces, each of which is explicitly isomorphic to G_0. By means of this decomposition, it is equivalent to give a measure μ on $\mathrm{Gal}(L(p^\infty)/L)$, or to give h distinct measures $\mu(\mathfrak{A}_i)$ on G_0, as follows:

(5.4.10) $\int\limits_{\mathrm{Gal}} f d\mu = \sum\limits_{i=1}^{h} \int\limits_{G_0} f(\mathfrak{A}_i^{-1} g_0) d\mu(\mathfrak{A}_i)(g_0).$

In order to construct the measures $\mu(\mathfrak{A}_i)$ on G_0, we proceed as follows. Recall the group (cf., 4.2)

(5.4.11) $G = (\mathcal{O} \otimes \mathbf{Z}_p)^\times \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times /$closure of $E(K)$ embedded diagonally

By means of the $(\mathfrak{P}, \bar{\mathfrak{P}})$ isomorphism (cf., 5.1.12)

(5.4.12) $(\mathcal{O}_L \otimes \mathbf{Z}_p)^\times \xrightarrow{\sim} (\mathcal{O} \otimes \mathbf{Z}_p)^\times \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times$

we have

(5.4.13) $G = (\mathcal{O}_L \otimes \mathbf{Z}_p)^\times /$closure of $E(K)$

whence a short exact sequence

(5.4.14) $0 \to E(L)/E(K) \to G \to G_0 \to 0,$

by means of which we can and will view continuous functions on G_0 as being *those* continuous functions on G which are *invariant* under the (finite) group $E(L)/E(K)$. We will first construct measures $\mu(\mathfrak{A}_i)$ on G, then take their direct images on G_0, i.e., given a continuous function f on G_0, and the forthcoming $\mu(\mathfrak{A}_i)$ on G, we define $\mu(\mathfrak{A}_i)$ on G_0 by the formula

(5.4.15) $\int\limits_{G_0} f d\mu(\mathfrak{A}_i) = \int\limits_{G} f d\mu(\mathfrak{A}_i)$

where the "f" on the right is "f viewed as $E(L)/E(K)$-invariant function on G."

The measure $\mu(\mathfrak{A}_i)$ on G is constructed as follows. We have already constructed, in 5.1, a $\mathfrak{c}\mathbf{N}_{L/K}(\mathfrak{A}_i)^{-1}$-polarized HBAV with $\Gamma_{00}(p^\infty)$-structure $(X(\mathfrak{A}_i), \lambda(\delta), i(\mathfrak{A}_i))$ over A, the valuation ring of incl(p) in $\mathbf{C}_p$. Extending its scalars to $D_p =$ integers in $\mathbf{C}_p$, we get a point at which we can evaluate the Eisenstein measure $\mu_{\mathfrak{c}\mathbf{N}_{L/K}(\mathfrak{A}_i)^{-1}}$ of 4.2, to obtain the desired D_p-valued measure on G:

$$(5.4.16)\quad \mu(\mathfrak{A}_i)\overset{\text{dfn}}{=}\mu_{\mathfrak{c}\mathbf{N}_{L/K}(\mathfrak{A}_i)^{-1}}((X(\mathfrak{A}_i),\lambda(\delta),i(\mathfrak{A}))_{D_p}).$$

5.5. In this section, we will show that when we construct the measure μ, using a *fixed* set $\mathfrak{A}_i^{-1}$ of prime-to-p representatives for the ideal classes, then parts (5.3.4–5–6) of the theorem are true.

Let χ be a fixed A_0-grossencharacter of L of conductor dividing p^∞, whose infinity type is

$$(5.5.1)\quad -k\sum\sigma-\sum d(\sigma)(\sigma-\bar{\sigma}),\qquad \text{with } k\geqq 1,\qquad \text{all } d(\sigma)\geqq 0.$$

Viewing χ as a D_p-valued character of $\operatorname{Gal}(L(p^\infty)/L)$, and restricting it to G_0, we get an expression

$$(5.5.2)\quad \chi((\alpha))=\chi_{\text{finite}}(\alpha)\cdot\frac{\prod\limits_\sigma \sigma(\bar{\alpha})^{d(\sigma)}}{\prod\limits_\sigma \sigma(\alpha)^{k+d(\sigma)}}$$

for any $\alpha\in(\mathcal{O}_L\otimes\mathbf{Z}_p)^\times$, where χ_{finite} is a *locally constant* character of $(\mathcal{O}_L\otimes\mathbf{Z}_p)^\times$ which satisfies

$$(5.5.3)\quad \chi_{\text{finite}}(e)\cdot\frac{\prod\limits_\sigma \sigma(\bar{e})^{d(\sigma)}}{\prod\limits_\sigma \sigma(e)^{k+d(\sigma)}}=1\qquad \text{for all } e\in E(L).$$

In terms of the $(\mathfrak{P},\bar{\mathfrak{P}})$-isomorphism (cf., 5.1.12)

$$(5.5.4)\quad (\mathcal{O}_L\otimes\mathbf{Z}_p)^\times \xrightarrow{\sim} (\mathcal{O}\otimes\mathbf{Z}_p)^\times\times(\mathcal{O}\otimes\mathbf{Z}_p)^\times,$$

we get the expression

$$(5.5.5)\quad \chi(x,y)=\chi_{\text{finite}}(x,y)\cdot\frac{\prod\limits_\sigma \sigma(y)^{d(\sigma)}}{\prod \sigma(x)^{k+d(\sigma)}}.$$

Let $\mathbf{F}(x,y)$ be the character of $(\mathcal{O}\otimes\mathbf{Z}_p)^\times\times(\mathcal{O}\otimes\mathbf{Z}_p)^\times$ defined by

$$\begin{aligned}(5.5.6)\quad \mathbf{F}(x,y)&=\frac{1}{\mathbf{N}(x)}\cdot\chi\left(\frac{1}{x},y\right)\\ &=\chi_{\text{finite}}\left(\frac{1}{x},y\right)\cdot\mathbf{N}(x)^{k-1}\cdot\prod_\sigma\sigma(x\,y)^{d(\sigma)}.\end{aligned}$$

According to the Definition (4.2.5) of the Eisenstein measure, we have, for any prime-to-p $\mathfrak{c}$, an equality in $V(\mathfrak{c},D_p)$:

$$\begin{aligned}(5.5.7)\quad \int_G \chi(x,y)\,d\mu_{\mathfrak{c}}&=G_{1,\mathbf{F}}=G_{1,\chi_{\text{finite}}\left(\frac{1}{x},y\right)\cdot\mathbf{N}(x)^{k-1}\prod\limits_\sigma\sigma(x\,y)^{d(\sigma)}}\\ (\text{by } 3.4.8)&=G_{k,\chi_{\text{finite}}\left(\frac{1}{x},y\right)\cdot\prod\limits_\sigma\sigma(x\,y)^{d(\sigma)}}\\ (\text{by } 3.4.14)&=\Big(\prod_\sigma\theta(\sigma)^{d(\sigma)}\Big)\Big(G_{k,\chi_{\text{finite}}\left(\frac{1}{x},y\right)}\Big).\end{aligned}$$

By definition of the measures $\mu(\mathfrak{A}_i)$, we thus have

$$(5.5.8)\quad \int_{G_0} \chi\, d\mu(\mathfrak{A}_i) = ((\prod_{\sigma} (\theta(\sigma))^{d(\sigma)})(G_{k,\chi_{\text{finite}}(\frac{1}{x},y)}))(X(\mathfrak{A}_i), \lambda(\delta), i(\mathfrak{A}_i)).$$

Referring to the definition of μ on $\operatorname{Gal}(L(p^\infty)/L)$ in terms of the $\mu(\mathfrak{A}_i)$, we find

$$(5.5.9)\quad \begin{aligned} \int_{\operatorname{Gal}(L(p^\infty)/L)} \chi\, d\mu &= \sum_{i=1}^{h} \int_{G_0} \chi(\mathfrak{A}_i^{-1} g_0)\, d\mu(\mathfrak{A}_i)(g_0) \\ &= \sum_{i=1}^{h} \chi(\mathfrak{A}_i)^{-1} \int_{G} \chi\, d\mu(\mathfrak{A}_i) \\ &= \sum_{i=1}^{h} \chi(\mathfrak{A}_i)^{-1}((\prod \theta(\sigma)^{d(\sigma)})(G_{k,\chi_{\text{finite}}(\frac{1}{x},y)}))(X(\mathfrak{A}_i), \lambda(\delta), i(\mathfrak{A}_i)). \end{aligned}$$

The algebraicity assertion (5.3.4) now follows directly from (3.5.8), (5.1.27) and (5.1.48), because the constants $c(\sigma)$ in (5.3.4) are none other than the constants $\sigma(c)$ occurring in (3.5.8).

In the notations of (3.5.7) and (3.5.8), let us denote

$$(5.5.10)\quad \begin{cases} x(\mathfrak{A}_i) = (X(\mathfrak{A}_i), \lambda(\delta), i(\mathfrak{A}_i)) \quad \text{over } A \\ \operatorname{Split}(\mathfrak{A}_i) = (H^1_{DR})(\bar{\Sigma}). \end{cases}$$

Then we can rewrite (5.5.9) as

$$(5.5.11)\quad \begin{aligned} &\frac{\int_{\text{Gal}} \chi\, d\mu}{\prod_{\sigma} c(\sigma)^{k+2d(\sigma)}} \\ &= \sum_{i=1}^{h} \chi(\mathfrak{A}_i)^{-1}\, \vartheta(\sum d(\sigma)\sigma, x(\mathfrak{A}_i), \omega(\mathfrak{A}_i), \operatorname{Split}(\mathfrak{A}_i))\, G_{k,\chi_{\text{finite}}(\frac{1}{x},y)}. \end{aligned}$$

Let us denote by $\tilde{F}$ the locally constant character of $(\mathcal{O} \times \mathbf{Z}_p)^\times \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times$

$$(5.5.12)\quad \tilde{F}(x,y) \overset{\text{dfn}}{=} \chi_{\text{finite}}\left(\frac{1}{x}, y\right),$$

extended by zero to all of $\mathcal{O} \otimes \mathbf{Z}_p \times \mathcal{O} \otimes \mathbf{Z}_p$.

Using (3.5.6), we find a *complex* expression for the algebraic number

$$(5.5.13)\quad \vartheta(\sum d(\sigma)\sigma, x(\mathfrak{A}_i), \omega(\mathfrak{A}_i), \operatorname{Split}(\mathfrak{A}_i))(G_{k,\chi_{\text{finite}}(\frac{1}{x},y)}) \in \bar{\mathbf{Q}}$$

= the value at $s=0$ off the entire function

$$\frac{1}{\prod_{\sigma} (\Omega(\sigma))^{k+2d(\sigma)}} \left(\frac{1}{\sqrt{\mathrm{d}_K}}\right) (-1)^{kg} \prod_{\sigma} \left(\frac{\pi^{d(\sigma)} \Gamma(k+d(\sigma)+s)}{\operatorname{Im}(\sigma(\delta))^{d(\sigma)}}\right)$$

$$\times \sum_{\substack{a \in \mathfrak{A}_i\left[\frac{1}{p}\right] \cap PV_p(\mathfrak{A}_i) \\ a \bmod E(K)}} \frac{P\tilde{F}(a) \cdot \prod_{\sigma} (\sigma(\bar{a}))^{d(\sigma)}}{\prod_{\sigma} (\sigma(a))^{k+2d(\sigma)} |\mathbf{N}_{L/\mathbf{Q}}(a)|^s}.$$

Notice that the terms in the series above are in fact *invariant under* $E(L)$, so that the sum is $[E(L): E(K)]$ times the same sum taken over "a's modulo $E(L)$."

Thus to establish both (5.3.5) and (5.3.6), we are reduced to proving the following "purely complex" statement.

(5.5.14) $\operatorname{Local}(\chi, \Sigma, \delta) \cdot \prod ((1-\check{\chi}(\overline{\mathfrak{P}}))(1-\chi(\overline{\mathfrak{P}}))) \cdot L(0, \chi)$

$=$ the value at $s=0$ of

$$\sum_{i=1}^{h} \chi(\mathfrak{A}_i)^{-1} \sum_{\substack{a \in \mathfrak{A}_i\left[\frac{1}{p}\right] \cap PV_p(\mathfrak{A}_i) \\ a \bmod E(L)}} \frac{P\tilde{F}(a) \cdot \prod_\sigma \sigma(\bar{a})^{d(\sigma)}}{\prod \sigma(a)^{k+d(\sigma)} |\mathbf{N}_{L/\mathbf{Q}}(a)|^s}.$$

This is a routine but tedious computation. Write the exact conductor of χ

(5.5.15) $\operatorname{Cond}(\chi) = \prod_i \mathfrak{P}_i^{a_i} \overline{\mathfrak{P}}_i^{b_i}$,

and write the ideal $\prod \mathfrak{P}_i^{a_i}$ as a product:

(5.5.16) $\prod \mathfrak{P}_i^{a_i} = (\alpha)\mathfrak{B} \quad \alpha \in L^\times, \quad \mathfrak{B}$ prime to p.

To fix ideas, let us first treat the case

(5.5.17) all $a_i \geqq 1$; i.e., χ is ramified at all the $\mathfrak{P}_i$.

Then the function $P\tilde{F}$ is supported in

(5.5.18) $(\prod \mathfrak{P}_i^{-a_i})\mathfrak{A}_i = (\alpha^{-1})\mathfrak{B}^{-1}\mathfrak{A}_i$,

where it is given by (in the notation of (5.2.22))

(5.5.19) $P\tilde{F}(\alpha^{-1}a) = P_{\delta_0}F(\alpha^{-1})\,\chi_{\text{finite}}(a) \quad$ for $a \in \mathfrak{B}^{-1}\mathfrak{A}_i$.

Thus we can rewrite

$$(5.5.20) \quad \sum_{i=1}^{h} \chi(\mathfrak{A}_i)^{-1} \sum_{\substack{a \in \mathfrak{A}_i\left[\frac{1}{p}\right] \cap PV_p(\mathfrak{A}_i) \\ a \bmod E(L)}} \frac{P\tilde{F}(a) \prod_\sigma \sigma(\bar{a})^{d(\sigma)}}{\prod_\sigma \sigma(a)^{k+d(\sigma)} |\mathbf{N}_{L/\mathbf{Q}}(a)|^s}$$

$$(\text{by } 5.5.18) \quad = \sum_{i=1}^{h} \chi(\mathfrak{A}_i)^{-1} \sum_{\substack{a \in \mathfrak{B}^{-1}\mathfrak{A}_i \\ a \bmod E(L)}} \frac{P\tilde{F}(\alpha^{-1}a) \prod_\sigma \sigma(\bar{\alpha}^{-1}\bar{a})^{d(\sigma)}}{\prod \sigma(\alpha^{-1}a)^{k+d(\sigma)} \cdot |\mathbf{N}_{L/\mathbf{Q}}(\alpha^{-1}a)|^s}$$

(by 5.5.19)

$$= \frac{P_{\delta_0}F(\alpha^{-1}) |\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s}{\left(\dfrac{\prod_\sigma \sigma(\bar{\alpha})^{d(\sigma)}}{\prod_\sigma \sigma(\alpha)^{k+d(\sigma)}}\right)} \sum_{i=1}^{h} \chi(\mathfrak{A}_i)^{-1} \sum_{\substack{a \in \mathfrak{B}^{-1}\mathfrak{A}_i \\ a \bmod E(L)}} \frac{\chi_{\text{finite}}(a) \prod_\sigma \sigma(\bar{a})^{d(\sigma)}}{\prod \sigma(a)^{k+d(\sigma)} |\mathbf{N}_{L/\mathbf{Q}}(a)|^s}$$

(by 5.2.26)

$$=\mathrm{Local}(\chi.\Sigma,\delta)\cdot\chi(\mathfrak{B})|\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s\sum_{i=1}^{h}\chi(\mathfrak{A}_i)^{-1}\sum_{\substack{a\in\mathfrak{B}^{-1}\mathfrak{A}_i\\ a\bmod E(L)}}\frac{\chi((a))}{|\mathbf{N}_{L/\mathbf{Q}}(a)|^s}$$

$$=\mathrm{Local}(\chi,\Sigma,\delta)|\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s\sum_{i=1}^{h}\chi(\mathfrak{B}^{-1}\mathfrak{A}_i)^{-1}\sum_{\substack{a\in\mathfrak{B}^{-1}\mathfrak{A}_i\\ \bmod E(L)}}\frac{\chi((a))}{|\mathbf{N}_{L/\mathbf{Q}}(a)|^s}$$

$$=\mathrm{Local}(\chi,\Sigma,\delta)|\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s L(s,\chi\text{ viewed as having conductor a power of }p)$$

$$=\mathrm{Local}(\chi,\Sigma,\delta)\cdot|\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s\times L(s,\chi)\times(\text{Euler factors at primes over }p).$$

Because we assumed χ to be ramified at all the $\mathfrak{P}$'s this is

$$=\mathrm{Local}(\chi,\Sigma,\delta)|\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s\cdot L(s,\chi)\cdot\prod_{\mathfrak{P}}(1-\chi(\overline{\mathfrak{P}})\,\mathbf{N}\,\overline{\mathfrak{P}}^{-s}).$$

To conclude the proof of (5.5.14) in this case, we have only to remark that under the assumption that χ is ramified at all the $\mathfrak{P}$'s, $\check{\chi}$ is ramified at all the $\overline{\mathfrak{P}}$'s, and hence

$$\check{\chi}(\overline{\mathfrak{P}})=0.$$

In the general case, when χ is unramified at *some* of the $\mathfrak{P}$'s (those $\mathfrak{P}_j$ with $a_j=0$ in 5.5.15) the functions $\tilde{F}$ and F (in the notations of (5.5.22)) no longer coincide. Let us denote by

(5.5.21) $\chi_1=$the grossencharacter χ on ideals prime to p, extended by zero to other integral ideals

$\chi_2=$the grossencharacter χ on ideals prime to those $\mathfrak{P}_i$ where $a_i\geqq 1$ and prime to all the $\overline{\mathfrak{P}}_k$'s, extended by zero to other integral ideals.

Thus we have

(5.5.22) $$L(s,\chi_2)=\sum_{\substack{\mathfrak{A}\text{ integral}\\ \text{prime to cond}(\chi)\\ \textit{and}\text{ to all }\overline{\mathfrak{P}}}}\chi(\mathfrak{A})\,\mathbf{N}\,\mathfrak{A}^{-s}=\left(\prod_{\mathfrak{P}}\left(\frac{1}{1-\frac{\chi(\mathfrak{P})}{\mathbf{N}(\mathfrak{P})^s}}\right)\right)L(s,\chi_1)$$

and hence

(5.5.23) $$L(s,\chi_2)=(\prod_{\mathfrak{P}}(1-\chi(\overline{\mathfrak{P}})\,\mathbf{N}\,\overline{\mathfrak{P}}^{-s}))\,L(s,\chi)$$

Thus we can *reformulate* (5.5.14) in terms of χ_2. It becomes

(5.5.24) $$\mathrm{Local}(\chi,\Sigma,\delta)\prod_{\mathfrak{P}}(1-\check{\chi}(\overline{\mathfrak{P}}))\,L(0,\chi_2)$$

$=$the value at $s=0$ of

$$\sum_{i=1}^{h}\chi(\mathfrak{A}_i)^{-1}\sum_{\substack{a\in\mathfrak{A}_i\left[\frac{1}{p}\right]\cap PV_p(\mathfrak{A}_i)\\ a\bmod E(L)}}\frac{P\tilde{F}(a)\prod_{\sigma}\sigma(\bar{a}))^{d(\sigma)}}{\prod_{\sigma}\sigma(a)^{k+d(\sigma)}|\mathbf{N}_{L/\mathbf{Q}}(a)|^s}\quad .$$

The function $P\tilde{F}$ is the partial Fourier transform of $\chi\left(\frac{1}{x},y\right)$, viewed as supported on $(\mathcal{O}\otimes\mathbf{Z}_p)^\times \times (\mathcal{O}\otimes\mathbf{Z}_p)^\times$ (and then extended by zero to $(\mathcal{O}\otimes\mathbf{Z}_p)\times(\mathcal{O}\otimes\mathbf{Z}_p)$). The function PF is the partial Fourier transform of $\chi\left(\frac{1}{x}, y\right)$, viewed as supported in

$$\Big(\prod_{a_i\geqq 1} \mathcal{O}_{\mathfrak{p}_i}^\times \times \prod_{a_j=0} \mathcal{O}_{\mathfrak{p}_j}\Big)\times(\mathcal{O}\otimes\mathbf{Z}_p)^\times$$

after being extended from $(\mathcal{O}\otimes\mathbf{Z}_p)^\times \times(\mathcal{O}\otimes\mathbf{Z}_p)^\times$ by (5.2.18–20).

The function $P\tilde{F}$ on $\mathfrak{A}_i\left[\frac{i}{p}\right]\cap PV_p(\mathfrak{A}_i)$ is supported in

$$(5.5.25)\quad \Big(\prod_{a_i\geqq 1}\mathfrak{P}_i^{-a_i}\Big)\Big(\prod_{a_j=0}\mathfrak{P}_j^{-1}\Big)\mathfrak{A}_i$$
$$=(\alpha^{-1})\mathfrak{B}^{-1}\Big(\prod_{a_j=0}\mathfrak{P}_j^{-1}\Big)\mathfrak{A}_i.$$

Because the ideal $\mathfrak{B}^{-1}(\prod_{a_j=0}\mathfrak{P}_j^{-1})\mathfrak{A}_i$ is prime to those $\mathfrak{P}_i$ with $a_i\neq 0$, and to all $\overline{\mathfrak{P}}_k$, the function $\chi_{2,\mathrm{finite}}$ (the ratio of χ_2 to its infinity type) is defined on this ideal. A routine Fourier transform computation shows that

(5.5.26) for $a\in\mathfrak{B}^{-1}(\prod_{a_j=0}\mathfrak{P}_j^{-1})\mathfrak{A}_i$, we have

$$P\tilde{F}(\alpha^{-1}a)=P_{\delta_0}F(\alpha^{-1})\chi_{2,\mathrm{finite}}(a)\cdot\prod_{a_j=0}\widehat{\mathrm{char}(\mathfrak{P}_j)}(a)$$

where

$$(5.5.27)\quad \widehat{\mathrm{char}(\mathfrak{P}_j)}(a)=\begin{cases}1-\dfrac{1}{\mathbf{N}\,\mathfrak{P}_j} & \text{if } \mathrm{ord}_{\mathfrak{P}_j}(a)\geqq 0\\[2ex] -\dfrac{1}{\mathbf{N}\,\mathfrak{P}_j} & \text{if } \mathrm{ord}_{\mathfrak{P}_j}(a)=-1.\end{cases}$$

Thus we may rewrite the second member of (5.5.24) as

$$(5.5.28)\quad \sum_{i=1}^{h}\chi(\mathfrak{A}_i)^{-1}\sum_{\substack{a\in\mathfrak{B}^{-1}(\prod_{a_j=0}\mathfrak{P}_j^{-1})\mathfrak{A}_i\\ a \bmod E(L)}}\frac{P\tilde{F}(\alpha^{-1}a)\prod_\sigma\sigma(\bar{\alpha}^{-1}\bar{a})^{d(\sigma)}}{\prod_\sigma\sigma(\alpha^{-1}a)^{k+d(\sigma)}|\mathbf{N}_{L/\mathbf{Q}}(\alpha^{-1}a)|^s}$$

$$(5.5.29)\quad =\frac{P_{\delta_0}F(\alpha^{-1})|\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s}{\left(\dfrac{\prod_\sigma\sigma(\bar{\alpha})^{d(\sigma)}}{\prod_\sigma\sigma(\alpha)^{k+d(\sigma)}}\right)}\Sigma\times(\mathfrak{A}_i)^{-1}\sum_{\substack{a\in\mathfrak{B}^{-1}(\prod_{a_j=0}\mathfrak{P}_j^{-1})\mathfrak{A}_i\\ a \bmod E(L)}}\frac{\chi_2((a))}{|\mathbf{N}_{L/\mathbf{Q}}(a)|^s}\prod_{a_j=0}\widehat{\mathrm{char}(\mathfrak{P}_j)}(a)$$

$$(5.5.30)\quad =\mathrm{Local}(\chi,\Sigma,\delta)|\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s\sum_{i=1}^{h}\chi(\mathfrak{B}^{-1}\mathfrak{A}_i)^{-1}$$
$$\cdot\sum_{\substack{a\in(\prod_{a_j=0}\mathfrak{P}_j^{-1})\mathfrak{B}^{-1}\mathfrak{A}_i\\ a \bmod E(L)}}\frac{\chi_2((a))}{|\mathbf{N}_{L/\mathbf{Q}}(a)|^s}\prod_{a_j=0}\widehat{\mathrm{char}(\mathfrak{P}_j)}(a).$$

To proceed further, we extend the functions $\widehat{\mathrm{char}(\mathfrak{P}_j)}$ to the set $\mathbf{I}$ of fractional ideals I of L of the form

$$(5.5.31)\quad I=\Big(\prod_{a_j\neq 0}\mathfrak{P}_j^{-1}\Big)\mathfrak{A}\qquad \begin{array}{l}\text{with } \mathfrak{A} \text{ an integral ideal, prime to those } \mathfrak{P}_i\\ \text{with } a_i\neq 0 \text{ and to all } \overline{\mathfrak{P}}_k\end{array}$$

given such an ideal $I\in\mathbf{I}$, we define

$$(5.5.32)\quad \widehat{\mathrm{char}(\mathfrak{P}_j)}(I)=\begin{cases}1-\dfrac{1}{\mathbf{N}\,\mathfrak{P}_j} & \text{if } I \text{ is } \mathfrak{P}_j\text{-integral}\\ -\dfrac{1}{\mathbf{N}\,\mathfrak{P}_j} & \text{if not.}\end{cases}$$

Thus we can rewrite (5.5.30) as

$$(5.5.33)\quad \mathrm{Local}(\chi,\Sigma,\delta)|\mathbf{N}_{L/\mathbf{Q}}(\alpha)|^s\sum_{i=1}^{h}|\mathbf{N}(\mathfrak{B}^{-1}\mathfrak{A}_i)|^s$$
$$\cdot\sum_{\substack{I\in\mathbf{I}\\ I\sim\mathfrak{B}\mathfrak{A}_i^{-1}}}\frac{\chi_2(I)}{\mathbf{N}(I)^s}\prod_{a_j\neq 0}\widehat{\mathrm{char}(\mathfrak{P}_j)}(I).$$

Because we are only interested in the value of (5.5.31) at $s=0$, we may put the "first two" s's in it to zero, and work with

$$(5.5.34)\quad \mathrm{Local}(\chi,\Sigma,\delta)\sum_{I\in\mathbf{I}}\frac{\chi_2(I)}{\mathbf{N}(I)^s}\prod_{a_j\neq 0}\widehat{\mathrm{char}(\mathfrak{P}_j)}(I).$$

Now let $\mathbf{I}(p)$ denote the set of prime-to-p integral ideals; then every element of $\mathbf{I}$ can be written uniquely in the form

$$(5.5.35)\quad I=\Big(\prod_{a_j=0}\mathfrak{P}_j^{n_j}\Big)I_0\qquad \text{with } I_0\in\mathbf{I}(p),\quad \text{and all } n_j\geqq -1.$$

We can factor (5.5.34) as the product

$$(5.5.36)\quad \mathrm{Local}(\chi,\Sigma,\delta)\cdot\sum_{I_0\in\mathbf{I}(p)}\frac{\chi(I_0)}{\mathbf{N}(I_0)^s}$$
$$\times\prod_{a_j=0}\sum_{n\geqq -1}\frac{\chi_2(\mathfrak{P}_j)^n}{\mathbf{N}(\mathfrak{P}_j)^{ns}}\widehat{\mathrm{char}(\mathfrak{P}_j)}((\mathfrak{P}_j)^n).$$

We easily evaluate the inner sums:

$$(5.5.37)\quad \sum_{n=-1}^{\infty}\frac{\chi_2(\mathfrak{P}_j)^n}{\mathbf{N}(\mathfrak{P}_j)^{ns}}\widehat{\mathrm{char}(\mathfrak{P}_j)}((\mathfrak{P}_j)^n))$$
$$=\frac{-1}{\mathbf{N}(\mathfrak{P}_j)}\sum_{n\geqq -1}\frac{\chi_2(\mathfrak{P}_j)^n}{\mathbf{N}(\mathfrak{P}_j)^{ns}}+\sum_{n\geqq 0}\frac{\chi_2(\mathfrak{P}_j)^n}{\mathbf{N}(\mathfrak{P}_j)^{ns}}$$
$$=\left(\frac{-1}{\mathbf{N}(\mathfrak{P}_j)}\cdot\frac{\chi_2(\mathfrak{P}_j)^{-1}}{\mathbf{N}(\mathfrak{P}_j)^{-s}}+1\right)\left(\sum_{n\geqq 0}\frac{\chi_2(\mathfrak{P}_j)^n}{\mathbf{N}(\mathfrak{P}_j)^{ns}}\right)$$
$$=\left(1-\frac{\mathbf{N}(\mathfrak{P}_j)^s}{\chi_2(\mathfrak{P}_j)\mathbf{N}(\mathfrak{P}_j)}\right)\left(\frac{1}{1-\dfrac{\chi_2(\mathfrak{P}_j)}{\mathbf{N}(\mathfrak{P}_j)^s}}\right)$$
$$=(1-\check{\chi}_2(\overline{\mathfrak{P}}_j)\mathbf{N}(\mathfrak{P}_j)^s)(1-\chi_2(\mathfrak{P}_j)\mathbf{N}(\mathfrak{P}_j)^{-s})^{-1}.$$

Also, the first sum in (5.5.34) is just $L(s, \chi_1)$:

$$(5.5.38)\quad \sum_{I_0 \in \mathbf{I}(p)} \frac{\chi(I_0)}{\mathbf{N}(I_0)^s} = L(s, \chi_1)$$

Combining (5.5.35) and (5.3.36), we see that (5.5.34) is just

$$(5.5.39)\quad \mathrm{Local}(\chi, \Sigma, \delta) \cdot L(s, \chi_1) \prod_{a_j = 0} \left(\frac{1 - \check{\chi}_2(\bar{\mathfrak{P}}_j)\, \mathbf{N}(\mathfrak{P}_j)^s}{1 - \chi_2(\mathfrak{P}_j)\, \mathbf{N}(\mathfrak{P}_j)^{-s}} \right)$$

whose value at $s = 0$ is

$$(5.5.40)\quad \mathrm{Local}(\chi, \Sigma, \delta) (\prod (1 - \check{\chi}(\bar{\mathfrak{P}}))) \left(\prod \left(\frac{1}{1 - \chi(\mathfrak{P})} \right) \right) L(0, \chi_1)$$

$$(\text{by } 5.5.22)\quad = \mathrm{Local}(\chi . \Sigma, \delta) \cdot \prod (1 - \check{\chi}(\bar{\mathfrak{P}})) \cdot L(0, \chi_2). \quad \text{QED}$$

5.6. In this section, we complete the proof of Theorem 5.30 by veryfying the functional equation (5.3.7). The key point is that the measure μ on $\mathrm{Gal}(L(p^\infty)/L)$ is *independent* of the auxiliary choice of representatives $\mathfrak{A}_i$ used in its definition (the independence because, by (4.1.2), the measure μ is uniquely determined by the knowledge of all $L_\mu(\chi)$ when χ runs over *all* p-power conductor grossencharacters of *any* fixed infinity type, and this knowledge is given, rather abundantly, by (5.3.6)).

Let ρ be a $D_p^\times$-valued character of $\mathrm{Gal}(L(p^\infty)/L)$. For each prime-to-$p$ fractional ideal $\mathfrak{A}$ of L, let $\rho_{\mathfrak{A}}$ denote the $D_p^\times$-valued function on $G_0 = (\mathcal{O}_L \otimes \mathbf{Z}_p)^\times /$ (closure of $E(L)$) defined by

$$(5.6.1)\quad \rho_{\mathfrak{A}}(g_0) = \rho(\mathfrak{A}^{-1} g_0) = \rho(\mathfrak{A})^{-1} \rho(g_0).$$

For any function H on G_0, we have already defined (cf., (4.2.12)) another function $\check{H}$ on G_0 by the formula

$$(5.6.2)\quad \check{H}(a) = \frac{1}{\mathbf{N}_{L/\mathbf{Q}}(a)} \cdot H(1/\bar{a}) \qquad \text{for } a \in (\mathcal{O}_L \otimes \mathbf{Z}_p)^\times.$$

The functional equation for the Eisenstein measure (cf., (4.2.14)) gives

$$(5.6.3)\quad \int_{G_0} \check{H}\, d\mu(\mathfrak{A}) \overset{\mathrm{dfn}}{=} \int_{G} \check{H}\, d\mu_{\mathfrak{c}\mathbf{N}_{L/K}(\mathfrak{A})^{-1}}(X(\mathfrak{A}, \lambda(\delta), i(\mathfrak{A}))$$

$$(\text{by } 4.2.14) = \mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\, \mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A})^{-1} \int_{G} H\, d\mu_{\mathfrak{c}^{-1}\mathbf{N}_{L/K}(\mathfrak{A})}(X(\mathfrak{A})^t, \lambda(\delta)^t, i(\mathfrak{A})^t).$$

But the dual of $(X(\mathfrak{A}), \lambda(\delta), i(\mathfrak{A}))$, as $\mathfrak{c}\mathbf{N}_{L/K}(\mathfrak{A})^{-1}$-polarized HBAV with $\Gamma_{00}(p^\infty)$-structure, is given by

$$(5.6.4)\quad (X(\mathfrak{A})^t, \lambda(\delta)^t, i(\mathfrak{A})^t) = (X(\mathfrak{c}\, \bar{\mathfrak{A}}^{-1}), \lambda(\delta), i(\mathfrak{c}\, \bar{\mathfrak{A}}^{-1}))$$

as follows immediately from the constructions of 5.1. Thus we can rewrite (5.6.3) as a functional equation

$$(5.6.5)\quad \int_{G_0} \check{H}\, d\mu(\mathfrak{A}) = \mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\, \mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A})^{-1} \int_{G_0} H\, d\mu(\mathfrak{c}\, \bar{\mathfrak{A}}^{-1}).$$

To apply this to ρ, we must calculate $(\rho_{\mathfrak{A}})_{,}^{\vee}$, as function on G_0, in terms of $\check{\rho}$ on $\mathrm{Gal}(L(p^\infty/L)$.

$$(5.6.6)\quad (\rho_{\mathfrak{A}})^{\vee}(a)=\frac{1}{\mathbf{N}_{L/\mathbf{Q}}(a)}\rho_{\mathfrak{A}}(1/\bar{a})$$

$$=\frac{1}{\mathbf{N}_{L/\mathbf{Q}}(a)}\rho(\mathfrak{A}^{-1})\,\rho(1/\bar{a})$$

while

$$(5.6.7)\quad \check{\rho}((a)\,\mathfrak{A}^{-1})=\frac{1}{\mathbf{N}_{L/\mathbf{Q}}(a)\,\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A})^{-1}}\rho((\bar{a})^{-1}(\bar{\mathfrak{A}}))$$

$$=\frac{\rho(\bar{\mathfrak{A}})}{\mathbf{N}_{L/\mathbf{Q}}(a)\,\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A})^{-1}}\cdot\rho(1/\bar{a})$$

$$(\text{by } 5.6.6)=\rho(\mathfrak{A})\,\rho(\bar{\mathfrak{A}})\,\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A})\,(\rho_{\mathfrak{A}})^{\vee}(a).$$

We now calculate

$$(5.6.8)\quad \int\limits_{\mathrm{Gal}(L(p^\infty)/L)}\check{\rho}\,d\mu=\sum_{i=1}^{h}\int\limits_{G_0}\check{\rho}((a)\,\mathfrak{A}_i^{-1})\,d\mu(\mathfrak{A}_i)\,(a)$$

$$(\text{by } 5.6.7)=\sum_{i=1}^{h}\rho(\mathfrak{A}_i)\,\rho(\bar{\mathfrak{A}}_i)\,\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A}_i)\int\limits_{G_0}(\rho_{\mathfrak{A}_i})^{\vee}d\mu(\mathfrak{A}_i)$$

$$(\text{by } 5.6.5)=\sum_{i=1}^{h}\rho(\mathfrak{A}_i)\,\rho(\bar{\mathfrak{A}}_i)\,\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A}_i)\,\mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\,(\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{A}_i)^{-1}\int\limits_{G_0}\rho_{\mathfrak{A}}\,d\mu(\mathfrak{c}\,\bar{\mathfrak{A}}_i^{-1})$$

$$=\sum_{i=1}^{h}\rho(\bar{\mathfrak{A}}_i)\,\mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\int\limits_{G_0}\rho(a)\,d\mu(\mathfrak{c}\,\bar{\mathfrak{A}}_i^{-1})$$

$$=\sum_{i=1}^{h}\rho(\mathfrak{c})\,\mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\int\limits_{G_0}\rho((a)\,\mathfrak{c}^{-1}\,\bar{\mathfrak{A}}_i)d\mu(\mathfrak{c}\,\mathfrak{A}_i^{-1})$$

$$=\rho(\mathfrak{c})\,\mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\int\limits_{\mathrm{Gal}}\rho\,d\mu.\quad \text{QED}$$

5.7. In this section, we verify the compatibility between the p-adic and the complex functional equations.

For any A_0-grossencharacter χ of L, whose conductor divides p^∞, and whose infinity type is

$$+k\sum_{\sigma\in\Sigma}\sigma-\sum_{\sigma\in\Sigma}d(\sigma)(\sigma-\bar{\sigma}),$$

we denote by

$$\textcircled{A}(\chi)\in\mathbf{C}_p,\qquad \textcircled{B}(\chi)\in\mathbf{C}$$

the quantities $\textcircled{A}$ and $\textcircled{B}$ defined in (5.3.4) and (5.3.5). Thus

$$(5.7.1)\quad \textcircled{A}(\chi)=\frac{L_\mu(\chi_p)}{\prod(c(\sigma))^{k+2d(\sigma)}}.$$

(5.7.2) $$\textcircled{B}(\chi)=\mathrm{Local}(\chi,\Sigma,\delta)[\text{units of } L\text{: units of } K]\cdot\prod(1-\check{\chi}(\overline{\mathfrak{P}}))(1-\chi(\overline{\mathfrak{P}}))$$
$$\times\frac{(-1)^{kg}}{\sqrt{d_K}\cdot\prod\limits_{\sigma}\mathrm{Im}(\sigma(\delta))^{d(\sigma)}}\cdot\frac{\prod\limits_{\sigma}(\pi^{d(\sigma)}\Gamma(k+d(\sigma))}{\prod\limits_{\sigma}(\Omega(\sigma)^{k+2d(\sigma)}}L_{\infty}(\chi)$$

Recall that $\check{\chi}$ is the A_0-grossencharacter defined by

(5.7.3) $$\check{\chi}(\mathfrak{A})=\frac{1}{\chi(\overline{\mathfrak{A}})\,\mathbf{N}\,\mathfrak{A}};$$

it is ramified precisely at the complex conjugates of those primes at which χ is ramified, and its infinity-type is

(5.7.4) $$-(2-k\sum_{\sigma\in\Sigma}\sigma-\sum_{\sigma\in\Sigma}(k+d(\sigma)-1)(\sigma-\bar{\sigma}).$$

Notice that

(5.7.5) $$k+2d(\sigma)=(2-k)+2(k+d(\sigma)-1).$$

The construction $\chi\mapsto\check{\chi}$ establishes a bijection

(5.7.6) $$\left\{\begin{matrix}A_0\text{-grossencharacters with}\\ k\geqq 1,\quad \text{all } d(\sigma)\geqq 0\end{matrix}\right\}\leftrightarrow\left\{\begin{matrix}A_0\text{-grossencharacters with}\\ k\leqq 1,\quad \text{all } k+d(\sigma)-1\geqq 0\end{matrix}\right\}$$

(5.7.7) **Theorem.** *For any A_0-grossencharacter χ* of L *of conductor dividing p^{∞}, we have the p-adic functional equation*

$$\textcircled{A}(\check{\chi})=\mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\cdot\chi(\mathfrak{c})\textcircled{A}(\chi).$$

Proof. By 5.7.5, this is just the functional equation 5.3.7 for L_{μ}, divided by $\prod\limits_{\sigma}(c(\sigma))^{k+2d(\sigma)}$ on each side.

(5.7.8) **Theorem.** *For any A_0-grossencharacter χ of L of conductor dividing p^{∞}, we have the complex functional equation*

$$\textcircled{B}(\check{\chi})=\mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\cdot\chi(\mathfrak{c})\cdot\textcircled{B}(\chi).$$

Before giving the rather long proof, notice that (5.7.7) and (5.7.8) taken together with (5.2.6), give

(5.7.9) **Corollary.** *The assertions* (5.3.4), (5.3.5), (5.3.6) *of Theorem* (5.3.0) *are also valid for all χ whose infinity types satisfy*

$$k\leqq 1,\qquad \text{all } k+d(\sigma)-1\geqq 0.$$

We now turn to the proof of (5.7.8). We will simply take the classical functional equation out of Tate's thesis, and compare the local terms which occur in it to the local terms which occur in our $\textcircled{B}(\chi)$. For ease of reference, we will adopr the *notations* of Tate's thesis as well – thus "c" will henceforth be the quasicharacter of the idele class group of L which corresponds to χ, with local components c_v at the finite places and c_σ at the archimedean places. We will also denote by χ_∞ the infinity type of χ, viewed as the character

$$(5.7.10)\quad \chi_\infty : L^\times \to \bar{\mathbf{Q}} \xhookrightarrow{\text{incl}(\infty)} \mathbf{C}$$

given by

$$(5.7.11)\quad \chi_\infty(a) = \prod_\sigma \frac{\sigma(\bar{a})^{d(\sigma)}}{\sigma(a)^{k+d(\sigma)}}.$$

Given a finite set S of finite places of L, and an ideal $\mathfrak{A}$ of L, we denote by

$$(5.7.12)\quad \mathfrak{A}^{\text{not}S} \overset{\text{dfn}}{=} \prod_{\mathfrak{q} \notin S} \mathfrak{q}^{\text{ord}_\mathfrak{q}(\mathfrak{A})}$$

the "prime to S part" of the ideal $\mathfrak{A}$. We will have particular occasion to apply this construction when S is the set

Ram (always meaning Ram(χ): Ram($\bar{\chi}$), when meant, will be written in full)

of places where our given χ is ramified.

The quasicharacter c of idele classes is a product

$$(5.7.13)\quad c = \prod_{\sigma \in \Sigma} c_\sigma \prod_v c_v$$

where the local quasicharacters are given by

$$(5.7.14)\quad \begin{cases} c_\sigma : \mathbf{C}^\times \to \mathbf{C}^\times \\ c_\sigma(r\,e^{i\theta}) = c_\sigma(z) = \dfrac{z^{k+d(\sigma)}}{\bar{z}^{d(\sigma)}} = (r^2)^{k/2}\, e^{i(k+2d(\sigma))\theta}, \end{cases}$$

(5.7.15) if $v \notin \text{Ram}$, $c_v : L_v^\times \to \mathbf{C}^\times$ is $c_v(a) = \chi(v)^{\text{ord}_v(a)}$
where $\chi(v)$ means χ("v viewed as prime ideal"),

(5.7.16) the c_v for $v \in \text{Ram}$ satisfy, for any $a \in L^\times$, the product formula
$$\prod_{v \in \text{Ram}} (c_v(a))\, \chi((a)^{\text{not Ram}}) = \chi_\infty(a).$$

We now select a "product" formula $f = \prod_\sigma f_\sigma \prod_v f_v$ in the Schwarz space of the adele group of L in such a way that

(5.7.17) at complex places σ, f_σ is the function "f_n" in the notation of [0], p. 318, with $n = k + 2d(\sigma)$
at $v \notin \text{Ram}$, f_v is the function "f_0" in the notation of [0], p. 320.

At each finite place v, let

(5.7.18) $\mathfrak{d}_L(v)=$ the exact power of "v viewed as prime ideal" dividing the absolute different $\mathfrak{d}_{L/\mathbf{Q}}$ of L.

The global zeta function $\zeta(f,c)$ is given by

$$(5.7.19)\quad \zeta(f,c)=\prod_{\sigma}((2\pi)^{d(\sigma)+1}\Gamma(k+d(\sigma)))\,L_{\infty}(\chi)\cdot *$$

where $*$ is given by

$$(5.7.20)\quad *=(\prod_{v\notin\mathrm{Ram}}(\mathbf{N}(\mathfrak{d}_L(v))^{-1/2}\chi^{-1}(\mathfrak{d}_L(v)))\prod_{v\in\mathrm{Ram}}\zeta(f_v,c_v).$$

Its alter-ego $\zeta(\hat{f},\hat{c})$ is given by

$$(5.7.21)\quad \zeta(\hat{f},\hat{c})=(\prod_{\sigma}((i)^{k+2d(\sigma)}\cdot(2\pi)^{k+d(\sigma)}\Gamma(1+d(\sigma)))\,L_{\infty}(\chi^{-1}\mathbf{N}^{-1})\cdot\hat{*}$$

where $\hat{*}$ is given by

$$(5.7.22)\quad \hat{*}=\prod_{v\in\mathrm{Ram}}\zeta(\hat{f}_v,\hat{c}_v).$$

We now combine the local functional equations

$$(5.7.23)\quad \zeta(f_v,c_v)=\rho_v(c_v)\,\zeta(\hat{f}_v,\hat{c}_v)$$

with the global functional equation

$$(5.7.24)\quad \zeta(f,c)=\zeta(\hat{f},\hat{c})$$

to obtain

$$\begin{aligned}(5.7.25)\quad &\prod_{\sigma}((i)^{k+2d(\sigma)}(2\pi)^{k+d(\sigma)}\Gamma(1+d(\sigma)))\cdot L_{\infty}(\chi^{-1}\mathbf{N}^{-1})\\ &=(\prod_{\sigma}((2\pi)^{d(\sigma)+1}\Gamma(k+d(\sigma)))\,L_{\infty}(\chi)\\ &\quad\cdot(\prod_{v\in\mathrm{Ram}}\rho_v(c_v))(\prod_{v\notin\mathrm{Ram}}(\mathbf{N}(\mathfrak{d}_L(v)))^{-1/2}\chi^{-1}(\mathfrak{d}_L(v))).\end{aligned}$$

We now combine this with the *trivial* functional equation

$$(5.7.26)\quad L_{\infty}(\chi^{-1}\mathbf{N}^{-1})=L_{\infty}(\check{\chi})$$

to rewrite (5.7.25) as

$$\begin{aligned}(5.7.27)\quad &\prod_{\sigma}(\pi^{k+d(\sigma)-1}\Gamma(1+d(\sigma)))\,L_{\infty}(\check{\chi})\\ &=\frac{\mathbf{N}(\mathfrak{d}_L^{\mathrm{not\,Ram}})^{-1/2}\chi^{-1}(\mathfrak{d}_L^{\mathrm{not\,Ram}})}{(2i)^{g(k-1)}(i)^{g}(-1)^{\Sigma d(\sigma)}}\cdot\prod_{v\in\mathrm{Ram}}\rho_v(c_v)\times\prod_{\sigma}(\pi^{d(\sigma)}\Gamma(k+d(\sigma)))\,L_{\infty}(\chi).\end{aligned}$$

We next break up the $\rho_v(c_v)$ according as the $v \in \mathrm{Ram}$ is a $\mathfrak{P}$ (one of the primes over p chosen by Σ and $\mathrm{incl}(p)$) or a $\overline{\mathfrak{P}}$. We define

$$(5.7.28) \quad \begin{cases} \mathrm{Local}(\chi, \mathfrak{P}\text{'s}) = \prod\limits_{\mathfrak{P} \in \mathrm{Ram}} \rho_{\mathfrak{P}}(c_{\mathfrak{P}}) \\ \mathrm{Local}(\chi, \overline{\mathfrak{P}}\text{'s}) = \prod\limits_{\overline{\mathfrak{P}} \in \mathrm{Ram}} \rho_{\overline{\mathfrak{P}}}(c_{\overline{\mathfrak{P}}}) \end{cases}$$

From the relation ([0], p. 315)

$$(5.7.29) \quad \rho(c_v)\,\rho_v(\hat{c}_v) = c_v(-1)$$

we get

$$(5.7.30) \quad \mathrm{Local}(\chi^{-1}\mathbf{N}^{-1}, \overline{\mathfrak{P}}\text{'s})\,\mathrm{Local}(\chi, \overline{\mathfrak{P}}\text{'s}) = \prod_{\overline{\mathfrak{P}} \in \mathrm{Ram}} c_{\overline{\mathfrak{P}}}(-1),$$

while by "transport of structure" we have

$$(5.7.31) \quad \mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s}) = \mathrm{Local}(\chi^{-1}\mathbf{N}^{-1}, \overline{\mathfrak{P}}\text{'s}),$$

and hence

$$(5.7.32) \quad \mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s}) = \frac{\prod\limits_{\overline{\mathfrak{P}} \in \mathrm{Ram}} c_{\overline{\mathfrak{P}}}(-1)}{\mathrm{Local}(\chi, \overline{\mathfrak{P}}\text{'s})}.$$

This allows us to rewrite (5.7.27) as

$$(5.7.33) \quad \prod_{\sigma} (\pi^{k+d(\sigma)-1}\,\Gamma(1+d(\sigma))) \cdot L_{\infty}(\check{\chi})$$
$$= \frac{\mathbf{N}(\mathfrak{d}_L^{\mathrm{not\,Ram}})^{-1/2}\,\chi^{-1}(\mathfrak{d}_L^{\mathrm{not\,Ram}}) \cdot \prod\limits_{\overline{\mathfrak{P}} \in \mathrm{Ram}} c_{\overline{\mathfrak{P}}}(-1)}{(2i)^{g(k-1)}(i)^g(-1)^{\Sigma d(\sigma)}}$$
$$\times \frac{\mathrm{Local}(\chi, \mathfrak{P}\text{'s})}{\mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s})} \cdot \prod_{\sigma} (\pi^{d(\sigma)}\,\Gamma(k+d(\sigma))) \cdot L_{\infty}(\chi).$$

Referring now to the Definition (5.7.2) of $\textcircled{B}(\chi)$, we see that (5.7.33) is equivalent to the following equality:

$$(5.7.34) \quad \textcircled{B}(\check{\chi}) = \frac{\mathrm{Local}(\check{\chi}, \Sigma, \delta)}{\mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s})} \cdot \frac{\mathrm{Local}(\chi, \mathfrak{P}\text{'s})}{\mathrm{Local}(\chi, \Sigma, \delta}$$
$$\times \frac{\mathbf{N}(\mathfrak{d}_L^{\mathrm{not\,Ram}})^{-1/2}\,\chi^{-1}(\mathfrak{d}_L^{\mathrm{not\,Ram}}) \prod\limits_{\overline{\mathfrak{P}} \in \mathrm{Ram}} c_{\overline{\mathfrak{P}}}(-1)}{(\prod\limits_{\sigma} (2i\,\mathrm{Im}(\sigma(\delta)))^{k-1}(i)^g(-1)^{\Sigma d(\sigma)}} \times \textcircled{B}(\chi).$$

Our problem is now to "recognize" (5.7.34) as being the asserted (5.7.8), which will require a closer look at the various local terms.

(5.7.35) **Lemma.** *The quantity* δ *given in* (5.3.3) *satisfies*

$$(2\delta) = \mathfrak{c}\,\mathfrak{d}_L.$$

Proof. By hypothesis, the image of $\Lambda^2_{\mathcal{O}}\mathcal{O}_L$ under

$$(5.7.36)\quad \langle u, v\rangle = \frac{\bar{u}\,v - u\,\bar{v}}{2\delta} = \mathrm{trace}_{L/K}\left(\frac{\bar{u}\,v}{2\delta}\right)$$

is precisely $\mathfrak{d}^{-1}\mathfrak{c}^{-1}$, which is to say

$$\mathrm{trace}_{L/K}\left(\frac{1}{2\delta}\,\mathfrak{c}\,\mathcal{O}_L\right) = \mathfrak{d}^{-1}, \qquad \text{whence } \left(\frac{1}{2\delta}\right)\mathfrak{c} = \mathfrak{d}_L^{-1}. \quad \text{QED}$$

(5.7.37) **Lemma.** *We have the formula*

$$\mathbf{N}(\mathfrak{d}_L^{\text{not Ram}})^{-1/2}\,\chi^{-1}(\mathfrak{d}_L^{\text{not Ram}}) = \frac{\mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\cdot\chi(\mathfrak{c})}{\chi((2\delta)^{\text{not Ram}})\sqrt{\mathbf{N}((2\delta)^{\text{not Ram}})}}.$$

Proof. Because $\mathfrak{c}$ is prime to p, hence to all primes where χ is ramified, the "not Ram" part of (5.7.36) is

$$(5.7.38)\quad (\mathfrak{d}_L^{\text{not Ram}})^{-1} = \frac{\mathfrak{c}}{(2\delta)^{\text{not Ram}}}$$

The result now follows upon applying $\chi\cdot\mathbf{N}^{1/2}$ to both sides, and remembering that, because $\mathfrak{c}$ is an ideal of K, we have

$$(5.7.39)\quad \sqrt{\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{c})} = \mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c}).$$

By (5.7.16), we have

$$\begin{aligned}(5.7.40)\quad \chi((2\delta)^{\text{not Ram}}) &= \frac{1}{\prod\limits_{v\in\text{Ram}} \mathrm{c}_v(2\delta)}\cdot\chi_\infty(2\delta)\\ &= \frac{1}{\prod\limits_{v\in\text{Ram}} c_v(2\delta)}\cdot\prod_\sigma \frac{\sigma(2\bar{\delta})^{d(\sigma)}}{\sigma(2\delta)^{k+d(\sigma)}}\\ &= \frac{(-1)^{\Sigma d(\sigma)}}{(\prod\limits_{v\in\text{Ram}} c_v(2\delta))\cdot(\prod\limits_\sigma \sigma(2\delta))^k} \qquad (\text{because } \bar{\delta} = -\delta).\end{aligned}$$

Substituting (5.7.37) and (5.7.40) into (5.7.34), and remembering that $2i\,\mathrm{Im}(\sigma(\delta)) = \sigma(2\delta)$ because δ is purely imaginary, we find

$$\begin{aligned}(5.7.41)\quad \textcircled{B}(\check{\chi}) &= \frac{\mathrm{Local}(\check{\chi}, \Sigma, \delta)}{\mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s})}\,\frac{\mathrm{Local}(\chi, \mathfrak{P}\text{'s})}{\mathrm{Local}(\chi, \Sigma, \delta)} \times \frac{1}{\sqrt{\mathbf{N}((2\delta)^{\text{not Ram}})}}\\ &\quad\times \prod_{v\in\text{Ram}} (c_v(2\delta))\cdot\prod_\sigma(-i\,\sigma(2\delta))\cdot\prod_{\overline{\mathfrak{P}}\in\text{Ram}} c_{\overline{\mathfrak{P}}}(-1)\times\mathbf{N}_{L/\mathbf{Q}}(\mathfrak{c})\,\chi(\mathfrak{c})\cdot\textcircled{B}(\chi).\end{aligned}$$

(5.7.42) **Lemma.** *We have the formula*

$$\prod_\sigma(-i\,\sigma(2\delta)) = \sqrt{\mathbf{N}_{L/\mathbf{Q}}(2\delta)}$$

Proof. Since $\sigma(\delta)$ is purely imaginary with positive imaginary part, each term $-i\sigma(2\delta)$ is a positive real number, and

$$\begin{aligned}\mathbf{N}_{L/\mathbf{Q}}(2\delta) &= \prod_{\sigma} \sigma(2\delta)\,\sigma(2\bar{\delta}) \\ &= \prod_{\sigma} (\sigma(2\delta)\cdot(-1)\,\sigma(2\delta)) \\ &= (\prod_{\sigma} (-i\sigma(2\delta)))^2 \quad \text{QED}\end{aligned}$$

Thus we can rewrite (5.7.41) as

$$(5.7.43)\quad \textcircled{\it B}(\check{\chi}) = \frac{\mathrm{Local}(\check{\chi}, \Sigma, \delta)}{\mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s})} \cdot \frac{\mathrm{Local}(\chi, \mathfrak{P}\text{'s})}{\mathrm{Local}(\chi, \Sigma, \delta)} \left(\frac{\prod\limits_{v \in \mathrm{Ram}} c_v(2\delta)}{\sqrt{\prod\limits_{v \in \mathrm{Ram}} |2\delta|_v}} \right)$$

$$\times (\prod_{\overline{\mathfrak{P}} \in \mathrm{Ram}} c_{\overline{\mathfrak{P}}}(-1))\, \mathbf{N}_{K/\mathbf{Q}}(\mathfrak{c})\, \chi(\mathfrak{c}) \cdot \textcircled{\it B}(\chi).$$

(5.7.44) **Key Lemma.** *We have the formula*

$$\frac{\mathrm{Local}(\chi, \Sigma, \delta)}{\mathrm{Local}(\chi, \mathfrak{P}\text{'s})} = \frac{\prod\limits_{\mathfrak{P} \in \mathrm{Ram}} c_{\mathfrak{P}}(2\delta)}{\prod\limits_{\mathfrak{P} \in \mathrm{Ram}} \sqrt{|2\delta|_{\mathfrak{P}}}}.$$

Let us admit this lemma for a moment, and use it to conclude the proof of theorem. Using (5.7.44) for $\check{\chi}$, and denoting by $\check{c}$ the quasicharacter corresponding to $\check{\chi}$, we obtain

$$(5.7.45)\quad \frac{\mathrm{Local}(\check{\chi}, \Sigma, \delta)}{\mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s})} = \frac{\prod\limits_{\mathfrak{P} \in \mathrm{Ram}(\check{\chi})} \check{c}_{\mathfrak{P}}(2\delta)}{\prod\limits_{\mathfrak{P} \in \mathrm{Ram}(\check{\chi})} \sqrt{|2\delta|_{\mathfrak{P}}}}.$$

By "transport of structure," we have

$$(5.7.46)\quad \check{c}_{\mathfrak{P}}(2\delta) = \frac{|\overline{2\delta}|_{\mathfrak{P}}}{c_{\overline{\mathfrak{P}}}(\overline{2\delta})} = \frac{|-2\delta|_{\overline{\mathfrak{P}}}}{c_{\overline{\mathfrak{P}}}(-2\delta)} = \frac{|2\delta|_{\overline{\mathfrak{P}}}}{c_{\overline{\mathfrak{P}}}(2\delta)\, c_{\overline{\mathfrak{P}}}(-1)}$$

But $\mathfrak{P}$ lies in $\mathrm{Ram}(\check{\chi})$ if and only if $\overline{\mathfrak{P}} \in \mathrm{Ram}(\chi)$, and

$$\sqrt{|2\delta|_{\mathfrak{P}}} = \sqrt{|\overline{2\delta}|_{\overline{\mathfrak{P}}}} = \sqrt{|-2\delta|_{\overline{\mathfrak{P}}}} = \sqrt{|2\delta|_{\overline{\mathfrak{P}}}},$$

so we can rewrite (5.7.45) as

$$(5.7.47)\quad \frac{\mathrm{Local}(\check{\chi}, \Sigma, \delta)}{\mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s})} = \frac{\prod\limits_{\overline{\mathfrak{P}} \in \mathrm{Ram}(\chi)} \sqrt{|2\delta|_{\overline{\mathfrak{P}}}}}{\prod\limits_{\overline{\mathfrak{P}} \in \mathrm{Ram}(\chi)} c_{\overline{\mathfrak{P}}}(2\delta)\, c_{\overline{\mathfrak{P}}}(-1)}$$

Combining (5.7.44) and (5.7.47), we find

$$(5.7.48)\quad \frac{\mathrm{Local}(\check{\chi}, \Sigma, \delta)}{\mathrm{Local}(\check{\chi}, \mathfrak{P}\text{'s})} \cdot \frac{\mathrm{Local}(\chi, \mathfrak{P}\text{'s})}{\mathrm{Local}(\chi, \Sigma, \delta)}$$

$$= \frac{\prod_{\mathfrak{P} \in \mathrm{Ram}(\chi)} \sqrt{|2\delta|_{\mathfrak{P}}} \prod_{\overline{\mathfrak{P}} \in \mathrm{Ram}(\chi)} \sqrt{|2\delta|_{\overline{\mathfrak{P}}}}}{\prod_{\mathfrak{P} \in \mathrm{Ram}(\chi)} c_{\mathfrak{P}}(2\delta) \prod_{\overline{\mathfrak{P}} \in \mathrm{Ram}(\chi)} c_{\overline{\mathfrak{P}}}(2\delta)\, c_{\overline{\mathfrak{P}}}(-1)}$$

$$= \frac{\prod_{v \in \mathrm{Ram}} \sqrt{|2\delta|_v}}{\prod_{v \in \mathrm{Ram}} c_v(2\delta) \cdot \prod_{\overline{\mathfrak{P}} \in \mathrm{Ram}} c_{\overline{\mathfrak{P}}}(-1)}.$$

Substituting into (5.7.43), we find (5.7.8).

We now turn to the proof of (5.7.44). We need a few lemmas.

(5.7.49) **Lemma.** *The function* χ_{finite} *on* $(\mathcal{O}_L \otimes \mathbf{Z}_p)^\times$ *defined by* (5.2.13) *is related to the p-adic components* $c_{\mathfrak{P}}, c_{\overline{\mathfrak{P}}}$ *of the quasicharacter c which corresponds to* χ *by the formula*

$$\chi_{\mathrm{finite}} = \prod_{\text{all } \mathfrak{P}} c_{\mathfrak{P}}^{-1} \cdot \prod_{\text{all } \overline{\mathfrak{P}}} c_{\overline{\mathfrak{P}}}^{-1} \qquad \text{on } (\mathcal{O}_L \otimes \mathbf{Z}_p)^\times.$$

Proof. Obvious from (5.2.13) and (5.7.16).

(5.7.50) **Lemma.** *The function* $F_{\mathfrak{p},1}$ on $\mathcal{O}_{\mathfrak{p}} \simeq \mathcal{O}_{L_{\mathfrak{P}}}$ *is given by*

$$F_{\mathfrak{p},1} = \begin{cases} c_{\mathfrak{P}} & \text{on } (\underline{\mathcal{O}}_{L_{\mathfrak{P}}})^\times, \quad \text{extended by zero to } \underline{\mathcal{O}}_{L_{\mathfrak{P}}} \text{ if } c_{\mathfrak{P}} \text{ is ramified} \\ 1 & \qquad\qquad\qquad\quad\ \text{if } c_{\mathfrak{P}} \text{ is unramified} \end{cases}$$

Proof. This follows from (5.7.49), and the definitions (5.2.18) and (5.2.15) of $F_{\mathfrak{p},1}$ in terms of χ_{finite}.

(5.7.51) **Lemma.** *The factor* $\rho_{\mathfrak{P}}(c_{\mathfrak{P}})$ *which occurs in Tate's local functional equation is given by the formula*

$$\rho_{\mathfrak{P}}(c_{\mathfrak{P}}) = \frac{\mathbf{N}(\mathfrak{d}_L(\mathfrak{P}))^{-1/2} \cdot \widehat{F_{\mathfrak{p},1}}(-1/\alpha)}{c_{\mathfrak{P}}(\alpha)}$$

where $\mathfrak{d}_L(\mathfrak{P})$ *is the exact power of* $\mathfrak{P}$ *dividing the absolute different, and where* α *is any element of* $L_{\mathfrak{P}}$ *satisfying*

$$\mathrm{ord}_{\mathfrak{P}}(\alpha) = \mathrm{ord}_{\mathfrak{P}}(\mathfrak{d}_L) + \mathrm{ord}_{\mathfrak{P}}(\mathrm{cond}(c_{\mathfrak{P}})).$$

Proof. This is just a "coordinate-free" reformulation of Tate's formula ([0], p.372).

(5.7.52) **Lemma.** *The quantity* $\delta_0 \in K \otimes \mathbf{Z}_p = \prod_{\mathfrak{p}} K_{\mathfrak{p}} \simeq \prod_{\mathfrak{P}} L_{\mathfrak{P}}$ *is the image in* $\prod_{\mathfrak{P}} L_{\mathfrak{P}}$ *of* $-1/2\delta$.

Proof. Via the isomorphism

$$(5.7.53)\quad L\otimes\mathbf{Z}_p\simeq\Big(\prod_{\mathfrak{P}}L_{\mathfrak{P}}\Big)\times\Big(\prod_{\bar{\mathfrak{P}}}L_{\bar{\mathfrak{P}}}\Big)\simeq\Big(\prod_{\mathfrak{p}}K_{\mathfrak{p}}\Big)\times\Big(\prod_{\mathfrak{p}}K_{\mathfrak{p}}\Big)\simeq(K\otimes\mathbf{Z}_p)\times(K\otimes\mathbf{Z}_p)$$

the element $2\delta\in L$ becomes the element

$$(2\delta(\mathfrak{P}\text{'s}),\ -2\delta(\mathfrak{P}\text{'s}))\in(K\otimes\mathbf{Z}_p)\times(K\otimes\mathbf{Z}_p),$$

where $2\delta(\mathfrak{P}\text{'s})$ is the image of 2δ under the isomorphism

$$\prod_{\mathfrak{P}}L_{\mathfrak{P}}\simeq\prod_{\mathfrak{p}}K_{\mathfrak{p}}\simeq K\otimes\mathbf{Z}_p.$$

The pairing $\langle\,,\rangle$

$$\langle\,,\rangle\colon\Lambda^2_{\mathcal{O}}\,\mathcal{O}_L\simeq\mathfrak{d}^{-1}\,\mathfrak{c}^{-1}$$

$$\langle u,v\rangle=\frac{\bar{u}\,v-u\,\bar{v}}{2\delta}$$

induces an isomorphism

$$\langle\,,\rangle\colon\Lambda^2_{\mathcal{O}\otimes\mathbf{Z}_p}(\mathcal{O}_L\otimes\mathbf{Z}_p)\xrightarrow{\sim}\mathfrak{d}^{-1}\,\mathfrak{c}^{-1}\otimes\mathbf{Z}_p$$

whose expression via (5.7.53) is

$$\langle(a,b),(c,d)\rangle=\frac{\mathrm{ad}-b\,c}{-2\delta(\mathfrak{P}\text{'s})}.$$

By definition, δ_0 is the quantity

$$\delta_0\overset{\mathrm{dfn}}{=}\langle(1,0),(0,1)\rangle.\quad\text{QED}$$

To prove (5.7.44), we simply compute. As in (5.2.10), we choose an element $a\in L$ which satisfies

$$(5.7.54)\quad\begin{array}{ll}\mathrm{ord}_{\mathfrak{P}}(a)=\mathrm{ord}_{\mathfrak{P}}(\mathrm{cond}(\chi)) & \text{at all } \mathfrak{P}\text{'s}\\ \mathrm{ord}_{\bar{\mathfrak{P}}}(a-1)\geqq 1+\mathrm{ord}_{\bar{\mathfrak{P}}}(\mathrm{cond}(\chi)) & \text{at all } \bar{\mathfrak{P}}\text{'s}.\end{array}$$

The ideal $\mathfrak{B}$ is such that

$$(5.7.55)\quad(a)\,\mathfrak{B}=\prod_{\mathfrak{P}}\mathfrak{P}^{\mathrm{ord}_{\mathfrak{P}}(\mathrm{cond}(\chi))}$$

is prime to all $\mathfrak{P}$'s and $\bar{\mathfrak{P}}$'s, hence

$$(5.7.56)\quad\mathfrak{B}^{-1}=(a)^{\mathrm{not\,Ram}}.$$

By definition of Local(χ,Σ,δ), we have

$$(5.7.57)\quad\mathrm{Local}(\chi,\Sigma,\delta)=\frac{\prod_{\mathfrak{P}\in\mathrm{Ram}}\hat{F}_{\mathfrak{p},1}\left(\dfrac{-1}{2\delta a}\right)\prod_{\mathrm{all}\,\bar{\mathfrak{P}}}F_{\mathfrak{p},2}\left(\dfrac{1}{a}\right)}{\chi(\mathfrak{B})\,\chi_\infty(a)}.$$

Because a is sufficiently near 1 at all $\overline{\mathfrak{P}}$'s, the terms $F_{\mathfrak{p},2}(1/a)$ are all 1, and the denominator can be simplified:

$$(5.7.58)\quad \frac{1}{\chi(\mathfrak{B})\chi_\infty(a)}=\frac{\chi((a)^{\text{not Ram}})}{\chi_\infty(a)}$$

$$=\frac{(\prod\limits_{v\in\text{Ram}} c_v(a))\,\chi((a)^{\text{not Ram}})}{(\prod\limits_{v\in\text{Ram}} c_v(a))\,\chi_\infty(a)}$$

$$=\frac{1}{\prod\limits_{v\in\text{Ram}} c_v(a)}$$

$$=\frac{1}{\prod\limits_{\mathfrak{P}\in\text{Ram}} c_{\mathfrak{P}}(a)},$$

the last equality because a is sufficiently near 1 at all the $\overline{\mathfrak{P}}$'s. Thus we can rewrite (5.7.57) as

$$(5.7.59)\quad \text{Local}(\chi,\Sigma,\delta)=\frac{\prod\limits_{\mathfrak{P}\in\text{Ram}} \hat{F}_{\mathfrak{p},1}\left(\frac{-1}{2\delta a}\right)}{\prod\limits_{\mathfrak{p}\in\text{Ram}} c_{\mathfrak{P}}(a)}.$$

If we use the quantity $2\delta a$ for the "α" of (5.7.51), we get an expression for Local (χ, $\mathfrak{P}$'s):

$$(5.7.60)\quad \text{Local}(\chi,\mathfrak{P}\text{'s})=\prod_{\mathfrak{P}\in\text{Ram}} \rho_{\mathfrak{P}}(c_{\mathfrak{P}})$$

$$=\prod_{\mathfrak{P}\in\text{Ram}}\left(\frac{\mathbf{N}(\mathfrak{d}_L(\mathfrak{P}))^{-1/2}\,\hat{F}_{\mathfrak{p},1}\left(\frac{-1}{2\delta a}\right)}{c_{\mathfrak{P}}(2\delta a)}\right).$$

Dividing, we get

$$(5.7.61)\quad \frac{\text{Local}(\chi,\Sigma,\delta)}{\text{Local}(\chi,\mathfrak{P}\text{'s})}=\frac{\prod\limits_{\mathfrak{P}\in\text{Ram}} c_{\mathfrak{P}}(2\delta)}{\prod\limits_{\mathfrak{P}\in\text{Ram}} \mathbf{N}(\mathfrak{d}_L(\mathfrak{P}))^{-1/2}}.$$

Because $(2\delta)=\mathfrak{d}_L\,\mathfrak{c}$ with $\mathfrak{c}$ prime to p, we have

$$(5.7.62)\quad \mathbf{N}(\mathfrak{d}_L(\mathfrak{P}))=(|2\delta|_{\mathfrak{P}})^{-1}$$

whence we get (5.7.44) as required. QED

5.8. In this final section, we explain the dependence of the p-adic L-function upon the choice (5.3.3) of δ. For each possible δ, let us denote by $\mu(\delta)$ the measure on $\text{Gal}(L(p^\infty)/L)$ whose existence and properties are given by Theorem (5.3.0). Given one δ, any other is of the form $\lambda\delta$, with $\lambda\in K^\times$ prime to p and totally positive. Given any such λ, we denote by

(5.8.0) $(\lambda^{-1}, 1) \in \mathrm{Gal}(L(p^\infty)/L)$

the image of $(\lambda^{-1}, 1) \in (\mathcal{O} \otimes \mathbf{Z}_p)^\times \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times$ under the map

(5.8.1)
$$\begin{tikzcd}
(\mathcal{O} \otimes \mathbf{Z}_p)^\times \times (\mathcal{O} \otimes \mathbf{Z}_p)^\times \arrow[r, "(\mathfrak{P}, \bar{\mathfrak{P}}\text{'s})", "\sim"'] & (\mathcal{O}_L \otimes \mathbf{Z}_0)^\times \arrow[r, two heads] & (\mathcal{O}_L \otimes \mathbf{Z}_p)^\times/(\text{closure of } E(L)) \arrow[d, "\wr"'] \\
 & & \mathrm{Gal}(L(p^\infty)/L(1) \arrow[d, phantom, "\cap"] \\
 & & \mathrm{Gal}(L(p^\infty)/L).
\end{tikzcd}$$

(5.8.2) **Theorem.** *For any $\lambda \in K^\times$ which is prime to p and totally positive, and any δ satisfying* (5.3.3), *the measures $\mu(\delta)$ and $\mu(\lambda\delta)$ on* $\mathrm{Gal}(L(p^\infty)/L)$ *constructed in* (5.3.0) *are related by the formula*

(5.8.3) for any continuous D_p-valued function $f(g)$ on $\mathrm{Gal}(L(p^\infty)/L)$,
$$\int_{\mathrm{Gal}(L(p^\infty)/L)} f(g)\, d\mu(\lambda\delta)(g) = \int_{\mathrm{Gal}(L(p^\infty)/L)} f((\lambda^{-1}, 1)\, g)\, d\mu(\delta)(g).$$

(5.8.4) **Corollary.** *The p-adic L-functions $L_{\mu(\delta)}$ and $L_{\mu(\lambda\delta)}$ are related by the formula*

(5.8.5) $L_{\mu(\lambda\delta)}(\rho) = \rho(\lambda^{-1}, 1)\, L_{\mu(\delta)}(\rho)$,

valid for any continuous $D_p^\times$-valued character of $\mathrm{Gal}(L(p^\infty)/L)$.

Proof. The corollary is simply (5.8.3) in the case $f = \rho$. By (4.1.12), the theorem follows from the truth of (5.8.5) for any set of ρ's which are stable by multiplication by arbitrary locally constant $D_p^\times$-valued characters of Gal. Thus it suffices to verify (5.8.5) for *all* A_0-grossencharacters of any *fixed* infinity type. In fact, our explicit formula (5.3.6), together with (5.7.44), will allow us to verify (5.8.5) explicitly when ρ is *any* A_0-grossencharacter χ whose infinity type $-k\Sigma\sigma - \Sigma d(\sigma)(\sigma - \bar{\sigma})$ satisfies

(5.8.6) $k \geqq 1, \quad \text{all } d(\sigma) \geqq 0.$

To see this, we agrue as follows. As in 5.1, we fix a nowhere-vanishing differential $\omega(\mathcal{O}_L)$ on $X(\mathcal{O}_L)$ over A. The constants $\Omega(\sigma)$ appearing in (5.3.5) are the ratio of this $\omega(\mathcal{O}_L)$ to the transcendental $\omega_{\mathrm{trans}}(\mathcal{O}_L)$:

(5.8.7) $\omega(\mathcal{O}_L) = \Omega \cdot \omega_{\mathrm{trans}}(\mathcal{O}_L).$

The quantity $\Omega = (\ldots, \Omega(\sigma), \ldots) \in (K \otimes \mathbf{C})^\times$ is *independent* of the choice of polarization imposed upon $X(\mathcal{O}_L)$.

However, the $\Gamma_{00}(p^\infty)$-structure carried by $X(\mathcal{O}_L)$ *does* depend very much on the choice of δ: let us denote it

(5.8.8) $i_\delta(\mathcal{O}_L).$

By (5.7.52), the transcendental expression for $i_\delta(\mathcal{O}_L)$ sits in the commutative diagram

(5.8.9)
$$\begin{tikzcd}
\mathfrak{d}^{-1}\otimes\mathbf{Z}_p \arrow[r, "\overset{i_\delta(\mathcal{O}_L)}{\sim}"] \arrow[d, dashed, "\wr"', "\times(-2\delta(\mathfrak{P}\text{'s}))"] & \mathcal{O}_L\otimes\mathbf{Z}_p \arrow[d, phantom, "\cup"] \\
\mathcal{O}\otimes\mathbf{Z}_p=\prod_{\mathfrak{p}}\mathcal{O}_{\mathfrak{p}} \arrow[r, "\sim"] & \prod_{\mathfrak{P}}\mathcal{O}_{L_{\mathfrak{P}}},
\end{tikzcd}$$

so that $i_\delta(\mathcal{O}_L)$ and $i_{\lambda\delta}(\mathcal{O}_L)$ are related by

(5.8.10) $\quad i_{\lambda\delta}(\mathcal{O}_L)=\lambda\, i_\delta(\mathcal{O}_L).$

The associated canonical differentials on $X(\mathcal{O}_L)$ over D_p are *defined* (cf., 1.10.11) in terms of these $\Gamma_{00}(p^\infty)$-structures, hence are related by

(5.8.11) $\quad \omega_{\mathrm{can}}(\lambda\,\delta, \mathcal{O}_L)=\lambda^{-1}\omega_{\mathrm{can}}(\delta;\mathcal{O}_L).$

Let us denote by $c(\delta)\in(\mathcal{O}\otimes D_p)^\times$ the ratio of $\omega(\mathcal{O}_L)$ to $\omega_{\mathrm{can}}(\delta;\mathcal{O}_L)$

(5.8.12) $\quad \omega(\mathcal{O}_L)=c(\delta)\cdot\omega_{\mathrm{can}}(\delta;\mathcal{O}_L).$

Combining (5.8.11) and (5.8.11), we see that

(5.8.13) $\quad c(\lambda\,\delta)=\lambda\, c(\delta).$

Now let us denote by

(5.8.14) $\quad \textcircled{B}_\delta(\chi)$

the quantity $\textcircled{B}$ of (5.3.5). Applying (5.3.6) (with δ), we obtain, for any A_0-grossencharacter χ whose infinity type satisfies (5.8.6), the formula

(5.8.15) $$\frac{L_{\mu(\delta)}(\chi_p)}{\prod_\sigma(\sigma(c(\delta)))^{k+2d(\sigma)}}=\textcircled{B}_\delta(\chi).$$

Applying it with $\lambda\,\delta$, we get

(5.8.16) $$\frac{L_{\mu(\lambda\delta)}(\chi)}{\prod(\sigma(\lambda\, c(\delta)))^{k+2d(\sigma)}}=\textcircled{B}_{\lambda\delta}(\chi).$$

Looking at the explicit formula for $\textcircled{B}(\chi)$, we obviously have

(5.8.17) $$\textcircled{B}_\delta(\chi)=\frac{\mathrm{Local}(\chi,\Sigma,\delta)}{\prod(\mathrm{Im}(\sigma(\delta))^{d(\sigma)}}\times(\text{independent of }\delta),$$

and hence

(5.8.18) $$\textcircled{B}_{\lambda\delta}(\chi)=\frac{\mathrm{Local}(\chi,\Sigma,\lambda\,\delta)}{\prod\mathrm{Im}(\sigma(\lambda\,\delta))^{d(\sigma)}}\cdot\frac{\prod\mathrm{Im}(\sigma(\delta))^{d(\sigma)}}{\mathrm{Local}(\chi,\Sigma,\delta)}\cdot\textcircled{B}_\delta(\chi)$$

Using (5.7.44), we can rewrite this in terms of the local quasicharacters $c_{\mathfrak{P}}$ attached to χ as

$$(5.8.19)\quad \textcircled{B}_{\lambda\delta}(\chi)=\frac{\prod\limits_{\mathfrak{P}\in\mathrm{Ram}} c_{\mathfrak{P}}(2\lambda\,\delta)}{\prod\limits_{\mathfrak{P}\in\mathrm{Ram}} c_{\mathfrak{P}}(2\delta)}\cdot\frac{1}{\prod\limits_{\sigma}\sigma(\lambda)^{d(\sigma)}}\cdot\textcircled{B}_{\delta}(\chi)$$

$$=\frac{\prod\limits_{\mathfrak{P}\in\mathrm{Ram}} c_{\mathfrak{P}}(\lambda)}{\prod\limits_{\sigma}\sigma(\lambda)^{d(\sigma)}}\textcircled{B}_{\delta}(\chi).$$

Because λ is prime to p, we have

$$(5.8.20)\quad \prod_{\mathfrak{P}\in\mathrm{Ram}} c_{\mathfrak{P}}(\lambda)=\prod_{\mathrm{all}\,\mathfrak{P}} c_{\mathfrak{P}}(\lambda)=\chi_{\mathrm{finite}}(\lambda^{-1},1),$$

and (5.8.19) becomes

$$(5.8.21)\quad \textcircled{B}_{\lambda\delta}(\chi)=\frac{\chi_{\mathrm{finite}}(\lambda^{-1},1)}{\prod\sigma(\lambda)^{d(\sigma)}}\textcircled{B}_{\delta}(\chi).$$

Substituting by (5.8.15) and (5.8.16), we get

$$(5.8.22)\quad \frac{L_{\mu(\lambda\delta)}(\chi)}{\prod(\sigma(\lambda c(\delta)))^{k+2d(\sigma)}}=\frac{\chi_{\mathrm{finite}}(\lambda^{-1},1)}{\prod\sigma(\lambda)^{d(\sigma)}}\cdot\frac{L_{\mu(\delta)}(\chi)}{\prod(\sigma(c(\delta)))^{k+2d(\sigma)}}$$

whence

$$(5.8.23)\quad L_{\mu(\lambda\delta)}(\chi)=\chi_{\mathrm{finite}}(\lambda^{-1},1)\cdot\prod_{\sigma}\lambda(\sigma)^{k+d(\sigma)}\cdot L_{\mu(\delta)}(\chi)$$

$$=\chi_{\mathrm{finite}}(\lambda^{-1},1)\,\chi_{\infty}(\lambda^{-1},1)\cdot L_{\mu(\delta)}(\chi)$$

$$=\chi(\lambda^{-1},1)\,L_{\mu(\delta)}(\chi).\quad \text{QED}$$

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