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Local-to-global extensions of representations of fundamental groups

LOCAL-TO-GLOBAL EXTENSIONS
OF REPRESENTATIONS
OF FUNDAMENTAL GROUPS

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Introduction.

The classification theory (cf. [Le]) of coherent modules with integrable connection (C.M.I.C.) in the punctured formal neighborhood of the point ∞ of $\mathbf{P}^1_{\mathbf{C}}$, shows that any such C.M.I.C. has a « canonical extension » to a C.M.I.C. on all of $\mathbf{P}^1_{\mathbf{C}}$-$\{0,\infty\}$ which has regular singularities at 0. If we replace $\mathbf{P}^1_{\mathbf{C}}$ by a projective smooth connected complex curve C of genus $g > 0$, and fix two distinct points $0, \infty$ of C, it remains true that any C.M.I.C. on the punctured formal neighborhood of ∞ in C extends to a C.M.I.C. on all of C-$\{0,\infty\}$ which has regular singularities at 0. However, the failure of C to be simply connected makes it seem unnatural to try to specify a particular such extension.

In this paper, we give the analogous theory of such extensions for the characteristic p analogues of C.M.I.C. in characteristic zero, namely lisse etale sheaves, or representations of pro-finite fundamental groups. In this analogy, a C.M.I.C. on the punctured formal neighborhood of a point ∞ in a complex curve $C_{\mathbf{C}}$ corresponds to a representation of the local Galois group at a rational point ∞ of a proper smooth geometrically connected curve $C_{\mathbf{K}}$ over a separably closed field K of characteristic $p > 0$. A C.M.I.C. on $C_{\mathbf{C}}$-$\{0,\infty\}$ corresponds in this analogy, to a representation of the pro-finite fundamental group of $C_{\mathbf{K}}$-$\{0,\infty\}$. The condition « regular singularities at 0 » corresponds to the condition « tame ramification at 0 ».

An especially interesting aspect of the characteristic p theory is the possibility of working over a field K which is not separably closed,

especially a finite field. As one might expect from looking at the complex situation, where one has a *canonical* extension in the $\mathbf{P}_\mathbf{C}^1$ case, one can construct a canonical extension in the $\mathbf{P}_\mathbf{K}^1$ case over any ground-field of characteristic p. For curves of higher genus, one can deduce a weak but useful extension theorem over any groundfield K at the expense of ignoring finitely many other points, simply by mapping the curve in question to $\mathbf{P}_\mathbf{K}^1$ in a reasonable way (cf. 1.7). However, in order to obtain the general extension theorem alluded to above on a curve of genus $g > 0$, we are forced to work over a separably closed groundfield, « jusqu'à nouvel ordre ».

The paper is divided into two chapters. The first is devoted to constructing the canonical extension in the $\mathbf{P}_\mathbf{K}^1$ case, where K is any field of characteristic $p > 0$ (strictly speaking, the theory we develop is also valid in characteristic zero, but there it is without interest). We construct our canonical extension by introducing an *a priori* notion of « special » finite etale covering of $\mathbf{G}_{m,K} = \mathbf{P}_\mathbf{K}^1 - \{0, \infty\}$, and showing that the category of such special coverings is equivalent, by the functor « restriction to the punctured formal neighborhood of ∞ », to the category of all finite etale coverings of this punctured neighborhood.

The first three sections of Chapter I develop the formalism of special coverings. The main result and some immediate corollaries are given in the fourth section. Section 5 gives the application to « canonical extensions » of sheaves. In Section 6, we use the canonical extension to give a global cohomological construction of the Swan representation. The final section gives an analogous, but less canonical, extension result for curves of higher genus, still over an arbitrary ground-field.

The second chapter is devoted to proving the extension theorem 2.16 for curves of any genus over a separably closed ground-field k of characteristic $p > 0$. We study this as a problem of « interpolating » to a global representation a finite number of local representations given at distinct points on the curve. The main technical result is (2.1.5), which roughly speaking says that p-groups pose no obstruction to this sort of interpolation theorem (this is a standard manifestation of the fact that the p-cohomological dimension of a smooth curve over a separably closed field of characteristic p is $\leqslant 1$, while « obstructions lie in an H^2 »). For the sake of completeness, we also give a rather strong interpolation theorem 2.1.4., originally proven by Harbater (cf. [Ha], 2.7), for representations to p-groups, again based on the vanishing of suitable H^2's.

It is a pleasure to acknowledge my overwhelming debt to Ofer Gabber, without whom this paper could not have been written. The results of the first chapter emerged in the course of many fruitful discussions with him, and those of the second chapter (except for (2.1.4)) are due to him alone (though I of course take full responsibility for any defects in their presentation).

1.

1.1. Tameness at Zero.

(1.1.1) Let K be a field, T an indeterminate, $A^1_K = \text{Spec}(K[T])$ the affine line over K, and $G_{m,K} = \text{Spec}(K[T,T^{-1}])$ the multiplicative group over K. For every integer $N \geq 1$, we denote by

$$[N]: G_{m,K} \to G_{m,K}$$

the K endomorphism « N » of the group-scheme $G_{m,K}$, given by $T \mapsto T^N$.

(1.1.2) Let $N \geq 1$ be an integer which is invertible in K.

A finite etale $G_{m,K}$-scheme

$$E \to G_{m,K}$$

is called « N-tame at 0 » if the pull-back $[N]^*(E)$ of E by $[N]$ extends to a finite etale covering $\tilde{E}_N$ of A^1_K. Notice that the extended covering $\tilde{E}_N$ is necessarily unique if it exists; it is just the normalization of A^1_K in $[N]^*(E)$:

$$
\begin{array}{ccccc}
E & \longleftarrow & [N]^*E & \hookrightarrow & \tilde{E}_N \\
\downarrow & & \downarrow & & \downarrow \\
G_{m,K} & \xleftarrow{[N]} & G_{m,K} & \hookrightarrow & A^1_K \ .
\end{array}
$$

The property of N-tameness at 0 is geometric in the sense that for any overfield L of K, E is N-tame at 0 if and only if $E \otimes_K L \to G_{m,L}$ is N-tame at 0.

To see this, we may replace E by $[N]^*E$ and reduce to the case $N = 1$. The only if direction is trivial ($\tilde{E} \otimes_K L$ extends $E \otimes_K L$).

Conversely, if $E \underset{K}{\otimes} L$ extends, it already extends over a finitely generated sub-extension L_0 of K. Let L_{00} be a maximal separable-over-K subfield of L_0. Then L_0 is a finite purely inseparable extension of L_{00}, so $A^1_{L_0} \to A^1_{L_{00}}$ and $G_{m,L_0} \to G_{m,L_{00}}$ are finite radicial, hence by « topological invariance of the etale site » (SGA 4, VIII, 1.1), the covering already extends over L_{00}. But L_{00} is a direct limit of smooth K-algebras, so formation of the normalization of A^1_K in E commutes with the extension of scalars $K \to L_{00}$. Denoting this normalization by $\tilde{E}$, we thus find that $\tilde{E} \underset{K}{\otimes} L_{00}$ is finite etale over $A^1_{L_{00}}$, whence $\tilde{E}$ is finite etale over A^1_K by descent (SGA 1, IX, 4.1), as required.

(1.1.3) A finite etale covering of $G_{m,K}$ is called « tame at 0 » if there exists an integer $N \geqslant 1$ invertible in K for which it is N-tame at zero.

The tame-at-0 finite etale coverings of $G_{m,K}$ form a full Galois subcategory of the Galois category (cf. SGA 1, Exp. V, § 4,5) of all finite etale coverings of $G_{m,K}$. Therefore for any geometric point (cf. SGA1, Exp. V, § 7)$\bar{x}$ of $G_{m,K}$, the category of tame-at-0 finite etale coverings is equivalent (by the functor $E \mapsto E_{\bar{x}}$) to the category of finite sets together with a continuous action of a suitable pro-finite quotient group $\pi_1(G_{m,K}, \bar{x})$(tame at 0) of $\pi_1(G_{m,K}, \bar{x})$.

(1.1.4) Let K^{sep} be a separable closure of K. The Galois group $\mathrm{Gal}(K^{sep}/K)$ operates on $G_{m,K^{sep}} = \mathrm{Spec}(K^{sep}[T, T^{-1}])$ through its action on K^{sep} alone; $\sigma \in \mathrm{Gal}$ maps $\Sigma a_i T^i$ to $\Sigma \sigma(a_i) T^i$, and $\sigma \mapsto \mathrm{Spec}(\sigma^{-1})$ is a left action of $\mathrm{Gal}(K^{sep}/K)$ on $G_{m,K^{sep}}$. For any finite etale $E \overset{\pi}{\to} G_{m,K^{sep}}$, and any $\sigma \in \mathrm{Gal}$, we denote by E^σ the finite etale covering of $G_{m,K^{sep}}$ which is the composite $E \overset{\pi}{\longrightarrow} G_{m,K^{sep}} \xrightarrow{\mathrm{Spec}(\sigma^{-1})} G_{m,K^{sep}}$.

Then $E \mapsto E^\sigma$ defines a left action of $\mathrm{Gal}(K^{sep}/K)$ on the category of finite etale coverings of $G_{m,K^{sep}}$. If we pick a geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to G_{m,K}$ and take for K^{sep} the separable closure of K in Ω, we have a well-known short exact sequence (SGA 1, IX, 6.1).

(1.1.4.1) $1 \to \pi_1(G_{m,K^{sep}}, \bar{x}) \to \pi_1(G_{m,K}, \bar{x}) \to \mathrm{Gal}(K^{sep}/K) \to 1$,

in which the action modulo inner automorphism of Gal on $\pi_1(G_{m,K^{sep}}, \bar{x})$ is induced by its action $E \mapsto E^\sigma$ on the category of finite etale coverings of $G_{m,K^{sep}}$.

(1.1.5) For any element $\sigma \in \mathrm{Gal}$, a finite etale covering E of $\mathbf{G}_{m,K^{\mathrm{sep}}}$ is tame at 0 (resp. N-tame-at-0 for a given N) if and only if E^σ is (for $\tilde{E}_N^{(\sigma)}$ extends $[N]^*(E^\sigma)$ iff $\tilde{E}_N$ extends $[N]^*(E)$). Therefore the kernel of the canonical projection

$$\pi_1(\mathbf{G}_{m,K^{\mathrm{sep}}},\bar{x}) \twoheadrightarrow \pi_1(\mathbf{C}_{m,K^{\mathrm{sep}}},\bar{x})(\text{tame at } 0)$$

is normal in $\pi_1(\mathbf{G}_{m,K},\bar{x})$, and (because tameness-at-0 is geometric) this kernel is equal to the corresponding kernel over K. Thus we obtain a short exact sequence of tame-at-0 $\pi_1's$

(1.1.5.1) $1 \rightarrow \pi_1(\mathbf{G}_{m,K^{\mathrm{sep}}},\bar{x})(\text{tame at } 0) \rightarrow \pi_1(\mathbf{G}_{m,K},\bar{x}) \ (\text{tame at } 0) \rightarrow$
$$\mathrm{Gal}\,(K^{\mathrm{sep}}/K) \rightarrow 1.$$

1.2. Monodromy and Geometric Monodromy.

(1.2.1) Let X be a connected scheme, $\bar{x}$ a geometric point of X, and $E \rightarrow X$ a finite etale covering of X. The fibre $E(\bar{x})$ over $\bar{x}$ is a finite set on which $\pi_1(X,\bar{x})$ acts continuously. The image of $\pi_1(X,\bar{x})$ in $\mathrm{Aut}\,(E(\bar{x}))$ is called the monodromy group of $E \rightarrow X$ at $\bar{x}$. If $\bar{y}$ is a second geometric point of X, any « chemin » from $\bar{x}$ to $\bar{y}$ induces a bijection $E(\bar{x}) \xrightarrow{\sim} E(\bar{y})$ which carries the monodromy group at $\bar{x}$ isomorphically to that at $\bar{y}$.

(1.2.2) Let K be a field, and suppose that X is a geometrically connected K-scheme. For any separably closed over-field L of K, and any geometric point $\bar{x}$ of $X \underset{K}{\otimes} L$, the image of $\pi_1(X \underset{K}{\otimes} L,\bar{x})$ in $\mathrm{Aut}\,(E(\bar{x}))$ is called the geometric monodromy group at $\bar{x}$. It is a subgroup of the monodromy group at $\bar{x}$. As above, this group is, for given L, independent of the base point $\bar{x}$ up to an isomorphism which is unique up to inner automorphism. It is also « independent of L », for if we denote by K^{sep} the separable closure of K in L, and by $\bar{y}$ the image of $\bar{x}$ in $X \underset{K}{\otimes} K^{\mathrm{sep}}$, the natural map of $\pi_1's$
$$\pi_1(X \underset{K}{\otimes} L,\bar{x}) \rightarrow \pi_1(X \underset{K}{\otimes} K^{\mathrm{sep}},\bar{y})$$

is surjective (for if $Z \rightarrow X \underset{K}{\otimes} K^{\mathrm{sep}}$ is a connected finite etale covering, then Z is a connected K^{sep}-scheme, so remains connected after any field extension (EGA IV, 4,5.21)).

(1.2.3) If $E \to X$ is a finite etale covering, then there exists a finite Galois extension K'/K such that the geometric monodromy group of $E \to X$ is *equal* to the monodromy group of $E \underset{K}{\otimes} K' \to X \underset{K}{\otimes} K'$. For if we denote by G the geometric monodromy group, then there exists a finite etale connected G-torsor $Z \to X \underset{K}{\otimes} K^{\text{sep}}$ over which $E \to X$ splits completely. Just descend this G-torsor to a finite etale connected G-torsor $Z_0 \to X \underset{K}{\otimes} K'$ over a finite Galois extension K' of K which still splits $E \to X$.

1.3. Special Coverings.

(1.3.1) Let K be a field of characteristic exponent p. We say that a finite etale covering $E \to G_{m,K}$ is « special » if it is tame at zero and if its geometric monodromy group has a unique p-Sylow subgroup. Notice that the class of finite groups having a unique p-Sylow subgroup is stable under formation of subgroups, quotients, and finite products, and that any extension of a group of order prime — to-p by such a group is again one. [When $p = 1$, i.e. when K has characteristic zero, then by the « 1-Sylow subgroup » of a finite group we mean the subgroup with one element. By the known structure of the fundamental group of $G_{m,K}$ when K has characteristic zero, *every* finite etale covering of $G_{m,K}$ is « special ».]

LEMMA 1.3.2. — *Let* K *be a field and* $E \to C_{m,K}$ *a finite etale covering. Then the following conditions are equivalent :*

1) $E \to G_{m,K}$ *is special,*

2) *there exists an overfield* L *of* K *such that* $E \underset{K}{\otimes} L \to G_{m,L}$ *is special,*

3) *for every overfield* L *of* K, $E \underset{K}{\otimes} L \to G_{m,L}$ *is special,*

4) *there exists an integer* $N \geqslant 1$ *prime to* p *such that the inverse image* $[N]^*(E)$ *of* E *by the* N'*th power map* $[N] : G_{m,K} \to G_{m,K}$ *extends to a finite etale covering* $\tilde{E}_N \to A_K^1$ *whose geometric monodromy group is a p-group :*

$$
\begin{array}{ccccc}
E & \longleftarrow & [N]^*E & \hookrightarrow & \tilde{E}_N \\
\downarrow & & \downarrow & & \downarrow \\
G_{m,K} & \underset{[N]}{\longleftarrow} & G_{m,K} & \hookrightarrow & A_K^1 \;.
\end{array}
$$

5) *for some integer* $N \geqslant 1$ *prime to* p, *the covering* $[N]^*E \to G_{m,K}$ *is special,*

6) *for every integer* $N \geqslant 1$ *prime to* p, *the covering* $[N]^*E \to G_{m,K}$ *(in the notations of* 4) *above) is special.*

7) *for every* $a \in K^\times = G_m(K)$, *the inverse image* Trans_a^*E *of* E *by the automorphism* $\text{Trans}_a : G_{m,K} \xrightarrow{\sim} G_{m,K}$ *defined by* $T \mapsto aT$ *is special :*

$$
\begin{array}{ccc}
E & \longleftarrow & \text{Trans}_a E \\
\downarrow & & \downarrow \\
G_{m,K} & \underset{\text{Trans}_a}{\longleftarrow} & G_{m,K}
\end{array}
$$

8) *There exists an integer* $N \geqslant 1$ *prime to* p, *and a finite Galois extension* K'/K *such that* K' *contains* N *distinct* N'th *roots of unity, such that the inverse image* $[N]^*(E \underset{K}{\otimes} K')$ *of* $E \to G_{m,K}$ *by the composite*

$$
G_{m,K'} \xrightarrow{\ [N]\ } G_{m,K^1} \to G_{m,K}
$$

extends to a finite etale covering $\tilde{E}_N \underset{K}{\otimes} K' \to A_{K'}^1$, *whose monodromy group is a p-group.*

Proof. — The equivalence of 1), 2), and 3) is just the invariance of both the geometric monodromy group and of « tameness at zero » under field extension. The implication 4) ⇒ 1) is obvious, and 1) ⇒ 4) holds because over an algebraically closed field L, for any integer $N \geqslant 1$ prime to p, the unique open normal subgroup of $\pi_1(G_{m,L}, \bar{x})$ of index N is the one corresponding to the N'th power covering $[N]$ of $G_{m,L}$ by itself. The equivalence 4) ⟺ 8) follows from (1.2.3). The implications 1) ⇒ 6) ⇒ 5) are obvious, and 5) ⇒ 4) by applying the implication 1) ⇒ 4) to the covering $[N]^*E$.

The implication 7) ⇒ 1) is trivial (take $a = 1$). To prove 1) ⇒ 7), we may suppose K algebraically closed, and that E satisfies 4) for some integer $N \geqslant 1$ prime to p. Given $a \in K^\times$, choose $b \in K^\times$ with $b^N = a$. Then $\text{Trans}_a^*(E)$ satisfies 4) with the same N (the extension is provided by the inverse image of $\tilde{E}_N$ by the automorphism $T \mapsto bT$ of A^1).

<div align="right">Q.E.D.</div>

(1.3.3) The special coverings of $C_{m,K}$ form a full Galois sub-category of the Galois category of all finite etale tame-at-zero coverings of $G_{m,K}$.

Therefore, for any geometric point $\bar{x}$ of $G_{m,K}$, the category of special coverings is equivalent (by the functor $E \mapsto E_{\bar{x}}$) to the category of finite sets together with a continuous action of a suitable quotient pro-finite group $\pi_1(G_{m,K}, \bar{x})$(special) of $\pi_1(G_{m,K}, \bar{x})$(tame at 0), itself a quotient of $\pi_1(G_{m,K}, \bar{x})$.

The quotient $\pi_1(G_{m,K^{sep}}, \bar{x})$(special) of $\pi_1(G_{m,K^{sep}}, \bar{x})$(tame at 0) is defined by purely group theoretic conditions (it is the maximal pro- « group with unique p-Sylow subgroup » quotient). Therefore the kernel of the projection

$$\pi_1(G_{m,K^{sep}}, \bar{x})(\text{tame at } 0) \twoheadrightarrow \pi_1(G_{m,K^{sep}}, \bar{x})(\text{special})$$

is invariant by any continuous automorphism of $\pi_1(G_{m,K^{sep}}, \bar{x})$. In particular, it is normal in $\pi_1(G_{m,K}, \bar{x})$(tame at 0), and (because specialness is geometric) this kernel is equal to the corresponding kernel over K. Therefore we obtain from the short exact sequence of tame-at-0 $\pi_1's$ a short exact sequence of special $\pi_1's$

(1.3.3.1) $\quad 1 \rightarrow \pi_1(G_{m,K^{sep}}, \bar{x})(\text{special}) \rightarrow \pi_1(G_{m,K}, \bar{x})$ (special) $\rightarrow$
$$\text{Gal}(K^{sep}/K) \rightarrow 1.$$

1.4. The Main Theorem.

For any field K, we denote by $K((T^{-1}))$ the field of finite-tailed Laurent series over K in the variable T^{-1}, i.e., $K((T^{-1}))$ is the fraction field of $K[[T^{-1}]]$. We will always view $K[T, T^{-1}]$ as the sub-ring of $K((T^{-1}))$ consisting of the Laurent polynomials in T^{-1}. Geometrically, the corresponding morphism

$$\text{Spec}(K((T^{-1}))) \rightarrow G_{m,K} = \text{Spec}(K[T, T^{-1}])$$

is the inclusion into $G_{m,K}$ of the punctured formal neighborhood of ∞ in $\mathbf{P}^1_K$.

MAIN THEOREM 1.4.1. — *Let K be a field. Then the inverse image functor*

$$\begin{pmatrix} \textit{special finite etale coverings} \\ \textit{of } G_{m,K} = \text{Spec}(K[T, T^{-1}]) \end{pmatrix} \rightarrow \begin{pmatrix} \textit{finite etale coverings} \\ \textit{of } \text{Spec}(K((T^{-1}))) \end{pmatrix}$$

is an equivalence of categories.

Proof. — Let us fix an integer $N \geqslant 1$ which is invertible in K, and a finite Galois extension K'/K such that K' contains N distinct N'th roots of unity. Let us denote by

$\mathscr{A}(N,K')$: the full subcategory of all finite etale coverings of $G_{m,K}$ whose inverse image under the composite $G_{m',K'} \xrightarrow{[N]} G_{m,K'} \to G_{m,K}$ extends to a finite etale covering of $A_{K'}^1$ whose monodromy group is a p-group.

$\mathscr{B}(N,K')$: the full subcategory of all finite etale coverings of $\text{Spec}(K((T^{-1})))$ whose inverse image under the composite map

$\text{Spec}(K'((T^{-1}))) \xrightarrow{[N]} \text{Spec}(K'((T^{-1}))) \to \text{Spec}(K((T^{-1})))$

has monodromy group a p-group.

The inverse image functor induces a functor

$$\mathscr{A}(N,K') \to \mathscr{B}(N,K')$$

for each pair (N,K') as above. By 1) $\Leftrightarrow$ 8) of 1.3.2, the category of special coverings of $G_{m,K}$ is the direct limit of the categories $\mathscr{A}(N,K')$. By the theory of local fields, the category of finite etale coverings of $\text{Spec}(K((T^{-1})))$ is the direct limit of the categories $\mathscr{B}(N,K')$.

Thus we are reduced to showing that for each (N,K') as above, the induced functor is an equivalence. For fixed (N,K'), the semi-direct product

$$G = |\mu_N(K') \ltimes \text{Gal}(K'/K)$$

operates on both $G_{m,K'} = \text{Spec}(K'[T,T^{-1}])$ and on $\text{Spec}(K'((T^{-1})))$, by the rule $g \mapsto \text{Spec}(g^{-1})$, where $g = (\zeta,\sigma)$ operates on the coordinate rings by

$$(\zeta,\sigma) : \Sigma a_i T^{-i} \to \Sigma \sigma(a_i) \zeta^{-i} T^{-i}.$$

It also operates on $A_{K'}^1 = \text{Spec}(K[T])$, by $g \to \text{Spec}(g^{-1})$, $g = (\zeta,\sigma)$ acting by

$$(\zeta,\sigma) : \Sigma a_i T^i \to \Sigma \sigma(a_i) \zeta^i T^i.$$

By elementary descent theory, and the uniqueness of the extension

to $\mathbf{A}^1$, the two functors

$$\mathscr{A}(N,K') \rightarrow \begin{pmatrix} \text{finite etale coverings } E \text{ of } \mathbf{G}_{m,K'} \text{ which extend to} \\ \text{finite etale coverings of } \mathbf{A}'_{K'}, \text{ and have monodromy} \\ \text{group a } p\text{-group, together with an action of } G \text{ on} \\ E \text{ covering its action on } \mathbf{G}_{m,K'}. \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} \text{finite etale coverings } \tilde{E} \text{ of } \mathbf{A}^1_{K'} \text{ with monodromy} \\ \text{group a } p\text{-group, together with an action of } G \text{ on} \\ \tilde{E} \text{ coverings its action on } \mathbf{A}^1_{K'}. \end{pmatrix}$$

defined by

$$E \mapsto [N]^*(E \underset{K}{\otimes} K') \text{ with its canonical } G\text{-action}$$

$$\Big\uparrow$$

the unique extension to $\mathbf{A}^1_{K'}$ of this data

are equivalences of categories.

Similarly, elementary descent theory shows that the functor

$$\mathscr{B}(N,K') \rightarrow \begin{pmatrix} \text{finite etale coverings } E \text{ of } \mathrm{Spec}(K'((T^{-1}))) \text{ whose} \\ \text{monodromy group is a } p\text{-group, with an action of } G \\ \text{on } E \text{ covering its action on } \mathrm{Spec}(K'((T^{-1}))) \end{pmatrix}$$

$$E \mapsto [N]^*(E \underset{K}{\otimes} K') \text{ with its canonical } G\text{-action}$$

is an equivalence of categories.

Now consider the inverse image functor

$$\begin{pmatrix} \text{finite etale coverings of} \\ \mathbf{A}^1_{K'} = \mathrm{Spec}(K'[T]) \quad \text{with} \\ \text{monodromy group a } p\text{-} \\ \text{group} \end{pmatrix} \rightarrow \begin{pmatrix} \text{finite etale coverings of} \\ \mathrm{Spec}(K'((T^{-1}))) \quad \text{with} \\ \text{monodromy group a } p\text{-} \\ \text{group.} \end{pmatrix}$$

This functor is visibly compatible with the action of the group G on both source and target. So if it is an equivalence, it automatically induces an equivalence between the « G-equivariant objects » of its source and target. Thus we are reduced to the case $(N,K') = (1,K)$, over the field K'. If the characteristic exponent $p = 1$, there is nothing to prove. It remains to

treat the case when p is a prime number, in which case the result holds in greater generality.

PROPOSITION 1.4.2. — *Let p be a prime number, R an F_p-algebra with connected spectrum. Then the rings R[T] and $R((T^{-1}))$ have connected spectra, and the universe image functor*

$$\begin{pmatrix} \text{finite etale coverings of} \\ A_R^1 = \text{Spec(R[T])} \quad \text{with} \\ \text{monodromy group a p-group} \end{pmatrix} \rightarrow \begin{pmatrix} \text{finite etale coverings of} \\ \text{Spec}(R((T^{-1}))) \text{ with mono-} \\ \text{dromy group a p-group} \end{pmatrix}$$

is an equivalence of categories.

Proof. — For the connectedness, one checks directly that for any ring R, the inclusions $R \hookrightarrow R[T] \hookrightarrow R((T^{-1}))$ induce bijections on idempotents. Let $\bar{x}$ be a geometric point of $\text{Spec}(R((T^{-1})))$, $\bar{y}$ its image in A_R^1. We must show that the dual map of fundamental groups

$$\pi_1 (\text{Spec}(R((T^{-1}))), \bar{x}) \rightarrow \pi_1(A_R^1, \bar{y})$$

induces an isomorphism of maximal pro-p quotients. Let us temporarily admit the truth of the following well-known lemma.

LEMMA 1.4.3. — *For any connected affine F_p-scheme Z, with geometric point $\bar{z}$, one has*

$$H^q(Z_{et}, F_p) \xleftarrow{\ \sim\ } H^q(\pi_1(Z, \bar{z}), F_p) \quad \text{for all } q \geqslant 0$$

$$H^q(Z_{et}, F_p) = 0 \quad \text{for } q \geqslant 2$$

so in particular one has

$$H^2(\pi_1(Z, \bar{z}), F_p) = 0.$$

(1.4.4) From the vanishing of $H^2(\pi_1(Z, \bar{z}), F_p)$, it follows (cf. 2.3.8.1) that the maximal pro-p quotient of $\pi_1(Z, \bar{z})$ is a free pro-p group. Thus both of the maximal pro-p quotients in question are free pro-p groups. But a homomorphism between free pro-p groups is an isomorphism if and only if it induces an isomorphism on the groups $H^1(-, F_p)$ of continuous character to F_p (cf. (2.3.7)). Because the continuous characters to F_p of the maximal pro-p quotient of $\pi_1(Z, \bar{z})$ are just the same as the continuous F_p-valued characters of $\pi_1(Z, \bar{z})$ itself, the interpretation of $H^1(Z_{et}, F_p)$ as

the group of these characters reduces us to showing that the map

$$H^1((A_R^1)_{et}, F_p) \rightarrow H^1(Spec(R((T^{-1})))_{et}, F_p)$$

is an isomorphism.

(1.4.5) For any F_p-algebra B, we denote by $F : B \rightarrow B$ the absolute Frobenius endomorphism $F(b) = b^p$, and by $\mathscr{P} = 1 - F$ the additive mapping $\mathscr{P}(b) = b - b^p$ of B to itself. The Artin-Schreier short exact sequence of F_p-sheaves on Z_{et}, $Z = Spec(A)$ any affine F_p-scheme,

$$0 \rightarrow F_p \rightarrow G_a \xrightarrow{\ \mathscr{P}\ } G_a \rightarrow 0$$

yields a long exact cohomology sequence in which (cf. SGA 4, IX 3.5 and VII 4.3)

$$H^i(Z_{et}, G_a) \xleftarrow{\ \sim\ } \Big| H^i(Z_{Zar}, \mathcal{O}_Z) = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i > 0, \end{cases}$$

whence we obtain

$$H^0(Z_{et}, F_p) \xrightarrow{\ \sim\ } \text{Ker} (\mathscr{P} : A \rightarrow A)$$

$$H^1(Z_{et}, F_p) \xrightarrow{\ \sim\ } A/\mathscr{P}A, \text{ the « Artin-Schreier quotient » of } A$$

$$H^q(Z_{et}, F_p) = 0 \quad \text{for } q \geqslant 2.$$

Thus we are reduced to observing that the inclusion of rings

$$R[T] \hookrightarrow R((T^{-1})) = R[T] \oplus T^{-1}R[[T^{-1}]]$$

induces a bijection on Artin-Schreier quotients. But this is clear, because the $\oplus$ decomposition written above is F-stable and F is T^{-1}-topologically nilpotent on $T^{-1}R[[T^{-1}]]$. Therefore $\mathscr{P} = 1 - F$ respects the $\oplus$ decomposition and is bijective on $T^{-1}R[[T^{-1}]]$.

(1.4.6) It remains to prove Lemma 1.4.3. Thus let Z be an affine connected F_p-scheme, $\bar{z}$ a geometric point of Z. We have already seen in 1.4.5 above that Artin-Schreier theory yields

$$H^q(Z_{et}, F_p) = 0 \quad \text{for } q \geqslant 2.$$

Now let $Z' \rightarrow Z$ be a finite etale connected Galois covering of Z, corresponding to an open normal subgroup of $\pi_1(Z, \bar{z})$. Then Z' is still

affine, so we have

$$H^q(Z'_{et}, F_p) = 0 \quad \text{for } q \geqslant 2.$$

The interpretation of $H^1(Z'_{et}, F_p)$ as the continuous homs to F_p of the open normal subgroup of $\pi_1(Z, \bar{z})$ corresponding to Z' shows that

$$\lim_{\to} H^1(Z'_{et}, F_p) = 0,$$

the direct limit taken over all open normal subgroup of $\pi_1(Z, \bar{z})$. For each $Z' \to Z$ as above, with covering group G, the Hochschild-Serre spectral sequence with F_p coefficients

$$E_2^{a,b} = H^a(G, H^b(Z'_{et}, F_p)) \Rightarrow H^{a+b}(Z_{et}, F_p)$$

has

$$\begin{cases} E_2^{a,0} = H^a(G, F_p) \quad \text{(because } Z' \text{ is connected)} \\ E_2^{a,1} = H^a(G, H^1(Z'_{et}, F_p)) \\ E_2^{a,b} = 0 \quad \text{if } b \neq 0,1. \end{cases}$$

Passing to the direct limit of these spectral sequences over all open normal subgroups of $\pi_1(Z, \bar{z})$, we obtain a spectral sequence

$$E_2^{a,b} \Rightarrow H^{a+b}(Z_{et}, F_p)$$

in which $E_2^{a,b} = 0$ for $b \neq 0$, and in which

$$E_2^{a,0} = \lim_{\to} H^a(G, F_p) \stackrel{\text{dfn}}{=} H^a(\pi_1(Z, \bar{z}), F_p). \qquad \text{Q.E.D.}$$

COROLLARY 1.4.7. — *Let* K *be a field,* Ω *an algebraically closed overfield of* $K((T^{-1}))$ *viewed as a geometric point* $\bar{x}$ *of* $\text{Spec}(K((T^{-1})))$, $K((T^{-1}))^{\text{sep}}$ *the separable closure of* $K((T^{-1}))$ *inside* Ω, *and* $\bar{y}$ *the image of* $\bar{x}$ *in* $C_{m,K}$. *The induced map of* π_1 *'s*

$$\text{Gal}(K((T^{-1}))^{\text{sep}}/K((T^{-1})))$$
$$= \pi_1(\text{Spec}(K((T^{-1}))), \bar{x}) \to \pi_1(G_{m,K}, \bar{y})(\text{special})$$

is an isomorphism. □

(1.4.8) Let K^{sep} denote the separable closure of K inside Ω. Then $K^{\text{sep}} \otimes_K K((T^{-1}))$ is the union of all the subfields $K'((T^{-1}))$ with K'/K a

finite Galois sub-extension of K^{sep}/K. Thus it is the fraction field of a henselien discrete valuation ring, hence has the same Galois theory as its completion $K^{sep}((T^{-1}))$. This remark allows us to choose a geometric point $\bar{x}_1$ of $\mathrm{Spec}\,(K^{sep}((T^{-1})))$ lying over $\bar{x}$, and then to interpret the short exact sequence of π_1's

$$1 \rightarrow \pi_1 (\mathrm{Spec}\,(K^{sep} \underset{K}{\otimes} K((T^{-1}))),\bar{x}) \rightarrow$$
$$\pi_1 (\mathrm{Spec}\,(K((T^{-1}))),\bar{x}) \rightarrow \mathrm{Gal}\,(K^{sep}/K) \rightarrow 1$$

as an exact sequence

$$(1.4.8.1)\quad 1 \rightarrow \pi_1 (\mathrm{Spec}\,(K^{sep}((T^{-1}))),\bar{x}_1) \rightarrow \pi_1 (\mathrm{Spec}\,(K((T^{-1}))),\bar{x}) \rightarrow$$
$$\mathrm{Gal}\,(K^{sep}/K) \rightarrow 1.$$

COROLLARY 1.4.9. — *The isomorphisms of the preceding corollary for* K *and* K^{sep} *sit is a commutative diagram*

$$1 \rightarrow \pi_1 (\mathrm{Spec}\,(K^{sep}((T^{-1}))),\bar{x}_1) \rightarrow \pi_1 (\mathrm{Spec}\,(K((T^{-1}))),\bar{x}) \rightarrow \mathrm{Gal}\,(K^{sep}/K) \rightarrow 1$$

$$1 \rightarrow \pi_1(G_{m,K^{sep}},\bar{y})(\text{special}) \rightarrow \pi_1(G_{m,K},\bar{y})(\text{special}) \rightarrow \mathrm{Gal}\,(K^{sep}/K) \rightarrow 1.$$

COROLLARY 1.4.10 (*Retraction Theorem*). —

1. *The composite homorphism*

$$\pi_1 (G_{m,K^{sep}},\bar{y}_1)(\textit{tame at } 0) \twoheadrightarrow \pi_1 (G_{m,K^{sep}},\bar{y}_1)(\textit{special})$$

$$\left(\Big\downarrow \begin{array}{l}\textit{inverse of the}\\ \textit{can. isom.}\end{array}\right.$$

$$\pi_1 (\mathrm{Spec}\,(K^{sep}((T^{-1}))),\bar{x}_1)$$

is the unique continuous retraction of the canonical homomorphism

$$\pi_1 (\mathrm{Spec}\,(K^{sep}((T^{-1}))),\bar{x}_1) \rightarrow \pi_1 (G_{m,K^{sep}},\bar{y}_1) \,(\textit{tame at } 0).$$

2. *The composite homomorphism*

$$\pi_1 (G_{m,K},\bar{y})(\textit{tame at } 0) \twoheadrightarrow \pi_1 (G_{m,K},\bar{y})(\textit{special})$$

$$\left(\Big\downarrow \begin{array}{l}\textit{inverse of the}\\ \textit{can. isom.}\end{array}\right.$$

$$\pi_1 (\mathrm{Spec}\,(K(T^{-1}))),\bar{x})$$

is the unique continuous retraction of the canonical homomorphism

$$\pi_1 (\text{Spec} (K((T^{-1}))), \bar{x}) \;\to\; \pi_1 (G_{m,K}, \bar{y})(\textit{tame at } 0)$$

which maps $\pi_1(G_{m,K^{\text{sep}}}, \bar{y}_1)$(*tame at* 0) *to* $\pi_1 (\text{Spec} (K^{\text{sep}}((T^{-1}))), \bar{x}_1)$ (*its restriction to this group is necessarily the retraction in 1. above*).

Proof. — The profinite group $\pi_1 (\text{Spec} (K^{\text{sep}}((T^{-1}))), \bar{x}_1)$ has a unique p-Sylow subgroup.

Q.E.D.

COROLLARY 1.4.11. — *The canonical homomorphism*

$$\pi_1 (\text{Spec} (K((T^{-1}))), \bar{x}) \;\to\; \pi_1 (G_{m,K}, \bar{y})$$

is injective, and admits a continuous retraction.

Proof. — Just compose with the projection onto $\pi_1(G_{m,K}, \bar{y})$(special).

Q.E.D.

COROLLARY 1.4.12 (Local monodromy at zero of special coverings). — *Let* K *be a separably closed field,* $\bar{x}$ *a geometric point of* Spec (K((T))), $\bar{y}$ *its image in* $G_{m,K}$, $E \to G_{m,K}$ *a special covering,*

$$\rho : \pi_1(G_{m,K}, \bar{y}) \;\twoheadrightarrow\; G \subset \text{Aut}(E(\bar{y}))$$

its monodromy representation, G(p) *the p-Sylow subgroup of* G, *and* $H = \rho(\pi_1(\text{Spec} (K((T))), \bar{x})) \subset G$ *the image of the inertia group at zero. Then* H *is a cyclic group of order* $\# G / \# G(p)$ (*and consequently* $G = G(p) \rtimes H$).

Proof. — By tameness at zero, H is a cyclic group of order prime to p, so it is isomorphic to its image in $G/G(p)$. Thus we may suppose G has order N prime to p. But the unique such quotient of $\pi_1(G_{m,K}, \bar{x})$ is $\mu_N(K)$, corresponding to the covering $[N]: G_{m,K} \to G_{m,K}$, and for this covering the assertion is obvious.

Q.E.D.

1.5. Canonical Extension of Lisse Sheaves.

(1.5.1) Let K be a field of characteristic p exponent, $\mathscr{F}$ a lisse etale sheaf of finite sets (resp. of finite groups, resp...) on $G_{m,K}$. We say that $\mathscr{F}$

is special if the finite etale covering of $G_{m,K}$ which represents $\mathscr{F}$ is special. Thus $\mathscr{F}$ is special if and only if it is tame at zero and its geometric monodromy group has a unique p-sylow subgroup.

(1.5.2) More usefully, if we pick a geometric point $\bar{x}$ of $G_{m,K}$, then $\mathscr{F}$ is special if and only if its monodromy homomorphism

$$\varphi : \pi_1(G_{m,K}, \bar{x}) \to \mathrm{Aut}\,(\mathscr{F}(\bar{x}))$$

factors through the quotient $\pi_1(G_{m,K}, \bar{x})(\text{special})$ of $\pi_1(G_{m,K}, \bar{x})$.

LEMMA 1.5.3. — *Let* K *be a field and* A *a finite ring. The category of lisse etale sheaves of left (resp. right)* A-*modules on* $G_{m,K}$ *which are special is a full Abelian subcategory of the category of all lisse etale sheaves of left (resp. right)* A-*modules on* $G_{m,K}$. *If* A *is commutative, it is stable under* $\otimes$ *and indeed under all « operations of linear algebra ».*

Proof. — In terms of a geometric point $\bar{x}$ of $G_{m,K}$, we are considering the full subcategory of all representation of $\pi_1(G_{m,K}, \bar{x})$ on finite left (resp. right) A-modules which factor through a particular quotient $\pi_1(G_{m,K}, \bar{x})(\text{special})$.

Q.E.D.

LEMMA 1.5.4. — *Let* K *be a field,* $\mathscr{F}$ *a lisse sheaf of finite Abelian groups on* $G_{m,K}$, $N \geqslant 1$ *an integer prime to* p, *and* $[N] : G_{m,K} \to G_{m,K}$ *the* N-*th power map* $T \mapsto T^N$. *Then the following conditions are equivalent :*

1) $\mathscr{F}$ *is special.*

2) $[N]^* \mathscr{F}$ *is special.*

3) $[N]_* \mathscr{F}$ *is special.*

Proof. — The equivalence 1) $\Leftrightarrow$ 2) is the already proven (1) $\Leftrightarrow$ 5) $\Leftrightarrow$ 6 of 1.3.2). To prove 1) $\Leftrightarrow$ 3), we may suppose K algebraically closed. By 1) $\Leftrightarrow$ 2) applied to $[N]_* \mathscr{F}$, we are reduced to showing that $\mathscr{F}$ is special if and only if $[N]^*[N]_* \mathscr{F}$ is special. But we have a canonical direct sum decomposition

$$[N]^*[N]_* \simeq \underset{\zeta \in \mu_N(K)}{\otimes} \mathrm{Trans}_\zeta^*(\mathscr{F}),$$

so the result follows from the equivalence 1) $\Leftrightarrow$ 7) of 1.3.2.

Q.E.D.

(1.5.5) Let A be a profinite adic commutative ring, e.g., a complete noetherian local ring with finite residue field. We say that a lisse « sheaf » $\mathcal{F}$ of finitely generated A-modules on $G_{m,K}$ is special if for every finite quotient ring A_0 of A by an open ideal, the lisse sheaf $\mathcal{F} \underset{A}{\otimes} A_0$ of A_0-modules is special.

THEOREM 1.5.6. — *Let* K *be a field,* A *a pro-finite adic commutative ring. Then the inverse image functor*

$$\begin{pmatrix} \text{special lisse sheaves of fin.} \\ \text{gen. A-modules on } G_{m,K} \end{pmatrix} \rightarrow \begin{pmatrix} \text{lisse sheaves of fin. gen. A-} \\ \text{modules on } \operatorname{Spec}(K((T^{-1}))) \end{pmatrix}$$

is an exact A-linear equivalence of categories compatible with

  1) *all constructions of linear algebra,*

  2) *change of pro-finite adic ring* $A \rightarrow A'$,

  3) *extension of ground-field* $K \hookrightarrow L$,

  4) $[N]^*$ *and* $[N]_*$, *for any integer* $N \geqslant 1$ *prime to* p,

  5) *inverse and direct image by translations* $T \mapsto aT$ *by* $a \in K^x = G_m(K)$.

Proof. — If A is finite this follows immediately from its set-theoretic analogue (1.4.1) and the previous lemmas 1.5.3-4. The case of a general A follows from the case of finite A's by passage to the inverse limit, over the discrete finite quotient rings of A.

Q.E.D.

COROLLARY 1.5.7. — *« The » quasi-inverse equivalence of categories*

$$\begin{pmatrix} \text{lisse sheaves of fin. gen.} \\ A\text{-}m o d u l e s \quad o n \\ \operatorname{Spec}(K((T^{-1}))) \end{pmatrix} \rightarrow \begin{pmatrix} \text{« special » lisse sheaves of} \\ \text{fin. gen. A-modules on } G_{m,K} \end{pmatrix},$$

denoted

$$\mathcal{F} \mapsto \mathcal{F}^{\text{can}},$$

is an exact A-linear functor, called « canonical extension » whose formation is compatible with

  1) *all constructions of linear algebra,*

  2) *change of pro-finite adic ring* $A \rightarrow A'$,

  3) *extension of ground-field* $K \rightarrow L$,

4) $[N]^*$ and $[N]_*$, for any integer $N \geqslant 1$ prime to p,

5) inverse and direct images by translations $T \mapsto aT$ by $a \in K^x = G_m(K)$.

In terms of a geometric point $\bar{x}$ of $\mathrm{Spec}\,(K((T^{-1})))$ with image $\bar{y}$ in $G_{m,K}$, the monodromy representation of $\mathscr{F}^{\mathrm{can}}$ at $\bar{y}$ is obtained from that of $\mathscr{F}$ at $\bar{x}$ by composition with the inverse of the canonical isomorphism

$$\pi_1\,(\mathrm{Spec}\,(K((T^{-1}))),\bar{x}) \xrightarrow{\sim} \pi_1(G_{m,K},\bar{y})\ (\text{special}). \qquad \square$$

COROLLARY 1.5.8. — Let R be an integral domain which is finitely generated as a $\mathbf{Z}$-algebra, and M a set of maximal ideals of R. For $m \in M$, denote by $\hat{R}_m$ the m-adic completion of R. Suppose we are given a family $\{\mathscr{F}_m\}_{m \in M}$ of lisse sheaves of free finitely generated $\hat{R}_m$-modules $\mathscr{F}_m$ on $\mathrm{Spec}\,(K((T^{-1})))$, which is compatible in the sense that

> for any geometric point $\bar{x}$ of $\mathrm{Spec}\,(K((T^{-1})))$, any element $\gamma \in \pi_1\,(\mathrm{Spec}\,(K((T^{-1}))),\bar{x})$, and any $m \in M$, the « reversed » characteristic polynomial
>
> $$\det\,(1 - T\gamma \,|\,(\mathscr{F}_m)_{\bar{x}}) \in \hat{R}_m[T]$$
>
> actually lies in $R[T]$, and in $R[T]$ is independent of $m \in M$.

Then their canonical extensions $\{\mathscr{F}_m^{\mathrm{can}}\}_{m \in M}$ on $G_{m,K}$ are compatible in the sense that

> for any geometric point $\bar{y}$ of $G_{m,K}$, any element $\gamma \in \pi_1(G_{m,K},\bar{y})$, and any $m \in M$, the « reversed » characteristic polynomial
>
> $$\det\,(1 - T\gamma \,|\,(\mathscr{F}_m^{\mathrm{can}})_{\bar{y}}) \in \hat{R}_m[T]$$
>
> actually lies in $R[T]$, and in $R[T]$ is independent of $m \in M$.

Proof. — To prove compatibility as above, it suffices to check for a single choice of geometric point (as one sees in joining any two by a « chemin »...). So we may fix a geometric point $\bar{x}$ of $\mathrm{Spec}\,(K((T^{-1})))$, and take for $\bar{y}$ its image in $G_{m,K}$. But in this case the monodromy representations ρ_m^{can} of the $\mathscr{F}_m^{\mathrm{can}}$ are obtained from the monodromy representations ρ_m of the $\mathscr{F}_m$ by composing with the inverse of the

canonical isomorphism

$$\pi_1\left(\mathrm{Spec}\left(K((T^{-1}))\right),\bar{x}\right) \xrightarrow{\sim} \pi_1(G_{m,K},\bar{y})(\text{special}),$$

so the assertion is obvious. Q.E.D.

1.6. Cohomological Construction of the Swan Representation (compare [Lau], 3.6, [Se-1], VI § 4).

(1.6.1) Let K be a separably closed field of characteristic $p > 0$, $j : G_{m,K} \to A^1_K$ the inclusion, ℓ a prime number $\ell \neq p$, and A a finite commutative local ring of residue characteristic ℓ. The functor

$$\begin{pmatrix} \text{lisse sheaves of finitely} \\ \text{generated A-modules on} \\ G_{m,K} \end{pmatrix} \to \begin{pmatrix} \text{finitely generated A-} \\ \text{modules} \end{pmatrix}$$

defined by

$$\mathscr{F} \mapsto H^1(A^1_K, j_! \mathscr{F})$$

is *exact* (the H^0 vanishes trivially and the H^2 vanishes because the cohomological dimension of a smooth affine curve over a separably closed field is ≤ 1).

(1.6.2) Composing the above functor with the functor « canonical extension » of (1.5.7) we obtain an exact functor

$$\begin{pmatrix} \text{lisse sheaf of finitely gene-} \\ \text{rated A-modules on} \\ \mathrm{Spec}\,(K((T^{-1}))) \end{pmatrix} \to \begin{pmatrix} \text{finitely generated A-} \\ \text{modules} \end{pmatrix},$$

defined by

$$\mathscr{F} \mapsto H^1(A^1_K, j_!(\mathscr{F}^{\mathrm{can}})).$$

(1.6.3) Fix a geometric point $\bar{x}$ of $\mathrm{Spec}\,(K((T^{-1})))$, and denote by I_∞ (« inertia group at ∞ ») the group $\pi_1(\mathrm{Spec}\,(K((T^{-1}))),\bar{x})$. In terms of I_∞, the above functor may be viewed as an exact functor

$$\begin{pmatrix} \text{continuous representations} \\ \text{of } I_\infty \text{ on finitely generated} \\ \text{A-modules} \end{pmatrix} \to \begin{pmatrix} \text{finitely generated A-} \\ \text{modules} \end{pmatrix}.$$

(1.6.4) If we further fix a finite quotient G of I_∞ and restrict this last functor to the full sub-category of representations of I_∞ which factor through G, we obtain an exact functor T

$$\begin{pmatrix} \text{finitely presented feft} \\ A[G]\text{-modules} \end{pmatrix} \to \begin{pmatrix} \text{finitely presented} \\ A\text{-modules} \end{pmatrix}.$$

(Recall that A is finite, so « finitely presented » is equivalent to « finitely generated »).

(1.6.5) We now apply to this last exact functor the following general lemma, whose proof is left to the reader. In it, $R = A$ and $S = A[G]$ (cf. [Ka], II).

LEMMA 1.6.6. — *Let* R *and* S *be not-necessarily-commutative rings (associative, with unit), and*

$$T : \begin{pmatrix} \text{finitely presented} \\ \text{left S-modules} \end{pmatrix} \to (\text{left R-modules})$$

an additive covariant right-exact functor. Then S *acts left-S-linearly on itself by right multiplication, so by functoriality* $T(S)$ *is an* (R,S)*-bimodule.*

We have a canonical isomorphism of functors

$$T(S) \underset{S}{\otimes} M \to T(M).$$

The functor T *is exact if and only if* $T(S)$ *is flat as a right S-module. If* $T(S)$ *is finitely presented as a right S-module, then* T *is exact if and only if* $T(S)$ *is a projective right S-module of finite presentation.*

(1.6.7) We now apply this lemma to our situation 1.6.4. Let us denote $\text{Reg}_{G;A}$ the regular representation $A[G]$ of G, viewed as a lisse sheaf of finitely generated A-modules on $\text{Spec}(K((T^{-1})))$. Then with $S = A[G]$, we have

1) $T(A[G]) = H^1(A_K^1, j_!((\text{Reg}_{G;A})^{\text{can}}))$

is a projective right $A[G]$-module of finite presentation.

2) For any left $A[G]$-module M of finite presentation viewed as a lisse sheaf of finitely generated A-modules $\mathscr{F}$ on $\text{Spec}(K((T^{-1})))$, we have a canonical isomorphism of A-modules

$$H^1(A_K^1, j_!((\text{Reg}_{G;A})^{\text{can}})) \underset{A[G]}{\otimes} M \xrightarrow{\sim} H^1(A_K^1, j_! \mathscr{F}^{\text{can}}).$$

Because for each A our functor T is exact, its formation commutes with arbitrary extensions of scalars $A \to A'$ of finite commutative local rings with residue characteristic ℓ. In particular, if $n \gg 0$, then A is a $\mathbf{Z}/\ell^n\mathbf{Z}$-algebra, so we have a canonical isomorphism of right A[G]-modules

$$T(\mathbf{Z}/\ell^n\mathbf{Z})[G]) \underset{\mathbf{Z}_\ell}{\otimes} A \;\xrightarrow{\sim}\; T(A[G]).$$

Passing to the inverse limit over n, we see that

a) $$T(\mathbf{Z}_\ell[G]) \overset{\mathrm{dfn}}{=} \varprojlim_n T((\mathbf{Z}/\ell^n\mathbf{Z})[G])$$

is a projective right $\mathbf{Z}_\ell[G]$-module of finite presentation.

b) For A any complete noetherian local ring with residue characteristic ℓ, and M any finitely generated left A[G]-module, corresponding to a lisse « sheaf » $\mathscr{F}$ of finitely generated A-modules on $\mathrm{Spec}\,(K((T^{-1})))$, we have a canonical isomorphism of A-modules

$$H^1(\mathbf{A}^1_K, j_!((\mathrm{Reg}_{G;\mathbf{Z}_\ell})^{\mathrm{can}})) \underset{\mathbf{Z}_\ell[G]}{\otimes} M \;\xrightarrow{\sim}\; H^1(\mathbf{A}^1_K, j_!\mathscr{F}^{\mathrm{can}}).$$

THEOREM 1.6.8. — *The cohomology group* $H^1(\mathbf{A}^1_K, j_!((\mathrm{Reg}_{G;\mathbf{Z}_\ell})^{\mathrm{can}}))$ *is a projective right* $\mathbf{Z}_\ell[G]$-*module of finite presentation, which is isomorphic to the Swan representation* Sw_G *(cf.* [Se-2], 19.1 *and* 19.2).

Proof. — We have already seen that this cohomology group is projective and finitely presented as a right $\mathbf{Z}_\ell[G]$-module. It remains to show that as $\bar{\mathbf{Q}}_\ell[G]$-module, it has the correct character. This amounts to checking that for any $\bar{\mathbf{Q}}_\ell$-irreducible representation M of G, with corresponding sheaf $\mathscr{F}$ on $\mathrm{Spec}\,(K((T^{-1})))$, we have

$$\dim_{\bar{\mathbf{Q}}_\ell} (H^1(\mathbf{A}^1_K, j_!\mathscr{F}^{\mathrm{can}})) = \mathrm{swan}_\infty (\mathscr{F}),$$

where « $\mathrm{swan}_\infty (\mathscr{F})$ » denotes the swan conductor ($= $ «$b(M)$» in the notations of [Se-2], 19.3). In view of the vanishing of the other H^i, this is equivalent to the formula

$$\chi(\mathbf{A}^1, j_!\mathscr{F}^{\mathrm{can}}) = -\,\mathrm{swan}_\infty (\mathscr{F}).$$

But $\chi(\mathbf{A}^1, j_!\mathscr{F}^{\mathrm{can}}) = \chi_{\mathrm{comp}}(\mathbf{A}^1, j_!\mathscr{F}^{\mathrm{can}}) = \chi_{\mathrm{comp}}(\mathbf{G}_m, \mathscr{F}^{\mathrm{can}})$, so we need

$$\chi_{\mathrm{comp}}(\mathbf{G}_{m,K}, \mathscr{F}^{\mathrm{can}}) = -\,\mathrm{swan}_\infty (\mathscr{F}).$$

But as $\mathscr{F}^{\text{can}}$ is lisse on $G_{m,K}$, tame at zero, and has finite monodromy (some quotient of G), this last formula is « Weil's formula », cf. ([Se-1], VI § 4) and ([Ra], I,4).

<div align="right">Q.E.D.</div>

1.7. Curves of Higher Genus.

(1.7.1) Let K be a field, C/K a proper smooth geometrically connected curve, and $P \in C(K)$ a K-rational point. Let us denote

F = K(C), the function field of C,
F_P = the completion of F at the discrete valuation defined by P.

For any Zariski open neighborhood U of P in C,

$$P \in U \subset C,$$

we have a natural « inclusion » morphism

$$\text{Spec}(\hat{\mathscr{O}}_{C,P}) \rightarrow U,$$

which over $U - \{P\}$ induces

$$\text{Spec}(F_P) \rightarrow U - \{P\}.$$

So for $\bar{x}$ a geometric point of $\text{Spec}(F_P)$, with image $\bar{y}$ in $U - \{P\}$, and image $\bar{z}$ in $\text{Spec}(K)$, we have induced homomorphisms sitting in a commutative triangle

$$\pi_1(\text{Spec}(F_P), \bar{x}) \rightarrow \pi_1(U-\{P\}, \bar{y})$$
$$\searrow \qquad \swarrow$$
$$\pi_1(\text{Spec}(K), \bar{z})$$

THEOREM 1.7.2. — *Given C/K and* $P \in C(K)$, *there exists a Zariski open neighborhood U of P in C such that for any geometric point* $\bar{x}$ *of* $\text{Spec}(F_P)$, *with image* $\bar{y} \in U - \{P\}$, *the homomorphism*

$$\pi_1(\text{Spec}(F_P), \bar{x}) \rightarrow \pi_1(U-\{P\}, \bar{y})$$

is injective and admits a continuous retraction compatible with the canonical projections of these groups onto $\pi_1(\text{Spec}(K), \bar{z})$.

Proof. — Choose any function $T \in F$ which has a simple pole at P, and take $U = C-\{$the other zeroes and poles of $f\}$. Then viewed as a morphism $T : C \to \mathbf{P}^1$, T is etale at P and induces an isomorphism $K((T^{-1})) \xrightarrow{\sim} F_P$ of completions, so we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Spec}\,(F_P) & \longrightarrow & U - \{P\} \\
\wr\downarrow & & \downarrow T \\
\mathrm{Spec}\,(K((T^{-1}))) & \longrightarrow & \mathbf{G}_{m,K} .
\end{array}
$$

Denoting by $\bar{x}_1$ and $\bar{y}_1$ the images of $\bar{x}$ in $\mathrm{Spec}\,(K((T^{-1})))$ and in $\mathbf{G}_{m,K}$ respectively, this gives rise to a commutative diagram of homomorphisms, the bottom horizontal one of which admits a continuous retraction, say (R)

$$
\begin{array}{ccc}
\pi_1\,(\mathrm{Spec}\,(F_P),\bar{x}) & \longrightarrow & \pi_1\,(U-\{P\},\bar{y}) \\
\text{(A)}\,\wr\downarrow & \overset{\text{(R)}}{\dashleftarrow} & \downarrow\,\text{(B)} \\
\pi_1\,(\mathrm{Spec}\,(K((T^{-1}))),\bar{x}_1) & \longrightarrow & \pi_1\,(\mathbf{G}_{m,K},\bar{y}_1)
\end{array}
$$

Then $(A)^{-1}(R)(B)$ provide the required retraction. Q.E.D.

Remarks. — 1) If U « works », so does any smaller open neighborhood of P.

2) If C has genus zero, the fact that $P \in C(K)$ shows that $C \simeq \mathbf{P}^1_K$, with $P \leftrightarrow \infty$. Then (1.4.10) we may take $U = \mathbf{P}^1_K - \{0\}$. However we cannot take $U = \mathbf{P}^1_K$ itself.

To fix ideas, suppose that K is separably closed. Then the prime-to-p completion of $\pi_1(\mathbf{P}^1_K - \{\infty\},\bar{y})$ is trivial, while the prime-to-p completion of $\pi_1\,(\mathrm{Spec}\,(K((T^{-1}))),\bar{x})$ is $\prod_{\ell \neq p} \mathbf{Z}_\ell\,(1)$. So there exist *no* surjective homomorphisms from $\pi_1(\mathbf{P}^1_K - \{\infty\},\bar{y})$ onto $\pi_1\,(\mathrm{Spec}\,(K((T^{-1}))),\bar{x})$, so certainly no retractions.

3) If C has genus $g \geqslant 1$, then we may always take U such that $U = C - D$ for a divisor D of degree $\leqslant 4g + 1$ which is disjoint from P (by R.R., for $n \geqslant 2g - 1$ we have $\ell(nP) = n + 1 - g$, so for $n \geqslant 2g$ there exist functions $f_n \in L(nP) - L((n-1)P)$; then $T = f_{2g+1}/f_{2g}$ has by construction a simple pole at P, and at most $4g + 1$ other zeroes and poles). If $C(K)$ contains a second rational point $Q \neq P$, we may take $U = C - Q - D$ with a divisor D of degree $\leqslant 2g$, by taking $T \in L(P+(2g-1)Q) - L((2g-1)Q)$ a difference which is again non-empty by R.R.

2.

2.1. Statement of the results.

(2.1.1) Let k be a separably closed field of characteristic $p > 0$, C/k a proper smooth geometrically connected curve, and

$$S = \{s_1, \ldots, s_n\} \subset C$$

a finite non-empty set of closed points of C. We denote by $\mathcal{O}^h_{C,s_i}$ the henselization of the local ring $\mathcal{O}_{C,s_i}$ by $K^h_{s_i}$ the fraction field of $\mathcal{O}^h_{C,s_i}$, and we define

$$C^h_{s_i} = \mathrm{Spec}\,(\mathcal{O}^h_{C,s_i}).$$

When no confusion can arise, we denote also by s_i the closed point of $C^h_{s_i}$. Because $\mathcal{O}^h_{C,s_i}$ is a discrete valuation ring, we have

$$C^h_{s_i} - s_i = \mathrm{Spec}\,(K^h_{s_i}).$$

We have a natural « inclusion » morphism of $C^h_{s_i} - s_i$ into $C - S$. Fix geometric points $\bar{x}$ of $C - S$ and $\bar{x}_i$ of each $\mathrm{Spec}\,(K^h_{s_i})$, and « chemins » in $C - S$ from the image of each $\bar{x}_i$ to $\bar{x}$. Then for each i we have an induced continuous homomorphism of pro-finite groups

$$\lambda_i : \pi_1(C^h_{s_i} - s_i, \bar{x}_i) \rightarrow \pi_1(C - S, \bar{x}).$$

(2.1.2) Let G a finite discrete group, $H \subset G$ a subgroup, and for $i = 1, \ldots, n$,

$$\rho_i : \pi_1(C^h_{s_i} - s_i, \bar{x}_i) \rightarrow G$$

a continuous group homomorphism. We say that the condition

$$\mathrm{Interp}\,(G, H, \rho_1, \ldots, \rho_n)$$

holds (with respect to our fixed choices of geometric points and chemins) if there exist a continuous group homomorphism

$$\rho : \pi_1(C - S, \bar{x}) \rightarrow G$$

and

<center>elements $h_1, \ldots, h_n \in H$</center>

such that for all $i = 1, \ldots, n$, the diagram

$$
\begin{array}{ccc}
\pi_1(C^h_{s_i} - s_i, \bar{x}_i) & \xrightarrow{\lambda_i} & \pi_1(C - S, \bar{x}) \\
{\scriptstyle h_i \rho_i h_i^{-1}} \searrow & & \swarrow {\scriptstyle \rho} \\
& G &
\end{array}
$$

commutes, where $h_i \rho_i h_i^{-1}$ denotes the homomorphism $\gamma_i \mapsto h_i \rho_i(\gamma_i) h_i^{-1}$. We say that

$$(\rho; h_1, \ldots, h_n)$$

is a solution to the interpolation problem $(G, H; \rho_1, \ldots, \rho_n)$.

(2.1.3) Notice that the group H acts on the set of solutions, by having $h \in H$ act as

$$(\rho; h_1, \ldots, h_n) \rightarrow (h\rho h^{-1}; h h_1, \ldots, h h_n).$$

In particular, if there exists a solution, then there exists another with $h_1 = e$, i.e., with $\rho.\lambda_1 = \rho_1$.

THEOREM 2.1.4 (Harbater). —

If G is a p-group, then Interp $(G, \{e\}; \rho_1, \ldots, \rho_n)$ *holds, and the number of solutions is* $(\#(G))^h$, *where h is the « p-rank » of C, i.e. h is the F_p-dimension of* $H^1(C_{et}, F_p)$.

THEOREM 2.1.5. — *Suppose we are given data* $(G, H; \rho_1, \ldots, \rho_n)$, *in which H is both a normal subgroup of G, and is a p-group. Denote by $\pi : G \rightarrow G/N$ the projection onto the quotient, so that we have an exact sequence of groups*

$$1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1.$$

Then we have the equivalence

$$\text{Interp } (G, H; \rho_1, \ldots, \rho_n) \Leftrightarrow \text{Interp } (G/H, \{e\}; \pi\rho_1, \ldots, \pi\rho_n).$$

THEOREM 2.1.6. —

Suppose that $\# S = 2$, and denote by $\pi_1(C-S,\bar{x})$ (tame at s_2) the quotient of $\pi_1(C-S,\bar{x})$ classifying finite etale coverings which are tame at s_2. Then the natural continuous homomorphism

$$\pi_1(C^h_{s_1} - s_1, \bar{x}) \rightarrow \pi_1(C-S,\bar{x})(\text{tame at } s_2)$$

$$\searrow \qquad \nearrow$$

$$\pi_1(C-S,\bar{x})$$

admits a continuous retraction (left inverse).

2.2. Elementary Exact Sequence.

(2.2.1) Let $\mathscr{F}$ be any etale abelian torsion sheaf on $C - S$. The Leray spectral sequence for the inclusion $j : C - S \rightarrow C$,

$$E_2^{p,q} = H^p(C, R^q j_* \mathscr{F}) \Rightarrow H^{p+q}(C-S, \mathscr{F})$$

has $E_2^{p,q} = 0$ unless $pq = 0$, because for $q \geq 1$ the sheaves $R^q j_* \mathscr{F}$ are supported at S. By (SGA4 IX 5.7 and X 5.2), $R^q j_* \mathscr{F}$ vanishes for $q \geq 2$, $E_2^{p,0}$ vanishes for $p \geq 3$, and $H^2(C-S, \mathscr{F}) = 0$, so we have a four-term exact sequence of etale cohomology groups

(2.2.1.1) $0 \rightarrow H^1(C, j_* \mathscr{F}) \rightarrow H^1(C-S, \mathscr{F}) +$

$$\oplus_i H^1(C^h_{s_i} - s_i, \mathscr{F}) \rightarrow H^2(C, j_* \mathscr{F}) \rightarrow 0.$$

(2.2.2) Case I : $\mathscr{F}$ is killed by a power of p. Then so is $j_* \mathscr{F}$, so by Artin-Schreier theory (SGA 4, X, 5.2) we have $H^2(C, j_* \mathscr{F}) = 0$, whence a short exact sequence

(2.2.2.1) $0 \rightarrow H^1(C, j_* \mathscr{F}) \rightarrow H^1(C-S, \mathscr{F}) \rightarrow \oplus_i H^1(C^h_{s_i} - s_i, \mathscr{F}) \rightarrow 0.$

For $\mathscr{F}$ the constant sheaf $\mathbf{F}_p$, we have $j_* \mathbf{F}_p = \mathbf{F}_p$, and an exact sequence

(2.2.2.2) $0 \rightarrow H^1(C, \mathbf{F}_p) \rightarrow H^1(C-S, \mathbf{F}_p) \rightarrow \oplus_i H^1(C^h_{s_i} - s_i; \mathbf{F}_p) = 0.$

(2.2.3) *Case II :* $\mathcal{F}$ is a constant sheaf A, with A a finite abelian group of order prime to p. Then $j_* A = A$, $H^2(C, j_* A) \xrightarrow{\sim} A(-1)$, and each $H^1(C^h_{s_i} - s_i, A) \xrightarrow{\sim} A(-1)$, so we have a four term exact sequence

(2.2.3.1) $0 \rightarrow H^2(C, A) \rightarrow H^1(C - S, A) \rightarrow$

$$\underset{i}{\oplus} A(-1) \xrightarrow{\Sigma \, \delta_i} A(-1) \rightarrow 0.$$

(2.2.3.2) The individual maps

$$H^1(C^h_{s_i} - s_i, A) \xrightarrow{\delta_i} H^2(C, A)$$
$$\wr | \qquad\qquad\qquad \wr |$$
$$A(-1) \qquad\qquad\qquad A(-1)$$

are each isomorphisms (for this, it suffices to check that each δ_i is surjective, but δ_i occurs « alone » in the analogous exact sequence for the inclusion of $C - s_i$ into C).

2.3. Review of pro-*p*-groups (cf. [Sh], Chpt III).

(2.3.1) Let $\mathscr{C}$ denote the category of pro-*p*-groups, with maps the continuous homomorphisms. Given any non-empty indexing set I, a family G_i, $i \in I$ of objects of $\mathscr{C}$, and an object H of $\mathscr{C}$, a family of maps $\varphi_i : G_i \rightarrow H$ is said to « tend to zero » if for every finite discrete quotient $H \xrightarrow{\pi} \bar{H}$ of H, the composite homomorphisms $\pi \circ \varphi_i : G_i \rightarrow \bar{H}$ are trivial for all but finitely many values of i.

(2.3.2) Given the G_i, $i \in I$ as above, they have a « restricted coproduct », i.e. an object G in $\mathscr{C}$ together with maps $\alpha_i : G_i \rightarrow G$ such that for any object H in $\mathscr{C}$, and any family of maps $\varphi_i : G_i \rightarrow H$ which tends to zero, there exists a unique map $\varphi : G \rightarrow H$ such that $\varphi \circ \alpha_i = \varphi_i$ for all $i \in I$. We denote this restricted coproduct G by $*_{(p)}(G_i)_{i \in I}$.

(2.3.3) By the universel mapping property, we have

(2.3.3.1) $H^1(*_{(p)}(G_i)_{i \in I}, F_p) \xrightarrow{\sim} \underset{i \in I}{\oplus} H^1(G_i, F_p)$.

(2.3.4) Given a set I, the free pro-*p*-group on I, denoted $F_p(I)$, is by definition the restricted coproduct of Z_p with itself I times (i.e. $G_i = Z_p$

for all $i \in I$). As a special case of 2.3.3.1 above we see that

$$(2.3.4.1) \qquad\qquad H^1(F_p(I), F_p) \;\to\; \bigoplus_{i \in I} F_p,$$

in particular we have

$$(2.3.4.2) \qquad\qquad \#(I) = \dim_{F_p}(H^1(F_p(I), F_p)).$$

(2.3.5) A pro-p group G is said to be a free pro-p group if it is isomorphic to $F_p(I)$ for some I. One knows that for a pro-p group G, the following conditions are equivalent (cf. [Sh], Theorem 15 and Cor. 2 of Prop. 23):

(1) G is a free pro-p group,

(2) for every surjective map $H_1 \twoheadrightarrow H_2$ of pro-p groups, $\mathrm{Hom}(G, H_1)$ maps onto $\mathrm{Hom}(G, H_2)$, i.e. any map of $G \to H_2$ lifts to H_1,

(3) same as (2) for those $H_1 \twoheadrightarrow H_2$ with kernel F_p,

(4) $H^2(G, F_p) = 0$.

(2.3.6) Given a family G_j, $j \in J$, of free pro-p groups, say $G_j \simeq F_p(I_j)$ for some set I_j, their restricted coproduct is again a free pro-p group, isomorphic to $F_p(I)$ with $I = \coprod_{j \in J} I_j$; this is obvious from the universal mapping property.

(2.3.7) If G_1 and G_2 are free pro$-p$ groups, one knows ([Sh], Prop. 23) that a map $\varphi : G_1 \to G_2$ is an isomorphism (resp. is surjective) if and only if the induced map

$$H^1(G_2, F_p) \;\to\; H^1(G_1, F_p)$$

is an isomorphism (resp. is injective).

(2.3.8) If G is any pro-finite group, not necessarily pro-p, we denote by
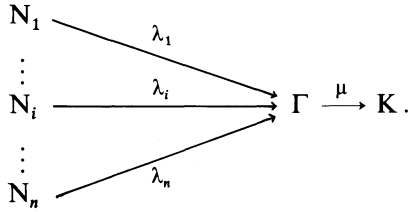
$$G \;\twoheadrightarrow\; G(p)$$

the projection of G onto its maximal pro-p quotient (i.e. for any pro-p group K, any continuous homomorphism $G \to K$ factors uniquely through $G \twoheadrightarrow G(p)$). Using criterion (3) for pro-p freeness, one sees that

$$(2.3.8.1) \quad H^2(G, F_p) = 0 \;\Rightarrow\; G(p) \text{ is a free pro-}p \text{ group},$$

while for any pro-finite G, the universal property of $G \twoheadrightarrow G(p)$ shows that

(2.3.8.2) $H^1(G(p),F_p) \xrightarrow{\sim} H^1(G,F_p)$.

LEMMA 2.3.9. — *Let $n \geq 1$ be an integer, and consider a diagram of continuous homomorphisms of pro-finite groups*

$$
\begin{array}{c}
N_1 \\
\vdots \\
N_i \\
\vdots \\
N_n
\end{array}
\quad
\begin{array}{c}
\lambda_1 \\
\lambda_i \\
\lambda_n
\end{array}
\quad \Gamma \xrightarrow{\mu} K .
$$

Suppose that

$$H^2(G,F_p) = 0 \qquad \text{for } G = N_1, \ldots, N_n, \Gamma, K ,$$

and suppose that the sequence

$$0 \to H^1(K,F_p) \xrightarrow{\mu^*} H^1(\Gamma,F_p) \xrightarrow{\oplus \lambda_i^*} \oplus H^1(N_i,F_p) \to 0$$

is exact.

Then

(1) *all the groups $N_i(p)$ for $i = 1, \ldots, n$, $\Gamma(p)$ and $K(p)$ are free pro-p groups,*

2) *the induced map $\mu(p) : \Gamma(p) \to K(p)$ is surjective, and admits a section (right inverse) $\alpha : K(p) \to \Gamma(p)$,*

(3) *the maps $\lambda_i(p) : N_i(p) \to \Gamma(p)$ for $i = 1, \ldots, n$, and the map $\alpha : K(p) \to \Gamma(p)$ define an isomorphism of free pro-p groups*

$$N_1(p) *_{(p)} N_2(p) *_{(p)} \ldots *_{(p)} N_n(p) *_{(p)} K(p) \xrightarrow{\sim} \Gamma(p).$$

Proof. — (i) *is 2.3.8.1, (2) is 2.3.7 and the fact that $K(p)$ is free pro-p so by 2.3.5 (1) $\Rightarrow$ (2) the section exists, and (3) is via the criterion 2.3.7.*
                                                                Q.E.D.

2.4. Proof of Theorem 2.1.4. (cf. [Ha], 2.7).

We will show that lemma 2.3.9 applies to the situation

(2.4.1)

$$\pi_1(C^h_{s_1} - s_1, \bar{x}_1) \xrightarrow{\lambda_1}$$
$$\vdots \qquad \pi_1(C - S, \bar{x}) \xrightarrow{\mu} \pi_1(C, \bar{x})$$
$$\pi_1(C^h_{s_n} - s_n, \bar{x}_n) \xrightarrow{\lambda_n}$$

where μ is the natural map induced by the inclusion of $C - S$ into C. The required exact sequence is precisely the one noted in 2.2.2.2. above. It remains only to see that all the groups above have $H^2(G, F_p) = 0$. For all but $\pi_1(C, \bar{x})$, we have the π_1 of a connected F_p-scheme which is affine, and for these the vanishing of $H^2(\pi_1, F_p)$ has already been established (1.4.3). The same Hochschild-Serre spectral sequence for the universal covering of any connected scheme Z with geometric point $\bar{z}$, with any finite abelian continuous $\pi_1(Z, \bar{z})$-module A as coefficients, has $E_2^{0,1} = 0$ and $E_2^{2,0} = H^2(\pi_1(Z, \bar{z}), A)$, so the exact sequence of terms of low degree gives an injective map

(2.4.2) $E_2^{2,0} = H^2(\pi_1(Z, \bar{z}, A) \hookrightarrow H^2(Z_{et}, A)$.

Taking $(Z, \bar{z}) = (C, \bar{x})$ and $A = F_p$, we get

(2.4.3) $H^2(\pi_1(C, \bar{x}), F_p) \hookrightarrow H^2(C_{et}, F_p) = 0$,

the final vanishing by Artin-Schreier theory (SGA 4, X, 5.2).

Thus we may apply the lemma. If we choose a section

$$\alpha : \pi_1(C, \bar{x})(p) \to \pi_1(C - S, \bar{x})(p),$$

then α and the $\lambda_i(p)$ exhibit $\pi_1(C - S, \bar{x})(p)$ as the restricted coproduct of the groups $\pi_1(C^h_{s_i} - s_i, \bar{x})(p)$ for $i = 1, \ldots, n$, and of $\pi_1(C, \bar{x})(p)$, and this last group $\pi_1(C, \bar{x})(p)$ is a free pro-p group on $h = \dim H^1(C_{et}, F_p)$ generators.

Q.E.D.

2.5. Proof of Theorem 2.1.5.

Recall the setting; we are given a finite group G, a normal subgroup $H \subset G$ which is a p-group, $\pi : G \to G/H$ is the projection, and we are given continuous homomorphisms $\rho_i : \pi_1(C^h_{s_i} - s_i, \bar{x}_i) \to G$.

The implication

$$\text{Interp } (G,H;\rho_1,\ldots,\rho_n) \Rightarrow \text{Interp } (G/H,\{e\};\pi\rho_1,\ldots,\pi\rho_n)$$

is trivial, for if $(\rho;h_1,\ldots,h_n)$ solves the first problem then $\pi \circ \rho$ solves the second. Suppose now that we wish to prove the other implication.

We first reduce to the case when H is abelian. For suppose that 2.1.5 holds universally when H is abelian. Then we may proceed by induction on $\#(H)$, the case $\#(H) = 1$ being obvious. Because H is a non-trivial p-group which is normal in G, its center $Z(H)$ is both non-trivial and normal in G. If $Z(H) = H$, there is by hypothesis nothing to prove, for H is abelian. If not, both $Z(H)$ and $H/Z(H)$ have order strictly lower than H.

We will apply the induction hypothesis in two steps. Denote by $\pi_0 : G \to G/Z(H)$ the projection. The induction hypothesis applied to the data $(G/Z(H),H/Z(H);\pi_0\rho_1,\ldots,\pi_0\rho_n)$, shows that

$$\text{Interp } (G/Z(H),H/Z(H);\pi_0\rho_1,\ldots,\pi_0\rho_n)$$

holds. By definition, then, there exist elements $\bar{h}_i \in H/Z(H)$ for $i = 1,\ldots,n$ such that

$$\text{Interp } (G/Z(H),\{e\};\bar{h}_1(\pi_0\rho_1)\bar{h}_1^{-1},\ldots,\bar{h}_n(\pi_0\rho_n)\bar{h}_n^{-1})$$

holds. Pick elements $h_i \in H$ with

$$\pi_0(h_i) = \bar{h}_i,$$

then

$$\text{Interp } (G/Z(H),\{e\};\pi_0 \circ (h_1\rho_1h_1^{-1}),\ldots,\pi_0 \circ (h_n\rho_nh_n^{-1}))$$

holds. Applying once again the induction hypothesis, this time to the situation

$$(G,Z(H);h_1\rho_1h_1^{-1},\ldots,h_n\rho_nh_n^{-1}),$$

we see that

$$\text{Interp } (G, Z(H); h_1\rho_1 h_1^{-1}, \ldots, h_n\rho_n h_n^{-1})$$

holds. By definition, there exist elements $z_1, \ldots, z_n$ in $Z(H)$ such that

$$\text{Interp } (G, \{e\}, z_1 h_1\rho_1 h_1^{-1} z_1^{-1}, \ldots, z_n h_n\rho_n h_n^{-1} z_n^{-1})$$

holds. As each $z_i h_i$ lies in H, this implies that

$$\text{Interp } (G, H; \rho_1, \ldots, \rho_n)$$

holds, as required.

Suppose now that $Z(H) = H$, i.e., that H is abelian. Our situation is summarized by the following commutative diagram:

$$1 \to \quad H \to G \xrightarrow{\pi} G/H \to 1$$

$$\pi_1(C^h_{s_i} - s_i, \bar{x}) \xrightarrow{\lambda_i} \pi_1(C - S, \bar{x})$$

with vertical maps ρ_i and $\bar{\rho}$.

The obstruction to lifting $\bar{\rho}$ to a continuous homomorphism $\rho : \pi_1(C - S, \bar{x}) \to G$ with $\bar{\rho} = \pi\rho$ is the cohomology class in $H^2(\pi_1(C - S, \bar{x}), H)$ of the pull-back by $\bar{\rho}$ of the extension $1 \to H \to G \to G/H \to 1$ where H is viewed as a continuous $\pi_1(C - S, \bar{x})$-module by means of the pulled-back extension. As already noted above (2.4.2), we have

$$H^2(\pi_1(C - S, \bar{x}), H) \hookrightarrow H^2(C - S, H) = 0,$$

the vanishing because $C - S$ is of cohomological dimension one for torsion sheaves, being a smooth affine curve over a separably closed field (SGA 4, IX 5.7, and X 5.2).

Therefore we may choose a lifting ρ of $\bar{\rho}$, which sits in a not necessarily commutative diagram

$$1 \to H \to G \xrightarrow{\pi} G/H$$

$$\pi_1(C^h_{s_i} - s_i, \bar{x}_i) \xrightarrow{\lambda_i} \pi_1(C - S, \bar{x}).$$

with maps ρ_i and ρ.

We know only that

$$\pi\rho\lambda_i = \pi\rho_i \qquad \text{for } i = 1, \ldots, n.$$

Therefore for each $i = 1, \ldots, n$, the « ratio » of ρ_i to $\rho\lambda_i$ is the unique mapping of sets

$$Z_i : \pi_1(C^h_{s_i} - s_i, \bar{x}_i) \to H$$

such that for all γ_i in $\pi_1(C^h_{s_i} - s_i, \bar{x}_i)$, we have

$$\rho_i(\gamma_i) = Z_i(\gamma_i) . (\rho\lambda_i)(\gamma_i).$$

Because both ρ_i and $\rho\lambda_i$ are homomorphisms, the function Z_i is easily checked to be a 1-cocycle from $\pi_1(C^h_{s_i} - s_i, \bar{x}_i)$ to H, for the action of $\pi_1(C^h_{s_i} - s_i, \bar{x})$ on H which is « conjugation by $\rho\lambda_i$ » (i.e. the restriction to $\pi_1(C^h_{s_i} - s_i, \bar{x}_i)$ by λ_i of the action of $\pi_1(C - S, \bar{x})$ on H already used above in proving the existence of ρ).

Suppose we could find a 1-cocycle

$$Z : \pi_1(C - S, \bar{x}) \to H,$$

for the above structure of $\pi_1(C - S, \bar{x})$-module on H, such that $Z_i = Z.\lambda_i$ for $i = 1, \ldots, n$. Then the function

$$Z\rho : \pi_1(C - S, \bar{x}) \to G$$
$$\gamma \mapsto Z(\gamma)\rho(\gamma)$$

would be a continuous homomorphism which satisfies $\rho_i = (Z\rho) \circ \lambda_i$ for $i = 1, \ldots, n$, and we would have $\text{Interp}(G, \{e\}; \rho_1, \ldots, \rho_n)$.

The next best thing is to find a 1-cocycle

$$Z : \pi_1(C - S, \bar{x}) \to H$$

such that for every $i = 1, \ldots, n$,

$$Z.\lambda_i \text{ is cohomologous to } Z_i.$$

For if $Z \circ \lambda_i$ and Z_i are cohomologous, then by definition there exists an element $h_i \in H$ such that for every element $\gamma_i \in \pi_1(C^h_{s_i} - s_i, \bar{x}_i)$, we have

$$(*) \qquad Z(\lambda_i(\gamma_i)) = h_i . Z_i(\gamma_i) . (h_i)^{-\gamma_i}.$$

A straightforward computation then shows that for $i = 1, \ldots, n$ we have

$$(Z\rho) \circ \lambda_i = h_i \rho_i h_i^{-1};$$

indeed because $\rho_i = (Z_i)(\rho \circ \lambda_i)$, this amounts to

$$Z(\lambda_i(\gamma_i))\rho(\lambda_i(\gamma_i)) = h_i Z_i(\gamma_i)\rho(\lambda_i(\gamma_i))h_i^{-1},$$

which, right multiplied by the inverse of $\rho(\lambda_i(\gamma_i))$, is precisely the cohomology relation $(*)$ above. Therefore, for such a Z, $(Z\rho; h_1, \ldots, h_n)$ is a solution of the interpolation problem $(G, H; \rho_i, \ldots, \rho_n)$.

It remains to show that we can find a 1-cocycle Z such that $Z \circ \lambda_i$ is cohomologous to the given 1-cocycle Z_i, for $i = 1, \ldots, n$. But this means precisely that the simultaneous restriction map

$$H^1(\pi_1(C - S, \bar{x}), H) \xrightarrow{\oplus \lambda_i} \bigoplus_i H^1(\pi_1(C_{s_i}^h - s_i, \bar{x}_1), H)$$

is surjective. This map is none other than the restriction map on etale H^1 for the finite locally constant Abelian sheaf $\underline{H}$ corresponding to H as $\pi_1(C - S, \bar{x})$-module,

$$H^1(C - S, \underline{H}) \to \bigoplus H^1(C_{s_i}^h - s_i, \underline{H}),$$

which we have already seen is surjective (2.2.1.1).

<div align="right">Q.E.D.</div>

2.6. Proof of Theorem 2.1.6.

The local Galois group $\pi_1(C_{s_i}^h - s_i, \bar{x}_1)$ sits in a well-known exact sequence

$$1 \to P \to \pi_1(C_{s_1}^h - s_1, \bar{x}_1) \to \prod_{\ell \neq p} Z_\ell(1) \to 1,$$

in which the wild inertia group P is the unique p-Sylow subgroup, and in which for any integer $N \geq 1$ prime to p, the unique discrete quotient of order N is the Galois group of the field extension obtained by adjoining the N'th root of any uniformizing parameter. Any finite discrete quotient

G of $\pi_1(C_{s_1}^h - s_1, \bar{x}_1)$ sits in an exact sequence

$$1 \to H \to G \to Z/NZ \to 1,$$

with H a p-group normal in G, and $N \geqslant 1$ an integer prime to p.

We wish to fill in the diagram (remember $\# S = 2$)

$$\pi_1(C_{s_1}^h - s_1, \bar{x}_1) \xrightarrow{\lambda_1} \pi_1(C - S, \bar{x}) \xrightarrow{\lambda_2} \pi_1(C_{s_2}^h - s_2; \bar{x}_2)$$

$$\text{id} \qquad \downarrow \rho$$

$$\pi_1(C_{s_1}^h - s_1, \bar{x}_1)$$

in such a way that

$$\rho \circ \lambda_1 = \text{id}, \quad \text{Im}(\rho \circ \lambda_2) \text{ is pro-prime to } p.$$

To do this, it suffices to find, for each finite discrete quotient G of $\pi_1(C_{s_1}^h - s_1, \bar{x}_1)$ as above, a map

$$\pi_1(C_{s_1}^h - s_1, \bar{x}_1) \xrightarrow{\lambda_1} \pi_1(C - S, \bar{x}) \xrightarrow{\lambda_2} \pi_1(C_{s_2}^h - s_2, \bar{x}_2)$$

$$\text{canonical} \atop \text{projection} \qquad \downarrow \rho_G$$
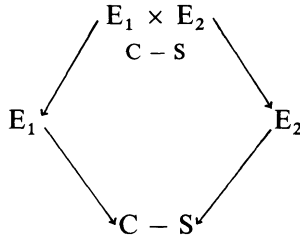
$$G$$

such that

$$\rho_G \circ \lambda_1 = \text{canonical projection}, \quad \text{Im}(\rho_G \circ \lambda_2) \text{ has order } N$$

and such that the collection of all ρ_G's is compatible. Because a directed inverse limit of *finite* non-empty sets is non-empty (Bourbaki *Top. Gen.* nouvelle édition, Ch. I, § 9, Prop. 8, p. 64), if we can show that for each individual G there exists at least one but at most finitely many ρ_G as above, then there exists some compatible system of ρ_G's, and we are done.

We first explain why there are, for a given G as above, only finitely many possible ρ_G. For if ρ_1 is one such, and if $E_1 \xrightarrow{\pi_1} C - S$ is the corresponding finite etale connected G-torsor, then for any other, say ρ_2, corresponding to a finite etale connected G-torsor $E_2 \to G - S$, the

fibre product

$$
\begin{array}{ccc}
 & \begin{array}{c} E_1 \times E_2 \\ C-S \end{array} & \\
E_1 & & E_2 \\
 & C-S &
\end{array}
$$

gives a finite etale not-necessarily connected G-torsor on E_1 which is *unramified* at all the missing points of E_1 (it is unramified at points over s_1, because ρ_1 and ρ_2 are equal on the inertia group at s_1); it is unramified at points over s_2 by Abhyankar's lemma, for both ρ_1 and ρ_2 map the inertia group at s_2 onto a cyclic group of order N prime to p. Therefore if we denote by C_1 the complete nonsingular model of E_1, and pick a geometric point $\bar{y}$ of E_1 lying over $\bar{x}$, the restriction of ρ_2 to $\pi_1(E_1,\bar{y})$ factors through $\pi_1(C_1,\bar{y})$. Because $\pi_1(C_1,\bar{y})$ is topologically finitely generated (SGA I, X, 2.6), and G is a finite group, the set $\mathrm{Hom}(\pi_1(C_1,\bar{y}),G)$ is finite. Therefore the restriction of ρ_2 to $\pi_1(E_1,\bar{y})$ is one of finitely many possible homomorphisms to G. Taking the intersection of all the possible kernels, we get an open subgroup of $\pi_1(E_1,\bar{y})$ on which any ρ_2 is trivial. As $\pi_1(E_1,\bar{y})$ is itself an open subgroup of $\pi_1(C-S,\bar{x})$. We see that there exists an open subgroup of $\pi_1(C-S,\bar{x})$ on which any possible ρ_2 is trivial. Therefore the set of possible ρ_2 is finite.

It remains to show that we can find a single ρ_G. For this, we use the structure of G as extension

$$
1 \to H \to G \to \mathbf{Z}/N\mathbf{Z} \to 1
$$

with H a p-group, and $N \geqslant 1$ prime to p. Because $(p,N)=1$, this extension splits; pick any splitting

$$
1 \to H \to G \overset{\pi}{\underset{s}{\rightleftarrows}} \mathbf{Z}/N\mathbf{Z} \to 1.
$$

Recall the exact sequence 2.2.3.1, with $A = \mathbf{Z}/N\mathbf{Z}$:

$$
0 \to H^1(C,\mathbf{Z}/N\mathbf{Z}) \to H^1(C-S,\mathbf{Z}/N\mathbf{Z}) \to
$$

$$
\bigoplus_{i=1,2} H^1(C^h_{s_i}-s_i,\mathbf{Z}/N\mathbf{Z}) \xrightarrow{\delta_1+\delta_2} H^2(C,\mathbf{Z}/N\mathbf{Z}) \to 0.
$$

As explained in 2.2.3.2, each of the maps δ_1 and δ_2 is an isomorphism. So given *any* element

$$\bar{\rho}_1 \in H^1(C_{s_1}^h - s_1, \mathbf{Z}/N\mathbf{Z}) = \mathrm{Hom}(\pi_1(C_{s_1}^h - s_1, \bar{x}), \mathbf{Z}/N\mathbf{Z})$$

there exists a *unique* element

$$\bar{\rho}_2 \in H^1(C_{s_2}^h - s_2, \mathbf{Z}/N\mathbf{Z}) = \mathrm{Hom}(\pi_1(C_{s_2}^h - s_2, \bar{x}_2), \mathbf{Z}/N\mathbf{Z})$$

for which
$$\delta_1(\bar{\rho}_1) + \delta_2(\bar{\rho}_2) = 0 \quad \text{in} \quad H^2(C, \mathbf{Z}/N\mathbf{Z}).$$

By the exactness, this means exactly that
$$\mathrm{Interp}(G/H, \{e\}; \bar{\rho}_1, \bar{\rho}_2)$$
holds.

Apply this with

$$\bar{\rho}_1 = \pi \circ \rho_1, \quad \text{for } \rho_1 \text{ the canonical projection of}$$

$$\pi_1(C_{s_1}^h - s_1, \bar{x}_1) \text{ onto its quotient } G.$$

This produces an element $\bar{\rho}_2 \in \mathrm{Hom}(\pi_1(C_{s_2}^h - s_2, \bar{x}_2), G/H)$, which we lift to an element
$$\rho_2 : \pi_1(C_{s_2}^h - s_2, \bar{x}_2) \to G$$

by defining
$$\rho_2 \stackrel{\mathrm{dfn}}{=} s \circ \bar{\rho}_2.$$
Then by construction,

$$\mathrm{Interp}(G/H, \{e\}; \pi\rho_1, \pi\rho_2)$$

holds. So by Theorem 2.1.5, we conclude that

$$\mathrm{Interp}(G, H; \rho_1, \rho_2)$$

holds, say with solution $(\rho; h_1, h_2)$. Then (cf. 2.1.3) $(h_1^{-1}\rho h_1; 1, h_1^{-1}h_2)$ is another solution. Writing it $(\rho_G; 1, h)$ this means that we have

$\rho_G \circ \lambda_1 = $ canonical projection
$\rho_G \circ \lambda_2 = h(s \circ \bar{\rho}_2)h^{-1}$, whose image has order N prime to p.

$$\text{Q.E.D.}$$

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