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INTERNAL RECONSTRUCTION OF UNIT-ROOT F-CRYSTALS VIA EXPANSION-COEFFICIENTS

BY NICHOLAS M. KATZ

with an Appendix

SOME RESULTS ON PARTIAL HODGE-WITT DECOMPOSITIONS

BY LUC ILLUSIE

TABLE OF CONTENTS

Introduction

One of Dwork's fundamental discoveries [Dw-1-2] was the existence of explicit p-adic limit formulas for certain "unit roots" of zeta functions of certain families of varieties over finite fields. This led him ([Dw-3], [Ka-1]) to the possibility of "isolating" the unit roots by constructing a suitable "unit root F-crystal", as a sub-F-crystal of the cohomology of the family, which would precisely account for the unit roots fibre-by-fibre. The *existence* of this "unit root F-crystal" implies the *existence* of p-adic limit formulas for the unit roots.

In this paper, we give an explicit *construction* of both the unit-root F-crystal in an H^i, and of the limit formulas to which it gives rise, in terms of the expansion-coefficients of the global i-forms.

If one views the unit-root F-crystal in an H^i as being the most accessible subquotient of the Newton-Hodge filtration (*cf.* [Ka-2]), it is natural to ask how to recover the *other* subquotients in terms of suitable "formal expansion coefficients" of suitable Hodge cohomology classes. Already in the case of the slope one subquotient of the H^2 of an ordinary surface, a solution to this problem seems to require entirely new ideas.

This paper is a natural sequel to our earlier papers [Ka-1-2-3-4]. We rely in an essential way upon the work of Illusie-Raynaud [Ill-Ray] on de Rham-Witt, which we apply to the fibres of the family in question. To our pleasant surprise, we are able to "get around" the present lack of an adequate theory of "de Rham-Witt with parameters". Perhaps our being able to do this is evidence for the existence of such a theory.

It is also worth pointing out that the limit formulas which Dwork discovered [Dw-1-2] involve "partial sums of series for periods of differential forms", while ours involve the expansion-coefficients of the forms themselves. It would be interesting to know if there is some general mechanism for passing from one sort of formula to the other.

Questions of the type discussed in this paper have also been considered by Barsotti's school, *cf.* [Cib], especially in the case $i=1$.

Illusie's appendix grew out of his reading of an earlier version of this paper, which he saw how to clarify and simplify in many ways. I thank him heartily.

1. The setting

Let us fix a prime number p, and a perfect field k of characteristic p. We denote by $W=W(k)$ the ring of Witt vectors of k, and by $\sigma: W \to W$ the absolute Frobenius automorphism of W.

For any W-scheme Z, we denote by Z_0 the k-scheme $Z \underset{W}{\otimes} k$ obtained from Z by reduction mod p.

We suppose given a smooth affine W-scheme $S=\operatorname{Spec}(A)$ with connected fibres, and a projective smooth S-scheme

$$\begin{array}{c} X \\ \downarrow{\scriptstyle f} \\ S \end{array}$$

with geometrically connected fibres, which we assume satisfies the condition (HLF) for all pairs of integers $i, j \geqq 0$, the Hodge cohomology groups $H^j(X, \Omega^i_{X/S})$ are locally free A-modules.

Because $S = \mathrm{Spec}(A)$ is connected, the ranks of the Hodge groups are constant on A; we denote them $h^{i,j}$. Because S is flat over **Z**, these ranks satisfy Hodge symmetry: $h^{i,j} = h^{j,i}$, the Hodge $\Rightarrow$ de Rham spectral sequence

$$E_1^{i,j} = H^j(X, \Omega^i_{X/S}) \quad \Rightarrow \quad H^{i+j}_{DR}(X/S)$$

degenerates at E_1, and its formation commutes with arbitrary change of base $S' \to S$. Moreover, if we denote by n the relative dimension of X/S, then we have a canonical trace isomorphism

$$H^{2n}_{DR}(X/S) \simeq H^n(X, \Omega^n_{X/S}) \xrightarrow{\sim} A,$$

and the resulting cup-product pairings

$$\begin{cases} H^j(X, \Omega^i_{X/S}) \times H^{n-j}(X, \Omega^{n-i}_{X/S}) \to H^n(X, \Omega^n_{X/S}) \simeq A, \\ H^{i+j}_{DR}(X/S) \times H^{2n-i-j}_{DR}(X/S) \longrightarrow H^{2n}_{DR}(X/S) \simeq A \end{cases}$$

are perfect dualities of locally free A-modules, whose formation commutes with arbitrary change of base $S' \to S$.

Let us denote by A_∞ the ring

$$A_\infty = \varprojlim A/p^n A,$$

and by S_∞ the formal scheme $Spf(A_\infty)$. The fundamental comparison theorem of crystalline cohomology provides a canonical horizontal isomorphism of graded A_∞-algebras

$$H^\bullet_{cris}(X_0/S_\infty) \simeq H^\bullet_{DR}(X/S) \underset{A}{\otimes} A_\infty.$$

Therefore, attached to every σ-linear ring-homomorphism

$$\Phi: \quad A_\infty \to A_\infty$$

whose reduction mod p is the absolute Frobenius on $A_0 = A/pA$, crystalline theory produces a horizontal algebra homomorphism

$$F_\Phi: \quad (H^\bullet_{DR}(X/S) \otimes A_\infty)^{(\Phi)} \to H^\bullet_{DR}(X/S) \otimes A_\infty,$$

which gives $H^i_{DR}(X/S) \otimes A_\infty$ the structure of "F-crystal on A". One knows that under the trace isomorphism

$$H^{2n}_{DR}(X/S) \otimes A_\infty \xrightarrow{\sim} A_\infty,$$

F_Φ, viewed as a Φ-linear endomorphism of A_∞, is just $p^n \Phi$.

For any perfect over-field K of k, and any k-homomorphism

$$s_0: \quad A_0 \to K,$$

there is a unique $W(k)$-homomorphism

$$s_\infty: \quad A_\infty \to W(K)$$

which satisfies

$$s_\infty \circ \Phi = \sigma \circ s_\infty,$$

called the "Φ-Teichmuller lifting of s_0". After the base change

$$A_\infty \to W(K)$$

by a Φ-Teichmuller point, we have a canonical isomorphism of $W(K)$-algebras

$$\begin{array}{ccc} (H^{\bullet}_{DR}(X/S) \underset{A}{\otimes} A_\infty) \underset{A_\infty}{\otimes} W(K) & \xrightarrow{\sim} & H^{\bullet}_{DR}(X \otimes W(K)/W(K)) \\ & & \wr\| \\ & & H^{\bullet}_{\mathrm{cris}}(X_{s_0}/W(K)) \end{array}$$

under which F_Φ specializes to the crystalline Frobenius

$$F_{\mathrm{cris}}: \quad H^{\bullet}_{\mathrm{cris}}(X_{s_0}/W(K))^{(\sigma)} \to H^{\bullet}_{\mathrm{cris}}(X_{s_0}/W(K)).$$

By a fundamental result of Mazur, the Newton polygon of each $H^i_{\mathrm{cris}}(X_{s_0}/W(K))$ lies above the Hodge polygon formed with the $\{h^{a,b}\}\, a+b=i$, and the two polygons have the same initial and terminal points. Because X_{s_0}/K is projective, the hard Lefschetz Theorem [Ka-Me] shows that the Newton slopes of $H^i(X_{s_0}/W(K))$ are symmetric under

$$\lambda \leftrightarrow i-\lambda.$$

That the Hodge slopes are similarly symmetric is nothing other than Hodge symmetry: $h^{a,b}=h^{b,a}$.

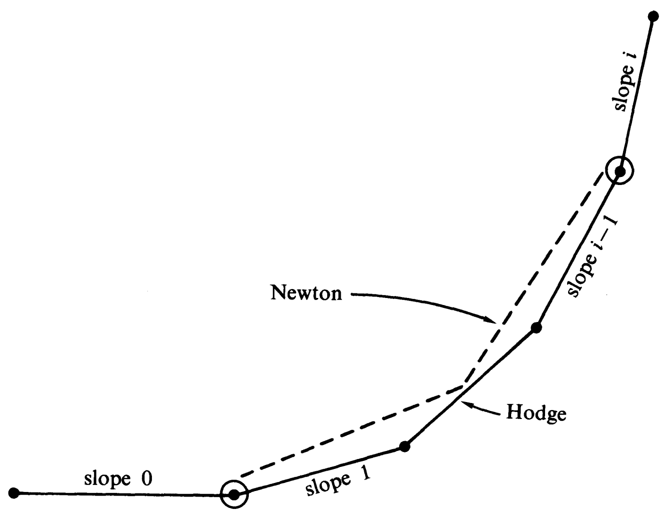
2. The condition HW(i)

One knows that the multiplicity of zero as a Newton slope of $H^i_{\mathrm{cris}}(X_{s_0}/W(K))$ is equal to the "stable rank" of the p-linear map F on

$$H^i_{\mathrm{cris}}(X_{s_0}/W(K)) \otimes K = H^i_{DR}(X_{s_0}/K),$$

and one knows that this stable rank is equal to the stable rank of F on $H^i(X_{s_0}, \mathcal{O}_{X_{s_0}})$. We will say that X/S satisfies the condition HW(i) if, for every $s_0 : A_0 \to K$ with K perfect, the p-linear Hasse-Witt operation F on $H^i(X_{s_0}, \mathcal{O}_{X_{s_0}})$ is an automorphism.

If X/S satisfies HW(i), then for every perfect-field fibre X_{s_0}/K of X_0/S_0, the Newton and Hodge polygons both have precisely $h^{0,\,i}$ sides of slope zero, and, by symmetry, both have precisely $h^{i,\,0} = h^{0,\,i}$ sides of slope i. Because Newton lies above Hodge, thanks to Mazur, the picture of the two polygons must be



The two circled points are both common break points of Newton and Hodge. Applying the theory of the Newton-Hodge filtration [Ka-2] to the F-crystal $H^i_{\mathrm{cris}}(X_0/S_\infty)$ on A, we see that when X/S satisfies HW(i), we have a filtration by sub-crystals

$$U_0 \subset U_{\leqq i-1} \subset H^i_{\mathrm{cris}}(X_0/S_\infty)$$

with :

U_0: a unit root F-crystal, locally free of rank $h^{0,\,i}$;

$U_{\leqq i-1}$: an F-crystal with all Newton and Hodge slopes $\leqq i-1$, locally free of rank $h^{0,\,i} + h^{1,\,i-1} + \ldots + h^{i-1,\,1}$;

$Q_{[1,\,i-1]} \overset{dfn}{=} U_{\leqq i-1}/U_0$: an F-crystal with all Newton and Hodge slopes in $[1, i-1]$, locally free of rank $h^{1,\,i-1} + \ldots + h^{i-1,\,1}$;

$Q_i \overset{dfn}{=} H^i_{\mathrm{cris}}/U_{\leqq i-1}$: an F-crystal with all Newton and Hodge slopes equal to i; locally free of rank $h^{i,\,0}$.

Because X/S is projective, the Hard Lefschetz Theorem shows that the slope i quotient Q_i is isogenous to the "twist" by p^i of the *dual* unit root F-crystal $U_0^\vee$ to U_0. [For $i=n$, usual Poincaré duality shows that these last two, Q_n and $U_0^\vee(n)$, are canonically isomorphic.]

3. The condition C(i)

Let us return to a perfect-field fibre X_{s0}/K of X_0/S_0. Because the Hodge $\Rightarrow$ de Rham spectral sequence degenerates at E_1, we know that every global i-form on X_{s0} is *closed*. Therefore the Cartier operator is defined on all of $H^0(X_{s0}, \Omega^i_{X_{s0}/K})$, as the composite

$$\begin{array}{c}
H^0(X_{s0}, \Omega^i_{X_{s0}/K}) \\
\downarrow \\
H^0(X_{s0}, \mathscr{H}^i_{DR}(X_{s0}/K)) \\
\| \\
H^0(X_{s0}^{(p)}, F_* \mathscr{H}^i_{DR}(X_{s0}/K)) \\
\wr\downarrow C \\
H^0(X_{s0}^{(p)}, \Omega^i_{X_{s0}^{(p)}/K}) \\
\| \\
H^0(X_{s0}, \Omega^i_{X_{s0}/K})^{(p)}
\end{array}$$

We will say that X/S satisfies C(i) if for every perfect field fibre X_{s0}/K of X_0/S_0, the p^{-1}-linear Cartier operator on $H^0(X_{s0}, \Omega^i_{X_{s0}/K})$ is an automorphism.

It is proved in Illusie's appendix that for X/S of the type we are considering, the conditions HW(i) and C(i) are equivalent. That they are for $i=n$ results from the following standard Lemma (*cf.* [Milne], 1.6) applied to the fibres of X_0/S_0.

LEMMA 3.1. — *Let* K *be a field of characteristic* $p>0$, Y/K *a projective smooth geometrically connected variety of dimension n. The (Serre) linear dual of the n-dimensional Hasse-Witt operation*

$$F:\quad H^n(Y^{(p)}, \mathcal{O}_{Y^{(p)}}) \to H^n(Y, \mathcal{O}_Y)$$

is the Cartier operator

$$C:\quad H^0(Y, \Omega^n_{Y/K}) \to H^0(Y^{(p)}, \Omega^n_{Y^{(p)}/K}).$$

Proof. — Because Y is n-dimensional, every n-form is closed, so C is certainly defined. If we view $H^n(Y, \mathcal{O}_Y)$ as $H^n(Y^{(p)}, F_*\mathcal{O}_Y)$, and view $H^0(Y, \Omega^n_{Y/K})$ as $H^0(Y^{(p)}), F_*\Omega^n_{Y/K})$, then the maps is question are obtained from applying "cohomology of $Y^{(p)}$" to the maps of locally free sheaves on $Y^{(p)}$

$$\begin{cases} \mathcal{O}_{Y^{(p)}} \overset{\text{“F”}}{\hookrightarrow} F_*\mathcal{O}_Y, \\ F_*\Omega^n_{Y/K} \twoheadrightarrow \mathcal{H}^n(F_*\Omega^\bullet_{Y/K}) \overset{C}{\underset{\sim}{\to}} \Omega^n_{Y^{(p)}/K}. \end{cases}$$

The maps of sheaves are themselves Serre-dual on $Y^{(p)}$, *via* the perfect pairings (check in local coordinates)

$$\begin{tikzcd} F_*\mathcal{O}_Y \times F_*\Omega^n_{Y/K} \arrow[r, "\text{“}C(f\omega)\text{”}"] \arrow[d, shift left=6, "C"] & \Omega^n_{Y^{(p)}/K} \arrow[d, equal] \\ \mathcal{O}_{Y^{(p)}} \times \Omega^n_{Y^{(p)}/K} \arrow[r, "\text{“}f\omega\text{”}"] \arrow[u, hook, shift left=6, "\text{“F”}"] & \Omega^n_{Y^{(p)}/K}. \end{tikzcd}$$

Therefore the result follows from Serre duality on $Y^{(p)}$.

Q.E.D.

4. Formal expansion

We henceforth suppose given an S-valued point $P \in X(S)$, which we view as a section of

$$\begin{tikzcd} X \arrow[d] \\ S \arrow[u, bend right, "P"'] \end{tikzcd}$$

We also suppose given a set of formal coordinates $(T_1, \ldots, T_n)$ along P, i. e., an isomorphism of pointed formal S-schemes

$$\operatorname{Spf}(A[[T_1, \ldots, T_n]]) \xrightarrow{\sim} \hat{X},$$

where $\hat{X}$ denotes the formal completion of X along P.

"Formal expansion along P" defines a morphism of complexes

$$\Omega^{\bullet}_{X/S} \to \Omega^{\bullet}_{A[[T]]/A}$$

from $\Omega^{\bullet}_{X/S}$ to the formal de Rham complex of the power series ring $A[[T_1, \ldots, T_n]]$ relative to A. For any integer $0 \leqq i \leqq n$, any global i-form $\omega \in H^0(X, \Omega^i_{X/S})$ has a formal expansion of the form

$$\omega \sim \sum_{\substack{K \subset \{1, \ldots, n\} \\ \# K = i}} \prod_{j \in K} \frac{dT_j}{T_j} \sum_{W} a(\omega; K, W) T^W$$

with coefficients

$$a(\omega; K, W) \in A$$

indexed by pairs (K, W) with K a subset of $\{1, \ldots, n\}$ of cardinality i, and $W = (w_1, \ldots, w_n) \in \mathbf{N}^n$ an n-tuple of non-negative integers satisfying

$$w_j \geqq 1 \quad \text{if} \quad j \in K.$$

LEMMA 4.1. — *Zariski locally on* S, *one can choose* $h^{i,0}$ *pairs of i-form expansion-indices* (K_v, W_v), $v = 1, \ldots, h^{i,0}$ *such that the map*

$$H^0(X, \Omega^i_{X/S}) \to (A)^{h^{i,0}}$$

defined by

$$\omega \mapsto (, \ldots, a(\omega; K_v, W_v), \ldots,)$$

is an isomorphism of A-*modules.*

Proof. — For a given choice of indices (K_v, W_v), the above map is an A-linear map between locally free A-modules of the same rank, whose formation commutes with arbitrary change of base.

Localizing on S, we may choose an A-basis

$$\{\omega_\alpha\}, \qquad \alpha = 1, \ldots, h^{i,\,0}$$

of $H^0(X/S, \Omega^i_{X/S})$. Then the above map is an isomorphism precisely where the determinant of the $h^{i,\,0} \times h^{i,\,0}$ matrix over A,

$$(a(\omega_\alpha;\, K_v,\, W_v))_{\alpha,\, v},$$

is invertible.

Therefore, it suffices to show that for any field-valued point $s : \mathrm{Spec}(L) \to S$, there exists a choice of indices $\{(K_v, W_v)\}_v$ for which the above determinant is non-zero.

At such a point s, the forms

$$\{\omega_\alpha(s)\}_s$$

form an L-basis of $H^0(X_s \Omega^i_{X_s/L})$. Because X_s is connected and $\Omega^i_{X_s/L}$ is locally free, the formal expansion map on X_s

$$H^0(X_s, \Omega^i_{X_s/L}) \to \prod_{\text{all } (K,\, W)} L$$

is injective (a global i-form on X_s which vanishes formally at a point of X_s is zero). Therefore the images of the $h^{i,\,0}$ elements $\{\omega_\alpha(s)\}_\alpha$ under this map form a set of $h^{i,\,0}$ linearly independent vectors in the vector space of ∞-tuples

$$\prod_{\text{all } (K,\, W)} L.$$

Therefore, the required non-vanishing results from the following elementary Lemma, applied to the vector space V of ∞-tuples

$$\prod_{\text{all } (K,\, W)} L.$$

and to the $h^{i,\,0}$-dimensional image of $H^0(X_s, \Omega^s_{X_s/L})$ in V.

LEMMA 4.2. — *Let* L *be a field,* V *an* L*-vector space,* $n \geqq 1$ *an integer, and* $H \subset V$ *an* n*-dimensional subspace.* If $\Lambda = \{\lambda\}$ *is a set of linear forms on* V *which define an embedding*

$$V \hookrightarrow L^\Lambda,$$

then there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$, *and a basis* $h_1, \ldots, h_n$ *of* H, *with*

$$\lambda_i(h_j) = \delta_{ij}.$$

Proof. — We proceed by induction on n, the case $n = 1$ being obvious. By induction, there exist $\lambda_1, \ldots, \lambda_{n-1} \in \Lambda$, and a basis $h_1 \ldots h_n$ of H, with

$$\lambda_i(h_j) = \delta_{ij} \qquad \text{for} \quad 1 \leqq i, j \leqq n-1.$$

Subtracting from h_n a suitable linear combination of $h_1, ., h_{n-1}$, we may further suppose

$$\lambda_i(h_n)=0 \qquad \text{for} \quad i=1, \ldots, n-1.$$

Choose any $\lambda_n \in \Lambda$ with $\lambda_n(h_n) \neq 0$. Replacing h_n by a non-zero scalar multiple, we have $\lambda_n(h_n)=1$. Replacing h_j *by* $h_j - \lambda_n(h_j)\, h_n$ for $j=1, \ldots, n-1$ we have

$$\lambda_i(h_j)=\delta_{i,\, j} \qquad \text{for} \quad 1 \leqq i, j \leqq n.$$

Q.E.D.

5. A basic Lemma on the formal de Rham complex

BASIC LEMMA 5.1. — *Let* R *be a ring which is flat over* **Z**, *and in which every prime number* $l \neq p$ *is invertible. Let* $n \geqq 1$ *be an integer*, $T_1, \ldots, T_n$ *indeterminates, i an integer* $0 \leqq i \leqq n$, *and*

$$\omega \in \Omega^i_{\mathbf{R}[[T_1, \ldots, T_n]]/\mathbf{R}}$$

a formal i-form,

$$\omega = \sum_{\substack{K \subset \{1, \ldots, n\} \\ \# K = i}} \prod_{k \in K} \frac{d T_k}{T_k} \sum_{\substack{W \in \mathbf{N}^n \\ w_j \geqq 1 \text{ if } j \in K}} a(\omega; K, W)\, T^W.$$

Suppose that ω *is closed. Then* ω *is exact if and only if its expansion coefficients* $a(\omega; K, W)$ *satisfy*

$$a(\omega; K, W) \in \begin{cases} \sum_{k \in K} w_k R, & \textit{if} \quad i \geqq 1, \\ 0.R=0 & \textit{if} \quad i=0. \end{cases}$$

Proof. — The "only if" direction is trivial. To establish the converse we argue as follows. There is a natural grading, by $\mathbf{N}^n$, of the formal de Rham complex, corresponding to the action of $(\mathbf{G}_m)^n$ on the n independent variables $T_1, \ldots, T_n$ (*cf*. [Ka-3], 5.8.1). For this grading, the form

$$\left(\prod_{k \in K} \frac{d T_k}{T_k}\right). T^W$$

is homogeneous of degree W, and the entire de Rham complex is the direct product over all W of its homogeneous subcomplexes of degree W. This allows us, when convenient, to treat only the case of a closed form ω which is homogeneous of some degree W.

On the other hand, the Lemma is obvious for $i=0$, any n, and for $i=n=1$, by direct calculation (*cf*. [Ka-3], 6.1). Proceeding by induction, we may assume that $1 \leqq i \leqq n$, and that the Lemma is already known for forms in $\leqq n-1$ variables.

Fix an element $W \in \mathbf{N}^n$, and consider a closed i-form ω which is homogeneous of degree W. If any of the $w_j = 0$, then ω does not involve the corresponding variable T_j, and the case of $\leqq n-1$ variables applies. Therefore we may assume $w_1, \ldots, w_n$ all non-zero. Renumbering if necessary, we may further assume that

$$\operatorname{ord}_p(w_1) \geqq \operatorname{ord}_p(w_2) \geqq \ldots \geqq \operatorname{ord}_p(w_n).$$

We wish to isolate the effect of the last variable. For ω an i-form of degree W, we may uniquely write

$$\omega = \alpha . T_n^{w_n} + \beta . T_n^{w_n} \frac{dT_n}{T_n},$$

with α and β forms of degree $(w_1, \ldots, w_{n-1})$ in $T_1, \ldots, T_{n-1}$. By hypothesis, β has all its expansion coefficients in $w_n R$, so we may write

$$\beta = w_n \gamma,$$

with γ a form in $(T_1, \ldots, T_{n-1})$, whence

$$\omega = \alpha T_n^{w_n} + w_n \gamma . T_n^{w_n} \frac{dT_n}{T_n}.$$

We will show that if such an ω is closed, then it is exact. If ω is closed, we readily calculate

$$0 = d\omega = (d\alpha) T_n^{w_n} + (-1)^i . \alpha . w_n . T_n^{w_n} \frac{dT_n}{T_n} + w_n (d\gamma) . T_n^{w_n} \frac{dT_n}{T_n},$$

whence, equating coefficients, we find

$$\begin{cases} d\alpha = 0, \\ d\gamma = (-1)^{i-1} \alpha. \end{cases}$$

Therefore we have

$$\omega = d((-1)^{i-1} . \gamma . T_n^{w_n}).$$

Q.E.D.

LEMMA 5.2. — *Let* R *be a ring,* $n \geqq 1$ *an integer,* $T_1, \ldots, T_n$ *indeterminates,* $W = (w_1, \ldots, w_n)$ *an n-tuple of non-negative integers, and* $\Omega^{\cdot}(W)$ *the homogeneous of degree* W *subcomplex of the formal de Rham complex of* $R[[T_1, \ldots, T_n]]$ *over* R. *If any of the* w_i *is invertible in* R, *then the complex* $\Omega^{\cdot}(W)$ *has no cohomology.*

Proof. — Renumbering if necessary, we may suppose w_n invertible in R. Then the last two paragraphs of the proof of 5.1 apply *mutatis mutandis* to the situation at hand.

Q.E.D.

COROLLARY 5.3. — *Let* p *be a prime number,* R *an* $\mathbf{F}_p$*-algebra,* $n \geqq 1$ *an integer,*

$T_1, \ldots, T_n$ *indeterminates, i an integer* $0 \leqq i \leqq n$, *and*

$$\omega \in \Omega^i_{R\,[[T_1, \ldots, T_n]]/R}$$

a formal i-form,

$$\omega = \sum_{\substack{K \subset \{1, \ldots, n\} \\ \# K = i}} \prod_{k \in K} \frac{d T_k}{T_k} \sum_{W \in \mathbf{N}^n} a(\omega;\, K,\, W)\, T^W.$$

Let us denote by ω_1 *and* ω_2 *the formal i-forms defined by*

$$\omega_1 = \sum_K \prod_{k \in K} \frac{d T_k}{T_k} \sum_{\substack{W \in \mathbf{N}^n \\ \text{some } w_i \text{ prime to } p}} a(\omega;\, K,\, W)\, T^W$$

$$\omega_2 = \sum_K \prod_{k \in K} \frac{d T_k}{T_k} \sum_{W \in \mathbf{N}^n} a(\omega;\, K,\, pW)\, T^{pW}.$$

so that

$$\omega = \omega_1 + \omega_2.$$

If ω *is closed, then* ω_1 *is exact, and the image of* ω *under the* R*-linear Cartier operator* C *is given by*

$$C(\omega) = \sum_K \prod_{k \in K} \frac{d T_k}{T_k} \sum_{W \in \mathbf{N}^n} a(\omega;\, K,\, pW)\, T^W$$

Proof. — The first assertion follows immediately from 5.2, for ω_1 is nothing other than the sum of the homogeneous components of ω whose degree W has some w_i invertible in R. The second assertion follows trivially from the first, by definition of the Cartier operator (*cf.* [Cart], II, (39) for the absolute case).

Q.E.D.

6. Formulation of the Main Theorem

Fix an integer $1 \leqq i \leqq n$, and suppose that X/S satisfies both HW(i) and C(i). Zariski localizing on S if necessary, suppose further that the sub-crystal $U_{\leqq i-1}$ and the quotient crystal Q_i of $H^i_{\mathrm{cris}}(X_0/S_\infty) \simeq H^i_{\mathrm{DR}}(X/S) \otimes A_\infty$ are both free A_∞-modules, and that there there exists a set of $h^{i,\,0}$ i-form expansion indices (K_v, W_v), $v = 1, \ldots, h^{i,\,0}$ for which the map

$$H^0(X, \Omega^i_{X/S}) \to (A)^{h^{i,\,0}},$$
$$\omega \mapsto (\ldots, a(\omega;\, K_v,\, W_v), \ldots)$$

is an isomorphism of A-modules, *cf.* 4.1.

Fix such a choice of i-form expansion indices (K_v, W_v), and denote by

$$\{\omega_\alpha\}, \qquad \alpha = 1, \ldots, h^{i,\,0}$$

the dual A-basis of $H^0(X, \Omega^i_{X/S})$, i. e.,

$$a(\omega_\alpha;\ K_v,\ W_v) = \delta_{\alpha,\,v}.$$

For every integer $m \geqq 0$, denote by $E(m)$ the $h^{i,\,0} \times h^{i,\,0}$ matrix of expansion coefficients defined by

$$E(m)_{v,\,\alpha} = a(\omega_\alpha;\ K_v,\ p^m W_v).$$

LEMMA 6.1. — *For every $m \geqq 0$, the matrix $E(m)$ is invertible over A_∞, i. e. it lies in $GL(h^{i,\,0}, A_\infty)$.*

Proof. — Because A_∞ is p-adically complete, this amounts to the statement that $E(m)$ mod p is invertible. By 5.3, $E(m)$ mod p is the matrix of the m'th iterate of the Cartier operator,

$$C^m:\quad H^0(X_0, \Omega^i_{X_0/S_0}) \to H^0(X_0^{(p^m)}, \Omega^i_{X_0^{(p^m)}/S_0}),$$

with respect to the bases $\{\omega_\alpha\}$ and $\{\omega_\alpha^{(p^m)}\}$. By the hypothesis $C(i)$, these iterates are all invertible.

Q.E.D.

In view of the interpretation of $E(m)$ mod p as the matrix of C^m, we have for every $m \geqq 0$ the mod p congruence

$$E(m+1) \equiv E(m)^{(p)} . E(1) \bmod p.$$

MAIN THEOREM 6.2. — *Hypotheses and notations as above, consider the composite A_∞-linear map pr between free A_∞-modules of rank $h^{i,\,0}$:*

$$\begin{tikzcd}
H^0(X, \Omega^i_{X/S}) \otimes A_\infty \arrow[r, hook] \arrow[dr, "pr"'] & H^i_{DR}(X/S) \otimes A_\infty \arrow[d] \\
 & Q_i
\end{tikzcd}$$

and the "formal expansion along P" map,

$$H^i_{DR}(X/S) \underset{A}{\otimes} A_\infty \to H^i_{DR}(A_\infty[[T_1, \ldots, T_n]]/A_\infty).$$

(1) *For any element* $\eta \in H^i_{DR}(X/S) \otimes A_\infty$, *the following conditions are equivalent:*

(*a*) η *lies in* $U_{\leqq i-1}$;

(*b*) η *dies in* $H^i_{DR}(A_\infty[[T]]/A_\infty)$, *i. e.*, η *dies under formal expansion.*

(2) *The map pr:* $H^0(X, \Omega^i_{X/S}) \otimes A_\infty \to Q_i$ *is an isomorphism; in other words we have an* A_∞*-direct sum decomposition*

$$U_{\leqq i-1} \oplus (H^0(X, \Omega^i_{X/S}) \otimes A_\infty) \xrightarrow{\sim} H^i_{DR}(X/S) \otimes A_\infty.$$

(3) *For any* σ*-linear* $\Phi: A_\infty \to A_\infty$, *and any integer* $v \geqq 1$, *denote by* $F(\Phi)^{(v)}$ *the matrix of the* v'*th "iterate" of* F_Φ *on* Q_i,

$$(F_\Phi)^{(v)}: \quad Q_i^{(\Phi^v)} \to Q_i,$$

with respect to the bases $\{pr(\omega_\alpha)^{(\Phi^v)}\}$ *and* $\{pr(\omega_\alpha)\}$.

For any W*-linear derivation* $D: A_\infty \to A_\infty$, *denote by* $\nabla(D)$ *the matrix of* $\nabla(D)$ *on* Q_i *with respect to the basis* $\{pr(\omega_\alpha)\}$.

Then we have the congruences of matrices

$$F(\Phi^v) \equiv p^{iv} E(m+v)^{-1} \Phi^v(E(m)) \bmod p^{m+v} A_\infty,$$
$$\nabla(D) \equiv E(m)^{-1} D(E(m)) \bmod p^m A_\infty$$

for all integers $v \geqq 1$, $m \geqq 1$.

Remark (6.3). — As a byproduct of part (3), we see that the sequences of matrices over A_∞

$$\begin{cases} E(m+1)^{-1}\Phi(E(m)), & m=1, 2, 3, \ldots, \\ E(m)^{-1} D(E(m)), & m=1, 2, 3, \ldots \end{cases}$$

both converge uniformly (i. e. for the p-adic topology) on A_∞. This fact alone is highly non-trivial. It is very reminiscent of Dwork's hypergeometric congruences ([Dw-2], 2.1) where expansion-coefficients are "replaced" by cleverly chosen partial sums. It would be interesting to understand the general mechanism of such replacements.

7. Proof of the Main Theorem

We now turn to the proof of the Main Theorem. Choose a σ-linear $\Phi: A_\infty \to A_\infty$ lifting absolute Frobenius. As explained in [Ka-2], if we denote by A_0 the connected smooth k-algebra A/pA, and by A_0^{perf} its perfection, then there is a unique lifting of the inclusion $A_0 \hookrightarrow A_0^{\text{perf}}$ to an inclusion $i(\Phi): A_\infty \hookrightarrow W(A_0^{\text{perf}})$, which sits in a commutative diagram

$$\begin{array}{ccc} A_\infty & \xrightarrow{i(\Phi)} & W(A_0^{\text{perf}}) \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\sigma} \\ A_\infty & \xrightarrow{i(\Phi)} & W(A_0^{\text{perf}}) \end{array}$$

Furthermore, we have

$$p\,A_\infty = i(\Phi)(A_\infty) \cap p\,W(A_0^{\text{perf}}),$$

simply because

$$A_\infty/p\,A_\infty = A_0 \hookrightarrow A_0^{\text{perf}} = W(A_0^{\text{perf}})/p\,W(A_0^{\text{perf}}).$$

Iterating, we find that for any integer $m \geqq 1$ we have

$$\begin{aligned} p^m A_\infty &= i(\Phi)(A_\infty) \cap p^m W(A_0^{\text{perf}}) \\ &= \text{the elements whose first } m \text{ Witt-vector} \\ &\qquad \text{coordinates in } A_0^{\text{perf}} \text{ all vanish.} \end{aligned}$$

Let K_0 denote the fraction field of A_0. Then the fraction field of A_0^{perf} is K_0^{perf}, so we can check vanishing of Witt vector components by viewing them in K^{perf}. Therefore we have

$$p^m A_\infty = i(\Phi)(A_\infty) \cap p^m W(K_0^{\text{perf}}),$$

i. e. p-adic divisibility in A_∞ is detected by p-adic divisibility at the Teichmüller representative of the generic point of A_0. Letting $m \to \infty$, we recover the injectivity of

$$A_\infty \xhookrightarrow{i(\Phi)} W(K_0^{\text{perf}}).$$

Applying these remarks component-by-component, we obtain:

SCHOLIE 7.1. — Let M be a free A_∞-module of finite rank, $N \subset M$ a free A_∞-submodule with free quotient. Then in order that an element $\eta \in M$ lie in N (resp. in $N + p^i M$ for an integer $i \geqq 1$) it is necessary and sufficient that it lie there after the extension of scalars $A_\infty \xhookrightarrow{i(\Phi)} W(K_0^{\text{perf}})$.

We now turn to the proof of (1). Let $\eta \in H^i_{DR}(X/S) \otimes A_\infty$. By the above scholie, η lies in $U_{\leqq i-1}$ if and only if this becomes true after the extension of scalars $A_\infty \xhookrightarrow{i(\Phi)} W(K_0^{\text{perf}})$. Choose a formal closed i-form $\hat{\eta} \in \Omega^i_{A_\infty[[T]]/A_\infty}$ whose cohomology class is the image of η under formal expansion, say

$$\hat{\eta}=\sum_{K}\prod_{j\in K}\frac{d\mathrm{T}_j}{\mathrm{T}_j}\sum_{W}a(\eta,\ \mathrm{K},\ \mathrm{W})\,\mathrm{T}^{\mathrm{W}}.$$

By the basic Lemma (5.1), such a form is formally exact if and only if

$$a(\eta,\ \mathrm{K},\ \mathrm{W})\equiv 0 \bmod p^{\nu}\mathrm{A}_{\infty},\qquad \text{where}\quad \nu=\min_{j\in \mathrm{K}}\mathrm{ord}_p(w_j).$$

Therefore the statement

$$\hat{\eta}=0 \qquad \text{in}\quad \mathrm{H}^i_{\mathrm{DR}}(\mathrm{A}_{\infty}[[\mathrm{T}]]/\mathrm{A}_{\infty})$$

holds if and only if it holds after the extension of scalars $\mathrm{A}_{\infty}\hookrightarrow \mathrm{W}(\mathrm{K}_0^{\mathrm{perf}})$.

Combining these remarks, we see that each of the conditions (*a*), (*b*) in part one holds for η if and only if it holds for η after the extension of scalars $\mathrm{A}_{\infty}\hookrightarrow \mathrm{W}(\mathrm{K}_0^{\mathrm{perf}})$.

Therefore to prove part (1) we are reduced to the case $\mathrm{A}=\mathrm{W}$(a perfect field).

THEOREM 7.2 (Illusie). — *Let* K *be a perfect field of characteristic* $p>0$, *and* $\mathrm{X}/\mathrm{W}(\mathrm{K})$ *a projective smooth* W*-scheme with geometrically connected fibres whose Hodge cohomology groups are torsion-free* W*-modules. If* X *satisfies* $\mathrm{C}(i)$, *then* $\mathrm{U}_{\leqq i-1}$ *is the kernel of formal expansion, i. e., part* (1) *of the Main Theorem* 6.2 *holds for* X/W.

Proof. — It is shown in Illusie's appendix (2.1, 2.2) that for X/W as above satisfying $\mathrm{C}(i)$, one has

$$\mathrm{H}^0(\mathrm{X}_0,\ \mathrm{BW}\Omega^i)=\mathrm{H}^1(\mathrm{X}_0,\ \mathrm{BW}\Omega^i)=0.$$

We will deduce from this vanishing that $\mathrm{U}_{\leqq i-1}$ is the kernel of formal expansion.

Over W(K), one knows ([Ka-2], 1.6.5) that the exact sequence

$$0\to \mathrm{U}_{\leqq i-1}\to \mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}/\mathrm{S})\to \mathrm{Q}_i\to 0$$

has a unique F-stable splitting. Because $\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}/\mathrm{W})$ is torsion free, this F-stable decomposition

(7.3.1) $$\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}/\mathrm{W})=\mathrm{U}_{\leqq i-1}\oplus \mathrm{Q}_i$$

is the unique F-stable decomposition of $\mathrm{H}^i_{\mathrm{DR}}$ of the form

$$\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}/\mathrm{W})=\begin{pmatrix}\text{F-stable submodule}\\ \text{on which F divides } p^{i-1}\end{pmatrix}\oplus\begin{pmatrix}\text{F-stable submodule}\\ \text{on which } p^i \text{ divides F}\end{pmatrix}.$$

Let us denote by X_0/K the special fibre of X/W. For each integer $n\geqq 1$, we denote by $\mathrm{W}_n\Omega^{\bullet}$ the de Rham-Witt complex of level n of X_0/K, and we denote by $\mathrm{W}.\Omega^{\bullet}$ the complex of pro-objects

$$\mathrm{W}.\Omega^j=\text{“}\varprojlim_n\text{”}\,\mathrm{W}_n\Omega^j.$$

With the notations of [Ill-Ray], IV, 4.1, consider the commutative diagram of exact sequences (a variant of diagram IV 4.1.1 of [Ill-Ray], in which their horizontal "$i+1$" becomes our vertical "i"):

(7.3.2)
$$\begin{tikzcd}
 & 0 \arrow[dr] & 0 \arrow[d] & & & \\
0 \arrow[dr] & & \tau_{\leqq i-1} W.\Omega^{\cdot} \arrow[d] \arrow[dr] & & & \\
0 \arrow[r] & W.\Omega^{\geqq i} \arrow[r] \arrow[dr] & W.\Omega^{\cdot} \arrow[r] \arrow[d] & W.\Omega^{<i} \arrow[r] \arrow[dr] & 0 & \\
 & & \tau'_{\geqq i} W.\Omega^{\cdot} \arrow[d] \arrow[dr] & & BW\Omega^i[1-i] \arrow[dl, equal] \arrow[dr] & \\
 & & 0 & BW\Omega^i[1-i] \arrow[dr] & & 0 \\
 & & & & 0 &
\end{tikzcd}$$

The long exact cohomology sequences of the horizontal and vertical short exact sequences above "cross" at $H^i(X_0, W.\Omega^{\cdot})$ in a diagram with exact rows and columns

(7.3.3)
$$\begin{tikzcd}
 & H^i(X_0, \tau_{\leqq i-1} W.\Omega^{\cdot}) \arrow[d, "(A)"] \arrow[dr, "\sim"] & \\
H^i(X_0, W.\Omega^{\geqq i}) \arrow[r, "(a)"] \arrow[dr, "\sim"'] & H^i(X_0, W\Omega^{\cdot}) \arrow[r, "(b)"] \arrow[d, "(B)"] & H^i(X_0, W.\Omega^{<i}) \\
 & H^i(X_0, \tau'_{\geqq i} W\Omega^{\cdot}) &
\end{tikzcd}$$

in which the slanted arrows are isomorphisms. By Chevalley's Lemma, the arrows (A) and (a) are injective, and the arrows (B) and (b) are surjective. Therefore we have a direct sum decomposition

$$(7.3.4) \quad H^i(X_0, W.\Omega^{\cdot}) = H^i(X_0, \tau_{\leqq i-1} W.\Omega^{\cdot}) \oplus H^i(X_0, W\Omega^{\geqq i})$$
$$= (\text{Kernel of (B)}) \oplus H^0(X_0, ZW\Omega^i).$$

By [Ill-Ray], III, 1.7.1-3-7 and II, proof of 3.5, we know that this decomposition is of the form

$$\mathrm{H}^i(\mathrm{X}_0,\ \mathrm{W}.\Omega^{\cdot})=\begin{pmatrix}\text{F-stable submodule}\\ \text{on which F divides } p^{i-1};\\ \mathrm{FV}'_{\leqq i-1}=\mathrm{V}'_{\leqq i-1}\mathrm{F}=p^{i-1}\end{pmatrix}\oplus\begin{pmatrix}\text{F-stable module}\\ \text{on which } p^i \text{ divides F};\\ \mathrm{F}=p^i\mathrm{F}_{\geqq i}\end{pmatrix}$$

via the endomorphisms $\mathrm{V}'_{\leqq i-1}$ of $\tau_{\leqq i-1}\mathrm{W}.\Omega^{\cdot}$ and $\mathrm{F}_{\geqq i}$ of $\mathrm{W}.\Omega^{\geqq i}$ of (*loc. cit.*).

Therefore, under the canonical isomorphism

$$\mathrm{H}^i(\mathrm{X}_0,\ \mathrm{W}.\Omega^{\cdot})\simeq\mathrm{H}^i_{\mathrm{cris}}(\mathrm{X}_0/\mathrm{W}(\mathrm{K}))\simeq\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}/\mathrm{S}),$$

the two decompositions above

$$\begin{array}{c}\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}/\mathrm{S})=\mathrm{U}_{\leqq i-1}\oplus\mathrm{Q}_i\\ \wr\Vert\\ \mathrm{H}^i(\mathrm{X}_0,\ \mathrm{W}.\Omega^{\cdot})=(\mathrm{Ker}(\mathrm{B}))\oplus\mathrm{H}^0(\mathrm{X}_0,\ \mathrm{ZW}.\Omega^i)\end{array}$$

must *coincide*. Therefore, we have

$$\mathrm{U}_{\leqq i-1}=\mathrm{Ker}(\mathrm{B}).$$

But as explained in [Ka-3], 8.1. Ker (B) is *equal* to the kernel of formal expansion. Therefore $\mathrm{U}_{\leqq i-1}$ = the kernel of formal expansion.

Q.E.D.

We now turn to the proof of part (2) of 6.2. We must prove that the natural inclusions induce an isomorphism of free A_∞-modules

$$\mathrm{U}_{\leqq i-1}\oplus(\mathrm{H}^0(\mathrm{X},\ \Omega^i_{\mathrm{X/S}})\otimes\mathrm{A}_\infty)\xrightarrow{\sim}\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X}/\mathrm{S})\otimes\mathrm{A}_\infty.$$

We first claim that the composite map

$$\begin{tikzcd} \mathrm{H}^0(\mathrm{X}, \Omega^i_{\mathrm{X/S}}) \otimes \mathrm{A}_\infty \arrow[r, hook] \arrow[dr, "(*)"'] & \mathrm{H}^i_{\mathrm{DR}}(\mathrm{X/S}) \otimes \mathrm{A}_\infty \arrow[d, "\text{formal expansion}"] \\ & \mathrm{H}^i_{\mathrm{DR}}(\mathrm{A}_\infty[[\mathrm{T}_1, \ldots, \mathrm{T}_n]]/\mathrm{A}_\infty) \end{tikzcd}$$

is injective. Because the matrix $\mathrm{E}(m)$ is invertible for any $m \geqq 1$ the expansion-indices $(\mathrm{K}_v, p^m \mathrm{W}_v)$ define an A_∞-isomorphism

$$\mathrm{H}^0(\mathrm{X/S}, \Omega^i_{\mathrm{X/S}}) \otimes \mathrm{A}_\infty \xrightarrow{\sim} (\mathrm{A}_\infty)^{h^{i,0}}$$
$$\omega \mapsto (\ldots, a(\omega; \mathrm{K}_v, p^m \mathrm{W}_v), \ldots).$$

But if a global i-form ω lies in the kernel of (*), then by the easy half of the Basic Lemma we have

$$a(\omega; \mathrm{K}_v, p^m \mathrm{W}_v) \in p^m \mathrm{A}_\infty,$$

whence

$$\mathrm{Ker}(*) \subset \bigcap_{m \geqq 1} p^m (\mathrm{H}^0_{\mathrm{DR}}(\mathrm{X}, \Omega^i_{\mathrm{X/S}}) \otimes \mathrm{A}_\infty) = 0,$$

because $\mathrm{H}^0(\Omega^i)$ is a free A_∞-module, and A_∞ is p-adically separated.

Therefore the map (*) is injective on $\mathrm{H}^0(\mathrm{X}, \Omega^i_{\mathrm{X/S}}) \otimes \mathrm{A}_\infty$, while by part (1), the map (*) kills $\mathrm{U}_{\leqq i-1}$. Therefore

$$\mathrm{U}_{\leqq i-1} \cap (\mathrm{H}^0(\mathrm{X}, \Omega^i_{\mathrm{X/S}}) \otimes \mathrm{A}_\infty) = 0,$$

and hence the map in question

$$\mathrm{U}_{\leqq i-1} \oplus (\mathrm{H}^0(\mathrm{X}, \Omega^i_{\mathrm{X/S}}) \otimes \mathrm{A}_\infty) \to \mathrm{H}^i_{\mathrm{DR}}(\mathrm{X/S}) \otimes \mathrm{A}_\infty$$

is *injective*.

To show that this map between free A_∞-modules of the same rank is an isomorphism, if suffices to check mod p, i.e. over A_0. Fix a σ-linear lifting $\Phi : \mathrm{A}_\infty \to \mathrm{A}_\infty$ of Frobenius. Then every perfect-field valued point $s_0 : \mathrm{A}_0 \to \mathrm{K}$ of S_0 lifts to a unique Φ-Teichmuller point $s_{0,\infty} : \mathrm{A}_\infty \to \mathrm{W}(\mathrm{K})$ whose reduction mod p is s_0. Therefore it suffices to show that the map in question is an isomorphism after extension of scalars by any Φ-Teichmüller point.

Thus we are reduced to proving (2) in the case when $A=W(K)$, with K a perfect field. Because A is now a discrete valuation ring, the A-submodule

$$U_{\leqq i-1}\oplus H^0(\Omega^i)\subset H^i_{DR}$$

is necessarily a lattice. In other words, the map

$$H^0(\Omega^i)\xrightarrow{pr_i} H^i_{DR}/U_{\leqq i-1}=Q_i$$

is necessarily an isogeny. Therefore there exists an integer $k\geqq 0$ such that

$$p^k Q_i\subset pr_i(H^0(\Omega^i))\subset Q_i.$$

We must show that $k=0$. Pick a basis $\{e_j\}$, $j=1,\ldots,h^{i,0}$, of Q_i. Then for any $k\gg 0$, there exist scalars $\lambda_{j,\alpha}(k)\in W$ such that

$$p^k e_\alpha=\sum_j \lambda_{j,\alpha}(k)\,pr_i(\omega_j)$$

inside Q_i. Because Q_i is torsion free, it suffices to show that

$$\lambda_{j,\alpha}(k)\equiv 0 \bmod p^k.$$

To show this, we argue as follows. Pick elements $\tilde{e}_j\in H^i_{DR}$ which project mod $U_{\leqq i-1}$ to the e_j's. Then we have

$$p^k\tilde{e}_\alpha\equiv\sum_j\lambda_{j\alpha}(k)\,\omega_j \bmod U_{\leqq i-1}.$$

For each $\tilde{e}_j$, choose a formal closed i-form

$$\eta_j\in\Omega^i_{W[[T_1,\ldots,T_n]]/W}$$

whose cohomology class in $H^i_{DR}(W[[T_1,\ldots,T_n]]/W)$ is that of the image of $\tilde{e}_j$ under formal expansion. Then by part (1), $U_{\leqq i-1}$ dies in formal expansion, whence we obtain

$$p^k\eta_\alpha=\sum_j\lambda_{j\alpha}(k)\,\omega_j \qquad \text{in} \quad H^i_{DR}(W[[T_1,\ldots,T_n]]/W).$$

By the easy half of the basic Lemma (5.1), this yields a congruence of expansion coefficients for any $m\geqq 1$

$$p^k a(\eta_\alpha, K_v, p^m W_v)\equiv\sum_j\lambda_{j\alpha}(k).\,a(\omega_j; K_v, p^m W_v) \bmod p^m.$$

Thus if we denote by $G(m)$ the $h^{i,0}\times h^{i,0}$ matrix over W given by

$$G(m)_{v,j}=a(\eta_j; K_v, p^m W_v)$$

we have the congruence, for any $m\geqq 1$

$$p^k G(m)\equiv E(m)(\lambda_{\alpha j}(k)) \bmod p^m.$$

Because $E(m)$ is invertible, this gives, for $m \geqq k$,

$$\lambda_{j\alpha}(k) \equiv 0 \bmod p^k,$$

as required. This concludes the proof of part (2).

It is now a simple matter to prove (3). We write the action of $(F_\Phi)^{(v)}$ on the basis $\{pr_i(\omega_\alpha)\}_\alpha$ of Q_i in the form

$$F_\Phi^v(\omega_\alpha^{(\Phi^v)}) \equiv \sum_j f_{j\alpha}^{(v)} \omega_j \bmod U_{\leqq i-1},$$

and for any W-linear derivation D of A_∞ to itself, we write

$$\nabla(D)(\omega_\alpha) \equiv \sum_j \nabla(D)_{j\alpha} \omega_j \bmod U_{\leqq i-1}.$$

Applying formal expansion, we kill the indeterminacy $U_{\leqq i-1}$, and obtain

$$\begin{cases} F_\Phi^v(\omega_\alpha^{(\Phi^v)}) = \sum_j f_{j\alpha}^{(v)} \omega_j & \text{in } H_{DR}^i(A_\infty[[T_1, \ldots, T_n]]/A_\infty), \\ \nabla(D)(\omega_\alpha) = \sum_j \nabla(D)_{j\alpha} \omega_j & \text{in } H_{DR}^i(A_\infty[[T_1, \ldots, T_n]]/A_\infty). \end{cases}$$

As explained in [Ka-3], 5.8.3 and in [Ka-4] respectively, the action of F_Φ and of $\nabla(D)$ on $H_{DR}^i(A_\infty[[T]]/A_\infty) = H_{cris}^i(A_0[[T]]/A_\infty)$ is induced by the particularly convenient endomorphisms of $\Omega_{A_\infty}^{\bullet}[[T]]/A_\infty$ given by the explicit formulas

$$\begin{cases} F_\Phi\left(\left(a\left(\prod_{j\in k}\frac{dT_j}{T_j}T^W\right)^{\Phi}\right)\right) = \Phi(a)p^{\;K}\left(\prod_{j\in K}\frac{dT_j}{T_j}\right)T^{pW}, \\ \nabla(D)\left(a\left(\prod\frac{dT_j}{T_j}\right)T^W\right) = D(a)\left(\prod\frac{dT_j}{T_j}\right)T^W. \end{cases}$$

Therefore, by the easy half of the basic Lemma (5.1) we have, for every $m \geqq 1$, and every $r \geqq 1$, the congruences

$$p^{ir}\Phi^r(a(\omega_\alpha; K_v, p^m W_v)) \equiv \sum f_{j\alpha}^{(r)} a(\omega_j; K_v, p^{m+r} W_v) \bmod p^{m+r},$$
$$D(a(\omega_\alpha; K_v, p^m W_v)) \equiv \sum \nabla(D)_{j\alpha} a(\omega_j; K_v, p^m W_v) \bmod p^m,$$

Assembling these into matrices, we find the congruences

$$p^{ir}\Phi^r(E(m)) \equiv E(m+r)F(\Phi)^r \bmod p^{m+r},$$
$$D(E(m)) \equiv E(m)\nabla(D) \bmod p^m.$$

Because $E(m)$ is invertible, we can rewrite these as the required congruences

$$\begin{cases} F(\Phi^r) \equiv E(m+r)^{-1} p^{ir}\Phi^r(E(m)) \bmod p^{m+r} A_\infty, \\ \nabla(D) \equiv E(m)^{-1} D(E(m)) \bmod p^m A_\infty. \end{cases}$$

Q.E.D.

8. An example: varieties with trivial canonical bundle

Let k be a perfect field of characteristic $p>0$, $\mathrm{S}=\mathrm{Spec}(\mathrm{A})$ a smooth affine $\mathrm{W}(k)$-scheme with connected fibres, and

$$\begin{tikzcd} \mathrm{X} \arrow[d] \\ \mathrm{S} \arrow[u, bend right, "\mathrm{P}"'] \end{tikzcd}$$

a pointed projective smooth S-scheme with geometrically connected fibres, of relative dimension n, whose canonical bundle $\Omega^n_{\mathrm{X/S}}$ is trivial. We suppose that X/S satisfies (HLF).

Denote by ω a nowhere vanishing global section of $\Omega^n_{\mathrm{X/S}}$. Let $(\mathrm{T}_1, \ldots, \mathrm{T}_n)$ be a set of local coordinates on X along P (these exist at least Zariski locally on S), and expand ω with respect to them:

$$\omega \sim \sum_{\mathrm{W}=(w_1, \ldots, w_n) \text{ all } w_i \geqq 1} a(\mathrm{W}) . \mathrm{T}^{\mathrm{W}} \prod_{i=1}^{n} \frac{d\mathrm{T}_i}{\mathrm{T}_i},$$

with expansion coefficients $a(\mathrm{W}) \in \mathrm{A}$. Because ω is nowhere vanishing, $a(1,1, \ldots, 1)$ is a unit in A, so we may and will assume

$$a(1,1, \ldots, 1)=1.$$

Then X/S satisfies $\mathrm{C}(n)$ if and only if

$$a(p, p, \ldots, p) \bmod p \text{ is invertible in } \mathrm{A}_0.$$

Assume that this is the case. As already noted, $\mathrm{C}(n) \Leftrightarrow \mathrm{HW}(n)$.

Therefore we may apply the Main Theorem. For any σ-linear lifting $\Phi : \mathrm{A}_\infty \to \mathrm{A}_\infty$ of Frobenius, and any W-linear derivation $\mathrm{D} : \mathrm{A} \to \mathrm{A}$, we have

$$\left\{ \begin{aligned} \mathrm{F}(\Phi^v) &\equiv p^{nv} \frac{\Phi^v(a(p^m, p^m, \ldots, p^m))}{a(p^{m+v}, p^{m+v}, \ldots, p^{m+v})} \bmod p^{m+v} \mathrm{A}_\infty, \\ \nabla(\mathrm{D}) &\equiv \frac{\mathrm{D}(a(p^m, p^m, \ldots, p^m))}{a(p^m, p^m, \ldots, p^m)} \bmod p^m \mathrm{A}_\infty. \end{aligned} \right.$$

Suppose that k is in fact a finite field. Then for any finite overfield $\mathbf{F}_q$ of k, any $\mathrm{W}(\mathbf{F}_q)$-valued point

$$s : \quad \mathrm{Spec}(\mathrm{W}(\mathbf{F}_q)) \to \mathrm{S},$$

we obtain by base change a new situation

$$\begin{tikzcd} \mathrm{X}_s \arrow[d] \\ \mathrm{Spec}(\mathrm{W}(\mathbf{F}_q)), \arrow[u, bend right, "\mathrm{P}_s"'] \end{tikzcd}$$

to which we may apply the Main Theorem.

Among the eigenvalues of Frobenius relative to $\mathbf{F}_q$

$$\mathrm{F}_{\mathbf{F}_q}: \quad \mathrm{H}^n_{\mathrm{cris}}(\mathrm{X}_{s0}/\mathrm{W}(\mathbf{F}_q)) \circlearrowleft$$

on $\mathrm{H}^n_{\mathrm{cris}}$, there is precisely one which is a p-adic unit, say

$$\lambda = \text{the unit eigenvalue} = \mathrm{F}_{\mathbf{F}_q} \,|\, \mathrm{U}_0,$$

and precisely one of the form $q^n \times$ (a unit), namely

$$q^n/\lambda = \text{the eigenvalue of slope } n = \mathrm{F}_{\mathbf{F}_q} \,|\, \mathrm{Q}_n.$$

Writing $q = p^v$, we see that $\mathrm{F}_{\mathbf{F}_q}$ is just $(\mathrm{F}_\Phi)^{(v)}$. The unique Φ is σ itself on $\mathrm{W}(\mathbf{F}_q)$, whence as a special case of the above congruences for $\mathrm{F}(\Phi^v)$, we find the congruences

$$q^n/\lambda \equiv q^n \left(\frac{a(q^m, \ldots, q^m)}{a(q^{m+1}, \ldots, q^{m+1})} \right)(s) \bmod (q^{m+1}),$$

or equivalently

$$\lambda \equiv \left(\frac{a(q^{m+1}, \ldots, q^{m+1})}{a(q^m, \ldots, q^m)} \right)(s) \bmod (q^{m+1-n})$$

for every $m \geqq 1$.

In the special case of an ordinary elliptic curve, this last congruence is (at least) 25 years old. As explained by Dwork in [Dw-1], pp. 257-258, it can be proven by combining the Dieudonné-Dwork integrality criterion with a 1958 result of Tate and Lazard giving the structure of one-parameter formal Lie groups of height one over complete mixed-characteristic discrete valuation rings with separably closed residue field.

We have now come full circle, for Dwork's theory of unit-root F-crystals began with this special case (*cf.* [Dw-1], p. 258 and p. 250).

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APPENDIX: SOME RESULTS ON PARTIAL HODGE-WITT DECOMPOSITIONS

BY LUC ILLUSIE [1]

Introduction

We prove the equivalence between the conditions $\mathrm{HW}(i)$ and $\mathrm{C}(i)$, and we prove the vanishing properties needed in the proof of [Ka 6], 6.2. We also give a transversality result generalizing assertion (2) of (*loc. cit.*).

Some of the results on transversality were obtained during a visit to Princeton University in February 1982, whose hospitality it is a pleasure to acknowledge. I would also like to thank N. Katz for many stimulating discussions.

Throughout this paper, k denotes a perfect field of characteristic $p>0$, $\mathrm{W}=\mathrm{W}(k)$ the ring of Witt vectors on k, K the fraction field of W, and X a proper and smooth k-scheme. Notations concerning crystalline cohomology and the de Rham-Witt complex are as in [Ill-Ray].

1. The equivalence between $\mathrm{HW}(i)$ and $\mathrm{C}(i)$

The following Lemma sums up well-known facts and easy consequences of [Ill-Ray]:

LEMMA 1.1. — *Let $i \in \mathbf{Z}$.*

(*a*) $\mathrm{H}^0(\mathrm{W}\Omega^i)$ *is a free, finitely generated* W-*module, with* F *bijective; there are canonical isomorphisms*

$$\mathrm{H}^0(\mathrm{W}\Omega^i)=\mathrm{H}^0(\mathrm{ZW}\Omega^i)=\mathrm{P}^i\mathrm{H}^i(\mathrm{X}/\mathrm{W}),$$

where P^* *denotes the filtration on* $\mathrm{H}^*(\mathrm{X}/\mathrm{W})$ *which is the abutment of the first de Rham-Witt spectral sequence.*

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(*b*) *In the first de Rham-Witt spectral sequence,* $H^1(W\Omega^i)=E_\infty^{i1}$, *in particular* $H^1(W\Omega^i)$ *is finitely generated over* W.

(*c*) $H^1(ZW\Omega^i)$ *is a finitely generated* W-*module* (*with* F *bijective*).

(*d*) *The sequence*

$$0 \to H^0(ZW\Omega^i) \to H^i(X/W) \to H^i(W\Omega^{<i}) \to 0$$

[*coming from the short exact sequences of pro-complexes*

$$0 \to ZW.\Omega^i[-i] \to t_{\leqq i}W.\Omega^{\bullet} \to W.\Omega^{<i} \to 0 \quad (^2)$$

and the natural identification $H^i(X, t_{\leqq i}W\Omega^{\bullet})=H^i(X/W)$] *is exact and admits a unique splitting which is stable under the operator* $V'_{\leqq i}$ of [Ill-Ray], III 1.7.1. *Moreover, the natural map* $H^1(W\Omega^{i-1}) \to H^i(W\Omega^{<i})$ *is injective.*

Proof. — By [Ill], II 2.17, 3.3, $H^0(W\Omega^i)$ is free and finitely generated over W and $H^0(W\Omega^i)=P^iH^i(X/W)$. Since $d: H^0(W\Omega^i) \to H^0(W\Omega^{i+1})$ is zero, we have $H^0(ZW\Omega^i) \xrightarrow{\sim} H^0(W\Omega^i)$, and therefore ([Ill-Ray], IV 1.3) F is bijective on $H^0(W\Omega^i)$, which proves (*a*). (*b*) is [Ill-Ray], II 3.11, which implies (*c*) since $H^1(ZW\Omega^i)=\varprojlim_{F} H^1(W\Omega^i)$ ([Ill-Ray], IV 1.3). In view of (*a*) and (*c*), the first assertion of (*d*) is contained in [Ill-Ray], IV 4.4, for $j=0$. Because of (*b*) and of the surjectivity of $H^i(X/W) \to H^i(W\Omega^{<i})$, the map $H^1(W\Omega^{i-1}) \to H^i(W\Omega^{<i})$ can be identified with the inclusion $P^{i-1}H^i/P^iH^i \to H^i/P^iH^i$, which proves the second assertion.

Q.E.D.

LEMMA 1.2. — *Let* $i \in \mathbf{Z}$, *and assume that* $H^{i+1}(X/W)$ *is torsion-free. Then* $H^0(W\Omega^i/F)=0$, *and the sequence*

$$0 \to H^0(W\Omega^i) \xrightarrow{V} H^0(W\Omega^i) \to H^0(W\Omega^i/V) \to 0$$

in exact.

Proof. — Indeed, by 1.1 (*d*), $H^1(W\Omega^i)$ injects into $H^{i+1}(X/W)$, hence has no p-torsion, so F and V are injective on $H^1(W\Omega^i)$.

Q.E.D.

LEMMA 1.3. — *Let* $i \in \mathbf{Z}$. *Assume that* $H^{i+1}(X/W)$ *is torsion-free and that* $H^0(\Omega^i)=H^0(Z\Omega^i)$. *Then the following conditions are equivalent:*

(i) $C: H^0(\Omega^i) \to H^0(\Omega^i)$ *is an isomorphism;*

(ii) $H^0(W\Omega^i)/p \to H^0(\Omega^i)$ *is an isomorphism.*

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(2) *See* [Ill-Ray] for the notations of truncations; $t_{\leqq i}$ is written $\tau_{\leqq i}$ in [Ka 6].

Proof. — By 1.1 (*a*) and 1.2, $H^0(W\Omega^i)/p = H^0(W\Omega^i)/V = H^0(W\Omega^i/V)$. On the other hand, by [Ill], I 3.11, we have

$$H^0(W\Omega^i/V) = \varprojlim_{C} H^0(Z_n\Omega^i).$$

The hypothesis $H^0(\Omega^i) = H^0(Z\Omega^i)$ implies $H^0(\Omega^i) = H^0(Z_n\Omega^i)$ for all $n \geqq 0$, so we find

$$H^0(W\Omega^i/V) = \varprojlim_{C} H^0(\Omega^i).$$

The equivalence (i) $\Leftrightarrow$ (ii) is now immediate.

Q.E.D.

The following definition is due to Ekedahl, *cf.* [Ill 1], 6.3.11.

DEFINITION 1.4. — *Let* $i \in \mathbf{Z}$. *We say* X *satisfies the condition* MO(i) (*Mazur-Ogus in degree* i) *if* $\sum_{a+b=i} h^{ab} = b_i$, *where* $b_i = \dim_K H^i(X/W) \otimes K$ *and where* $h^{ab} = \dim_k H^b(\Omega^a)$.

It is readily seen that MO(i) is equivalent to the conjunction of the following conditions: (a) $H^i(X/W)$ and $H^{i+1}(X/W)$ are torsion-free, (b) for $a+b=i$, $E_1^{ab} = E_\infty^{ab}$ in the Hodge to de Rham spectral sequence $H^b(\Omega^a) \Rightarrow H^*_{DR}(X/k)$. If X admits a proper and smooth lifting X over W satisfying (HLF) ([Ka 6], § 1), i.e. such that $H^s(X, \Omega^r_X)$ is torsion-free for all (r, s), then X satisfies MO(i) for all i.

In the sequel, for J an interval of **Z**, we denote by $H^n(X/W)_J$ the part of $H^n(X/W) \otimes K$ where the Frobenius $\mathscr{F}$ acts with slopes in J. We will denote by Nwt_n the Newton polygon of $\mathscr{F}$ on $H^n(X/W) \otimes K$, and Hdg_n the Hodge polygon made with the Hodge numbers $h^{0\,n}$, $h^{1,\,n-1}$, ..., i.e. having slope zero $h^{0\,n}$ times, slope 1 $h^{1,\,n-1}$ times, etc. In general, one has the inequalities ([B-O], [Maz 1], [Maz 2], [Ny])

$$(1.5) \qquad \mathrm{Nwt}_n \geqq \mathrm{Hdg}_n^{\mathrm{abs}} \geqq \mathrm{Hdg}_n,$$

where $\mathrm{Hdg}_n^{\mathrm{abs}}$ is the Hodge polygon made with the abstract Hodge numbers of the F-cristal ($H^n(X/W)$/torsion, $\mathscr{F}$), and Nwt_n and $\mathrm{Hdg}_n^{\mathrm{abs}}$ have the same end points. An important result of Ekedahl (*cf.* [Ill 1], 6.3.13) asserts that if X satisfies MO(i), then $\mathrm{Hdg}_i = \mathrm{Hdg}_i^{\mathrm{abs}}$ (so in particular Nwt_i and Hdg_i have the same end points).

LEMMA 1.6. — *Let* $i \in \mathbf{Z}$. *Assume* X *satisfies* MO(i). *Then the following conditions are equivalent:*

(i) $C: H^0(\Omega^i) \to H^0(\Omega^i)$ *is an isomorphism* [*by* MO(i), $H^0(\Omega^i) = H^0(Z\Omega^i)$ *so that* C *is defined on all of* $H^0(\Omega^i)$];

(ii) $\dim H^i(X/W)_{[i]} = h^{i\,0}$;

(ii bis) Nwt_i *and* Hdg_i *have the same segment of slope* i.

Since Nwt_i and Hdg_i have the same end point, it is clear that one has (ii) $\Leftrightarrow$ (ii *bis*). On the other hand, by [Ill], II 3.5, we have

$$\dim H^i(X/W)_{[i]} = rk\, H^0(W\Omega^i),$$

and by 1.1 (*a*) we have

$$rk\, \mathrm{H}^0(\mathrm{W}\Omega^i) = \dim \mathrm{H}^0(\mathrm{W}\Omega^i)/p \qquad \text{and} \qquad \mathrm{H}^0(\mathrm{W}\Omega^i)/p = \mathrm{H}^0(\mathrm{W}\Omega^i)/\mathrm{V}.$$

Since $\mathrm{MO}(i)$ implies that $\mathrm{H}^i(\mathrm{X}/\mathrm{W})$ and $\mathrm{H}^{i+1}(\mathrm{X}/\mathrm{W})$ are torsion-free, it follows from 1.2 and the exact sequence

$$0 \to \mathrm{W}.\Omega^{i-1}/\mathrm{F} \xrightarrow{d\mathrm{V}} \mathrm{W}.\Omega^i/\mathrm{V} \to \Omega^i \to 0$$

([Ill], I 3.19.1) that the natural map $\mathrm{H}^0(\mathrm{W}\Omega^i)/p \to \mathrm{H}^0(\Omega^i)$ is injective. Therefore the conclusion follows from 1.3.

Q.E.D.

Proposition 1.7. — *Let* $i \in \mathbf{Z}$. *Assume that* X/k *is projective* (*and smooth*), *that* X/k *satisfies* $\mathrm{MO}(i)$ *and that* $h^{0\,i} = h^{i\,0}$. *Then the following conditions are equivalent :*

(i) "$\mathrm{HW}(i)$" $\mathrm{F}: \mathrm{H}^i(\mathcal{O}) \to \mathrm{H}^i(\mathcal{O})$ *is an isomorphism;*

(ii) "$\mathrm{C}(i)$" $\mathrm{C}: \mathrm{H}^0(\Omega^i) \to \mathrm{H}^0(\Omega^i)$ *is an isomorphism.*

Proof. — In general, when $\mathrm{H}^i(\mathrm{X}/\mathrm{W})$ and $\mathrm{H}^{i+1}(\mathrm{X}/\mathrm{W})$ are torsion-free, $\dim \mathrm{H}^i(\mathrm{X}/\mathrm{W})_{[0]}$ is the stable rank of F on $\mathrm{H}^i(\mathcal{O})$, *cf.* [Ka 5], 3.3.4. Thus, (i) says $\dim \mathrm{H}^i(\mathrm{X}/\mathrm{W})_{[0]} = h^{0\,i}$. Since X is projective, one has $\dim \mathrm{H}^i(\mathrm{X}/\mathrm{W})_{[0]} = \dim \mathrm{H}^i(\mathrm{X}/\mathrm{W})_{[i]}$ by the Hard Lefschetz Theorem [Ka-Me]. Therefore (i) $\Leftrightarrow$ (ii) follows from 1.6.

Q.E.D.

Remark 1.8. — The hypotheses of 1.7 are verified for example if X admits a projective, smooth lifting X/W satisfying (HLF). However, it is not known whether the Hodge symmetry $h^{ij} = h^{ji}$ holds when it is only assumed X/k is projective and satisfies $\mathrm{MO}(i)$ for all i.

2. The vanishing of $\mathrm{H}^0(\mathrm{BW}\Omega^i)$ and $\mathrm{H}^1(\mathrm{BW}\Omega^i)$

Lemma 2.1. — *One has* $\mathrm{H}^0(\mathrm{BW}\Omega^i) = 0$ *for all* i.

Proof. — Consider the exact sequence [Ill-Ray], IV 1.2.2

$$0 \to \mathrm{H}^0(\mathrm{ZW}\Omega^{i-1}) \xrightarrow{(1)} \mathrm{H}^0(\mathrm{W}\Omega^{i-1}) \to \mathrm{H}^0(\mathrm{BW}\Omega^i) \to \mathrm{H}^1(\mathrm{ZW}\Omega^{i-1}) \xrightarrow{(2)} \mathrm{H}^1(\mathrm{W}\Omega^{i-1}) \to \dots$$

By 1.1 (*a*), (1) is an isomorphism, and by 1.1 (*c*) and [Ill-Ray], IV 2.12, (2) is injective, hence the conclusion.

Q.E.D.

Proposition 2.2. — *Let* $i \in \mathbf{Z}$. *Assume that* X *satisfies* $\mathrm{MO}(i)$ (1.4). *Then one has the equivalence*

$$\mathrm{C}(i) \Leftrightarrow (\mathrm{H}^1(\mathrm{BW}\Omega^i) = 0).$$

Proof. — Assume that X satisfies C (i). We shall prove $H^1(BW\Omega^i)=0$ in two steps.

We will first prove that $H^1(BW\Omega^i)$ is of finite type over W, i. e. that $H^1(BW\Omega^i)_u=0$ in the notation of [Ill-Ray], IV 2.15. By [Ill-Ray], IV 2.15.4, this amounts to proving that $H^1(ZW\Omega^i)_u=0$ and that $H^2(ZW\Omega^{i-1})_u=0$. We already know that $H^1(ZW\Omega^i)_u=0$ [1.1 (c)].

By [Ill-Ray], IV 2.12, $H^2(ZW\Omega^{i-1})_u=0$ is equivalent to $H^2(W\Omega^{i-2})_{V-u}=0$, i.e. to $T^{i-2,\,2}=0$, where

(2.2.1) $$T^{ab}:=\dim H^b(W\Omega^a)/(V^{-\infty}Z+V)=\dim H^b(W\Omega^a)_{V-u}$$

is the dimension of the formal group associated to the unipotent quotient of $H^b(W\Omega^a)$ (*cf.* [Ill-Ray], II 3.1 and [Ill 1], 3.1.3). By [1.1 (a), (b)] we know that

$$T^{i\,0}=T^{i-1,\,1}=0.$$

Therefore $T^{i-2,\,2}=(\Delta_2 T)^{i\,0}$, where

(2.2.2) $$(\Delta_2 T)^{ab}:=T^{ab}-2T^{a-1,\,b+1}+T^{a-2,\,b+2}.$$

In order to prove that $(\Delta_2 T)^{i\,0}=0$, we will use some partial consequences of Ekedahl's theory of "diagonal decomposition" [Ek] ([Ill 1], § 6). Recall Ekedahl's definition of the Hodge-Witt numbers h^{ab}_W ([Ill 1], 6.3.1),

(2.2.3) $$h^{ab}_W:=m^{ab}+(\Delta_2 T)^{ab}$$

where

$$m^{ab}=\dim H^b(W\Omega^a)/(p\text{-tors}+V)+\dim H^{b+1}(W\Omega^{a-1})/(p\text{-tors}+F).$$

By Ekedahl's basic inequality $h^{ab}_W\leqq h^{ab}$ ([Ill 1], 6.3.10), MO(i) implies that $h^{i\,0}_W=h^{i\,0}$. Therefore we must prove $h^{i\,0}=m^{i\,0}$. Recall that the Hodge polygon M_i made with the numbers $(m^{0\,i}, m^{1,\,i-1}, \ldots, m^{i\,0})$ is the highest Hodge polygon lying below Nwt_i ([Ill 1], 6.2) (so in particular $\mathrm{Nwt}_i\geqq M_i\geqq \mathrm{Hdg}_i$). By 1.6, C$(i)$ means that Nwt_i and Hdg_i have the same segment of slope i. In view of the above interpretation of M_i, this means that $h^{i\,0}=m^{i\,0}$.

We now prove that $H^1(BW\Omega^i)=0$. We first claim that the sequence (*cf.* [Ill-Ray], IV 1.2.2).

(*) $$0\to H^1(ZW\Omega^{i-1})\xrightarrow{a} H^1(W\Omega^{i-1})\xrightarrow{b} H^1(BW\Omega^i)\to 0,$$

made from the H^1 terms in the long cohomology sequence for

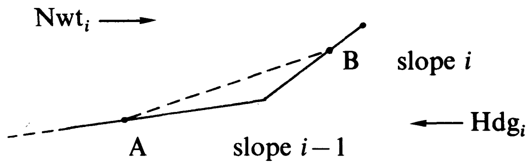
$$0\to ZW.\Omega^{i-1}\to W.\Omega^{i-1}\to BW.\Omega^i\to 0,$$

is exact. Since $H^1(ZW\Omega^{i-1})_u=0$ [1.1 (c)], a is injective ([Ill-Ray], IV 2.12). We have proven above that $H^2(ZW\Omega^{i-1})_u=0$, so b is surjective (*loc. cit.*). So (*) is exact. Since the terms of (*) are finitely generated over W, and F operates on (*) with F bijective on $H^1(ZW\Omega^{i-1})$ and p-adically nilpotent on $H^1(BW\Omega^i)$, (*) admits a unique F-stable splitting. Since $H^1(W\Omega^{i-1})$ injects into $H^i(X/W)$ [1.1 (*d*)] and $H^i(X/W)$ is torsion-free by MO(i), it follows that the terms of ($*$) are free of finite type over W. Now by [Ill-Ray], IV 2.3, we have

$$rk\, H^1(BW\Omega^i) = \dim H^i(X/W)_{]i-1,\, i[}.$$

But since Nwt_i and Hdg_i have the same segment of slope i, Nwt_i has no slopes in $]i-1, i[$, i.e. $rk\ H^1(BW\Omega^i)=0$, hence $H^1(BW\Omega^i)=0$, since $H^1(BW\Omega^i)$ is free.

Conversely, suppose that $H^1(BW\Omega^i)=0$. Then $h^{i\,0}=m^{i\,0}$, as follows from the first part of the proof [$H^1(BW\Omega^i)_u=0$ would suffice for this]. In other words, the polygons M_i and Hgd_i have the same segment of slope i. Since Nwt_i and M_i "kiss" ([Ill 1], 6.2), this implies that Nwt_i and Hdg_i look like



where the part of Nwt_i between A and B is precisely the part whose slopes lie in $]i-1, i[$. But since $H^1(BW\Omega^i)=0$, this part has horizontal length zero, so $\dim H^i(X/W)_{[i]}=h^{i\,0}$, which proves C($i$) (1.6).

Q.E.D.

3. A transversality result

This section is independant of paragraphs 1 and 2. The results it contains are not used in [Ka 6].

3.1. We shall discuss the following questions. On $H^n(X/W)$ we have the filtration P^* (resp. P_*) coming from the first de Rham-Witt spectral sequence (resp. the conjugate spectral sequence) ([Ill], II 3.1.2) ([Ill-Ray], III 2.1.5). If X admits a proper and smooth lifting $\tilde{X}/W$, then $H^n(X/W)\simeq H^n_{DR}(\tilde{X}/W)$, and we can consider the Hodge filtration $\mathrm{Fil}^*_{\tilde{X}}$ on $H^n(X/W)$ coming from the Hodge to de Rham spectral sequence of $\tilde{X}/W$. For fixed i and n we thus have maps

$$(3.1.1)\qquad P_{i-1}H^n(X/W)\oplus P^iH^n(X/W)\to H^n(X/W),$$

$$(3.1.2)\qquad P_{i-1}H^n(X/W)\oplus \mathrm{Fil}^i_{\tilde{X}}H^n(X/W)\to H^n(X/W).$$

By [Ka 6], 6.2 (2) and 7.3.4, if $\tilde{X}$ satisfies (HLF) and C(i), then for $i=n$ both (3.1.1) and (3.1.2) are isomorphisms. One cannot expect (3.1.1) and (3.1.2) to be injective in general, as the example of a supersingular K3 shows ([Ill-Ray], IV 2.17). However, we shall give "polygonal criteria" ensuring injectivity or bijectivity of (3.1.1) and (3.1.2). These generalize ([Ka 6], *loc. cit.*) and some results of [Ka 3].

3.2. Let $i \in \mathbf{Z}$. Consider again the commutative diagram with exact rows and columns ([Ka 6], 7.3.2):

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & t_{\leqq i-1} W.\Omega^{\cdot} \arrow[r, equal] \arrow[d] & t_{\leqq i-1} W.\Omega^{\cdot} \arrow[d] & \\
0 \arrow[r] & W.\Omega^{\geqq i} \arrow[r] \arrow[d, equal] & W.\Omega^{\cdot} \arrow[r] \arrow[d] & W.\Omega^{<i} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W.\Omega^{\geqq i} \arrow[r] & t'_{\geqq i} W.\Omega^{\cdot} \arrow[r] \arrow[d] & BW.\Omega^{i}[-i+1] \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd} \tag{3.2.1}$$

The resulting exact sequence

$$0 \to W.\Omega^{\geqq i} \oplus t_{\leqq i-1} W.\Omega^{\cdot} \to W.\Omega^{\cdot} \to BW.\Omega^{i}[-i+1] \to 0 \tag{3.2.2}$$

yields a long exact sequence of cohomology

$$\ldots \to H^n(W\Omega^{\geqq i}) \oplus H^n(t_{\leqq i-1} W\Omega^{\cdot}) \xrightarrow{a} H^n(X/W) \xrightarrow{b} H^{n-i+1}(BW\Omega^{i}) \xrightarrow{(d_1,\, d_2)} \ldots \tag{3.2.3}$$

This is an exact sequence of profinite W-modules on which the Frobenius $\mathscr{F}$ and, for X purely of dimension N, the operator $\mathscr{V}$ induced by $V'_{\leqq N}$ ([Ill-Ray], III 1.7.1) operate. One has $\mathscr{F}\mathscr{V} = \mathscr{V}\mathscr{F} = p^N$, and on $H^{n-i+1}(BW\Omega^i)$, one has $\mathscr{F} = p^{i-1} F_1$, $\mathscr{V} = p^{N-i} V_1$, with F_1 and V_1 topologically nilpotent ([Ill-Ray], IV 2). The image of a is $P^i H^n(X/W) + P_{i-1} H^n(X/W)$.

Proposition 3.3. — *Let $i, j \in \mathbf{Z}$ and $n = i + j$.*

(i) *If* $H^j(BW\Omega^i)_u = 0$ (*notation of* [Ill-Ray], IV 2.15), *the map a in* (3.2.3) *is injective, i. e.*

$$H^n(W\Omega^{\geqq i}) \xrightarrow{\sim} P^i H^n(X/W), \qquad H^n(t_{\leqq i-1} W\Omega^{\cdot}) \xrightarrow{\sim} P_{i-1} H^n(X/W)$$

and $$P^i \cap P_{i-1} = 0.$$

(ii) *If* $H^j(BW\Omega^i)_u = H^{j+1}(BW\Omega^i)_u = 0$, *then the sequence*

$$0 \to H^n(W\Omega^{\geqq i}) \oplus H^n(t_{\leqq i-1} W\Omega^{\cdot}) \xrightarrow{a} H^n(X/W) \xrightarrow{b} H^{j+1}(BW\Omega^i) \to 0 \tag{3.3.1}$$

is exact and admits a canonical splitting stable under $\mathscr{F}$ *and* $\mathscr{V}$.

Proof of (i). — We have to show the vanishing of the coboundary map

$$(d_1, d_2): \mathrm{H}^j(\mathrm{BW}\,\Omega^i) \to \mathrm{H}^n(\mathrm{W}\,\Omega^{\geqq i}) \oplus \mathrm{H}^n(t_{\leqq i-1}\,\mathrm{W}\,\Omega^{\cdot}).$$

By construction, d_1 (resp. d_2) is the coboundary map of the long exact sequence associated to the lower row (resp. right column) of (3.2.1). It follows that d_1 is the composition of the maps

$$\mathrm{H}^j(\mathrm{BW}\,\Omega^i) \xrightarrow{(1)} \mathrm{H}^j(\mathrm{ZW}\,\Omega^i) \to \mathrm{H}^n(\mathrm{W}\,\Omega^{\geqq i})$$

induced by the inclusions $\mathrm{BW}.\Omega^i[-i] \to \mathrm{ZW}.\Omega^i[-i] \to \mathrm{W}\Omega^{\geqq i}$. The hypothesis $\mathrm{H}^j(\mathrm{BW}\,\Omega^i)_u = 0$ means that $\mathrm{H}^j(\mathrm{ZW}\,\Omega^i)_u = 0$ and that $\mathrm{H}^{j+1}(\mathrm{ZW}\,\Omega^{i-1})_u = 0$ ([Ill-Ray], VI 2.15.4); the fact that $\mathrm{H}^j(\mathrm{ZW}\,\Omega^i)_u = 0$ implies that (1) vanishes ([Ill-Ray], IV 2.12), so $d_1 = 0$. In the same way, d_2 factorizes as

$$\mathrm{H}^j(\mathrm{BW}\,\Omega^i) \xrightarrow{(2)} \mathrm{H}^{j+1}(\mathrm{ZW}\,\Omega^{i-1}) \to \mathrm{H}^n(t_{\leqq i-1}\,\mathrm{W}\,\Omega^{\cdot})$$

where (2) is the coboundary map coming from

$$0 \to \mathrm{ZW}.\Omega^{i-1} \to \mathrm{W}.\Omega^{i-1} \to \mathrm{BW}.\Omega^i \to 0,$$

and by (*loc. cit.*) $\mathrm{H}^{j+1}(\mathrm{ZW}\,\Omega^{i-1})_u = 0$ implies that (2) vanishes, hence $d_2 = 0$.

Proof of (ii). — The exactness of (3.3.1) follows from (i), and implies the diagram obtained from (3.2.1) by applying $\mathrm{H}^n(\mathrm{X}, -)$ and passing to the inverse limit is exact. Since the maps in this diagram are compatible with $\mathscr{F}$ and $\mathscr{V}$, we have only to show that the surjections

(3) $$\mathrm{H}^n(\mathrm{W}\,\Omega^{\cdot}) \to \mathrm{H}^n(\mathrm{W}\,\Omega^{<i}),$$

(4) $$\mathrm{H}^n(\mathrm{W}\,\Omega^{<i}) \to \mathrm{H}^{j+1}(\mathrm{BW}\,\Omega^i),$$

have natural sections compatible with $\mathscr{F}$ and $\mathscr{V}$. That (3) admits such a section is contained in [Ill-Ray], IV 4.4 [since the hypotheses of (ii) imply $\mathrm{H}^j(\mathrm{ZW}\,\Omega^i)_u = \mathrm{H}^{j+1}(\mathrm{ZW}\,\Omega^i)_u = 0$]. The same assertion for (4) follows from the fact that the natural sequence (of finitely generated W-modules on which F and V operate)

(3.3.2) $$0 \to \mathrm{H}^{j+1}(\mathrm{ZW}\,\Omega^{i-1}) \to \mathrm{H}^{j+1}(\mathrm{W}\,\Omega^{i-1}) \to \mathrm{H}^{j+1}(\mathrm{BW}\,\Omega^i) \to 0$$

is exact ([Ill-Ray], IV 2.12) hence has unique splitting stable under F and V [note that for $j=0$, (3.3.2) is the sequence (*) of the proof of (2.2)].

Q.E.D.

COROLLARY 3.4. — *Let* $i, j \in \mathbf{Z}$ *and* $n = i + j$. *If* $\mathrm{H}^j(\mathrm{BW}\,\Omega^i)_u = \mathrm{H}^{j+1}(\mathrm{BW}\,\Omega^i) = 0$, *the map a of* (3.2.3):

$$\mathrm{H}^n(\mathrm{W}\,\Omega^{\geqq i}) \oplus \mathrm{H}^n(t_{\leqq i-1}\,\mathrm{W}\,\Omega^{\cdot}) \to \mathrm{H}^n(\mathrm{W}\,\Omega^{\cdot})$$

is an isomorphism.

In other words, (3.1.1) *is an isomorphism, and* $H^n(W\Omega^{\geq i}) \xrightarrow{\sim} P^i H^n(X/W)$, $H^n(t_{\leq i-1} W\Omega^{\cdot}) \xrightarrow{\sim} P_{i-1} H^n(X/W)$.

In view of 2.1 *and* 2.2, *one recovers* ([Ka 6], 7.3.4) *when* $j=0$.

3.5. Polygonal criteria. — The hypotheses of 3.3 (i) or (ii) amount to the vanishing of certain of the invariants T^{ab} (2.2.1): as recalled above, by [Ill-Ray], IV 2.15.4, we have the equivalence

$$(3.5.1) \qquad (H^j(BW\Omega^i)_u=0) \Leftrightarrow (H^j(ZW\Omega^i)_u=H^{j+1}(ZW\Omega^{i-1})_u=0),$$

and by [Ill-Ray], IV 2.12, we have the equivalence

$$(3.5.2) \qquad H^j(ZW\Omega^i)_u=0 \Leftrightarrow T^{i-1,\,j}=0.$$

Ekedahl [Ek] has given the following interpretation of the T^{ij}'s in terms of the Hodge polygons associated with the numbers m^{ab} and h_W^{ab} (2.2.3).

Proposition 3.5.3. — *For* $n\in\mathbf{Z}$, *let* M_n (*resp.* $\mathrm{Hdg}\,W_n$) *denote the Hodge polygon made with the numbers* $(m^{0\,n}, \ldots, m^{n\,0})$ [*resp.* $(h_W^{0\,n}, \ldots, h_W^{n\,0})$] [cf. *proof of* (2.2) *and* [Ill 1], 6.2]. *Then, for* $i+j=n$ *one has*

$$T^{ij}=y_{i+1}(M_n)-y_{i+1}(\mathrm{Hdg}\,W_n),$$

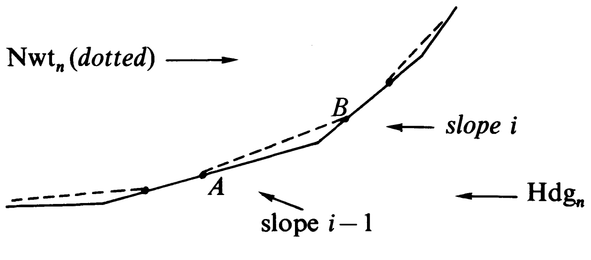
where $(0,\ y_r(P))$ *is the point where the line of slope* r *of the polygon* P *meets the y-axis.*

Proof. — Integrate the double difference equation (2.2.3).

Q.E.D.

If X satisfies MO(n), then $h_W^{ij}=h^{ij}$ for $i+j=n$ by Ekedahl's inequality ([Ill 1], 6.3.10). Therefore the vanishing of $T^{a,\,n-a}$ means that the segment of M_n of slope $a+1$ is contained in the segment of Hdg_n of the same slope, or equivalently that the Newton polygon Nwt_n of $H^n(X/W)$ touches the segment of Hdg_n of slope $a+1$. In view of (3.5.1) and (3.5.2), we find:

Corollary 3.5.4. — *Let* $i,\ j\in\mathbf{Z}$ *and* $i+j=n$. *If* X *satisfies* MO(n), *then* $H^{j+1}(BW\Omega^i)_u=0$ *means that* Nwt_n *and* Hdg_n *look like*



where AB *is precisely the part of* Nwt_n *whose slopes lie in* $]i-1,\ i[$.

In summary, then, if X satisfies MO(n), and if we have $H^j(BW\Omega^i)_u = H^{j+1}(BW\Omega^i)_u = 0$ for a pair of integers (i, j) with $i+j=n$, then the split exact sequence (3.3.1) gives a de Rham-Witt style interpretation of the Newton-Hodge decomposition associated to the break-points A and B [Ka 2] (p^i divides $\mathscr{F}$ on $H^n(W\Omega^{\geq i})$, $\mathscr{F}$ divides p^{i-1} on $H^n(t_{\leq i-1} W\Omega^{\cdot})$, and the slopes of $H^{j+1}(BW\Omega^i)$ lie in $]i-1, i[$).

3.6. For the rest of this paper, we suppose given a proper and smooth lifting $\tilde{X}$ of X over W. Then, as recalled in 3.1, we have a canonical isomorphism

$$(3.6.1) \qquad H^*(X/W) \xleftarrow{\sim} H^*_{DR}(\tilde{X}/W) := H^*(\tilde{X}, \Omega^{\cdot}_{\tilde{X}/W}).$$

In fact, by [Ill], II 1.4) there is an isomorphism in the derived category $D(X, \text{pro-}W)$ of sheaves of pro-W-modules on X

$$(3.6.2) \qquad \Omega^{\cdot}_{\tilde{X}/W} \otimes \mathbf{Z}/p^{\cdot} \xrightarrow{\sim} W.\Omega^{\cdot}_X,$$

which yields (3.6.1) by applying $H^*(\tilde{X}, -)$ and passing to the limit, since by Grothendieck's comparison theorem $H^*_{DR}(\tilde{X}/W) = \varprojlim H^*(\tilde{X}, \Omega^{\cdot}_{\tilde{X}/W} \otimes \mathbf{Z}/p^n)$.

By definition, the Hodge filtration (associated to $\tilde{X}$) on $H^*(X/W)$, denoted $\mathrm{Fil}^{\cdot}_{\tilde{X}} H^*(X/W)$ [or, for brevity, $\mathrm{Fil}^{\cdot} H^*(X/W)$ when $\tilde{X}$ is fixed], is the filtration on the abutment of the spectral sequence

$$(3.6.3) \qquad E_1^{ij} = H^j(\tilde{X}, \Omega^i) \Rightarrow H^*(X/W) (\xrightarrow{\sim} H^*_{DR}(\tilde{X}/W)).$$

i. e.

$$\mathrm{Fil}^i H^*(X/W) = \mathrm{Im}(H^*(\tilde{X}, \Omega^{\geq i}) \to H^*(X/W)).$$

Notice that (3.6.3) is the inverse limit of the Hodge to de Rham spectral sequences of the $\tilde{X}_n/W_n$, where $\tilde{X}_n = \tilde{X} \otimes W_n$. When $\tilde{X}$ satisfies (HLF), (3.6.3) degenerates at E_1.

Thanks to (3.6.2), the conjugate spectral sequence over W_n defined in ([Ill-Ray], III 2.1) can be identified with the second spectral sequence of $\Omega^{\cdot}_n := \Omega^{\cdot}_{\tilde{X}/W} \otimes \mathbf{Z}/p^n$, so that the (limit) conjugate spectral sequence of (*loc. cit.*) can be identified with the corresponding limit spectral sequence

$$(3.6.4) \qquad E_2^{ij} = \varprojlim H^i(\tilde{X}, \mathscr{H}^j \Omega^{\cdot}_n) \Rightarrow H^*(X/W).$$

In what follows, we will write $H^i(\tilde{X}, \mathscr{H}^j \Omega^{\cdot})$ for the inverse limit $\varprojlim H^i(\tilde{X}, \mathscr{H}^j \Omega^{\cdot}_n)$ [warning: the group $H^i(\tilde{X}, \mathscr{H}^j \Omega^{\cdot})$ is *not* in general the ith hypercohomology group of $\tilde{X}$ with value in the Zariski sheaf $\mathscr{H}^j \Omega^{\cdot}_{\tilde{X}}$].

In particular, we have

$$P_i H^*(X/W) = \mathrm{Im}(H^*(\tilde{X}, t_{\leq i} \Omega^{\cdot}) \to H^*(X/W))$$

where $H^*(\tilde{X}, t_{\leqq i}\Omega^{\cdot}) := \varprojlim H^*(\tilde{X}, t_{\leqq i}\Omega_n^{\cdot})$.

By analogy with what is done in [Ill-Ray], we are led to consider also the W-modules

$$H^*(\tilde{X}, Z\Omega^i) := \varprojlim H^*(\tilde{X}, Z\Omega_n^i), \qquad H^*(\tilde{X}, B\Omega^i) := \varprojlim H^*(\tilde{X}, B\Omega_n^i).$$

Just as with $H^j(\tilde{X}, \Omega^i)$, these W-modules depend on $\tilde{X}$, as opposed to $H^*(\tilde{X}, \mathscr{H}^i\Omega^{\cdot})$, which depends only on X. Not much is known about their structure. What one can say is this:

(*a*) $H^*(\tilde{X}, Z\Omega_n^i)$ and $H^*(\tilde{X}, B\Omega_n^i)$ are finitely generated W_n-modules. Indeed, the argument in [Ka 3], § 8, shows that the p-adic filtration on $Z\Omega_n^i$ (resp. $B\Omega_n^i$) has an associated graded which is locally free of finite type over X.

(*b*) For all i, j, $H^j(\tilde{X}, Z\Omega^i)$ [resp. $H^j(\tilde{X}, B\Omega^i)$] is free of finite type over W modulo a torsion subgroup which is annihilated by a power of p. In view of the known structure of $H^*(\tilde{X}, \mathscr{H}^*\Omega^{\cdot})$ ($\xrightarrow{\sim} H^*(X, \mathscr{H}^* W\Omega^{\cdot})$), this follows from the fact that $H^*(\tilde{X}, \Omega^*)$ is finitely generated over W.

Consider now, for $i \in \mathbf{Z}$, the diagram analogous to (3.2.1):

$$\tag{3.6.5}\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & t_{\leqq i-1}\Omega^{\cdot} \arrow[r, equal] \arrow[d] & t_{\leqq i-1}\Omega^{\cdot} \arrow[d, "\beta"] & \\
0 \arrow[r] & \Omega^{\geqq i}_{\cdot} \arrow[r] \arrow[d, equal] & \Omega^{\cdot} \arrow[r] \arrow[d] & \Omega^{<i}_{\cdot} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^{\geqq i}_{\cdot} \arrow[r, "\alpha"] & t'_{\geqq i}\Omega^{\cdot} \arrow[r] \arrow[d] & B\Omega^i_{\cdot}[-i+1] \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

and the resulting exact sequence

$$\tag{3.6.6} 0 \to \Omega^{\geqq i}_{\cdot} \oplus t_{\leqq i-1}\Omega^{\cdot} \to \Omega^{\cdot} \to B\Omega^i_{\cdot}[-i+1] \to 0.$$

By the above remarks, the exact sequence of cohomology of (3.6.6) yields, by passing to the limit, a long exact sequence of profinite W-modules

$$\tag{3.6.7} \ldots \to H^n(\tilde{X}, \Omega^{\geqq i}) \oplus H^n(\tilde{X}, t_{\leqq i-1}\Omega^{\cdot}) \xrightarrow{a} H^n(X/W) \xrightarrow{b} H^{n-i+1}(\tilde{X}, B\Omega^i) \xrightarrow{(d_1,\, d_2)} \ldots$$

The image of a is $\mathrm{Fil}^i H^n(X/W) \oplus P_{i-1} H^n(X/W)$.

PROPOSITION 3.7. — *Suppose that* $\tilde{X}$ *satisfies* (HLF). *Let* $i, j \in \mathbf{Z}$ *and* $n=i+j$.

(i) *If* $H^j(BW\Omega^i)_u = H^{j+1}(BW\Omega^i)_u = 0$, *the map* a *in* (3.6.7) *is injective, i.e. we have*

$$H^n(\tilde{X}, \Omega^{\geqq i}) \tilde{\rightarrow} \mathrm{Fil}^i H^n(X/W),$$

$$H^n(\tilde{X}, t_{\leqq i-1}\Omega^{\cdot}) \tilde{\rightarrow} P_{i-1} H^n(X/W),$$

and $\mathrm{Fil}^i \cap P_{i-1} = 0$. *Moreover, the cokernel of* a *is free, of finite type over* W.

(ii) *If* $H^j(BW\Omega^i)_u = H^{j+1}(BW\Omega^i)_u = H^{j+2}(BW\Omega^{i+1})_u = 0$, *the sequence*

$$0 \rightarrow H^n(\tilde{X}, \Omega^{\geqq i}) \oplus H^n(\tilde{X}, t_{\leqq i-1}\Omega^{\cdot}) \rightarrow H^n(X/W) \rightarrow H^{j+1}(\tilde{X}, B\Omega^i) \rightarrow 0$$

is exact.

Proof. — It is enough to prove (i). By (HLF), the sequence

$$0 \rightarrow H^n(\tilde{X}, \Omega^{\geqq i}) \rightarrow H^n(X/W) \rightarrow H^n(\tilde{X}, \Omega^{<i}) \rightarrow 0$$

defined by the middle row of (3.6.5) is exact, with free, finitely generated terms. Similarly, by 3.3 (ii) (plus the remarks in 3.6), the sequence

$$0 \rightarrow H^n(\tilde{X}, t_{\leqq i-1}\Omega^{\cdot}) \rightarrow H^n(X/W) \rightarrow H^n(\tilde{X}, t'_{\geqq i}\Omega^{\cdot}) \rightarrow 0$$

defined by the middle column of (3.6.5) [with $H^n(\tilde{X}, t'_{\geqq i}\Omega^{\cdot}) := \varprojlim H^n(X, t'_{\geqq i}\Omega^{\cdot}_r)$] is exact, with free, finitely generated terms. We have to show that the map

$$(d_1, d_2): H^j(\tilde{X}, B\Omega^i) \rightarrow H^n(\tilde{X}, \Omega^{\geqq i}) \oplus H^n(\tilde{X}, t_{\leqq i-1}\Omega^{\cdot})$$

vanishes and that the cokernel of a is torsion-free. This amounts to showing the injectivity of the maps

$$\alpha: H^n(X, \Omega^{\geqq i}) \rightarrow H^n(X, t'_{\geqq i}\Omega^{\cdot}),$$

$$\beta: H^n(X, t_{\leqq i-1}\Omega^{\cdot}) \rightarrow H^n(X, \Omega^{<i})$$

coming from the lower row and right column of (3.6.5), together with the fact that the (common) cokernel of α and β is torsion-free. Recall that a linear map $u: L \rightarrow M$ between free, finitely generated W-modules is injective with torsion-free cokernel if and only if $u \otimes_{\mathbf{Z}} \mathbf{F}_p$ is injective. Therefore we have to prove the injectivity of $\alpha \otimes \mathbf{F}_p$ and $\beta \otimes \mathbf{F}_p$.

3.7.1. *Injectivity of* $\alpha \otimes \mathbf{F}_p$. — We have a commutative square

$$\begin{tikzcd} \mathrm{H}^n(\tilde{\mathrm{X}}, \Omega^{\geq i})/p \arrow[r, "\alpha\otimes\mathbf{F}_p"] \arrow[d] & \mathrm{H}^n(\mathrm{X}, t'_{\geq i}\Omega^{\cdot})/p \arrow[d] \\ \mathrm{H}^n(\tilde{\mathrm{X}}, \Omega^{\geq i}/p) \arrow[r] & \mathrm{H}^n(\tilde{\mathrm{X}}, t'_{\geq i}\Omega^{\cdot}/p)(:=\varprojlim \mathrm{H}^n(\tilde{\mathrm{X}}, t'_{\geq i}\Omega^{\cdot}_r/p)) \end{tikzcd}$$

with injective vertical maps, so it suffices to show that the lower map is injective. There is an obvious isomorphism $\Omega^{\geq i}_{\cdot}/p \to \Omega_{\mathrm{X}}^{\geq i}$ and, by (3.6.2), an isomorphism in D(X, pro-W)

$$t'_{\geq i}\Omega^{\cdot}_{\cdot}/p \xrightarrow{\sim} t'_{\geq i}\mathrm{W}.\Omega^{\cdot}/p$$

(since the components of the pro-complexes $t'_{\geq i}\Omega^{\cdot}_{\cdot}$ and $t'_{\geq i}\mathrm{W}.\Omega^{\cdot}$ are p-torsion-free). It is easily checked that these isomorphisms sit in a commutative diagram

$$(*)\qquad\begin{tikzcd} \Omega^{\geq i}_{\cdot}/p \arrow[rrr] \arrow[d, "\simeq"'] & & & t'_{\geq i}\Omega^{\cdot}_{\cdot}/p \arrow[d, "\simeq"] \\ \Omega_{\mathrm{X}}^{\geq i} \arrow[r] & \Omega^{\cdot}_{\mathrm{X}} \arrow[r, "(1)", "\simeq"'] & \mathrm{W}.\Omega^{\cdot}/p \arrow[r] & t'_{\geq i}\mathrm{W}.\Omega^{\cdot}/p \end{tikzcd}$$

where (1) is the inverse of the canonical isomorphism ([Ill], I 3.15). Finally, the natural projection

$$\mathrm{L}_i:=(0\to \mathrm{W}.\Omega^{i-1}/\mathrm{F}\xrightarrow{d\mathrm{V}}\mathrm{W}.\Omega^{i}/p\xrightarrow{d}\mathrm{W}.\Omega^{i+1}/p\to\dots)\to\Omega_{\mathrm{X}}^{\geq i}$$

is a quasi-isomorphism ([Ill, I 3.2.0), and its composition with the lower row of (∗) is the map

$$\begin{tikzcd} \mathrm{L}_i \arrow[d, "f"'] & 0 \arrow[r] & \mathrm{W}.\Omega^{i-1}/\mathrm{F} \arrow[r, "d\mathrm{V}"] \arrow[d, "d\mathrm{V}"] & \mathrm{W}.\Omega^{i}/p \arrow[r] \arrow[d, equal] & \mathrm{W}.\Omega^{i+1}/p \arrow[r] \arrow[d, equal] & \dots \\ t'_{\geq i}\mathrm{W}.\Omega^{\cdot}/p & 0 \arrow[r] & \mathrm{BW}.\Omega^{i}/p \arrow[r] & \mathrm{W}.\Omega^{i}/p \arrow[r] & \mathrm{W}.\Omega^{i+1}/p \arrow[r] & \dots \end{tikzcd}$$

To sum up, we are reduced to proving that the map

$$\varprojlim \mathrm{H}^n(\mathrm{X}, f):\mathrm{H}^n(\mathrm{X}, \mathrm{L}_i)\to\mathrm{H}^n(\mathrm{X}, t'_{\geq i}\mathrm{W}\Omega^{\cdot}/p)\qquad [:=\varprojlim \mathrm{H}^n(\mathrm{X}, t'_{\geq i}\mathrm{W}_r\Omega^{\cdot}/p)]$$

is injective. For this, we argue as follows. We have a short exact sequence

$$0\to\mathrm{L}_i\xrightarrow{f} t'_{\geq i}\mathrm{W}.\Omega^{\cdot}/p\to\mathrm{BW}.\Omega^{i}/\mathrm{V}_1\,\mathrm{BW}.\Omega^{i}[-i+1]\to 0,$$

where V_1 is the operator on $\mathrm{BW}.\Omega^i$ induced from V on $\mathrm{W}.\Omega^{i-1}$ ([Ill-Ray], IV 2.1). So the desired injectivity is equivalent to the surjectivity of the natural map

$$\mathrm{H}^{n-1}(\mathrm{X}, t'_{\geqq i}\mathrm{W}\Omega^{\cdot}/p) \to \mathrm{H}^{j}(\mathrm{BW}\Omega^{i}/\mathrm{V}_1) \qquad [:=\varprojlim \mathrm{H}^{j}(\mathrm{BW}.\Omega^{i}/\mathrm{V}_1)].$$

Consider the commutative square

$$(**) \qquad \begin{tikzcd} \mathrm{H}^{n-1}(\mathrm{X}, t'_{\geqq i}\mathrm{W}\Omega^{\cdot}) \arrow[r] \arrow[d] & \mathrm{H}^{j}(\mathrm{BW}\Omega^{i}) \arrow[d] \\ \mathrm{H}^{n-1}(\mathrm{X}, t'_{\geqq i}\mathrm{W}\Omega^{\cdot}/p) \arrow[r] & \mathrm{H}^{j}(\mathrm{BW}\Omega^{i}/\mathrm{V}_1) \end{tikzcd}$$

By 3.3 (ii), its upper row is surjective. Since the exact sequence (3.3.1) is canonically split and $\mathrm{H}^n(\mathrm{X}/\mathrm{W})$ is torsion-free [by (HLF)], $\mathrm{H}^{j+1}(\mathrm{BW}\Omega^{i})$ is free of finite type over W and in particular has no V-torsion. So the right vertical map in (**) is surjective, and therefore the lower row is surjective.

3.7.2. *Injectivity of* $\beta \otimes \mathbf{F}_p$. — Again, it is enough to show that the map

$$\mathrm{H}^{n}(\mathrm{X}, t_{\leqq i-1}\Omega^{\cdot}/p)\,[:=\varprojlim \mathrm{H}^{n}(\mathrm{X}, t_{\leqq i-1}\Omega^{\cdot}_{r}/p)) \longrightarrow \mathrm{H}^{n}(\mathrm{X}, \Omega^{<i}/p]$$

is injective. As above, one checks that the isomorphisms

$$\Omega^{\leqq i-1}_{\cdot}/p \xrightarrow{\sim} \Omega^{\leqq i-1}_{\mathrm{X}} \qquad \text{and} \qquad t_{\leqq i-1}\Omega^{\cdot}_{\cdot}/p \xrightarrow{\sim} t_{\leqq i-1}\mathrm{W}.\Omega^{\cdot}/p$$

in $\mathrm{D}(\mathrm{X}, \text{pro-W})$ coming from (3.6.2) sit in a commutative square

$$\begin{tikzcd} t_{\leqq i-1}\Omega^{\cdot}_{\cdot}/p \arrow[r] \arrow[d] & \Omega^{\leqq i-1}_{\cdot}/p \arrow[d] \\ t_{\leqq i-1}\mathrm{W}.\Omega^{\cdot}/p \arrow[r] & \Omega^{\leqq i-1}_{\mathrm{X}} \end{tikzcd}$$

The composition of the lower map with the inverse of the quasi-isomorphism

$$\mathrm{K}_{i-1} := (\mathrm{W}.\mathcal{O}/p \to \mathrm{W}.\Omega^{1}/p \to \ldots \to \mathrm{W}.\Omega^{i-2}/p \to \mathrm{W}.\Omega^{i-1}/\mathrm{V} \to 0) \to \Omega^{\leqq i-1}_{\mathrm{X}}$$

([Ill], I 3.20) is the map

$$\begin{array}{ccc} t_{\leqq i-1}\mathrm{W}.\Omega^{\cdot}/p & & 0\to \mathrm{W}.\mathcal{O}/p\to \mathrm{W}.\Omega^1/p\to \ldots\to \mathrm{W}.\Omega^{i-2}/p\to \mathrm{ZW}.\Omega^{i-1}/p\to 0\\ \downarrow{\scriptstyle g} & & \quad\ \| \qquad\quad \| \qquad\qquad\qquad \| \qquad\qquad \downarrow \\ \mathrm{K}_{i-1} & & 0\to \mathrm{W}.\mathcal{O}/p\to \mathrm{W}.\Omega^1/p\to \ldots\to \mathrm{W}.\Omega^{i-2}/p\to \mathrm{W}.\Omega^{i-1}/\mathrm{V}\to 0.\end{array}$$

So we are reduced to showing that the map

$$\varprojlim \mathrm{H}^n(\mathrm{X}, g): \mathrm{H}^n(\mathrm{X}, t_{\leqq i-1}\mathrm{W}\Omega^{\cdot}/p)\,[: =\varprojlim \mathrm{H}^n(\mathrm{X}, t_{\leqq i-1}\mathrm{W}_r\Omega^{\cdot}/p)] \to \mathrm{H}^n(\mathrm{X}, \mathrm{K}_{i-1})$$

is injective. Since F is an automorphism of $\mathrm{ZW}.\Omega^{i-1}$ ([Ill], I 3.22) [Ill-Ray], III 1.3), the map $\mathrm{ZW}.\Omega^{i-1}/p \to \mathrm{W}.\Omega^{i-1}/\mathrm{V}$ is injective, and we have a short exact sequence

$$0 \to t_{\leqq i-1}\mathrm{W}.\Omega^{\cdot}/p \to \mathrm{K}_{i-1} \to \mathrm{BW}.\Omega^i/\mathrm{V}_1\,[-i+1] \to 0.$$

We must show that the coboundary map

$$\mathrm{H}^j(\mathrm{X}, \mathrm{BW}\Omega^i/\mathrm{V}_1) \xrightarrow{d} \mathrm{H}^n(\mathrm{X}, t_{\leqq i-1}\mathrm{W}\Omega^{\cdot}/p)$$

is zero. The above exact sequence is a quotient of the right column of (3.2.1), so we have a commutative square

$$\begin{array}{ccc} \mathrm{H}^j(\mathrm{X}, \mathrm{BW}\Omega^i) & \xrightarrow{d_2} & \mathrm{H}^n(\mathrm{X}, t_{\leqq i-1}\mathrm{W}\Omega^{\cdot}) \\ \downarrow & & \downarrow \\ \mathrm{H}^j(\mathrm{X}, \mathrm{BW}\Omega^i/\mathrm{V}_1) & \xrightarrow{d} & \mathrm{H}^n(\mathrm{X}, t_{\leqq i-1}\mathrm{W}\Omega^{\cdot}/p) \end{array}$$

with d_2 as in (3.2.3). By 3.3 (ii), $d_2=0$. Since the left vertical map is surjective, as we observed at the end of 3.7.1, it follows that $d=0$, which completes the proof of 3.7.

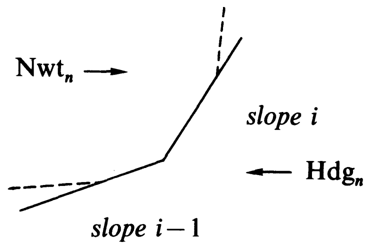
Q.E.D.

COROLLARY 3.8. — *Suppose that* $\tilde{\mathrm{X}}$ *satisfies* (HLF). *Let* $i, j \in \mathbf{Z}$ *and* $n=i+j$. *If* $\mathrm{H}^j(\mathrm{BW}\Omega^i)_u = \mathrm{H}^{j+1}(\mathrm{BW}\Omega^i)=0$, *the map a in* (3.6.7)

$$\mathrm{H}^n(\tilde{\mathrm{X}}, \Omega^{\geqq i}) \oplus \mathrm{H}^n(\tilde{\mathrm{X}}, t_{\leqq i-1}\Omega^{\cdot}) \to \mathrm{H}^n(\mathrm{X}/\mathrm{W}).$$

is an isomorphism [*in orther words,* (3.1.2) *is an isomorphism, and the maps* $\mathrm{H}^n(\tilde{\mathrm{X}}, \Omega^{\geqq i}) \to \mathrm{Fil}^i\,\mathrm{H}^n(\mathrm{X}/\mathrm{W})$, $\mathrm{H}^n(\tilde{\mathrm{X}}, t_{\leqq i-1}\Omega^{\cdot}) \to \mathrm{P}_{i-1}\,\mathrm{H}^n(\mathrm{X}/\mathrm{W})$ *are isomorphisms*).

Proof. — By 3.4 and 3.7 (i), it is enough to show that $\mathrm{rk}\ \mathrm{H}^n(\tilde{\mathrm{X}}, \Omega^{\geqq i}) = \mathrm{rk}\ \mathrm{H}^n(\mathrm{W}\Omega^{\geqq i})$. By (HLF) $\mathrm{rk}\ \mathrm{H}^n(\tilde{\mathrm{X}}, \Omega^{\geqq i}) = h^{ij} + \ldots h^{n\,0}$. Since $\mathrm{H}^{j+1}(\mathrm{BW}\Omega^i)=0$, Nwt_n has no slopes in the open interval $]i-1, i[$, so 3.5.4 says that Nwt_n and Hdg_n have a common breakpoint:



Therefore

$$h^{ij}+\ldots+h^{n\,0}=\dim \mathrm{H}^n(\mathrm{X}/\mathrm{W})_{\geqq i}=\mathrm{rk}\,\mathrm{H}^n(\mathrm{W}\,\Omega^i) \quad (3.3),$$

so

$$\mathrm{rk}\,\mathrm{H}^n(\tilde{\mathrm{X}}, \Omega^{\geqq i})=\mathrm{rk}\,\mathrm{H}^n(\mathrm{W}\,\Omega^{\geqq i}).$$

Q.E.D.

Remark 3.8.1. — For $i=n$, the hypothesis $\mathrm{H}^j(\mathrm{BW}\,\Omega^i)_u=0$ is automatically satisfied, in fact we have $\mathrm{H}^0(\mathrm{BW}\,\Omega^n)=0$ (2.1), so 3.8 gives the punctual case of ([Ka 6], 6.2 (2)).

As a particular case of 3.7 (i), we have:

COROLLARY 3.9. — *Let* $i\in\mathbf{Z}$. *Assume that* $\tilde{\mathrm{X}}$ *satisfies* (HLF) *and that* $\mathrm{H}^1(\mathrm{BW}\,\Omega^i)_u=0$. *Then we have a commutative diagram*

$$\begin{tikzcd}
 & & \mathrm{H}^0(\tilde{\mathrm{X}}, \Omega^i) \arrow[d] \arrow[dr, hook] & & \\
0 \arrow[r] & \mathrm{H}^i(\mathrm{X}, t_{\leq i-1}\,\mathrm{W}\,\Omega^{\cdot}) \arrow[r] & \mathrm{H}^i(\mathrm{X}/\mathrm{W}) \arrow[r] & \mathrm{H}^0(\mathrm{X}, \mathscr{H}^i\,\mathrm{W}\,\Omega^{\cdot}) \arrow[r] & 0
\end{tikzcd}$$

in which the bottom row is exact and in which the oblique map is injective with free, finitely generated cokernel.

Remark 3.10. — Suppose that X has geometrically connected fibres and let $x\in\tilde{\mathrm{X}}(\mathrm{W})$ be a W-valued point. By [Ka 3], 8.1, we know that the natural map

$$\mathrm{H}^0(\mathrm{X}, \mathscr{H}^i\,\mathrm{W}\,\Omega^{\cdot}) \xrightarrow{\sim} \mathrm{H}^0(\tilde{\mathrm{X}}, \mathscr{H}^i\,\Omega^{\cdot}) \rightarrow \mathrm{H}^i_{\mathrm{DR}}(\hat{\mathrm{X}_x}/\mathrm{W})$$

is injective. Thus, under the hypotheses of 3.9, we find that the map

$$\mathrm{H}^0(\tilde{\mathrm{X}}, \Omega^i) \rightarrow \mathrm{H}^i_{\mathrm{DR}}(\hat{\mathrm{X}_x}/\mathrm{W})$$

is injective, i.e. a non-zero global i-form cannot be formally exact at any point. This generalizes ([Ka 3], 5.9.3), since in the case X is of pure relative dimension 1, the condition $H^1(BW\Omega^i)_u = 0$ is automatically satisfied ([Ill-Ray], IV 2.15.6) (the conjugate spectral sequence degenerates at E_2 for dimension reasons, so X is Hodge-Witt, *cf.* [Ill-Ray], IV 4.6). This fact should be connected to possible higher dimensional analogues of the limit formulas of [Ka 3], 6.2.

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