AN ESTIMATE FOR CHARACTER SUMS

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In this note, we give estimates for a class of character sums that occur as eigenvalues of adjacency matrices of certain graphs constructed by F. R. K. Chung. Her situation is as follows. We are given a finite field $F$, an integer $n \geq 1$, an extension field $E$ of $F$ of degree $n$, and an element $x$ in $E$ that generates $E$ over $F$, i.e., an element $x$ such that $E$ is $F(x)$.

**Theorem 1.** Let $\chi$ be any nontrivial complex-valued multiplicative character of $E^\times$ (extended by zero to all of $E$), and $x$ in $E$ any element that generates $E$ over $F$. Then

$$\left| \sum_{t \in F} \chi(t - x) \right| \leq (n - 1)\sqrt{\#(F)}.$$

It turns out to be easier to consider the following more general situation. $F$ is a finite field, $n \geq 1$ is an integer, and $B$ is a finite etale $F$-algebra of dimension $n$ over $F$ (i.e., over a finite extension $K$ of $F$, there exists an isomorphism of $K$-algebras $B \otimes_F K \cong K \times K \times \cdots \times K$). We assume given an element $x$ in $B$ that is regular in the sense that its characteristic polynomial $\det_F(T - x | B)$ in the regular representation of $B$ on itself has $n$ distinct eigenvalues. (In terms of the above isomorphism $B \otimes_F K \cong K \times K \times \cdots \times K$, $x$ is regular if and only if $x \otimes 1 \cong (x_1, \ldots, x_n)$ with all distinct components $x_i$. Or equivalently, $x$ is regular if and only if $B$ is equal to the $F$-subalgebra $F[x]$ generated by $x$. In the special case when $B$ is a field $F$, the element $x$ is regular if and only if $F(x) = E$.)

**Theorem 2.** Let $\chi$ be any nontrivial complex-valued multiplicative character of $B^\times$ (extended by zero to all of $B$), and $x$ in $B$ any regular element. Then

$$\left| \sum_{t \in F} \chi(t - x) \right| \leq (n - 1)\sqrt{\#(F)}.$$

**Proof.** The basic idea is that the theorem is an immediate consequence of Weil’s estimates for one-variable character sums in the case when the $F$-algebra $B$ is completely split, and that one can reduce to this case by thinking geometrically about suitable Lang torsors.

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We begin by explaining how to view the problem geometrically. Given any finite-dimensional commutative $F$-algebra $A$, we denote by $A$ the smooth affine scheme over $F$ given by “$A$ as algebraic group over $F$”; concretely, for any $F$-algebra $R$, the group $A(R)$ of $R$-valued points of $A$ is $A \otimes_F R$. We denote by $A^\times$ the open subscheme of $A$ given by “$A^\times$ as algebraic group over $F$”; concretely, for any $F$-algebra $R$, the group $A^\times(R)$ of $R$-valued points of $A$ is $(A \otimes_F R)^\times$. These concepts will be applied to the cases $A = B$ and $A = F$.

It will be important in what follows to think of $A^\times$ as a smooth commutative group scheme over $F$, but to think of $A$ only as an ambient scheme (not as a group scheme) containing $A^\times$ as an open subscheme.

Because $B^\times$ is a smooth, geometrically connected commutative group scheme over the finite field $F$, the Lang isogeny $1 - \text{Frob}_F : B^\times \to B^\times$ makes $B^\times$ into a $B^\times$-torsor over itself, the “Lang torsor” $L$. Let us now fix a prime number $l \neq \text{char}(F)$, an algebraic closure $\overline{Q}_l$ of $Q_l$, and an isomorphism of fields $C \simeq \overline{Q}_l$. This isomorphism allows us to view $\chi$ as a $\overline{Q}_l$-valued character of $B^\times$, by which it makes sense to push out the Lang torsor $L$ to obtain a lisse rank one $\overline{Q}_l$-sheaf $L_\chi$ on $B^\times$ which is pure of weight zero. If we denote by $j : B^\times \to B$ the inclusion, we may form the extension by zero $j_!L_\chi$ on $B$. Now consider the morphism of $F$-schemes of $f : F \to B$ defined by $f(t) := t - x$, and the pullback sheaf $\mathcal{F} := f^*(j_!L_\chi)$ on $F$. The sheaf $\mathcal{F}$ is lisse of rank one and pure of weight zero on the open set $f^{-1}(B^\times)$, and zero outside. The sheaf $\mathcal{F}$ is everywhere tamely ramified, simply because on $f^{-1}(B^\times)$ it is lisse of order dividing that of $\chi$, hence of order prime to the characteristic of $F$.

In terms of this data, the character sum in question is given by

$$\sum_{t \in F} \chi(t - x) = \sum_{t \in f^{-1}(B^\times)(F)} \text{Trace}(\text{Frob}_F | \mathcal{F}),$$

and by the Lefschetz Trace Formula this last sum is equal to

$$\sum_i (-1)^i \text{Trace}(\text{Frob}_F | H^i_{\text{comp}}(f^{-1}(B^\times) \otimes_F \overline{F}, \mathcal{F})).$$

By Weil (but expressed in the language of Deligne’s paper [De]) we know that the above cohomology groups $H^i_{\text{comp}}$ are mixed of weight $\leq i$. For dimension reasons, $H^i_{\text{comp}}$ vanishes for $i > 2$, and $H^0_{\text{comp}}$ vanishes because $\mathcal{F}$ is lisse on the incomplete curve $f^{-1}(B^\times) \otimes_F \overline{F}$. It thus remains only to establish the following two facts:

(a) $H^2_{\text{comp}}(f^{-1}(B^\times) \otimes_F \overline{F}, \mathcal{F}) = 0$,
(b) $\dim H^1_{\text{comp}}(f^{-1}(B^\times) \otimes_F \overline{F}, \mathcal{F}) = n - 1$.

Both of these facts are geometric, i.e., they concern the situation over the algebraic closure of $F$, and hence it suffices to verify them universally in the case when the $F$-algebra $B$ is completely split. (The key point here is that our hypothesis that $\chi$ is nontrivial is stable under finite extension of scalars.)
Indeed, after extension of scalars from $F$ to any finite extension field $K$, the pullback to $(B^\times) \otimes_F K$ of $L_\chi$ is $L_{\tilde{\chi}}$, where $\tilde{\chi}$ is the character of $(B \otimes_F K)^\times$ obtained from $\chi$ by composition with the norm homomorphism $\text{Norm}_{K/F}$ from $(B \otimes_F K)^\times$ to $B^\times$. Because this norm map is surjective, the character $\tilde{\chi}$ is nontrivial provided that $\chi$ is nontrivial.)

Suppose now that $B$ is simply the $n$-fold self product of $F$ with itself. Then a nontrivial character $\chi$ of $B^\times$ is simply an $n$-tuple $(\chi_1, \ldots, \chi_n)$ of characters of $F^\times$, not all of which are trivial, the regular element $x$ is just an $n$-tuple $(x_1, \ldots, x_n)$ with all distinct components $x_j$, the open set $f^{-1}(B^\times)$ is just the complement $F - \{x_1, \ldots, x_n\}$ of the $n$ distinct points $x_i$ in $F$, the sheaf $\mathcal{F}$ is just the tensor product of the sheaves $[t \mapsto t - x_i]^* L_{x_i} | F - \{x_1, \ldots, x_n\}$, and the sum in question is

$$\sum_{t \in F - \{x_1, \ldots, x_n\}} \chi_1(t - x_1) \chi_2(t - x_2) \cdots \chi_n(t - x_n).$$

By assumption, at least one of the $\chi_i$ is nontrivial. For such an index $i$, the sheaf $[t \mapsto t - x_i]^* L_{x_i}$ is tamely but nontrivially ramified at $x_i$, while all the other factors $[t \mapsto t - x_j]^* L_{x_j}$ with $j \neq i$ are lisse at $x_i$ (by the hypothesis that all the $x_j$ are distinct). Therefore, the sheaf $\mathcal{F}$ is nontrivially ramified at the point $x_i$. Because $\mathcal{F}$ is lisse of rank one on $F - \{x_1, \ldots, x_n\}$, its coinvariants under the inertia group $I_{x_i}$ must vanish, and a fortiori its covariants under the entire $\pi_1\text{geom}$ of $F - \{x_1, \ldots, x_n\}$ must also vanish, i.e., its $H^2_{\text{comp}}$ vanishes. Once we have the vanishing of all the $H^i_{\text{comp}}$ save for $i = 1$, the asserted dimension formula $\dim H^1_{\text{comp}} = n - 1$ is then equivalent to the Euler characteristic formula

$$\sum_i (-1)^i \dim H^i_{\text{comp}}((F - \{x_1, \ldots, x_n\}) \otimes_F \mathcal{F}) = 1 - n,$$

which holds because $\mathcal{F}$ is lisse of rank one and everywhere tame on the open curve $(F - \{x_1, \ldots, x_n\}) \otimes_F \mathcal{F}$, whose Euler characteristic is $1 - n$. Q.E.D.

Remarks and Questions. (1) If we drop the hypothesis that the element $x$ be regular, then Theorem 2 remains valid for characters $\chi$ of $B^\times$ whose restriction to $F^\times$ is nontrivial. The proof proceeds along the same lines as above, reducing to the completely split case in which $\chi$ is simply an $n$-tuple $(\chi_1, \ldots, \chi_n)$ of characters of $F^\times$, with the property that their product $\prod \chi_i$ is nontrivial on $F^\times$. Now one gets the vanishing of $H^2_{\text{comp}}$ by observing that the sheaf $\mathcal{F}$ is nontrivially ramified at $\infty$ (as an $I_{\infty}$-representation, $\mathcal{F}$ is isomorphic to $L_{\prod \chi_i}$, and the constant "$n - 1"$ actually improves to "(the number of distinct $x_i) - 1"). Indeed, in the case of the choice $x := 0$, the character sum in question is exactly $\sum_{t \in F^\times} \chi(t)$. (Alternately, one could apply Theorem 2 directly to the (automatically finite etale) subalgebra $B_0 := F[x]$ of $B$ generated by $x$ over $F$, to the regular element $x$ of $B_0$, and to the nontrivial (because nontrivial on $F^\times$) character $\chi | (B_0)^\times$.)
What happens if we also drop the hypothesis that $B$ be etale? Suppose that we are given an arbitrary $n$-dimensional commutative $F$-algebra $A$, a multiplicative character $\chi$ of $A^\times$ (extended by zero to all of $A$) whose restriction to $F^\times$ is nontrivial, and an element $x$ in $A$. It seems plausible that the estimate

$$\left\| \sum_{t \in F} \chi(t - x) \right\| \leq (n - 1) \sqrt{\#(F)}$$

should still hold. For example, in the case when $A$ is the algebra of dual numbers $F[x]/(x^2)$, the character sums in question are none other than the usual Gauss sums attached to the field $F$.

**References**

