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in: Journal für die reine und angewandte Mathematik | Journal für die reine und angewandte Mathematik | Article
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On a certain class of exponential sums

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Abstract. In a recent note ([1]), Birch and Bombieri consider the following family of exponential sums over finite fields $\mathbb{F}_q$; for $\alpha_1, \alpha_2 \in \mathbb{F}_q^\times$, and ψ a non-trivial additive character of $\mathbb{F}_q$,

$$S(q, \alpha_1, \alpha_2, \psi) \overset{\text{dfn}}{=} \sum_{\substack{x, y, z, t \in \mathbb{F}_q^\times \\ \frac{\alpha_1}{xy} + \frac{\alpha_2}{zt} = 1}} \psi(x+y+z+t).$$

They prove the existence of constants c_0 and c_1 such that for any finite field $\mathbb{F}_q$ of characteristic $p \geqq c_0$, one has

$$|S(q, \alpha_1, \alpha_2, \psi)| \leqq c_1 q^{\frac{3}{2}}.$$

In this note, we will explain how a quite general class of exponential sums "with the same shape" can be similarly estimated, with no exceptional characteristics (i.e., $c_0 = 1$) and with a completely explicit c_1 (e.g., $c_1 = 8$ for the above sums).

For us, the key structural feature of the above sums is that the equation defining the variety of summation,

$$\frac{\alpha_1}{xy} + \frac{\alpha_2}{zt} = 1, \quad xyzt \neq 0$$

is of the form $f = \beta$, where $\beta \neq 0$ and where f is a sum of inverse monomials in disjoint sets of variables, i.e., each variable occurs in precisely one monomial.

Thus we consider an integer $n \geqq 1$, a second integer $r \geqq 1$, and a partition

$$n = n_1 + n_2 + \cdots + n_r$$

of n as the sum of r integers n_i, with each $n_i \geqq 1$. For each $i = 1, \ldots, r$, we introduce n_i indeterminates,

$$X_{i,j}, \quad j = 1, \ldots, n_i.$$

We fix a collection of strictly positive integers

$$b_{i,j} \geqq 1 \quad \text{for} \quad 1 \leqq i \leqq r,\ 1 \leqq j \leqq n_i,$$

a finite field $\mathbb{F}_q$, a non-trivial additive character ψ of $\mathbb{F}_q$, a collection of (possibly trivial) multiplicative characters of $\mathbb{F}_q^\times$

$$\chi_{ij}, \quad \text{for} \quad 1 \leqq i \leqq r,\ 1 \leqq j \leqq n_i,$$

and r elements

$$\alpha_1, \ldots, \alpha_r \in \mathbb{F}_q^\times .$$

For each $\beta \in \mathbb{F}_q$, we denote by V_β the subvariety of $(\mathbb{G}_m \otimes \mathbb{F}_q)^n$ (with coordinates the X_{ij}) defined by the equation

$$\beta = \sum_i \frac{\alpha_i}{\prod_j (X_{ij})^{b_{ij}}}$$

and by S_β the exponential sum

$$S_\beta = \sum_{x \in V_\beta(\mathbb{F}_q)} \psi(\sum_{i,j} x_{ij}) \prod_{i,j} \chi_{ij}(x_{ij}).$$

Theorem. *Hypotheses and notations as above, for $\beta \neq 0$ we have*

$$|S_\beta| \leqq C_1 (\sqrt{q})^{n-1},$$

with

$$C_1 = \prod_{i=1}^{r} \left(1 + \sum_{j=1}^{n_i} b_{ij}\right) - 1 .$$

The proof is based upon the simple observation that the additive Fourier transform of S_β with respect to β is an r-fold product of multiple Kloosterman sums. Explicitly, one finds by elementary calculation the identity, for variable $t \in \mathbb{F}_q$,

$$\sum_{\beta \in \mathbb{F}_q} \psi(t\beta) S_\beta = \prod_{i=1}^{r} Kl(i, \alpha_i t),$$

where

$$Kl(i, a) \overset{\text{dfn}}{=} \sum_{x_{i,1}, \ldots, x_{i,n_i} \in \mathbb{F}_q^\times} \prod_j \chi_{ij}(x_{ij})\, \psi\left(\sum_j x_{ij} + \frac{a}{\prod_j x_{ij}^{b_{ij}}}\right),$$

for arbitrary $a \in \mathbb{F}_q$.

Let us now pick a prime number $l \neq \mathrm{char}(\mathbb{F}_p)$, and an l-adic place λ of the field $E=\mathbb{Q}$ (ζ_p, values of χ_{ij}). Denote by $Kl(i)$ the lisse E_λ-sheaf on $\mathbb{G}_m \otimes \mathbb{F}_q$ which is denoted

$$Kl(\psi;\, \mathbf{1},\, \chi_{i,1},\, \ldots,\, \chi_{i,n_i};\, 1,\, b_{i,1},\, \ldots,\, b_{i,n_i})$$

in ([4] 4. 1. 1). Let $j\colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ be the natural inclusion, and consider the sheaf $j_* Kl(i)$ on $\mathbb{A}^1 \otimes \mathbb{F}_q$. According to ([4] 4. 1. 1) and ([4] 7. 3. 2 (3)), its trace function at rational points $t \in \mathbb{A}^1(\mathbb{F}_q)$ is given by

$$\mathrm{tr}(F_t \,|\, (j_* Kl(i))_{\bar{t}}) = (-1)^{n_i}\, Kl(i, t).$$

In terms of the automorphisms $T_{\alpha_i}\colon x \mapsto \alpha_i x$ of $\mathbb{A}^1 \otimes \mathbb{F}_q$, we may rewrite the above identity as

$$\mathrm{tr}(F_t \,|\, (T_{\alpha_i}^* j_* Kl(i))_{\bar{t}}) = (-1)^{n_i}\, Kl(i, \alpha_i t).$$

In terms of the E_λ-sheaf $\mathfrak{J}$ on $\mathbb{A}^1 \otimes \mathbb{F}_q$ defined by

$$\mathfrak{J} = \bigotimes_{i=1}^{r} T_{\alpha_i}^* (j_* Kl(i)),$$

we thus have, for every $t \in \mathbb{A}^1(\mathbb{F}_q)$,

$$\begin{aligned}\mathrm{tr}(F_t \,|\, \mathfrak{J}_{\bar{t}}) &= (-1)^n \prod_{i=1}^{r} Kl(i, \alpha_i t)\\ &= (-1)^n \sum_{\beta} \psi(t\beta)\, S_\beta.\end{aligned}$$

By Fourier inversion, we obtain

$$q \cdot S_\beta = \sum_t \psi(-t\beta)\, \mathrm{tr}(F_t \,|\, \mathfrak{J}_{\bar{t}}).$$

In terms of the additive character $x \mapsto \psi(-\beta x)$ of $\mathbb{F}_q$ and the corresponding Artin-Schreier sheaf $\mathfrak{L}_{\psi(-\beta x)}$ on $\mathbb{A}^1 \otimes \mathbb{F}_q$, the Lefschetz trace formula applied to $\mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}$ yields

$$(-1)^n q S_\beta = \sum_{i=0}^{2} (-1)^i\, \mathrm{tr}(F \,|\, H^i_{\mathrm{comp}}(\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q,\, \mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)})).$$

It remains only to study the individual cohomology groups in question.

Lemma 1. *The sheaf $j^* \mathfrak{J}$ on $\mathbb{G}_m \otimes \mathbb{F}_q$ is lisse of rank*

$$\prod_{i=1}^{r} \left(1 + \sum_{j=1}^{n_i} (\textit{the prime-to-}p \textit{ part of } b_{ij})\right),$$

pure of weight $n = \sum n_i$, and tame at zero.

Proof. Indeed, each $Kl(i)$ on $\mathbb{G}_m \otimes \mathbb{F}_q$ is lisse of rank

$$1 + \sum_j (\text{prime-to-}p \text{ part of } b_{ij}),$$

pure of weight n_i, and tame at zero (cf. [4] 4. 1. 1). Q.E.D.

Lemma 2. *The stalk at zero of $\mathfrak{J}$ is one-dimensional.*

Proof. By ([4] 7. 3. 2), each factor $T_{\alpha_i}^*$ $(j_* Kl(i))$ has one-dimensional stalk at zero. Q.E.D

Lemma 3. *For every β, $H_c^0(\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q, \mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}) = 0$.*

Proof. Because $\mathfrak{L}_{\psi(-\beta x)}$ is lisse and non-zero on $\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q$, the lemma for any single β is equivalent to the injectivity of the canonical map $\mathfrak{J} \to j_* j^* \mathfrak{J}$. This injectivity is obvious from rewriting the individual factors $T_{\alpha_i}^*\ j_* Kl(i)$ in the definition of $\mathfrak{J}$ as $j_*((T_{\alpha_i} | \mathbb{G}_m)^*\ (Kl(i))) = j_*$ (a lisse sheaf on $\mathbb{G}_m$). Q.E.D.

Lemma 4. *Every ∞-break of $\mathfrak{J}$ (as representation of I_∞) is <1.*

Proof. Indeed, each $Kl(i)$ has rank $\varrho_i \geqq 1 + n_i \geqq 2$, and all its ∞-breaks are $\frac{1}{\varrho_i} \leqq \frac{1}{2} < 1$ (cf. [4] 4. 1. 1). Q.E.D.

Lemma 5. *For $\beta \neq 0$, we have*

(a) *every ∞-break of $\mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}$ is* 1.

(b) $H_c^2(\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q, \mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}) = 0$.

(c) $\mathrm{Swan}_\infty(\mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}) = \mathrm{rank}\,(j^* \mathfrak{J})$.

Proof. For $\beta \neq 0$, the ∞-break of $\mathfrak{L}_{\psi(-\beta x)}$ is 1, so (a) follows from lemma 4. We have (a) $\Rightarrow$ (c) trivially, and (a) $\Rightarrow$ (b) because $\mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}$ is totally wild at ∞, so has vanishing H_c^2. Q.E.D.

Lemma 6. *For $\beta \neq 0$, $H_c^1(\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q, \mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)})$ has dimension*

$$h_c^1 = \mathrm{rank}\,(j^* \mathfrak{J}) - 1,$$

and it is mixed of weight $\leqq n+1$.

Proof. For $\beta \neq 0$, H_c^1 is the only non-vanishing cohomology group, so

$$h_c^1 = -\chi_c(\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q, \mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}).$$

Because $\mathfrak{J}$ has one-dimensional stalk at zero, the exact sequence

$$0 \to j_! j^* \mathfrak{J} \to \mathfrak{J} \to \begin{pmatrix} \text{one-dim'l,} \\ \text{conc. at zero} \end{pmatrix} \to 0$$

gives

$$\begin{aligned} \chi_c(\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q, \mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}) &= 1 + \chi_c(\mathbb{G}_m \otimes \bar{\mathbb{F}}_q, j^* \mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}) \\ &= 1 - \mathrm{Swan}_\infty(\mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)}) \\ &= 1 - \mathrm{rank}\,(j^* \mathfrak{J}). \end{aligned}$$

As for the weights, the above exact sequence exhibits the H_c^1 in question as a quotient of

$$H_c^1(\mathbb{G}_m \otimes \bar{\mathbb{F}}_q, j^*\mathfrak{I} \otimes \mathfrak{L}_{\psi(-\beta x)});$$

because $j^*\mathfrak{I} \otimes \mathfrak{L}_{\psi(-\beta x)}$ is lisse on $\mathbb{G}_m \otimes \mathbb{F}_q$ and pure of weight n, this last group is mixed of weight $\leqq n+1$ by Deligne's fundamental estimate ([3] 3. 3. 1). Q.E.D.

For $\beta \neq 0$, lemmas 3 and 5 give

$$(-1)^{n-1} q \cdot S_\beta = \text{trace of } F \text{ on } H_c^1(\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q, \mathfrak{I} \otimes \mathfrak{L}_{\psi(-\beta x)});$$

combining this with lemma 6 yields the estimate

$$|S_\beta| \leqq (\operatorname{rank}(j^*\mathfrak{I}) - 1)\sqrt{q}^{\,n-1},$$

and by lemma 1 we have

$$\operatorname{rank}(j^*\mathfrak{I}) = \prod_{i=1}^{r}\left(1 + \sum_{j=1}^{n_i} (\text{prime-to-}p \text{ part of } b_{ij})\right).$$

This concludes the proof of (a slight sharpening of) the theorem. Q.E.D.

Remarks. (1) For $\beta = 0$, we obtain the estimate

$$\begin{aligned}|S_0| &\leqq \operatorname{rank}(j^*\mathfrak{I}) \cdot \sqrt{q}^{\,n+1} + (\operatorname{rank}(j^*\mathfrak{I}) - 1) \cdot \sqrt{q}^{\,n-1} \\ &\leqq (1 + c_1)(\sqrt{q}^{\,n+1} + \sqrt{q}^{\,n}).\end{aligned}$$

To show this, it suffices to show that

$$\dim H_c^i(\mathbb{A}^1 \otimes \bar{\mathbb{F}}_q, \mathfrak{I}) \leqq \begin{cases} \operatorname{rank}(j^*\mathfrak{I}) & \text{for } i = 2, \\ \operatorname{rank}(j^*\mathfrak{I}) - 1 & \text{for } i = 1. \end{cases}$$

For this, we argue as follows. The Euler-Poincaré formula gives

$$h_c^2 - h_c^1 = 1 - \operatorname{Swan}_\infty(\mathfrak{I}),$$

while in terms of the break-decomposition of $\mathfrak{I}$ as I_∞-representation,

$$\mathfrak{I} = \bigoplus_{x \geqq 0} \mathfrak{I}(x)$$

we have the trivial inequality

$$h_c^2 \leqq \dim \mathfrak{I}(0) = \operatorname{rank}(j^*\mathfrak{I}) - \sum_{x > 0} \dim(\mathfrak{I}(x)).$$

Because all ∞-breaks of $\mathfrak{I}$ are < 1, we have

$$\operatorname{Swan}_\infty(\mathfrak{I}) = \sum_{x > 0} x \dim(\mathfrak{I}(x)) \leqq \sum_{x > 0} (\dim \mathfrak{I}(x)),$$

whence

$$h_c^2 + \operatorname{Swan}_\infty(\mathfrak{I}) \leqq \operatorname{rank}(j^*\mathfrak{I}).$$

In view of the Euler-Poincaré formula, this yields

$$1 + h_c^1 \leqq \operatorname{rank}(j^*\mathfrak{I}).$$

Q.E.D.

(2) If we keep $\beta \neq 0$, we can ask how the sum S_β depends upon β and also upon the r non-zero quantities $\alpha_1, \ldots, \alpha_r$. Let us denote by T the $r+1$-dimensional torus $(\mathbb{G}_m \otimes \mathbb{F}_q)^{r+1}$ over $\mathbb{F}_q$, with coordinates $\alpha_1, \ldots, \alpha_r, \beta$. Then we may form the sheaf

$$\mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)} = \left(\bigoplus_{i=1}^{r} T_{\alpha_i}^{*}(j^{*} Kl(i)) \right) \otimes \mathfrak{L}_{\psi(-\beta x)}$$

on $\mathbb{A}_T^1$. By Deligne's semicontinuity theorem ([5] 2. 1. 2) applied to $\mathfrak{J}$ and the projection pr: $\mathbb{A}_T^1 \to T$, the "Fourier transform" sheaves $R^i \operatorname{pr}_!(\mathfrak{J} \otimes \mathfrak{L}_{\psi(-\beta x)})$ are all lisse on T. They vanish for $i \neq 1$, and the remaining sheaf $R^1 \operatorname{pr}_!$ provides a lisse sheaf on T of rank $= (\operatorname{rank}(j^{*}\mathfrak{J}) - 1)$, mixed of weight $\leqq n+1$, whose trace at any point $(\alpha_1, \ldots, \alpha_r, \beta) \in T(k)$, k a finite extension of $\mathbb{F}_q$, is equal to

$$(-1)^{n-1} (\# k) \sum_{x \in V_\beta(k)} (\psi \circ \operatorname{tr}) (\sum_{i,j} x_{ij}) \prod_{i,j} (\chi_{ij} \circ \mathbb{N}) (x_{i,j})$$

where trace and norm are with respect to the extension $k/\mathbb{F}_q$, and where V_β now denotes the subvariety of $(\mathbb{G}_m \otimes k)^n$ defined by the equation

$$\beta = \sum_i \frac{\alpha_i}{\prod_j X_{ij}^{b_{ij}}}.$$

Except in some very special and atypical cases (e.g., $r=1$), this sheaf will *not* be pure of weight $n+1$.

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Eingegangen 15. Januar 1986