Proceedings of Symposia in Pure Mathematics
Volume 28, 1976

An Overview of Deligne's Work on Hilbert's Twenty-First Problem

Nicholas M. Katz

Abstract Hilbert's twenty-first problem on the existence of differential equations with regular singular points and prescribed monodromy is interpreted as a GAGA-type problem of algebraic-analytic comparison. Deligne's solution is outlined. Several open questions are raised.

The setting of the problem

Let X be a complete connected non-singular curve over $\mathbb{C}$, whose underlying complex manifold is thus a compact Riemann surface. Let U be a non-empty Zariski open set in X, the complement in X of a finite (possibly empty) set of closed points. The underlying complex manifold U^{an} is thus a finitely punctured Riemann surface.

Consider a linear homogeneous differential equation of rank n on U. When $X = \mathbb{P}^1$ and $U \subset \mathbb{P}^1 - \{\infty\}$, this simply means an $n \times n$ system

$$\frac{d}{dz}\vec{f} + A(z)\cdot\vec{f} = 0$$

$$\begin{cases} \vec{f} = \begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} \\ A(z) \text{ an } n \times n \text{ matrix of \underline{rational} functions of } z, \\ \qquad \text{holomorphic on } U. \end{cases}$$

As we are so fond of emphasizing to engineering students, this includes the case of the n'th order linear homogeneous equation

$$\left(\frac{d}{dz}\right)^n f + a_1(z)\cdot\left(\frac{d}{dz}\right)^{n-1} f + \ldots + a_n(z)f = 0,$$

by taking for the matrix $A(z)$ the particular choice

$$A(z) = \begin{pmatrix} 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ldots & 0 \\ \vdots & & & & \\ 0 & \ldots & \ldots & 0 & -1 \\ a_n(z) & \ldots & \ldots & \ldots & a_1(z) \end{pmatrix}$$

In case X has higher genus, we have no global coordinate z at our disposal, so we are led to define a differential equation on U to be a pair (M,∇) consisting of a locally free coherent algebraic sheaf M on U together with a connection $\nabla\colon M \longrightarrow M\otimes \Omega^1_{U/\mathbb{C}}$ (this simply means an additive mapping which satisfies the "product rule": for $f\in \mathcal{O}_U$ and $m\in M$, $\nabla(fm)=m\otimes df+f\otimes\nabla(m)$). In the example of the $n\times n$ system on an open set in $\mathbb{P}^1$, the locally free coherent algebraic sheaf M is simply $(\mathcal{O}_U)^n$, and the connection ∇ is the map $(\mathcal{O}_U)^n \longrightarrow (\Omega^1_{U/\mathbb{C}})^n$ given by

$$\nabla(\vec{f}\,) = d\vec{f} + A(z)\cdot\vec{f}\,dz$$

It is sometimes convenient to view a connection ∇ as a way of having derivations of $\mathcal{O}_U$ act on M, viewing ∇ as an $\mathcal{O}_U$-linear map $\underline{\mathrm{Der}}(U/\mathbb{C}) \longrightarrow \underline{\mathrm{End}}_{\mathbb{C}}(M)$, $D \longrightarrow \nabla(D)$ where $\nabla(D)\colon M \longrightarrow M$ is the composite

$$\begin{array}{ccc} M & \xrightarrow{\ \nabla\ } & M\otimes\Omega^1_U \\ & \searrow^{\nabla(D)} & \big\downarrow \ \mathrm{id}\otimes\text{contract with } D \\ & & M \end{array}$$

Thus, in the above example, $\nabla(\frac{d}{dz})$ is given by

$$\nabla(\tfrac{d}{dz})\vec{f} = \frac{d\vec{f}}{dz} + A(z)\vec{f}$$

Notice that the solutions of the differential equation become the sections of $M^{\nabla} \overset{\mathrm{dfn}}{=\!=}$ the kernel of $\nabla\colon M \longrightarrow M\otimes\Omega^1_U$. If we fix a point $z_0\in U$, then the space S of germs of local holomorphic solutions near (in the analytic sense) z_0 is an $n = \mathrm{rank}\,(M)$-dimensional $\mathbb{C}$-space, by the local existence theorem for differential equations. Given any loop γ in U^{an} starting and ending at z_0, "analytic continuation of solutions along γ" defines an automorphism of S. It is immediate that this automorphism is "multiplicative" in γ, and that it depends only on the homotopy class of γ in $\pi_1(U^{an}, z_0)$. In this way the fundamental group $\pi_1(U^{an}, z_0)$ acts on the $\mathbb{C}$-space S, defining the monodromy representation of the differential equation.

For example, if we take the differential equation

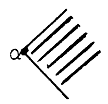
$$z\frac{df}{dz} = \alpha f \qquad \alpha \in \mathbb{C}$$

on the punctured complex "plane" $\mathbb{C}-\{0\}$, its solution is the function $z^\alpha = \exp(\alpha \log z)$, which under analytic continuation along a path γ which loops once counterclockwise around the origin (e.g. $z \longrightarrow e^{2\pi i\theta} z$, $0 \leq \theta \leq 1$) is transformed into $e^{2\pi i\alpha} \cdot z^\alpha$. The fundamental group of $\mathbb{C}-\{0\}$ is isomorphic to $\mathbb{Z}$, with generator γ, and the monodromy representation in $\mathbb{C}^\times = GL(1, \mathbb{C})$ is given by $\gamma \longrightarrow e^{2\pi i\alpha}$

Let us now recall the notion of a "regular singular point" of a differential equation. Classically, one considered an n'th order differential equation

$$\left(\frac{d}{dz}\right)^n f + a_1(z)\left(\frac{d}{dz}\right)^{n-1} f + \ldots + a_n(z) f = 0$$

on an open set $U \subset \mathbb{P}^1$. A point $\alpha \in \mathbb{P}^1 - U$ is called a regular singular point of this equation if in any punctured angular sector around α



the local holomorphic solutions satisfy a growth estimate

$$\begin{array}{lll} \text{if } \alpha \text{ finite} & |f(z)| = O(|z-\alpha|^{-N}) & \text{for some } N \geq 0, \text{ as } z \longrightarrow \alpha \\ \text{if } \alpha = \infty & |f(z)| = O(|z|^N) & \text{for some } N \geq 0, \text{ as } z \longrightarrow \infty. \end{array}$$

This notion is apparently analytic, rather than algebraic, in character. However, Fuchs discovered that in fact the notion is purely algebraic. He proved that a necessary and sufficient condition that α be a regular singular point is that the functions

$$\begin{cases} (z-\alpha)^i \cdot a_i(z), & i=1,\dots,n \quad \text{if } \alpha \text{ finite} \\ z^i a_i(z) & i=1,\dots,n \quad \text{if } \alpha = \infty \end{cases}$$

should all be <u>holomorphic</u> at $z = \alpha$.

We may rephrase Fuch's criterion as follows. Let t be a uniformizing parameter at α (e.g. $t = z - \alpha$ if α finite, $t = 1/z$ if $\alpha = \infty$), and denote by D the differential operator $t\frac{d}{dt}$ Rewrite the equation as a monic equation in D:

$$D^n f + b_1(t)\, D^{n-1} f + \dots + b_n(t) \cdot f = 0.$$

Then $t=0$ is a regular singular point if and only if all of the functions $b_i(t)$ are holomorphic at $t=0$.

The notion of a regular singular point of a "fancy" differential equation (M, ∇) on U is defined as before in terms of the growth of local holomorphic solution vectors $\vec{f}$ near the singular point, where we measure the growth by expressing $\vec{f}$ in a local basis of any locally free coherent algebraic sheaf $\overline{M}$ on X which <u>extends</u> M (on a <u>curve</u>, such an $\overline{M}$ always exists, for <u>any</u> locally free M). In this more general setting, Fuch's algebraic criterion for regular singular points, as rephrased by Deligne, is that there <u>exist</u> a locally free coherent algebraic sheaf $\overline{M}$ on X extending M, such that near any singular point α of the equation, the derivation $D = t\frac{d}{dt}$ (t: uniformizing parameter at α) should act (through ∇) <u>stably</u> on $\overline{M}$, i.e. $\nabla(D)(\overline{M}) \subset \overline{M}$.

Notice the compatibility with Fuch's criterion for n'th order equations: If we convert the equation

$$D^n f + b_1(t)\, D^{n-1} f + \dots + b_n(t) f = 0$$

into system form, it becomes

$$\nabla(D)\, \vec{f} = D\vec{f} + \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ \vdots & & & & \\ 0 & & & 0 & -1 \\ b_n(t) & & & & b_1(t) \end{pmatrix} \vec{f},$$

so that if the $b_i(t)$ are all holomorphic at $t=0$, then indeed $\nabla(D)$ acts stably on n-tuples of functions holomorphic at $t=0$.

Statement of the Problem

Hilbert's twenty-first problem asks whether any finite-dimensional complex representation of the fundamental group $\pi_1(U^{an})$ can be obtained as the monodromy representation of a differential equation on U with regular singular points.

One could ask the same question without requiring regular singular points, but the simplest examples show that then there will be too many equations with given monodromy. For example, take $U = \mathbb{A}^1$ (thus $U^{an} = \mathbb{C}$), and the trivial one-dimensional representation of $\pi_1(U^{an}) = 0$. For any polynomial $P(z) \in \mathbb{C}[z]$, the differential equation

$$\frac{df}{dz} = P(z) \cdot f$$

has solution

$$f = \exp\left(\int_0^z P(t)\, dt\right)$$

which is an entire (singlevalued) function of z, so without monodromy. But as differential equations on the algebraic variety $\mathbb{A}^1$, these are pairwise non-isomorphic; only the choice $P \equiv 0$ gives regular singular points. For this reason, one insists on regular singular points.

In the case $U = \mathbb{P}^1 - \{0, 1, \infty\}$, the triply punctured Riemann sphere, the affirmative solution of Hilbert's problem for two-dimensional representations of $\pi_1(U^{an})$ goes back to Riemann, and amounts to the theory of the hypergeometric equation. Indeed, Hilbert's twenty-first problem is often referred to as the Riemann problem. Traditionally, it was viewed as a problem in "function theory," and in that setting has been solved repeatedly, by such men as Birkhoff, Plemelj, and most recently by Röhrl. But as Lipman

Bers remarked in his talk on uniformization, if a problem is worth solving, it's worth solving several times.

Algebro-Geometric Perspective

From the perspective of algebraic geometry, the point of the problem is to describe in purely algebraic terms on U (differential equations on U with regular singular points) a purely topological invariant of U^{an} (the finite dimensional complex representations of its fundamental group).

The earliest result of this kind was the realization that complete connected nonsingular curves over $\mathbb{C}$ are identical with compact Riemann surfaces, the identification provided by the fact that a rational function on the curve is the same as a meromorphic function on the Riemann surface.

This fact was generalized by Chow and Kodaira, who proved that any compact complex connected surface with two algebraically independent (over $\mathbb{C}$) meromorphic functions was a nonsingular projective algebraic surface. The obvious generalization of this result to more than two dimensions is false, and its "counterexamples" have been studied these last few years by Artin and Moishezon as "algebraic spaces".

Around the same time, Chow proved that any closed analytic subset of $\mathbb{P}^n$ (i.e. defined locally by the vanishing of analytic functions) was in fact algebraic, and (by considering the graph) that any analytic map between such was in fact algebraic.

The final step in this direction was Serre's GAGA theorem according to which if X is any projective algebraic variety over $\mathbb{C}$, then the coherent algebraic sheaves $\mathcal{F}$ on X are identical with the coherent analytic sheaves $\mathcal{F}^{an}$ on X^{an} (i.e. $\mathcal{F} \to \mathcal{F}^{an}$ is an equivalence of categories) and the cohomology groups of such sheaves are the "same" whether computed algebraically or analytically $(H^i(X,\mathcal{F}) \xrightarrow{\sim} H^i(X^{an},\mathcal{F}^{an}))$.

For example, if we take X to be non-singular, and $\mathcal{F} = \Omega^p_{X/\mathbb{C}}$ the sheaf of germs of holomorphic p-forms, then we get a purely algebraic description of the Hodge cohomology groups

$$\begin{array}{ccc} H^q(X, \Omega^p_{X/\mathbb{C}}) & \xrightarrow{\sim} & H^q(X^{an}, \Omega^p_{X^{an}/\mathbb{C}}) \\ & & \wr\| \text{ (Dolbeaut)} \\ & & \text{harmonic forms of type } (p,q) \end{array}$$

If, instead of a single Ω^p, we take the entire de Rham complex $\Omega^{\bullet}_{X/\mathbb{C}}$, then we get a purely algebraic description of the complex cohomology groups in terms of algebraic "hypercohomology."

$$\begin{array}{ccc} \mathbb{H}^i(X, \Omega^{\cdot}_{X/\mathbb{C}}) & \xrightarrow{\sim} & \mathbb{H}^i(X^{an}, \Omega^{\cdot}_{X^{an}/\mathbb{C}}) \\ & & \wr\uparrow \text{ (holomorphic Poincaré Lemma)} \\ & & H^i(X^{an}, \mathbb{C}) \end{array}$$

<u>The general notion of a differential equation</u>

Let X be <u>any</u> non-singular connected algebraic variety over $\mathbb{C}$. A "differential equation" on X is by definition a pair (M, ∇) consisting of a locally free coherent algebraic sheaf M on X, together with an integrable connection

$$\nabla : M \longrightarrow M \otimes \Omega^1_{X/\mathbb{C}}$$

This means that ∇ is an additive mapping which satisfies the product rule $\nabla(fm) = m \cdot df + f\nabla(m)$ for all $f \in \mathcal{O}_X$, $m \in M$, and is integrable in the sense that if we make $D \in \underline{\mathrm{Der}}\,(X/\mathbb{C})$ act on M as the composite

$$\begin{array}{ccc} M & \xrightarrow{\ \nabla\ } & M \otimes \Omega^1_{X/\mathbb{C}} \\ & \searrow^{\nabla(D)} & \downarrow \text{id} \otimes \text{ contract with } D \\ & & M \end{array}$$

then we have $\nabla([D_1, D_2]) = [\nabla(D_1), \nabla(D_2)]$. In local coordinates, this is a "completely integrable Pfaffian system."

Analogously, we define a differential equation on a complex manifold as an integrable connection on a locally free coherent analytic sheaf.

A holomorphic version of the problem, and its solution

Let V be an arbitrary connected complex manifold. We may ask whether any finite dimensional complex representation of $\pi_1(V)$ arises as the monodromy representation associated to a differential equation on V. The answer is easily seen to be "yes," as follows.

On any complex manifold, the usual Frobenius existence theorem says that the functor "sheaf of germs of sections killed by ∇" is an equivalence of categories between the category of analytic differential equations and the category of locally constant sheaves of complex vector spaces. (The inverse functor is to tensor a locally constant sheaf E with the holomorphic structural sheaf, and to endow the tensor product $E \otimes \mathcal{O}^{an}$ with the connection $id \otimes d$.)

Finally, on any "reasonable" topological space, picking a base point defines an equivalence of categories between the category of locally constant sheaves of anythings (for example, of finite dimensional complex vector spaces) and the category of representations of the fundamental group on those anythings.

Combining these two equivalences, we see that the functor "associated monodromy representation" defines an equivalence of categories between the category of all differential equations on a connected complex manifold V and the category of finite-dimensional complex representations of its fundamental group.

A compact form of the problem and its solution

Suppose that X is a projective non-singular connected algebraic variety over $\mathbb{C}$. Then Serre's GAGA theorem tells us that the functor $(M, \nabla) \longrightarrow (M^{an}, \nabla^{an})$ is an equivalence of categories between the algebraic differential equations (on X) and the analytic differential equations (on X^{an}). Combining this with the holomorphic solution to the problem, we see that the category of algebraic differential equations on X is equivalent to the category of finite dimensional complex representations of $\pi_1(X^{an})$. We should remark here that because X is supposed compact, there are no "missing points," and hence there are no regularity conditions to impose at them. Thus we have solved the compact case of the twenty-first problem "just" by using GAGA. Even in the case when X is one-dimensional (i.e. X^{an} is a compact Riemann surface), this is of some interest (cf. open problem 1) below).

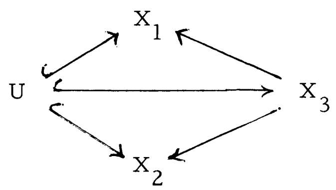
The non-compact case

As Grothendieck was the first to emphasize, the key result in all such questions is Hironaka's resolution of singularities, in the following form.

Let U be a smooth quasi-projective (Zariski-open in a projective) variety over $\mathbb{C}$. Then there exists a projective smooth variety $X/\mathbb{C}$ such that $U \subset X$ as an open dense set, and such that the closed set $D = X - U$ in X is a union of smooth divisors (subvarieties of codimension one) D_i which cross transversely. Furthermore, if

$$U \begin{array}{l} \hookrightarrow X_1 \\ \hookrightarrow X_2 \end{array}$$

are two such compactifications, there always exists a third which "dominates" both:



In this setup, we denote by $\underline{\mathrm{Der}}_D(X/\mathbb{C})$ the sub-sheaf of the tangent sheaf consisting of those derivations which preserve the ideals of each of the D_i. At a point on X where r of the divisors, say $D_1, \ldots, D_r$, cross, we may choose local coordinates $x_1, \ldots, x_n$ such that D_i is defined by $x_i = 0$ for $i \leq r$. There, $\underline{\mathrm{Der}}_D(X/\mathbb{C})$ is free on $x_i \frac{\partial}{\partial x_i}$, $i \leq r$ and $\frac{\partial}{\partial x_j}$, $j > r$. The linear dual of $\underline{\mathrm{Der}}_D(X/\mathbb{C})$ is noted $\Omega^1_{X/\mathbb{C}}(\log D)$. It is locally free on $\frac{dx_1}{x_1}, \ldots, \frac{dx_r}{x_r}$, $dx_{r+1}, \ldots, dx_n$ near this same point.

In case U is connected and <u>one-dimensional</u>, the X is unique — it's just the complete non-singular model of the function field of U, the sheaf $\mathrm{Der}_D(X/\mathbb{C})$ is the subsheaf of $\underline{\mathrm{Der}}\,(X/\mathbb{C})$ of sections which <u>vanish</u> at the points of $X - U$, and $\Omega^1_{X/\mathbb{C}}(\log D)$ is the sheaf of one forms with at worst first order poles at the points of D.

<u>Deligne's definition of regular singular points in the general case</u>

In terms of the sheaves $\Omega^1_{X/\mathbb{C}}(\log D)$ and $\underline{\mathrm{Der}}_D(X/\mathbb{C})$, we may rephrase Deligne's version of Fuch's criterion for regular singular points in the one-variable case as follows: In order that an algebraic differential equation (M, ∇) on an open curve U have regular singular points at all points of $D = X - U$, it is necessary and sufficient that there exist a locally free coherent algebraic sheaf $\overline{M}$ on X extending M, such that the action of $\underline{\mathrm{Der}}\,(U/\mathbb{C})$ on M extends to an action of $\underline{\mathrm{Der}}_D(X/\mathbb{C})$ on $\overline{M}$ (equivalently, that the arrow $\nabla: M \longrightarrow M \otimes \Omega^1_{U/\mathbb{C}}$ extends to an arrow $\overline{\nabla}: \overline{M} \longrightarrow \overline{M} \otimes \Omega^1_{X/\mathbb{C}}(\log D)$.

Deligne now takes over this definition verbatim for a differential equation on <u>any</u> connected non-singular quasi-projective variety U over $\mathbb{C}$, where X is some Hironaka compactification of U of the type discussed above. We should remark that in the higher dimensional case, it is not a priori clear that there exists any <u>locally free</u> coherent algebraic sheaf $\overline{M}$ on X which prolongs M on U; we will prove this. And we will admit here that this definition does not depend on the auxiliary choice of X.

<u>Deligne's solution</u>

We can now prove that Hilbert's twenty-first problem has an affirmative solution for any quasi-projective smooth connected variety U over $\mathbb{C}$. Suppose we're given a representation of the fundamental group of U^{an}. Then as explained above, it arises as the monodromy of a unique <u>analytic</u> differential equation (M^{an}, ∇^{an}) on U^{an}. Let us admit for a moment the following analytic lemma:

<u>Key Lemma</u> Let (M^{an}, ∇^{an}) be any analytic differential equation on U^{an}, and let $U \hookrightarrow X$ be a compactification as above. Then there exists a locally free coherent <u>analytic</u> sheaf $\overline{M^{an}}$ on X^{an} which extends M^{an}, such that ∇^{an} extends to an arrow $\overline{\nabla^{an}}: \overline{M^{an}} \longrightarrow \overline{M^{an}} \otimes \left(\Omega^1_{X/\mathbb{C}}(\log D)\right)^{an}$.

Then we may apply GAGA to the pair $\left(\overline{M^{an}}, \overline{\nabla^{an}}\right)$, to conclude the existence of a locally free coherent algebraic sheaf $\overline{M}$ on X and an arrow $\overline{\nabla}: \overline{M} \longrightarrow \overline{M} \otimes \Omega^1_{X/\mathbb{C}}(\log D)$ such that $\left(\overline{M^{an}}, \overline{\nabla^{an}}\right) = \left(\overline{M}, \overline{\nabla}\right)^{an}$. If we define (M, ∇) on U to be the <u>restriction</u> to U of $(\overline{M}, \overline{\nabla})$, it is immediate that (M, ∇) has regular singular points, and that $(M, \nabla)^{an} \simeq (M^{an}, \nabla^{an})$, so that (M, ∇) gives rise to the given representation of $\pi_1(U^{an})$.

<u>Proof of the key lemma</u>

The idea is to use the local monodromy of the equation to construct an extension of M^{an} <u>locally along</u> D which has the desired properties, and to do so in a sufficiently canonical way that these local extensions patch together to give the needed global extension $\overline{M^{an}}$ of M^{an}. [In the one-variable case, D consists of isolated points, so the extension problem is local around each point of D, and there is no patching problem. But in several variables, where we can slide along D from one point to another, there _is_ a patching problem.]

Let us begin by constructing a local extension. At a point "0" of D where r of the divisors D_i cross, say $D_1, \ldots, D_r$, we may choose local coordinates $x_1, \ldots, x_n$ such that D_i is defined by $x_i = 0$ for $i=1, \ldots, r$. In a small coordinate polydisc V around "0" defined by $|x_j| < \varepsilon$ for $j=1, \ldots, n$, the open manifold $U^{an} \cap V$ is the product of r punctured discs $0 < |x_i| < \varepsilon$, $i=1, \ldots, r$, and of $n-r$ discs $|x_k| < \varepsilon$, $k = r+1, \ldots, n$. The restriction of (M^{an}, ∇^{an}) to $V \cap U^{an}$ is an analytic differential equation on $V \cap U^{an}$, so corresponds to a representation ρ of $\pi_1(V \cap U^{an})$ in a finite dimensional complex vector space L.

The important thing about this local situation is that $\pi_1(V \cap U^{an})$ is the _free Abelian_ group on the r generators γ_i = turning once counterclockwise around D_i. So the representation ρ is specified by the r _commuting_ automorphisms $\rho(\gamma_i)$ of L, $i=1, \ldots, r$. A consideration of Jordan normal form shows that there are unique endomorphisms B_j of the representation space L such that

1) $\exp(2\pi i B_j) = \rho(\gamma_j)$, $j=1, \ldots, r$
2) the eigenvalues of B_j have real parts in the strip $-1 < \mathrm{Re} \leq 0$
3) the B_j mutually commute

In terms of this data, we may construct an extension to V of $(M^{an}, \nabla^{an}) | V \cap U^{an}$ as follows. We define

$$\overline{M^{an}} \overset{\text{dfn}}{=} L \otimes_{\mathbb{C}} \mathcal{O}_V$$

and we define the "connection with logarithmic poles"

$$\begin{array}{ccc} \overline{\nabla^{an}}: \overline{M^{an}} & \longrightarrow & \overline{M^{an}} \underset{\mathcal{O}_V}{\otimes} \Omega^1_V (\log D) \\ \| & & \| \\ L \underset{\mathbb{C}}{\otimes} \mathcal{O}_V & & L \otimes \Omega^1_{V/\mathbb{C}}(\log D) \end{array}$$

by defining

$$\overline{\nabla^{an}}(\ell \otimes f) = f\left(-\sum_{i=1}^{r} B_i \ell \otimes \frac{dx_i}{x_i}\right) + \ell \otimes df$$

To see that this pair $\left(\overline{M^{an}}, \overline{\nabla^{an}}\right)$ does in fact extend $(M^{an}, \nabla^{an}) | V \cap U^{an}$, we need only check that its restriction to $V \cap U^{an}$ has the correct monodromy (by the "holomorphic case" of the Hilbert problem). But a fundamental solution matrix is given explicitly by

$$\prod_{i=1}^{r} x_i^{B_i} \overset{\text{dfn}}{=} \exp\left(\sum_{i=1}^{r} B_i \log x_i\right)$$

which does indeed have the correct monodromy.

This construction shows the <u>local</u> existence of locally free extensions of M^{an} which are <u>stable</u> under $\underline{\text{Der}}_D(X^{an}/\mathbb{C})$. Among <u>all</u> locally free extensions to V of $M^{an} | V \cap U^{an}$, we may <u>characterize</u> the one constructed above by the following property:

* In any basis of it as an $\mathcal{O}_V$-module, a fundamental solution matrix has the form

$$H(x) \cdot \prod_{i=1}^{r} x_i^{B_i}$$

with $H(x) \in GL(n, \mathcal{O}_V)$ and with the B_i commuting matrices, the real parts of whose eigenvalues lie in $-1 < \text{Re} \leq 0$. To see that * indeed <u>characterizes</u> our extension $\overline{M^{an}}$, we argue as follows. Any <u>other</u> extension of M^{an} is of the form $A(x) \cdot \overline{M^{an}}$ where $A(x)$ is an invertible $n \times n$ matrix of functions analytic in $V \cap U^{an}$, with possibly essential singularities along the D_i. The extension $A(x)\overline{M^{an}}$ depends only on the class of $A(x)$ in the double-coset space

$$GL(n, \mathcal{O}_V)\backslash GL(n, \mathcal{O}_{V\cap U^{an}})/GL(n, \mathcal{O}_V).$$

So suppose that in terms of such an extension $A(x)\overline{M^{an}}$, the fundamental solution matrix is of the form $K(x)\prod x_i^{C_i}$, with $K(x) \in GL(n, \mathcal{O}_V)$ and with the C_i commuting matrices with all eigenvalues lying in $-1 < \mathrm{Re} \leq 0$. If we equate the fundamental solution matrices, we obtain the equation

$$K(x)\prod_{i=1}^{r} x_i^{C_i} = A^{-1}(x)\cdot H(x)\cdot \prod x_i^{B_i}.$$

By considering the effect of analytic continuation along γ_j, we see that

$$\exp(2\pi i C_j) = \exp(2\pi i B_j).$$

By the unicity of the logarithm whose eigenvalues lie in $-1 < \mathrm{Re} \leq 0$, we conclude that $C_j = B_j$ for $j=1, \ldots, r$, hence that $K(x) = A^{-1}(x)H(x)$. Thus $A(x) \in GL(n, \mathcal{O}_V)$, and our "other" extension $A(x)\overline{M^{an}}$ is just $\overline{M^{an}}$ itself.

It remains to check that these well-defined local extensions patch together, or equivalently that they are compatible with localization. If we restrict our extension to a small enough polydisc $V' \subset V$ around a point where only the divisors $D_1, \ldots, D_s$ (s < r cross, and choose local coordinates $y_1, \ldots, y_n$ adopted to the situation (D_i is defined in V' by $y_i = 0$, $i=1, \ldots, s$), we must check that a fundamental solution matrix is of the form

$$(\text{matrix in } GL(n, \mathcal{O}_{V'})) \times \prod y_i^{B_i}.$$

In terms of the coordinates $x_1, \ldots, x_n$ on V, the fundamental solution matrix was

$$\prod_{i=1}^{r} x_i^{B_i}.$$

Since for $i=1, \ldots, s$, both $x_i = 0$ and $y_i = 0$ define D_i near P, there are invertible functions u_i, $i=1,\dots,s$ near P such that $x_i = u_i y_i$, $i=1,\dots,s$. As the divisors $D_{s+1},\dots,D_r$ do not pass through P, the functions x_{s+1}, $\dots,x_r$ are themselves invertible near P. So in a small enough polydisc V' around P, we may take the logarithms of the invertible functions $u_1,\dots,u_s$, $x_{s+1},\dots,x_r$, i.e. there are functions $z_1,\dots,z_r$ holomorphic on V' such that

$$u_i = \exp(z_i) \qquad i=1,\dots,s$$

$$x_i = \exp(z_i) \qquad i=s+1,\dots,r.$$

Then

$$\prod_{i=1}^{r} x_i^{B_i} = \prod_{i=1}^{s} (u_i y_i)^{B_i} \prod_{i=s+1}^{r} x_i^{B_i} = \exp\left(\sum_{i=1}^{r} B_i z_i\right) \prod_{i=1}^{s} y_i^{B_i}$$

and $\exp(\sum z_i B_i) \in GL(n, \mathcal{O}_{V'})$ as desired. QED

Remark The proof shows clearly that the notion of regular singular points in several variables can be expressed as a growth condition on the local solutions in (angular sectors of) the punctured polydiscs at infinity, in complete analogy with the one-variable case (cf. [2] for more details).

More of Deligne's results — Comparison theorems

Thus far we have proven that for any connected non-singular quasi-projective variety U over $\mathbb{C}$, any finite-dimensional complex representation arises as the monodromy representation of an algebraic differential equation on U with regular singular points. It is natural to ask if this differential equation is unique. In categorical terms, we have proven that the functor

$$\begin{array}{l}\text{Algebraic D.E.'s on U with} \\ \text{regular singular points}\end{array} \longrightarrow \text{analytic D.E.'s on } U^{an} \simeq \text{rep'ns of } \pi_1(U^{an})$$

is essentially surjective, and we ask if it is an equivalence of categories (or equivalently, given its surjectivity, whether it is fully faithful). Concretely, given two algebraic differential equations (M,∇) and (M',∇') on U with regular singular points, we are asking if the map

$$\mathrm{Hom}\,((M,\nabla),\ (M',\nabla')) \longrightarrow \mathrm{Hom}\,((M,\nabla)^{an},\ (M',\nabla')^{an})$$

is an isomorphism.

Consider the "internal hom" differential equation $(M''= \underline{\mathrm{Hom}}(M,M'),\nabla'')$, defined by $(\nabla''(D)\varphi)(m) = \nabla'(D)(\varphi(m)) - \varphi(\nabla(D)(m))$ for $D \in \underline{\mathrm{Der}}\,(U/\mathbb{C})$, $\varphi \in \mathrm{Hom}\,(M,M')$, and $m \in M$; its horizontal (= killed by ∇'') sections are exactly the D. E.-maps from (M,∇) to (M',∇'), and it will have regular singular points if both (M,∇) and (M',∇') do. So we may rephrase the question as follows: given a differential equation (M,∇) with regular singular points, is the map

$$\begin{matrix}\text{global horizontal}\\ \text{sections of } M \text{ on } U\end{matrix} \longrightarrow \begin{matrix}\text{global horizontal}\\ \text{sections of } M^{an} \text{ on } U^{an}\end{matrix}$$

an isomorphism?

The answer is easily seen to be yes, for in terms of the particular extension $\overline{M}$ we constructed, any global analytic horizontal section of M^{an} extends to a holomorphic global section of $\overline{M}^{an}$, which by GAGA is a global section of $\overline{M}$, and so restricts to give a global section of M, as required.

In fact, there is something more that we can say. Let (M,∇) be an algebraic differential equation on U with regular singular points, and let $(M^{an})^{\nabla}$ denote the locally constant sheaf of germs of local holomorphic horizontal sections. Because the connection is <u>integrable</u>, we can extend the map

$$\nabla: M \longrightarrow M \otimes \Omega^1_{V/\mathbb{C}}$$

to an entire complex

$$M \xrightarrow{\nabla} M \otimes \Omega^1_{U/\mathbb{C}} \xrightarrow{\nabla\otimes 1 + 1\otimes d} M \otimes \Omega^2_{U/\mathbb{C}} \longrightarrow \ldots\ ,$$

the "De Rham complex of (M,∇)," noted $(M\otimes \Omega^{\cdot}_{U/\mathbb{C}}, \nabla)$.

We can consider the hypercohomology groups

$$\mathbb{H}^i(U, (M \otimes \Omega^{\cdot}_{U/\mathbb{C}}, \nabla))$$

(for U affine, these are the cohomology groups of the complex of global sections of the De Rham complex). We have a canonical map

$$\mathbb{H}^i(U, (M \otimes \Omega^{\cdot}_{U/\mathbb{C}}, \nabla)) \longrightarrow \mathbb{H}^i(U^{an}, (M^{an} \otimes \Omega^{\cdot}_{U^{an}/\mathbb{C}}, \nabla^{an}))$$

$$\wr\Big| \quad \text{(an isomorphism by the holomorphic Poincaré lemma)}$$

$$H^i(U^{an}, (M^{an})^{\nabla})$$

which for $i=0$ is just the map comparing algebraic and analytic global horizontal sections. The "something more" is that this map is an <u>isomorphism</u> for <u>all</u> i. The simplest proof is to remark that for the extension $(\overline{M}, \overline{\nabla})$ we explicitly constructed, (the "quasi-canonical" extension in Deligne's terminology) we have isomorphisms

$$\mathbb{H}^i(X, (\overline{M} \otimes \Omega^{\cdot}_{X/\mathbb{C}}(\log D), \overline{\nabla})) \xrightarrow{\sim} \mathbb{H}^i(U, (M \otimes \Omega^{\cdot}_{U/\mathbb{C}}, \nabla))$$

<u>and</u>

$$\mathbb{H}^i(X^{an}, \overline{M}^{an} \otimes \Omega^{\cdot}_{X^{an}/\mathbb{C}}(\log D), \overline{\nabla}) \xrightarrow{\sim} \mathbb{H}^i(U^{an}, M^{an} \otimes \Omega^{\cdot}_{U/\mathbb{C}}, \nabla).$$

The desired result now follows by applying GAGA to see that the canonical map

$$\mathbb{H}^i(X, (\overline{M} \otimes \Omega^{\cdot}_{X/\mathbb{C}}(\log D), \overline{\nabla})) \longrightarrow \mathbb{H}^i(X^{an}, (\overline{M}^{an} \otimes \Omega^{\cdot}_{X^{an}/\mathbb{C}}, \overline{\nabla}))$$

is an isomorphism.

If we take the <u>trivial</u> representation of $\pi_1(U^{an})$, the corresponding locally constant sheaf is the constant sheaf $\mathbb{C}$ itself, and the corresponding algebraic differential equation with regular singular points is $\mathcal{O}_U$ with the "trivial" connection furnished by exterior differentiation $d: \mathcal{O}_U \longrightarrow \Omega^1_{U/\mathbb{C}}$. The comparison theorem in this special case asserts that the map

$$\begin{array}{ccc} \mathbb{H}^i(U, \Omega^{\cdot}_{U/\mathbb{C}}) & \longrightarrow & \mathbb{H}^i(U^{an}, \Omega^{\cdot}_{U^{an}/\mathbb{C}}) \\ & & \wr\downarrow \\ & & H^i(U^{an}, \mathbb{C}) \end{array}$$

is an isomorphism (e.g., on a smooth affine U, closed modulo exact global algebraic differential forms calculate the complex cohomology). This theorem, first proven by Grothendieck, was the starting point for the systematic application of Hironaka's resolution result to studying the cohomology of open varieties (classically, the problem of understanding "integrals of the second kind"). In fact, the first "modern" attempt to study such questions was that of Atiyah and Hodge [1], which essentially concerned itself with the cohomology of a variable Zariski open set U in a fixed projective smooth X. Only after Hironaka's result, though, did it become apparent that one should instead study the cohomology of a fixed open smooth U with the aid of an auxiliary Hironaka compactification $U \hookrightarrow X'$.

The Moral

To an algebraic geometer, it is that so far as non-singular algebraic varieties U over $\mathbb{C}$ are concerned, the apparently topological notions of locally constant sheaf on U^{an}, and of the cohomology of U^{an} with coefficients in such a sheaf, are in fact the purely algebraic notions of differential equation on U with regular singular points, and of the hypercohomology groups of U with coefficients in their de Rham complexes.

What we don't know

Here are three open problems that lie well within the scope of the traditional theory of differential equations. The first of them was raised by Lipman Bers in the course of a stimulating discussion at the Symposium. The second is a favorite of mine. The third was raised by Monsky.

1. Let X^{an} be a compact Riemann surface of genus $g \geq 2$. Then its universal covering is the upper half-plane $\mathfrak{h}$, and X^{an} is obtained as the quotient of $\mathfrak{h}$ by a fuchsian group $G \subset SL(2,R)/\pm 1$; $X^{an} = \mathfrak{h}/G$, and G _is_ the fundamental group $\pi_1(X^{an})$. As an _abstract_ group, $\pi_1(X^{an})$ is generated by $2g$ elements $a_i, b_i, i=1,\ldots,g$, with the single relation $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1$. Viewing X^{an} as $\mathfrak{h}/G$ gives us an embedding of $\pi_1(X^{an})$ in $SL(2,\mathbb{R})/\pm 1$, which lifts in 2^{2g} ways to an embedding of $\pi_1(X^{an})$ in $SL(2,\mathbb{R})$. Thus we have 2^{2g} representations of $\pi_1(X^{an})$ in $SL(2,\mathbb{R})$, which may be realized as follows. There are 2^{2g} choices of line bundle $\mathcal{L}$ on X such that $\mathcal{L}^{\otimes 2} \simeq \Omega^1_{X/\mathbb{C}}$. _Each_ of these allows us to write $H^1(X, \Omega^1_{X/\mathbb{C}}) = H^1(X, \mathcal{L}^{\otimes 2}) = H^1(X, \underline{Hom}(\mathcal{L}^{-1}, \mathcal{L})) = Ext^1(X; \mathcal{L}^{-1}, \mathcal{L})$, under which isomorphism the canonical generator of $H^1(X, \Omega^1_{X/\mathbb{C}})$ ("the cohomology class of a point") gives rise to a rank two vector bundle $M(\mathcal{L})$ which sits in a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow M(\mathcal{L}) \longrightarrow \mathcal{L}^{-1} \longrightarrow 0$$

It is classical that there is one and only one connection ∇ on $M(\mathcal{L})$ such that the monodromy representation is a _real_ representation; _this_ representation gives an embedding $\pi_1(X^{an}) \longrightarrow SL(2,\mathbb{R})$, and as $\mathcal{L}$ varies we recover the 2^{2g} embeddings described above.

Let X'^{an} be _another_ compact Riemann surface of the _same_ genus $g \geq 2$. Viewing it as $\mathfrak{h}/G'$, where $G' \subset SL(2,\mathbb{R})/\pm 1$ is _another_ fuchsian group, we obtain as before 2^{2g} embeddings of $\pi_1(X'^{an})$ in $SL(2,\mathbb{R})$. But of course the groups $\pi_1(X^{an})$ and $\pi_1(X'^{an})$ are _isomorphic_ as abstract groups, so we could compose with this group-isomorphism to produce 2^{2g} embeddings of $\pi_1(X^{an}) \longrightarrow SL(2,\mathbb{R})$, this time with image in $SL(2,\mathbb{R})/\pm 1$ the fuchsian group corresponding to X' rather than to X. By the affirmative solution of the twenty-first problem, we know that these may be realized as the monodromy representations of certain rank two differential equations

(M_i, ∇_i), $i=1,\ldots,2^{2g}$.

Is it true that ∇_i is the unique connection on M_i such that the monodromy representation is real? Can the bundles M_i that arise in this way (for variable X') be characterized algebraically? Can bundles which admit connections whose monodromy representation is real be characterized algebraically? (The "compact form" of this last question has been answered by Seshadri and Narasimhan, who show that it is precisely the "stable" bundles of degree zero (a purely algebraic notion) which admit connections with irreducible unitary monodromy, and that this unitary connection is unique.) If these questions could be answered affirmatively, one could try to resuscitate the original Poincaré "proof" of uniformization by the "continuity principle."

2. Can we characterize algebraically those differential equations with regular singular points (M, ∇) on an open curve U whose monodromy representations factor through finite groups. This can be done rather strikingly for the hypergeometric equation, but already for second-order equations on $\mathbb{P}^1 - \{4 \text{ or more points}\}$ the question is completely open and extremely interesting [cf. 7]. It should be remarked that this question is treated in Forsythe "in principle," but even in cases when one knows the answer ahead of time, it seems hopeless to ever carry out Forsythe's test procedure.

3. What is the role of equations with irregular singular points. (It would be absurd to ignore the equation $f' - f = 0$, which has an irregular singularity at ∞.) What is the meaning of the cohomology groups $H^i_{DR}(U, (M \otimes \Omega^{\cdot}_{U/\mathbb{C}}, \nabla))$ when (M, ∇) has irregular singular points? Are they finite-dimensional? The finite-dimensionality in the case when U is a curve is known, and due essentially to Birkhoff (his theory of canonical forms for irregular singularities) — Deligne gives a finally not-so-different proof, where he also shows that the algebraic "index" $\dim H^0 - \dim H^1$ is different from its analytic counterpart.

and that this difference measures the irregularity of the singularities.

References

1. Atiyah, M., Hodge, W.: Integrals of the second kind on an algebraic variety. Annals of Math. 62, 56-91 (1955).

2. Deligne, P.: Equations differentielles à points singuliers réguliers. Lecture Notes in Mathematics 163, Berlin-Heidelberg-New York: Springer 1970.

3. Forsythe, A.R.: Theory of Differential Equations, Vol. IV, Cambridge, 1900-1902.

4. Grothendieck, A.: On the de Rham cohomology of algebraic varieties. Publ. Math. I.H.E.S. 29 (1966).

5. Hironaka, H.: Resolution of singularities of an algebraic variety over a field of characteristic zero I, II, Annals of Math. 79, 109-326 (1964).

6. Ince, E.L.: Ordinary Differential Equations, New York, Dover, 1956 (esp. 356-372, 389-393).

7. Katz, N.M.: Algebraic Solutions of Differential Equations, Inventiones Math. 18, 1-118 (1972).

8. Manin, Y.: Moduli Fuchsiani, Ann. Scuola Norm. Sup. Pisa, Ser III, 19 (13-126 (1965).

9. Monsky, P.: Finiteness of de Rham Cohomology, Amer. J. Math., XCIV, 237-245 (1972).

10. Narasimhan, M.S., and Seshadri, C.S.: Stable and unitary vector bundles on a compact Riemann surface. Annals of Math., 82, 540-567 (1965).

11. Serre, J.-P.: Géometrie algébrique et géometrie analytique. Ann. Inst. Fourier Grenoble 6, 1-42 (1956).