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Estimates for Nonsingular Mixed Character Sums

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1 Introduction and statement of the main result

Let k be a finite field, p its characteristic, q its cardinality,

$$\psi : (k, +) \to \mathbb{Z}[\zeta_p]^\times \subset \mathbb{C}^\times$$

a nontrivial additive character of k, and

$$\chi : k^\times \to \mathbb{Z}[\zeta_{q-1}]^\times \subset \mathbb{C}^\times$$

a nontrivial multiplicative character of k. We extend χ to k by defining $\chi(0) = 0$.

We wish to consider character sums over $\mathbb{A}^n$, $n \geq 1$, of the following form. We are given a polynomial $f(x) := f(x_1, \ldots, x_n)$ in $k[x_1, \ldots, X_n]$ of degree $d \geq 1$, and we are given a second polynomial $g(X) := g(x_1, \ldots, x_n)$ in $k[x_1, \ldots, x_n]$ of degree $e \geq 1$. We are interested in understanding when the sum

$$\sum_{x \in k^n} \psi(f(x))\chi(g(x))$$

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has "square root" cancellation, i.e. when we can exhibit an explicit constant $C = C(n, d, e)$ and prove the estimate

$$\left|\sum_{x\in k^n} \psi(f(x))\chi(g(x))\right| \leq C(\#k)^{n/2}.$$

In this paper, we will exhibit one particularly nice class of pairs (f, g) for which such estimates hold. The general problem of understanding for which pairs (f, g) one has, or should have, such estimates is far from being understood.

Let us first recall the notion of a "Deligne polynomial". A polynomial $f = f(x_1, \ldots, x_n)$ in $n \geq 1$ variables over k of degree $d \geq 1$ is called a Deligne polynomial if its degree d is prime to p and if its highest degree term, f_d, is a homogeneous form of degree d in n variables which is nonzero, and whose vanishing, if $n \geq 2$, defines a smooth hypersurface in the projective space $\mathbb{P}^{n-1}$.

For $f = f(x_1, \ldots, x_n)$ a Deligne polynomial of degree d, one has Deligne's fundamental estimate [3, 8.4]

$$\left|\sum_{x\in k^n} \psi(f(x))\right| \leq (d-1)^n(\#k)^{n/2}.$$

If $g = g(x_1, \ldots, x_n)$ is a Deligne polynomial of degree e, such that $g = 0$ defines a smooth hypersurface in $\mathbb{A}^n$, then one has the analogous estimate [6, Theorem. 1]

$$\left|\sum_{x\in k^n} \chi(g(x))\right| \leq (e-1)^n(\#k)^{n/2}.$$

Our main result is that if f and g above are suitably transverse, then we have a good estimate for the mixed sum. To state the estimate, we define the constant

$$\begin{aligned} C(n, d, e) &:= (-1)^n \times \text{coef. of } L^n \text{ in } \quad \frac{(1+L)^{n+1}}{(1+L)(1+dL)(1+eL)} \\ &= \text{the value at } (x, y) = (d, e) \text{ of } \quad \frac{x(x-1)^n - y(y-1)^n}{x-y} \\ &= \sum_{a+b=n} (d-1)^a(e-1)^b \quad + \sum_{a+b=n-1} (d-1)^a(e-1)^b. \end{aligned}$$

Recall also that, given an integer w, a number $\alpha \in \mathbb{C}$ is said to be pure of weight w (relative to q) if it and all its $Aut(\mathbb{C}/\mathbb{Q})$-conjugates have absolute value $q^{w/2}$. Such an α is necessarily algebraic over $\mathbb{Q}$. A polynomial $P(T) \in 1 + T\mathbb{C}[T]$ is said to be pure of weight

w if all its reciprocal roots are pure of weight w; it is said to be mixed of weight $\leq w$ if each of its reciprocal roots α is pure of some integer weight $w_\alpha \leq w$.

Theorem 1.1. Suppose that $f = f(x_1, \ldots, x_n)$ and $g = g(x_1, \ldots, x_n)$ are Deligne polynomials over k of degrees d and e, respectively. If $n \geq 2$, suppose in addition that the smooth hypersurfaces in $\mathbb{P}^{n-1}$ defined by $f_d = 0$ and by $g_e = 0$ are transverse, i.e., their intersection is smooth of codimension 2 in $\mathbb{P}^{n-1}$. Then we have the following results.

(1) We have the estimate

$$\left|\sum_{x \in k^n} \psi(f(x))\chi(g(x))\right| \leq C(n, d, e)(\#k)^{n/2}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree $\leq C(n, d, e)$, which is mixed of weight $\leq n$.

(2) If $P(T)$ has degree $= C(n, d, e)$, then $P(T)$ is pure of weight n.

(3) If $g = 0$ defines a nonsingular hypersurface in $\mathbb{A}^n$, then $P(T)$ has degree $= C(n, d, e)$, and is pure of weight n. □

We are indebted to Steve Sperber for the observation that the ideas that go into proving this theorem lead in a straightforward way to a theorem dealing with the following more general situation. Instead of (f, g), we give ourselves an integer $r \geq 1$, and $r+1$ Deligne polynomials $(f, g_1, \ldots, g_r)$ in n variables over k, of degrees $(d, e_1, \ldots, e_r)$. If $n \geq 2$, we assume that the $r + 1$ smooth hypersurfaces in $\mathbb{P}^{n-1}$ defined by the vanishing of their highest degree forms are transverse, in the sense that for any integer j with $r + 1 \geq j \geq 1$, the intersection of any j of them is smooth of codimension j in $\mathbb{P}^{n-1}$ if $j \leq n - 1$, and is empty if $j \geq n$. Then we get the following result.

Theorem 1.2. For any r- tuple of nontrivial multiplicative characters $(\chi_1, \ldots, \chi_r)$, we have the bound

$$\left|\sum_{x \in k^n} \psi(f(x)) \prod_i \chi_i(g_i(x))\right| \leq C(n, d, e_1, \ldots, e_r)(\#k)^{n/2},$$

where $C(n, d, e_1, \ldots, e_r)$ is defined as

$$C(n, d, e_1, \ldots, e_r) := (-1)^n \times \text{coef. of } L^n \text{ in } \quad \frac{(1+L)^{n+1}}{(1+L)(1+dL)\prod_i(1+e_iL)}. \qquad \square$$

We will discuss the proof of this more general result in the appendix.

2 Statement of a second version of the main result

In this section, we give a generalization in the spirit of [9, 5.1.1] and [6, Theorems 3,4]. Let X/k be a projective, smooth, and geometrically connected k-scheme of dimension $n \geq 1$, given with a projective embedding $X \hookrightarrow \mathbb{P}^N_k := \mathbb{P}$. We fix integers $d \geq 1$ and $e \geq 1$, both prime to p. We are given a linear form

$$Z \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)),$$

a degree d form

$$F \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)),$$

and a degree e form

$$G \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(e)),$$

all on the ambient projective space $\mathbb{P}$. Assume that the following four transversality hypotheses hold.

(1) $X \cap Z$ is lisse of codimension 1 in X.
(2) $X \cap Z \cap F$ is lisse of codimension 1 in $X \cap Z$ (:= empty, if $n = 1$).
(3) $X \cap Z \cap G$ is lisse of codimension 1 in $X \cap Z$ (:= empty, if $n = 1$).
(4) $X \cap Z \cap F \cap G$ is lisse of codimension 2 in $X \cap Z$ (:= empty, if $n \leq 2$).

To this data, we attach the smooth affine k-scheme

$$V := X - X \cap Z = X[1/Z],$$

and the functions

$$f := F/Z^d : V \to \mathbb{A}^1_k$$

and

$$g := G/Z^e : V \to \mathbb{A}^1_k.$$

We denote by $c(X)$ the total Chern class of X, and by L the class of $\mathcal{O}_X(1)$. We define the constant $C(X, d, e)$ by

$$C(X, d, e) := (-1)^n \int_X \frac{c(X)}{(1 + L)(1 + dL)(1 + eL)}.$$

Thus when X is $\mathbb{P}^n$ with the identity embedding of itself into $\mathbb{P} = \mathbb{P}^n$, $C(X,d,e)$ is the constant $C(n,d,e)$ of the first section. When X is a complete intersection in $\mathbb{P}^{n+r}$ of multidegree $(a_1,\ldots,a_r)$, then

$$\begin{aligned} C(X,d,e) &:= (-1)^n \int_{\mathbb{P}^{n+r}} \frac{a_1\ldots a_r L^r (1+L)^{n+r+1}}{(1+L)(1+dL)(1+eL)\prod_i (1+a_i L)} \\ &= \text{coef. of } L^n \text{ in } \frac{a_1\ldots a_r (1+L)^{n+r+1}}{(1+L)(1+dL)(1+eL)\prod_i (1+a_i L)} \end{aligned}$$

Theorem 2.1. Suppose that (X,Z,F,G) are as above. Then we have the following results.

(1) We have the estimate

$$\left|\sum_{x\in V(k)} \psi(f(x))\chi(g(x))\right| \le C(X,d,e)(\#k)^{n/2}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree $\le C(X,d,e)$, which is mixed of weight $\le n$.

(2) If $P(T)$ has degree $= C(X,d,e)$, then $P(T)$ is pure of weight n.

(3) If $X\cap G$ is smooth of codimension 1 in X, or equivalently if $g=0$ is smooth of codimension 1 in V, then $P(T)$ has degree $= C(X,d,e)$, and is pure of weight n. □

Thus when X is $\mathbb{P}^n$ with the identity embedding of itself into $\mathbb{P} = \mathbb{P}^n$, this theorem is just Theorem 1.1.

3 Proof of Theorem 2.1; the strategy

As is customary in such questions, we choose a prime number $\ell \neq p$ and choose an embedding of $\mathbb{Q}(\zeta_p, \zeta_{q-1})$ into $\overline{\mathbb{Q}}_\ell$, so that we can view all our characters, both additive and multiplicative, as having values in $\overline{\mathbb{Q}}_\ell^\times$, and so that we can apply ℓ-adic cohomology.

On the smooth, geometrically connected, affine variety $V[1/g]$ of dimension n, we have the lisse, rank one, Artin-Schreier sheaf $\mathcal{L}_{\psi(f)}$, the lisse, rank one, Kummer sheaf $\mathcal{L}_{\chi(g)}$, and their lisse, rank one, tensor product $\mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}$, cf. [2, 1.4.2, 1.4.3]. Each of these lisse sheaves is pure of weight 0. By the Lefschetz Trace Formula [5], we have

$$\sum_{x\in V(k)} \psi(f(x))\chi(g(x)) = \sum_i (-1)^i Trace(Frob_k | H^i_c(V[1/g]\otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}).$$

By Deligne's Weil II result [4, 3.3.4], the (reversed characteristic polynomial of $Frob_k$ on the) cohomology group H^i_c above is mixed of weight $\leq i$. By the dual of the Lefschetz affine theorem, H^i_c vanishes for $i < n$, cf. [11, Exposé XVIII, Theorem 3.2.5 and Exposé XIV, Corollary 3.2].

Let us admit temporarily the following theorem, and explain how it implies Theorem 2.1.

Theorem 3.1. Suppose that (X, Z, F, G) are as in Theorem 2.1. Then we have the following results.

(1) $H^i_c := H^i_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$ vanishes for $i \neq n$.

(2) If $X \cap G$ is smooth of codimension 1 in X, or equivalently if $g = 0$ is smooth of codimension 1 in V, then we have the following results.

(2a) H^n_c has dimension $C(X, d, e)$, and is pure of weight n.

(2b) The "forget supports" map is an isomorphism

$$H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) \cong H^n(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}). \qquad \square$$

Using this, we prove Theorem 2.1 as follows. Over the affine space

$$\mathbb{A} := H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \times H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \times H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(e)),$$

we have the product $X \times \mathbb{A}$, the closed subscheme of this product consisting of points $(x \in X, Z, F, G)$ where $L(x)G(x) = 0$, and its open complement $\mathcal{V}[1/g]_{univ}$, consisting of points $(x \in X, Z, F, G)$ where $L(x)G(x)$ is invertible. We view $\mathcal{V}[1/g]_{univ}$ as fibered over $\mathbb{A}$, say

$$\pi_{univ} : \mathcal{V}[1/g]_{univ} \to \mathbb{A}.$$

The triples $(Z, F, G) \in \mathbb{A}$ which satisfy our four transversality conditions with respect to X form a dense open set $\mathcal{U} \subset \mathbb{A}$. Over this open set $\mathcal{U} \subset \mathbb{A}$, the pullback $\mathcal{V}[1/g]$ of $\mathcal{V}[1/g]_{univ}$ is an affine smooth $\mathcal{U}$-scheme, say

$$\pi : \mathcal{V}[1/g] \to \mathcal{U}.$$

with geometrically connected fibres of dimension n, whose fiber over a point (Z, F, G) is $V[1/g] = X[1/LG]$.

On $\mathcal{V}[1/g]$, we have the lisse sheaf $\mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}$. The sheaf

$$\mathcal{N} := R^n \pi_! (\mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

is then a sheaf of perverse origin on $\mathcal{U}$, cf. [8, Introduction and Corollary 5]. For a sheaf of perverse origin, one knows [8, Propositions 11, 12] that the stalk at any point has rank at most the generic rank, and that the open set $\mathcal{U}_{lisse}$ where the sheaf is lisse consists precisely of the points $\mathcal{U}_{max}$ where the stalk has this maximum rank.

The stalk of $\mathcal{N}$ at a k-valued point $(Z, F, G) \in \mathcal{U}(k)$ is the cohomology group

$$H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}).$$

The supplementary condition on (Z, F, G) that $X \cap G$ be smooth of codimension 1 in X defines a dense open set $\mathcal{U}_1 \subset \mathcal{U}$. By the second part of Theorem 3.1, the stalk of $\mathcal{N}$ at any point of $\mathcal{U}_1$ has rank $C(X, d, e)$, and this stalk is pure of weight n. [Let us note in passing that this proves part (3) of Theorem 2.1.]

Therefore the generic rank of $\mathcal{N}$ must be $C(X, d, e)$. So for any k-valued point $(Z, F, G) \in \mathcal{U}(k)$, we have

$$dim H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) \leq C(X, d, e).$$

As this group is mixed of weight $\leq n$, and all other H^i_c vanish, we have

$$\sum_{x \in V(k)} \psi(f(x))\chi(g(x)) = (-1)^n Trace(Frob_k | H^n_c),$$

so we get the estimate

$$\left| \sum_{x \in V(k)} \psi(f(x))\chi(g(x)) \right| \leq C(X, d, e)(\#k)^{n/2}.$$

This proves part (1) of Theorem 2.1.

On the dense open set $\mathcal{U}_1$, $\mathcal{N}$ is punctually pure of weight n, and has constant rank $C(X, d, e)$. Thus we have the inclusion $\mathcal{U}_1 \subset \mathcal{U}_{lisse} = \mathcal{U}_{max}$. Now the sheaf $\mathcal{N}$ is mixed, by [4, 3.3.3], so its restriction to $\mathcal{U}_{lisse}$ is a lisse sheaf which is mixed. Such a sheaf on a lisse k-scheme is a successive extension of pure lisse sheaves, by [4, 3.4.1], so the weights that occur, and their multiplicities, can be read by looking at any single point in $\mathcal{U}_{lisse}(\overline{k})$. Taking a point in $\mathcal{U}_1(\overline{k})$, we conclude that $\mathcal{N}|\mathcal{U}_{max}$ is pure of weight n. This proves the second assertion of Theorem 2.1.

4 Proof of part (1) of Theorem 3.1

Let us recall the situation. We have X/k a projective, smooth, and geometrically connected k-scheme of dimension $n \geq 1$, given with a projective embedding $X \hookrightarrow \mathbb{P}^N_k := \mathbb{P}$. And we have homogeneous forms (Z, F, G) of prime-to-p degrees $1, d, e$, respectively, in the ambient $\mathbb{P}$, subject to various transversality conditions. We must show that

$$H^i_c := H^i_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

vanishes for $i \neq n$. As already noted earlier, H^i_c vanishes for $i < n$ by the dual of the Lefschetz affine theorem. So it remains to show that H^i_c vanishes for $i > n$.

We first treat the case where χ^e is trivial. In this case, we argue as follows. Consider the finite flat covering $V_e := V[g^{1/e}]$ of V gotten by taking the eth root of g, say

$$\rho : V_e \to V.$$

Concretely, V_e is the closed subscheme of $V \times \mathbb{A}^1$, with coordinate t on $\mathbb{A}^1$, of equation $t^e = g$. The direct image sheaf $\rho_\star \overline{\mathbb{Q}}_\ell$ on V has a direct sum decomposition, as the direct sum of the constant sheaf on V with various Kummer sheaves on $V[1/g]$, extended by zero. More precisely, denote by

$$j : V[1/g] \subset V$$

the inclusion. We have a direct sum decomposition on V

$$\rho_\star \overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell \bigoplus \oplus_{\Lambda^e triv, \Lambda\ nontriv}\, j_! \mathcal{L}_{\Lambda(g)}.$$

By the projection formula, we see that for each i,

$$H^i_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

is a direct summand of

$$\begin{aligned} H^i_c(V_e \otimes_k \overline{k}, \rho^\star \mathcal{L}_{\psi(f)}) &= H^i_c(V \otimes_k \overline{k}, \rho_\star \rho^\star \mathcal{L}_{\psi(f)}) = H^i_c(V \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \rho_\star \overline{\mathbb{Q}}_\ell) \\ &= H^i_c(V \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}) \bigoplus \oplus_{\Lambda^e triv, \Lambda\ nontriv} H^i_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\Lambda(g)}). \end{aligned}$$

[We note for later use that this same projection formula argument shows that for each i, the ordinary cohomology group

$$H^i(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

is a direct summand of the ordinary cohomology group

$$H^i(V_e \otimes_k \overline{k}, \rho^\star \mathcal{L}_{\psi(f)}).]$$

So it suffices to show that the cohomology groups

$$H^i_c(V_e \otimes_k \overline{k}, \rho^\star \mathcal{L}_{\psi(f)})$$

vanish for $i > n$. We will see that this results from the "nonsingular" case ($\epsilon = \delta = -1$) of [7, Theorem 4]. For this, we argue as follows. We began with $X \subset \mathbb{P}$. To fix ideas, think of this ambient $\mathbb{P} = \mathbb{P}^N$ as having homogeneous coordinates $(X_0, \ldots, X_N)$. In the projective space $\mathbb{P}^{N+1}$, with homogeneous coordinates $(T, X_0, \ldots, X_N)$, we consider the closed subscheme X_e defined by the equations which defined X, together with the new equation

$$T^e - G = 0.$$

Then $(X_0, \ldots, X_N)$ define a map from X_e to X, which makes X_e a finite flat covering of X of degree e. [A more intrinsic way to view X_e is as follows. On X, we have the invertible $\mathcal{O}_X$-module $\mathcal{M} := \mathcal{O}_X(1)$, and the global section $m := G$ of $\mathcal{M}^{\otimes e}$. Then X_e represents the functor on X-schemes which attaches to an X-scheme $\pi : Y \to X$ the set

$$\{z \in H^0(Y, \pi^\star \mathcal{M}) |\ z^e = \pi^\star m \text{ in } H^0(Y, \pi^\star \mathcal{M}^{\otimes e}).]$$

Inside X_e, V_e is the open set $X_e - X_e \cap Z = X_e[1/Z]$, and $\rho^\star \mathcal{L}_{\psi(f)}$ on V_e is just $\mathcal{L}_{\psi(f)}$, for f the "same" function F/Z^d, but now viewed on $V_e = X_e[1/Z]$. A rereading of [7, Lemma 10 made cohomological, Corollary 14(1), and the first paragraph of the proof of Theorem 16], shows that the asserted vanishing of the cohomology groups

$$H^i_c(V_e \otimes_k \overline{k}, \rho^\star \mathcal{L}_{\psi(f)})$$

for $i > n$ is proven (though not explicitly stated!) in [7], provided that the following three conditions hold.

(1) X_e is Cohen-Macaulay and equidimensional of dimension n.

(2) $X_e \cap Z$ is smooth of dimension $n - 1$.

(3) $X_e \cap Z \cap F$ is smooth of dimension $n - 2$ (:= empty, if $n = 1$).

To see that these three conditions hold, we argue as follows. To show X_e is Cohen-Macaulay and equidimensional of dimension n, we argue as follows. The scheme X_e is the finite flat covering of X defined by taking the eth root of G. The open set $X_e[1/G] \subset X$ is finite etale over $X[1/G]$, so is itself smooth. And over an open neighborhood U of a point $x \in X$ where $G(x) = 0$, the covering X_e is a hypersurface in the smooth scheme $U \times \mathbb{A}^1$, so is Cohen-Macaulay, cf. [1, Chapter III, Corollary 4.5]. To see that $X_e \cap Z$ is smooth of dimension $n-1$, view it as the covering of $X \cap Z$ defined by taking the eth root of G. By hypothesis $G = 0$ defines a smooth hypersurface in the smooth scheme $X \cap Z$ of dimension $n-1$, and e is prime to p, so the total space $X_e \cap Z$ of this covering is itself smooth. Similarly, $X_e \cap Z \cap F$ is the covering of $X \cap Z \cap F$ defined by taking the eth root of G, and we argue as above, now using the assumed smoothness of both $X \cap Z \cap F$ and of $X \cap Z \cap F \cap G$. This concludes the proof of the first part of Theorem 3.1, in the case when χ^e is trivial.

We now explain how to reduce the general case to this one. The asserted vanishing of the cohomology groups is a geometric statement, so we may extend scalars at will from the original finite field k to any finite extension. Our first task is to show that after such an extension, we can find a particularly nice coordinate system $(Y_0, \ldots, Y_N)$ in the ambient $\mathbb{P}$, which is suitably transverse to the situation (X, Z, F, G). We will inductively find these homogeneous coordinates, or rather the hyperplanes they define. We start by defining

$$Y_0 := Z.$$

We wish to find a coordinate system $(Z = Y_0, Y_1, \ldots, Y_N)$ in $\mathbb{P}$ such that the following conditions hold.

(1) X is transverse to the coordinate system $(Y_0, \ldots, Y_N)$, in the sense that for any subset $I \subset \{0, 1, \ldots, N\}$, the intersection $X \bigcap \cap_{i \in I} Y_i$ is smooth of dimension $dimX - \#I$ (:=empty if $\#I > dimX$).

(2) If $X \cap G$ is smooth, then it is transverse to the coordinate system $(Y_0, \ldots, Y_N)$.

(3) Each of $X \cap Z$, $X \cap Z \cap F$, $X \cap Z \cap G$, and $X \cap Z \cap F \cap G$, viewed as a closed smooth subscheme of $\mathbb{P} \cap Z$, is transverse to the coordinate system $(Y_1, \ldots, Y_N)$.

It is standard that, over $\overline{k}$, given any finite list of smooth, equidimensional subschemes $W_i \subset \mathbb{P}$, we can find a hyperplane $Y_1 = 0$ in $\mathbb{P}$ which is transverse to each W_i, in the sense that $W_i \cap Y_1$ is smooth of codimension 1 in W_i (:= empty, if $dim(W_i) = 0$). We apply this with the list taken to be Z, X, $X \cap Z$, $X \cap Z \cap F$, $X \cap Z \cap G$, $X \cap Z \cap F \cap G$, and, in the second part of Theorem 3.1, $X \cap G$ itself. This produces the desired Y_1. To define Y_2, we consider this list of W_i's, augmented by adding their intersections, when nonempty, with Y_1. We then continue, at each step keeping the terms on our previous list of smooth subschemes of $\mathbb{P}$ and adding on their intersections, when nonempty, with the previously obtained hyperplane. In this way, we get the desired coordinate system $(Z = Y_0, Y_1, \ldots, Y_N)$ in the ambient $\mathbb{P}$, defined over some finite extension of k, which is suitably transverse to our original situation. Thus it suffices to treat the case where our original coordinate system $(X_0, \ldots, X_N)$ has $Z = X_0$ and is suitably transverse to (X, Z, F, G) as above.

Pick a prime-to-p integer r such that χ^r is trivial (e.g., one might take r to be $\#k - 1$). Consider the "rth power map"

$$[r] : \mathbb{P} \to \mathbb{P}, (X_0, \ldots, X_N) \mapsto (X_0^r, \ldots, X_N^r).$$

It is finite and flat of degree r^N, and finite etale over the dense open set where all Y_i are invertible.

Lemma 4.1. We have the following results.

(1) Suppose we are given a closed subscheme $W \subset \mathbb{P}$ which is smooth and equidimensional, and which is transverse to the coordinate system $(X_0, \ldots, X_N)$, in the sense that for any subset $I \subset \{0, 1, \ldots, N\}$, the intersection $W \bigcap \cap_{i \in I} X_i$ is smooth of dimension $dim W - \#I$ (:=empty if $\#I > dim W$). Then its inverse image W_r in the covering $[r] : \mathbb{P} \to \mathbb{P}$, is smooth.

(2) For any closed subscheme $W \subset \mathbb{P}$, the intersection $W_r \cap Z$ is the inverse image of $W \cap Z \subset \mathbb{P} \cap Z$ under the "rth power map"

$$[r : Z] : \mathbb{P} \cap Z \to \mathbb{P} \cap Z, (X_1, \ldots, X_N) \mapsto (X_1^r, \ldots, X_N^r).$$

(2) For any closed subscheme $W \subset \mathbb{P}$ such that $W \cap Z$ is smooth, $W_r \cap Z$ is smooth. □

Proof. (1) Since k is perfect, it suffices to show that W_r is a regular scheme. Over a $\overline{k}$-valued point w of W where all the X_i are invertible, our covering is finite étale. Over a $\overline{k}$-valued point w of W where precisely the $X_i, i \in I$ vanish, with $\#I \geq 1$, pick some index j with X_j invertible at w, and consider the functions $x_i := X_i/X_j$. By the transversality hypothesis, these x_i are part of a system of parameters at w. Our covering over an open neighborhood of w is an etale covering of degree $r^{N-\#I}$ of the finite flat covering obtained by extracting the rth roots of the x_i. In this finite flat covering, there is a unique point over w, whose local ring is visibly regular. Thus W_r is a regular scheme.

(2) This is a tautology.

(3) By (2), this results from (1) applied to $W \cap Z \subset \mathbb{P} \cap Z$ and the map

$$[r : Z] : \mathbb{P} \cap Z \to \mathbb{P} \cap Z, (X_1, \ldots, X_N) \mapsto (X_1^r, \ldots, X_N^r). \qquad \blacksquare$$

We now consider the pullback of our situation (X, Z, F, G) by the map $[r] : \mathbb{P} \to \mathbb{P}$. We obtain $(X_r, Z_r = Z^r, F_r, G_r)$. Here $F_r(X_i) := F(X_i^r)$, $G_r(X_i) := G(X_i^r)$. We have $Z_r = Z^r$ because by construction we have $Z = X_0$. We put $V_r := X_r[1/Z_r] = X_r[1/Z]$, $f_r := F_r/Z^{rd}$, and $g_r := G_r/Z^{re}$. We have a finite flat map

$$[r]_{V[1/g]} : V_r[1/g_r] \to V[1/g]$$

of degree r^N. By the projection formula, for each i the cohomology group

$$H_c^i(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

is a direct summand of the cohomology group

$$H_c^i(V_r[1/g_r] \otimes_k \overline{k}, \mathcal{L}_{\psi(f_r)} \otimes \mathcal{L}_{\chi(g_r)}).$$

[We remark for later use that this same projection formula argument shows that for each i the ordinary cohomology group

$$H^i(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

is a direct summand of the ordinary cohomology group

$$H^i(V_r[1/g_r] \otimes_k \overline{k}, \mathcal{L}_{\psi(f_r)} \otimes \mathcal{L}_{\chi(g_r)}).]$$

We claim that the data (X_r, Z, F_r, G_r) satisfies all the transversality conditions of section 2, but now with degrees (d, e) replaced by degrees (dr, er). First of all, X_r is geometrically connected, because at any of the finitely many points where exactly n of the X_i intersect X, the covering $[r]_X : X_r \to X$ is fully ramified. But if X_r were not geometrically connected, each of its connected components would map onto X.

The transversality hypotheses of section 2 are that $X \cap Z$, $X \cap Z \cap F$, $X \cap Z \cap G$, and $X \cap Z \cap F \cap G$, are all smooth of the correct dimension (:=empty, if that dimension is negative). Their inverse images under $[r] : \mathbb{P} \to \mathbb{P}$ are the schemes $X_r \cap Z$, $X_r \cap Z \cap F_r$,

$X_r \cap Z \cap G_r$, and $X_r \cap Z \cap F_r \cap G_r$. That these inverse images (and also $X_r \cap G_r$, if $X \cap G$ is assumed smooth) are all smooth of the correct dimension (:=empty, if that dimension is negative) results from Lemma 4.1.

But in this situation, χ^{er} is trivial, so the cohomology groups

$$H^i_c(V_r[1/g_r] \otimes_k \overline{k}, \mathcal{L}_{\psi(f_r)} \otimes \mathcal{L}_{\chi(g_r)})$$

vanish for $i \neq n$. And hence their direct summands

$$H^i_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

vanish for $i \neq n$. This concludes the proof of part (1) of Theorem 3.1.

5 Proof of part (2) of Theorem 3.1

Let us recall the situation. We start with (X, Z, F, G), but now we assume that $X \cap Z$, $X \cap Z \cap F$, $X \cap Z \cap G$, $X \cap Z \cap F \cap G$, and *in addition* $X \cap G$, are all smooth of the correct dimension (:=empty, if that dimension is negative). We first show that

$$H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

is pure of weight n.

We first explain how to reduce to the case when χ^e is trivial. Exactly as in the previous section, we pick a prime-to-p integer r so that χ^r is trivial, extend scalars so that $(Z = X_0, X_1, \ldots, X_N)$ is a suitably transverse coordinate system, and pass to the situation (X_r, Z, F_r, G_r), for which all of these smoothness assumptions still hold. Our cohomology group

$$H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

is a direct factor of

$$H^n_c(V_r[1/g_r] \otimes_k \overline{k}, \mathcal{L}_{\psi(f_r)} \otimes \mathcal{L}_{\chi(g_r)}).$$

So we are reduced to proving that $H^n_c(V_r[1/g_r] \otimes_k \overline{k}, \mathcal{L}_{\psi(f_r)} \otimes \mathcal{L}_{\chi(g_r)})$ is pure of weight n.

So it suffices to considering the situation (X, Z, F, G) of the paragraph above, but under the additional hypothesis that χ^e is trivial. In this case, we return to the considerations of the first part of section 4, where we introduced the covering V_e defined by taking the eth root of g, and saw that our cohomology group

$$H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$$

was a direct factor of

$$H^n_c(V_e \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}).$$

In this situation, the key observation is that X_e is in fact smooth. Indeed, it is the covering of X defined by extracting the eth root of G. But e is prime to p, X is smooth, and $X \cap G$ is smooth, so it follows that X_e is regular, and hence smooth. It is geometrically connected, because it is fully ramified over X at any point of $X \cap G$. We have already seen in the first part of section 4 that $X_e \cap Z$ and $X_e \cap Z \cap F$ are both smooth of the correct dimension. So the purity of

$$H^n_c(V_e \otimes_k \overline{k}, \mathcal{L}_{\psi(f)})$$

now results from [9, 5.1.1(2)].

To conclude the proof of part (2a) of Theorem 3.1, it remains to compute the dimension of

$$H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}).$$

Since this is the only nonvanishing cohomology group, its dimension is equal to $(-1)^n \times$ the Euler characteristic

$$\chi_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}).$$

By standard arguments of reducing the L function mod various primes λ of $\mathbb{Z}[\zeta_p, \zeta_{\#k-1}]$ of residue characteristic $\neq p$ which divide the order of χ and considering the degree of the resulting mod λ L-function, we see that this Euler characteristic is independent of the particular choice of χ, and is the same with χ replaced by the trivial character:

$$\chi_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) = \chi_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}).$$

On the other hand, we have

$$\chi_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}) = \chi_c(V \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}) - \chi_c(V \cap (g=0) \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}).$$

Now $\chi_c(V \otimes_k \overline{k}, \mathcal{L}_{\psi(f)})$ is the additive character euler characteristic attached to the situation (X, L, F), with $f = F/Z_d$ on $V = X[1/L]$. Similarly, $\chi_c(V \cap (g = 0) \otimes_k \overline{k}, \mathcal{L}_{\psi(f)})$ is the additive character euler characteristic attached to the situation $(X \cap G, L, F)$, with $f = F/Z_d$ on $V \cap (g = 0) = (X \cap G)[1/L]$. So from [9, 5.1.1 and Remarque on page 166], we have the formulas

$$\begin{aligned}\chi_c(V \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}) &= \int_X \frac{c(X)}{(1+L)(1+dL)},\\ \chi_c(V \cap (g=0) \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}) &= \int_{X \cap G} \frac{c(X \cap G)}{(1+L)(1+dL)}\\ &= \int_X \frac{eLc(X)}{(1+L)(1+dL)(1+eL)}.\end{aligned}$$

Subtracting, we find

$$\chi_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}) = \int_X \frac{c(X)}{(1+L)(1+dL)(1+eL)} := (-1)^n C(X, d, e),$$

as required.

It remains to prove part (2b) of Theorem 3.1, that the “forget supports” map is an isomorphism

$$H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) \cong H^n(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}).$$

The right hand group $H^n(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)})$ is, up to a Tate twist, the Poincaré dual of $H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\overline{\psi}(f)} \otimes \mathcal{L}_{\overline{\chi}(g)})$, so (by part (2a) of Theorem 3.1, applied with $\overline{\psi}$ and $\overline{\chi}$) it has the same dimension, $(-1)^n C(X, d, e)$, as the left hand group. Therefore it suffices to show that the “forget supports” map is injective:

$$H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) \hookrightarrow H^n(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}).$$

For this we first reduce to the case when χ^e is trivial, by passing to the covering V_r and looking at the commutative diagram

$$\begin{array}{ccc} H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) & \xrightarrow{\textit{forget}} & H^n(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) \\ \cap & & \cap \\ H^n_c(V_r[1/g_r] \otimes_k \overline{k}, \mathcal{L}_{\psi(f_r)} \otimes \mathcal{L}_{\chi(g_r)}) & \xrightarrow{\textit{forget}} & H^n(V_r[1/g_r] \otimes_k \overline{k}, \mathcal{L}_{\psi(f_r)} \otimes \mathcal{L}_{\chi(g_r)}). \end{array}$$

So it suffices to treat the case when χ^e is trivial. In this case, we pass to the covering V_e, and look at the commutative diagram

$$\begin{array}{ccc} H^n_c(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) & \xrightarrow{\text{forget}} & H^n(V[1/g] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \mathcal{L}_{\chi(g)}) \\ \cap & & \cap \\ H^n_c(V_e \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}) & \xrightarrow{\text{forget}} & H^n(V_e \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}). \end{array}$$

This bottommost "forget supports" map is in fact bijective, by [9, 5.1.1, part (0)].

6 Appendix: the case of $r \geq 1$ g's

We begin by stating the generalization of Theorem 1.2 analogous to Theorem 2.1. As in that theorem, X/k is a projective, smooth, and geometrically connected k-scheme of dimension $n \geq 1$, given with a projective embedding $X \hookrightarrow \mathbb{P}^N_k := \mathbb{P}$. We fix integers $d \geq 1$ and $e_1, \ldots, e_r \geq 1$, all prime to p. We are given a linear form

$$Z \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)),$$

a degree d form

$$F \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)),$$

and, for $i = 1, \ldots, r$, a degree e_i form

$$G_i \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(e_i)),$$

all on the ambient projective space $\mathbb{P}$. We assume that the following transversality hypotheses hold.

(1) $X \cap Z$ is lisse of codimension 1 in X.

(2) $X \cap Z \cap F$ is lisse of codimension 1 in $X \cap Z$ (:= empty, if $n = 1$).

(3) For any nonempty subset $I \subset \{1, \ldots, r\}$, $X \cap Z \cap \cap_{i \in I} G_i$ is lisse of codimension $\#I$ in $X \cap Z$ (:= empty, if $\#I \geq n$).

(4) For any nonempty subset $I \subset \{1, \ldots, r\}$, $X \cap Z \cap F \cap \cap_{i \in I} G_i$ is lisse of codimension $1 + \#I$ in $X \cap Z$ (:= empty, if $1 + \#I \geq n$).

To this data, we attach the smooth affine k-scheme

$$V := X - X \cap Z = X[1/Z],$$

and the functions

$$f := F/Z^d : V \to \mathbb{A}^1_k$$

and

$$g_i := G_i/Z^{e_i} : V \to \mathbb{A}^1_k.$$

We denote by $c(X)$ the total Chern class of X, and by L the class of $\mathcal{O}_X(1)$. We define the constant $C(X, d, e_1, \dots, e_r)$ by

$$C(X, d, e_1, \dots, e_r) := (-1)^n \int_X \frac{c(X)}{(1+L)(1+dL)\prod_i(1+e_iL)}.$$

Thus when X is $\mathbb{P}^n$ with the identity embedding of itself into $\mathbb{P} = \mathbb{P}^n$, $C(X, d, e_1, \dots, e_r)$ is the constant $C(n, d, e_1, \dots, e_r)$ of Theorem 1.2.

We have the following generalization of Theorem 2.1.

Theorem 6.1. Suppose that $(X, Z, F, G_1, \dots, G_r)$ are as above. Then we have the following results.

(1) We have the estimate

$$\left|\sum_{x\in V(k)} \psi(f(x)) \prod_i \chi_i(g_i(x))\right| \leq C(X, d, e_1, \dots, e_r)(\#k)^{n/2}.$$

The associated L function is a polynomial $P(T)$ (for n odd) or a reciprocal polynomial $1/P(T)$(for n even) of degree $\leq$ $C(X, d, e_1, \dots, e_r)$, which is mixed of weight $\leq n$.

(2) If $P(T)$ has degree $= C(X, d, e_1, \dots, e_r)$, then $P(T)$ is pure of weight n.

(3) If, for any nonempty subset $I \subset \{1, \dots, r\}$, $X \cap \cap_{i\in I} G_i$ is lisse of codimension $\#I$ in X (:= empty, if $\#I > n$), then $P(T)$ has degree $= C(X, d, e_1, \dots, e_r)$, and is pure of weight n. □

Exactly as in Section 3, Theorem 6.1 follows from the following generalization of Theorem 3.1.

Theorem 6.2. Suppose that $(X, Z, F, G_1, \dots, G_r)$ are as in Theorem 6.1. Then we have the following results.

(1) $H^i_c := H^i_c(V[1/\prod_i g_i] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes (\otimes_i \mathcal{L}_{\chi_i(g_i)}))$ vanishes for $i \neq n$.

(2) If, for any nonempty subset $I \subset \{1, \dots, r\}$, $X \cap \cap_{i \in I} G_i$ is lisse of codimension $\#I$ in X (:= empty, if $\#I > n$), then we have the following results.

(2a) H^n_c has dimension $C(X, d, e_1, \dots, e_r)$, and is pure of weight n.

(2b) The "forget supports" map is an isomorphism

$$H^n_c(V[1/\prod_i g_i] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes (\otimes_i \mathcal{L}_{\chi_i(g_i)}))$$
$$\cong H^n(V[1/\prod_i g_i] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes (\otimes_i \mathcal{L}_{\chi_i(g_i)})).$$

□

To prove the first part of Theorem 6.2, it suffices, exactly as in section 4, to prove the vanishing of H^i_c for $i > n$. We first reduce to the case when all $\chi_i^{e_i}$ are trivial. Extending scalars, we can find a coordinate system $(Z = Y_0, Y_1, \dots, Y_N)$ in the ambient $\mathbb{P}$ which is transversal to X, to each $X \cap \cap_{i \in I} G_i$ which is smooth, to $X \cap Z$, $X \cap Z \cap F$, to every nonempty $X \cap Z \cap \cap_{i \in I} G_i$, and to every nonempty $X \cap Z \cap F \cap \cap_{i \in I} G_i$. So it suffices to treat the case when the original coordinate system $(Z = X_0, X_1, \dots, X_N)$ has all these transversality properties. Then with $q := \#k$, we consider the "$q-1$'th power mapping"

$$[q-1] : \mathbb{P} \to \mathbb{P}, (X_0, \dots, X_N) \mapsto (X_0^{q-1}, \dots, X_N^{q-1}).$$

It is finite and flat of degree $(q-1)^N$, and finite etale over the dense open set where all X_i are invertible. Exactly as in section 4, it suffices to treat the pullback situation $(X_{q-1}, Z, F_{q-1}, G_{1,q-1}, \dots, G_{r,q-1})$ by this map. This completes the reduction to the case when all $\chi_i^{e_i}$ are trivial.

When all the $\chi_i^{e_i}$ are trivial, we pass to the covering $X_{e_1,\dots,e_r}$ of X defined by extracting, for each $i = 1, \dots, r$, the e_ith root of G_i. On this covering, we have the pullbacks Z and F of their namesakes on X. Exactly as in section 4, the "nonsingular" case ($\epsilon = \delta = -1$) of [7, Theorem 4], applied now to the data $(X_{e_1,\dots,e_r}, Z, F)$, gives the vanishing of H^i_c for $i > n$.

To prove the second part of Theorem 6.2, we observe that under the additional transversality hypotheses, the covering $X_{e_1,\dots,e_r}$ of the previous paragraph is itself smooth, so the purity of H^n_c again results from [9, 5.1.1(2)]. Exactly as in section 5, the dimension of H^n_c is $(-1)^n \times$ the Euler characteristic

$$\chi_c(V[1/\prod_i g_i] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)} \otimes \otimes_i(\mathcal{L}_{\chi_i(g_i)})) = \chi_c(V[1/\prod_i g_i] \otimes_k \overline{k}, \mathcal{L}_{\psi(f)}).$$

The asserted formula for this Euler formula now follows by inclusion-exclusion from the formulas of [9, 5.1.1 and Remarque on page 166]. This proves part (2a). The proof of part (2b) is entirely analogous to the proof of part (2b) of Theorem 3.1 given in Section 5.

References

[1] Altman, A., and S. Kleiman. "Introduction to Grothendieck Duality Theory." *Lecture Notes in Mathematics*, Vol. 146. Berlin: Springer, 1970.

[2] Deligne, P. "Applications de la Formule des Traces aux Sommes Trigonométriques." Pp. 168–232 in SGA 4 1/2, cited below.

[3] Deligne, P. "La Conjecture de Weil." *Publications Mathématiques de l'IHÉS* 43 (1974): 273–307.

[4] Deligne, P. "La Conjecture de Weil II." *Publications Mathématiques de l'IHÉS* 52 (1981): 313–428.

[5] Grothendieck, A. "Formule de Lefschetz et Rationalité des Fonctions L, Séminaire Bourbaki." *Societe Mathematique De France* 9, no. 279 (1995): 41–55.

[6] Katz, N. "Estimates for Nonsingular Multiplicative Character Sums." *International Mathematics Research Notices* 7 (2002): 333–349.

[7] Katz, N. "Estimates for "Singular" Exponential Sums." *International Mathematics Research Notices* 16 (1999): 875–899.

[8] Katz, N. "A Semicontinuity Result for Monodromy Under Degeneration." *Forum Mathematicum* 15, no. 2 (2003): 191–200.

[9] Katz, N. *Sommes Exponentielles*, Rédigé par G. Laumon Astérisque 79 (1980): 1–207.

[10] Cohomologie Etale. Séminaire de Géométrie Algébrique du Bois Marie SGA 4 1/2. par P. Deligne, avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie, et J. L. Verdier. *Lecture Notes in Mathematics*, Vol. 569. Berlin: Springer-Verlag, 1977.

[11] Théorie des Topos et Cohomologie Etale des Schémas, Tome 3. Séminaire de Géométrie Algébrique du Bois Marie 1963–1964 (SGA 4). Dirigé par M. Artin, A. Grothendieck, J. L. Verdier. *Lecture Notes in Mathematics*, Vol. 305. Berlin: Springer-Verlag, 1973.