APPENDIX: E-POLYNOMIALS, ZETA-EQUIVALENCE, AND POLYNOMIAL-COUNT VARIETIES

NICHOLAS M. KATZ

Given a noetherian ring R, we denote by (Sch/R) the category of separated R-schemes of finite type, morphisms being the R-morphisms. We denote by $K_0(Sch/R)$ its Grothendieck group. By definition, $K_0(Sch/R)$ is the quotient of the free abelian group on elements $[X]$, one for each separated R-scheme of finite type, by the subgroup generated by all the relation elements

$$[X] - [Y], \text{ whenever } X^{red} \cong Y^{red},$$

and

$$[X] - [X \setminus Z] - [Z], \text{ whenever } Z \subset X \text{ is a closed subscheme.}$$

It follows easily that if X is a finite union of locally closed subschemes Z_i, then in $K_0(Sch/R)$ we have the inclusion-exclusion relation

$$[X] = \sum_i [Z_i] - \sum_{i<j} [Z_i \cap Z_j] + \dots.$$

For any ring homomorphism $R \longrightarrow R'$ of noetherian rings, the "extension of scalars" morphism from (Sch/R) to (Sch/R') which sends X/R to $X \otimes_R R'/R'$, extends to a group homomorphism from $K_0(Sch/R)$ to $K_0(Sch/R')$.

Suppose A is an abelian group, and ρ is an "additive function" from (Sch/R) to A, i.e., a rule which assigns to each $X \in (Sch/R)$ an element $\rho(X) \in A$, such that $\rho(X)$ depends only on the isomorphism class of X^{red}, and such that whenever $Z \subset X$ is a closed subscheme, we have

$$\rho(X) = \rho(X - Z) + \rho(Z).$$

Then ρ extends uniquely to a group homomorphism from $K_0(Sch/R)$ to A, by defining $\rho(\sum_i [X_i]) = \sum_i \rho(X_i)$.

When R=$\mathbb{C}$, we have the following simple lemma, which we record now for later use.

Lemma 1. *Every element of $K_0(Sch/\mathbb{C})$ is of the form $[S] - [T]$, with S and T both projective smooth (but not necessarily connected) $\mathbb{C}$-schemes.*

Proof. To show this, we argue as follows. It is enough to show that for any separated $\mathbb{C}$-scheme of finite type X, $[X]$ is of this type. For then $-[X] = [T] - [S]$, and

$$[S_1] - [T_1] + [S_2] - [T_2] = [S_1 \sqcup S_2] - [T_1 \sqcup T_2],$$

and the disjoint union of two projective smooth schemes is again one. [Indeed, if we embed each in a large projective space, say $S_i \subset \mathbb{P}^{N_i}$ and pick a point $a_i \in \mathbb{P}^{N_i} \setminus S_i$, then $S_1 \times a_2$ and $a_1 \times S_2$ are disjoint in $\mathbb{P}^{N_1} \times \mathbb{P}^{N_2}$.]

We first remark that for any X as above, $[X]$ is of the form $[V] - [W]$ with V and W affine. This follows from inclusion-exclusion by taking a finite covering of X by affine open sets, and noting that the disjoint union of two affine schemes of finite type is again an affine scheme of finite type. So it suffices to prove our claim for affine X. Embedding X as a closed subscheme of some affine space $\mathbb{A}^N$ and using the relation

$$[X] = [\mathbb{A}^N] - [\mathbb{A}^N \setminus X],$$

it now suffices to prove our claim for smooth quasiaffine X. By resolution, we can find a projective smooth compactification Z of X, such that $Z \setminus X$ is a union of smooth divisors D_i in Z with normal crossings. Then by inclusion-exclusion we have

$$[X] = [Z] - \sum_i [D_i] + \sum_{i,j} [D_i \cap D_j] + \dots .$$

In this expression, each summand on the right hand side is projective and smooth. Taking for S the disjoint union of the summands with a plus sign and for T the disjoint union of the summands with a minus sign, we get the desired expression of our $[X]$ as $[S] - [T]$, with S and T both projective and smooth. □

Now take for R a finite field $\mathbb{F}_q$. For each integer $n \geq 1$, the function on $(Sch/\mathbb{F}_q)$ given by $X \mapsto \#X(\mathbb{F}_{q^n})$ is visibly an additive function from $(Sch/\mathbb{F}_q)$ to $\mathbb{Z}$. Its extension to $K_0(Sch/\mathbb{F}_q)$ will be denoted

$$\gamma \mapsto \#\gamma(\mathbb{F}_{q^n}).$$

We can also put these all functions together, to form the zeta function. Recall that the zeta function $Z(X/\mathbb{F}_q, t)$ of $X/\mathbb{F}_q$ is the power series (in fact it is a rational function) defined by $Z(X/\mathbb{F}_q, t) = exp(\sum_{n \geq 1} \#X(\mathbb{F}_{q^n}) t^n/n)$. Then $X \mapsto Z(X/\mathbb{F}_q, t)$ is an additive function with values in the multiplicative group $\mathbb{Q}(t)^\times$. We denote by

$$\gamma \mapsto Zeta(\gamma/\mathbb{F}_q, t)$$

its extension to $K_0(Sch/\mathbb{F}_q)$. We say that an element $\gamma \in K_0(Sch/\mathbb{F}_q)$ is *zeta-trivial* if $Zeta(\gamma/\mathbb{F}_q, t) = 1$, i.e., if $\#\gamma(\mathbb{F}_{q^n}) = 0$ for all $n \geq 1$. We say that two elements of $K_0(Sch/\mathbb{F}_q)$ are *zeta-equivalent* if they have the same zeta functions, i.e., if their difference is zeta-trivial.

We say that an element $\gamma \in K_0(Sch/\mathbb{F}_q)$ is *polynomial-count* if there exists a (necessarily unique) polynomial $P_{\gamma/\mathbb{F}_q}(t) = \sum_i a_i t^i \in \mathbb{C}[t]$ such that for every finite extension $\mathbb{F}_{q^n}/\mathbb{F}_q$, we have

$$\#\gamma(\mathbb{F}_{q^n}) = P_{\gamma/\mathbb{F}_q}(q^n).$$

If $\gamma/\mathbb{F}_q$ is polynomial-count, its counting polynomial $P_{\gamma/\mathbb{F}_q}(t)$ lies in $\mathbb{Z}[t]$. [To see this, we argue as follows. On the one hand, from the series definition of the zeta function, and the polynomial formula for the number of rational points, we have

$$(td/dt)log(Z(\gamma/\mathbb{F}_q, t)) = \sum_i a_i q^i t/(1 - q^i t).$$

As the zeta function is a rational function, say $\prod_i (1 - \alpha_i t)/\prod_j (1 - \beta_j t)$ in lowest terms, we first see by comparing logarithmic derivatives that each of its zeroes and poles is a non-negative power of $1/q$. Thus for some integers b_n, the zeta function is of the form $\prod_{n \geq 0} (1 - q^n t)^{-b_n}$. Again comparing logarithmic derivatives, we see that we have $a_n = b_n$ for each n.]

Equivalently, an element $\gamma \in K_0(Sch/\mathbb{F}_q)$ is polynomial-count if it is zeta-equivalent to a $\mathbb{Z}$-linear combination of classes of affine spaces $[\mathbb{A}^i]$, or, equivalently, to a $\mathbb{Z}$-linear combination of classes of projective spaces $[\mathbb{P}^i]$ (since $[\mathbb{A}^i] = [\mathbb{P}^i] - [\mathbb{P}^{i-1}]$, with the convention that $\mathbb{P}^{-1}$ is the empty scheme). If $\gamma/\mathbb{F}_q$ is polynomial-count, then so is its extension of scalars from $\mathbb{F}_q$ to any finite extension field, with the *same* counting polynomial. [But an element $\gamma/\mathbb{F}_q$ which is not polynomial-count can become polynomial-count after extension of scalars, e.g., a nonsplit torus over $\mathbb{F}_q$, or, even more simply, the zero locus of a square-free polynomial $f(z) \in \mathbb{F}_q[z]$ which does not factor completely over $\mathbb{F}_q$.]

Now let R be a ring which is finitely generated as a $\mathbb{Z}$-algebra. We say that an element $\gamma \in K_0(Sch/R)$ is zeta-trivial if, for every finite field k, and for every ring homomorphism $\phi : R \longrightarrow k$, the element $\gamma_{\phi,k}/k$ in $K_0(Sch/k)$ deduced from γ by extension of scalars is zeta-trivial. And we say that two elements are *zeta-equivalent* if their difference is zeta-trivial.

We say that an element $\gamma \in K_0(Sch/R)$ is *strongly polynomial-count* with (necessarily unique) counting polynomial $P_{\gamma/R}(t) \in \mathbb{Z}[t]$ if, for every finite field k, and for every ring homomorphism $\phi : R \longrightarrow k$, the element $\gamma_{\phi,k}/k$ in $K_0(Sch/k)$ deduced from γ by extension of scalars is polynomial-count with counting polynomial $P_{\gamma/R}(t)$.

We say that an element $\gamma \in K_0(Sch/R)$ is *fibrewise polynomial-count* if, for every ring homomorphism $\phi : R \longrightarrow k$, the element $\gamma_{\phi,k}/k$ in $K_0(Sch/k)$ deduced from γ by extension of scalars is polynomial-count (but we allow its counting polynomial to vary with the choice of (k, ϕ)).

All of these notions, zeta-triviality, zeta equivalence, being strongly or fibrewise polynomial-count, are stable by extension of scalars of finitely generated rings.

We now pass to the complex numbers $\mathbb{C}$. Given an element $\gamma \in K_0(Sch/\mathbb{C})$, by a "spreading out" of $\gamma/\mathbb{C}$, we mean an element $\gamma_R \in K_0(Sch/R)$, R a subring of $\mathbb{C}$ which is finitely generated as a $\mathbb{Z}$-algebra, which gives back $\gamma/\mathbb{C}$ after extension of scalars from R to $\mathbb{C}$. It is standard that such spreadings out exist, and that given two spreadings out $\gamma_R \in K_0(Sch/R)$ and $\gamma_{R'} \in K_0(Sch/R')$, then over some larger finitely generated ring R'' containing both R and R', the two spreadings out will agree in $K_0(Sch/R'')$.

We say that an element $\gamma \in K_0(Sch/\mathbb{C})$ is zeta-trivial if it admits a spreading out $\gamma_R \in K_0(Sch/R)$ which is zeta-trivial. One sees easily, by taking spreadings out to a common R, that the zeta-trivial elements form a subgroup of $K_0(Sch/\mathbb{C})$.

We say that two elements are zeta-equivalent if their difference is zeta-trivial. We say that an element is strongly polynomial-count, with counting polynomial $P_\gamma(t) \in \mathbb{Z}[t]$, (respectively fibrewise polynomial-count) if it admits a spreading out which has this property.

Given $X/\mathbb{C}$ a separated scheme of finite type, its *E-polynomial* $E(X; x, y) \in \mathbb{Z}[x, y]$ is defined as follows. The compact cohomology groups ${H_c}^i(X^{an}, \mathbb{Q})$ carry Deligne's mixed Hodge structure, cf. [De-Hodge II] and [De-Hodge III, 8.3.8], and one defines

$$E(X; x, y) = \sum_{p,q} e_{p,q} x^p y^q,$$

where the coefficients $e_{p,q}$ are the virtual Hodge numbers, defined in terms of the pure Hodge structures which are the associated gradeds for the weight filtration on the compact cohomology as follows:

$$e_{p,q} := \sum_i (-1)^i h^{p,q}(gr_W^{p+q}({H_c}^i(X^{an}, \mathbb{C}))).$$

Notice that the value of $E(X; x, y)$ at the point $(1, 1)$ is just the (compact, or ordinary, they are equal, by [Lau]) Euler characteristic of X. One knows that the formation of the E-polynomial is additive (because the excision long exact sequence is an exact sequence in the abelian category of mixed Hodge structures, cf. [De-Hodge III, 8.3.9]). So we can speak of the E-polynomial $E(\gamma; x, y)$ attached to an element $\gamma \in K_0(Sch/\mathbb{C})$.

Theorem 1. *We have the following results.*

(1) *If $\gamma \in K_0(Sch/\mathbb{C})$ is zeta-trivial, then*
$$E(\gamma; x, y) = 0.$$
(2) *If $\gamma_1 \in K_0(Sch/\mathbb{C})$ and $\gamma_2 \in K_0(Sch/\mathbb{C})$ are zeta-equivalent, then*
$$E(\gamma_1; x, y) = E(\gamma_2; x, y).$$
In particular, if X and Y in $(Sch/\mathbb{C})$ are zeta-equivalent, then
$$E(X; x, y) = E(Y; x, y).$$
(3) *If $\gamma \in K_0(Sch/\mathbb{C})$ is strongly polynomial-count, with counting polynomial $P_\gamma(t) \in \mathbb{Z}[t]$, then*
$$E(\gamma; x, y) = P_\gamma(xy).$$
In particular, if $X \in (Sch/\mathbb{C})$ is strongly polynomial-count, with counting polynomial $P_X(t) \in \mathbb{Z}[t]$, then
$$E(X; x, y) = P_X(xy).$$

Proof. Assertion (2) is an immediate consequence of (1), by the additivity of the E-polynomial. Statement (3) results from (2) as follows. If $\gamma \in K_0(Sch/\mathbb{C})$ is strongly polynomial-count, with counting polynomial $P_\gamma(t) = \sum_i a_i t^i \in \mathbb{Z}[t]$, then by definition γ is zeta-equivalent to $\sum_i a_i[\mathbb{A}^i] \in K_0(Sch/\mathbb{C})$. So we are reduced to noting that $E(\mathbb{A}^i; x, y) = x^i y^i$, which one sees by writing $[\mathbb{A}^i] = [\mathbb{P}^i] - [\mathbb{P}^{i-1}]$ and using the basic standard fact that $E(\mathbb{P}^i; x, y) = \sum_{0 \le j \le i} x^j y^j$. So it remains only to prove assertion (1) of the theorem. By lemma 1, every element $\gamma \in K_0(Sch/\mathbb{C})$ is of the form $[X] - [Y]$, with X and Y are projective smooth $\mathbb{C}$-schemes. So assertion (1) results from the following theorem, which is proven, but not quite stated, in [Wang]. [What Wang proves is that "K-equivalent" projective smooth connected $\mathbb{C}$-schemes have the same Hodge numbers, through the intermediary of using motivic integration to show that K-equivalent projective smooth connected $\mathbb{C}$-schemes are zeta-equivalent.] □

Theorem 2. *Suppose X and Y are projective smooth $\mathbb{C}$-schemes which are zeta-equivalent. Then*
$$E(X; x, y) = E(Y; x, y).$$

Proof. Pick spreadings out $\mathcal{X}/R$ and $\mathcal{Y}/R$ over a common R which are zeta-equivalent. At the expense of inverting some nonzero element in R, we may further assume that both $\mathcal{X}/R$ and $\mathcal{Y}/R$ are projective and smooth, and that R is smooth over $\mathbb{Z}$. We denote the structural morphisms of $\mathcal{X}/R$ and $\mathcal{Y}/R$ by
$$f : \mathcal{X} \longrightarrow Spec(R), g : \mathcal{Y} \longrightarrow Spec(R).$$

One knows [Ka-RLS, 5.9.3] that, for any finitely generated subring $R \subset \mathbb{C}$, there exists an integer $N \ge 1$ such that for all primes ℓ which are prime to N, there exists a finite extension $E/\mathbb{Q}_\ell$, with ring of integers $\mathcal{O}$ and an injective ring homomorphism from R to $\mathcal{O}$. Fix one such prime number ℓ, which we choose larger than both $dim(X)$ and $dim(Y)$, and one such inclusion of R into $\mathcal{O}$.

Over $Spec(R[1/\ell])$, the $\mathbb{Q}_\ell$-sheaves $R^i f_\star \mathbb{Q}_\ell$ and $R^i g_\star \mathbb{Q}_\ell$ are lisse, and pure of weight i [De-Weil II, 3.3.9]. By the Lefschetz Trace Formula and proper base change, for each finite field k, and for each k-valued point ϕ of $Spec(R[1/\ell])$, we have

$$Zeta(\mathcal{X}_{k,\phi}/k,t) = \prod_i det(1 - tFrob_{k,\phi}|R^i f_\star \mathbb{Q}_\ell)^{(-1)^{i+1}}$$

and

$$Zeta(\mathcal{Y}_{k,\phi}/k,t) = \prod_i det(1 - tFrob_{k,\phi}|R^i g_\star \mathbb{Q}_\ell)^{(-1)^{i+1}}$$

By the assumed zeta-equivalence, we have, for each finite field k, and for each k-valued point ϕ of $Spec(R[1/\ell])$, the equality of rational functions

$$Zeta(\mathcal{X}_{k,\phi}/k,t) = Zeta(\mathcal{Y}_{k,\phi}/k,t).$$

Separating the reciprocal zeroes and poles by absolute value, we infer by purity that for every i, we have

$$det(1 - tFrob_{k,\phi}|R^i f_\star \mathbb{Q}_\ell) = det(1 - tFrob_{k,\phi}|R^i g_\star \mathbb{Q}_\ell).$$

Therefore by Chebotarev the virtual semisimple representations of $\pi_1(Spec(R[1/\ell]))$ given by $(R^i f_\star \mathbb{Q}_\ell)^{ss}$ and $(R^i g_\star \mathbb{Q}_\ell)^{ss}$ are equal:

$$(R^i f_\star \mathbb{Q}_\ell)^{ss} \cong (R^i g_\star \mathbb{Q}_\ell)^{ss}.$$

Now make use of the inclusion of R into $\mathcal{O}$, which maps $R[1/\ell]$ to E. The pullbacks $X_\mathcal{O}$ and $\mathcal{Y}_\mathcal{O}$ of $\mathcal{X}/R$ and $\mathcal{Y}/R$ to $\mathcal{O}$ are proper and smooth over $\mathcal{O}$. Thus their generic fibres, X_E and $\mathcal{Y}_E$ are projective and smooth over E, of dimension strictly less than ℓ, and they have *good reduction*. Via the chosen map from $Spec(E)$ to $Spec(R[1/\ell])$, we may pull back the representations $R^i f_\star \mathbb{Q}_\ell$ and $R^i g_\star \mathbb{Q}_\ell$ of $\pi_1(Spec(R[1/\ell]))$ to $\pi_1(Spec(E))$, the galois group $Gal_E := Gal(E^{sep}/E)$. Their pullbacks are the etale cohomology groups $H^i(X_{E^{sep}}, \mathbb{Q}_\ell)$ and $H^i(\mathcal{Y}_{E^{sep}}, \mathbb{Q}_\ell)$ respectively, viewed as representations of Gal_E. These representations of Gal_E need not be semisimple, but their semisimplifications are isomorphic:

$$H^i(X_{E^{sep}}, \mathbb{Q}_\ell)^{ss} \cong H^i(\mathcal{Y}_{E^{sep}}, \mathbb{Q}_\ell)^{ss}.$$

By a fundamental result of Fontaine-Messing [F-M, Theorems A and B] (which applies in the case of good reduction, $E/\mathbb{Q}_\ell$ unramified, and dimension less than ℓ) and Faltings [Fal, 4.1] (which treats the general case, of a projective smooth generic fibre), we know that $H^i(X_{E^{sep}}, \mathbb{Q}_\ell)$ and $H^i(\mathcal{Y}_{E^{sep}}, \mathbb{Q}_\ell)$ are Hodge-Tate representations of Gal_E, with Hodge-Tate numbers exactly the Hodge numbers of the complex projective smooth varieties X and Y respectively (i.e., the dimension of the Gal_E-invariants in $H^a(X_{E^{sep}}, \mathbb{Q}_\ell)(b) \otimes \mathbb{C}_\ell$ under the semilinear action of Gal_E is the Hodge number $H^{b,a-b}(X)$, and similarly for Y). By an elementary argument of Wang [Wang, 5.1], the semisimplification of a Hodge-Tate representation is also Hodge-Tate, with the same Hodge-Tate numbers. So the theorem of Fontaine-Messing and Faltings tells us that for all i, $H^i(X)$ and $H^i(Y)$ have the same Hodge numbers. This is precisely the required statement, that $E(X;x,y) = E(Y;x,y)$. □

The reader may wonder why we introduced the notion of being fibrewise polynomial-count, for an element $\gamma \in K_0(Sch/\mathbb{C})$. In fact, this notion is entirely superfluous, as shown by the following Theorem.

Theorem 3. *Suppose $\gamma \in K_0(Sch/\mathbb{C})$ is fibrewise polynomial-count. Then it is strongly polynomial-count.*

Proof. Write γ as $[X]-[Y]$, with X and Y projective smooth $\mathbb{C}$-schemes. Repeat the first paragraph of the proof of the previous theorem. Extending R if necessary, we may assume that the element $[\mathcal{X}/R]-[\mathcal{Y}/R] \in K_0(Sch/R)$ is fibrewise polynomial-count. So for each finite field k and each ring homomorphism $\phi : R \longrightarrow k$, there exists a polynomial $P_{k,\phi} = \sum_n a_{n,k,\phi} t^n \in \mathbb{Z}[t]$ such that

$$Zeta(\mathcal{X}_{k,\phi}/k, t)/Zeta(\mathcal{Y}_{k,\phi}/k, t) = \prod_n (1 - (\#k)^n t)^{-a_{n,k,\phi}}.$$

Writing the cohomological expressions of the zeta functions and using purity, we see that the coefficient $a_{n,k,\phi}$ is just the difference of the $2n$'th ℓ-adic Betti numbers of $\mathcal{X}_{k,\phi} \otimes \overline{k}$ and $\mathcal{Y}_{k,\phi} \otimes \overline{k}$, which is in turn the difference of the ranks of the two lisse sheaves $R^{2n} f_\star \mathbb{Q}_\ell$ and $R^{2n} g_\star \mathbb{Q}_\ell$. This last difference is independent of the particular choice of (k, ϕ). □

References

[De-Hodge II] Deligne, Pierre, Thorie de Hodge II. Publ. Math.IHES 40 (1971), 5-47.

[De-Hodge III] Deligne, Pierre, Thorie de Hodge III. Publ. Math.IHES 44 (1974), 5-77.

[De-Weil II] Deligne, Pierre, La conjecture de Weil II. Publ. Math. IHES 52 (1981), 313-428.

[Fal] Faltings, Gerd, p-adic Hodge theory. J. Amer. Math. Soc. 1 (1988), no. 1, 255–299.

[F-M] Fontaine, Jean-Marc, and Messing, William, p-adic periods and p-adic tale cohomology. Current trends in arithmetical algebraic geometry (Arcata, Calif., 1985), 179–207, Contemp. Math., 67, Amer. Math. Soc., Providence, RI, 1987.

[Ka-RLS] Katz, Nicholas M., Rigid local systems. Annals of Mathematics Studies, 139. Princeton University Press, Princeton, NJ, 1996.

[Lau] Laumon, Grard, Comparaison de caractristiques d'Euler-Poincar en cohomologie l-adique. C. R. Acad. Sci. Paris Sr. I Math. 292 (1981), no. 3, 209-212.

[Wang] Wang, Chin-Lung, Cohomology theory in birational geometry. J. Differential Geom. 60 (2002), no. 2, 345–354