APPENDIX: LEFSCHETZ PENCILS WITH IMPOSED SUBVARIETIES

NICHOLAS M. KATZ

ABSTRACT. In this appendix, which is entirely expository, we give some basic facts about the existence of Lefschetz pencils with imposed subvarieties. These facts are certainly well known to the experts, but we are unaware of a suitable reference. We thank de Caltaldo and Kollar for a very helpful conversation.

1. INTRODUCTION

We work over a field k. We are given a projective, smooth, geometrically connected k-scheme X/k of dimension $n \geq 2$, a projective embedding $X \subset \mathbb{P}$, and a closed subscheme $Z \subset X$ which is smooth, each of whose connected components $Z_i \subset X$ satisfies the inequality

$$dim(Z_i) < codim_X(Z_i) - 1.$$

We will show

Theorem 1.1. *There exists an integer $d_0 = d_0(Z, X, \mathbb{P})$ such that for any degree $d \geq d_0$, and for any extension field E/k with $\#E$ infinite, there exist E-rational Lefschetz pencils of degree d hypersurface sections of X all of which contain Z.*

Thus when k is a finite field, one may have to pass to a finite extension to obtain such a Lefschetz pencil.

2. INCIDENCE VARIETIES AND DUAL VARIETIES

We denote by $\mathcal{O}_X(1)$ the pullback to X of $\mathcal{O}_{\mathbb{P}}(1)$ by the given projective embedding. We denote by Hyp_d the vector space $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))$, and by $PHyp_d$ the projective space of lines in Hyp_d. Thus $PHyp_d$ is the space of degree d hypersurfaces in $\mathbb{P}$.

For a scheme T, and a closed subscheme $W \subset T$, we denote by $I_T(W) \subset \mathcal{O}_T$ the sheaf of ideals defining W. The closed subschemes of T form a monoid with unit the empty subscheme in the obvious way: $W_1 + W_2$ is the closed subscheme whose sheaf of ideals $I_T(W_1 + W_2)$ is

Date: May 13, 2006.

the image in $\mathcal{O}_T$ of $I_T(W_1) \otimes_{\mathcal{O}_T} I_T(W_2)$ under the multiplication map $f \otimes g \mapsto fg$.

We apply these considerations in the following way. For any k-scheme S/k, we denote by X_S and Z_S the base changes to S of X/k and Z/k respectively. We denote by

$$\pi_S : X_S \to S$$

the structural morphism. Given a point $x \in X(S)$, we denote by $[x]$ the corresponding section of X_S/S, viewed as a closed subscheme of X_S. We say that two points x_1, x_2 in $X(S)$ are everywhere disjoint if the schemes $[x_1], [x_2]$ are disjoint in X_S, or equivalently if for all geometric points $\phi : Spec(L) \to S$ of S, the points $x_{1,\phi}, x_{2,\phi}$ in $X(L)$ are distinct.

When Z is nonempty, we will be interested in the ideal sheaves on X_S, for varying S, of the form

$$I_{X_S}(Z_S + a[x_1] + b[x_2] + c[z]),$$

where a, b, c are nonnegative integers, $x_1, x_2 \in (X \setminus Z)(S)$ are everywhere disjoint, and $z \in Z(S)$. When Z is empty, we will be interested in the ideal sheaves

$$I_{X_S}(a[x_1] + b[x_2]),$$

where a, b are nonnegative integers, and $x_1, x_2 \in X(S)$ are everywhere disjoint.

Lemma 2.1. *Fix an integer $D \geq 1$. There exists an integer $d_1 = d_1(Z, X, D)$ with the following properties.*

(1) *Suppose Z is nonempty. For any k-scheme S/k, any three nonnegative integers a, b, c all $\leq D$, any ideal sheaf $I_{X_S}(Z_S+a[x_1]+b[x_2] + c[z])$ as above, and any $d \geq d_1$, we have*

$$R^i\pi_{S\star}(I_{X_S}(Z_S + a[x_1] + b[x_2] + c[z])(d)) = 0$$

for $i \geq 1$, and $R^0\pi_{S\star}(I_{X_S}(Z_S+a[x_1]+b[x_2]+c[z])(d))$ is a locally free $\mathcal{O}_S$ module of finite rank whose formation commutes with arbitrary change of base on S.

(2) *Suppose Z isempty. For any k-scheme S/k, any two nonnegative integers a, b both $\leq D$, any ideal sheaf $I_{X_S}(a[x_1] + b[x_2])$ as above, and any $d \geq d_1$, we have*

$$R^i\pi_{S\star}(I_{X_S}(a[x_1] + b[x_2])(d)) = 0$$

for $i \geq 1$, and $R^0\pi_{S\star}(I_{X_S}(a[x_1] + b[x_2])(d))$ is a locally free $\mathcal{O}_S$ module of finite rank whose formation commutes with arbitrary change of base on S.

Proof. We first prove (1). The ideal sheaf $I_{X_S}(Z_S + a[x_1] + b[x_2] + c[z])$ is flat over S. Indeed, this is tautological for $I_{X_S}(Z_S)$, as it began life over the field k. Then looking locally on S and on X_S, one sees that in the short exact sequence

$$0 \to I_{X_S}(Z_S + a[x_1] + b[x_2] + c[z]) \to I_{X_S}(Z_S) \to Quot \to 0,$$

the term $Quot$ is a sheaf, supported on the disjoint sections $[x_1], [x_2], [z]$, of locally free S modules.

Consider now the universal case, when the base S_{univ} is

$$((X \setminus Z) \times (X \setminus Z) - Diag) \times Z$$

and the three sections are the tautological ones. Then Serre vanishing [Ha, III, 5.2] gives the existence of a d_1 such that we have the asserted vanishings in the universal case for the finitely many ideal sheaves in question. It then follows [Mum-AV, page 53, Cor. 4] that we have the same vanishing after any base change from the universal base S_{univ} to any geometric point of that base. The asserted vanishing then follows over any noetherian base S from [Mum-AV, page 53, Cor. 3] and Nakayama's lemma, and then over any base by first reducing to the affine case, say $S = Spec(A)$, and then writing A as the direct limit of its finitely generated subrings. To get the local freeness of the R^0's for $d \geq d_1$, we start with the case $a = b = c = 0$, in which case the result is obvious, as then the R^0 is the pullback to S of the finite-dimensional k-vector space $H^0(X, I_X(Z)(d))$. Then we use exact sequences of the form

$$0 \to I_{X_S}(Z_S+(a+1)[x_1]+b[x_2]+c[z]) \to I_{X_S}(Z_S+a[x_1]+b[x_2]+c[z]) \to Quot \to 0,$$

in which Quot is a locally free $\mathcal{O}_S$ module on $[x_1] \cong S$, and

$$0 \to I_{X_S}(Z_S+a[x_1]+b[x_2]+(c+1)[z]) \to I_{X_S}(Z_S+a[x_1]+b[x_2]+c[z]) \to Quot \to 0,$$

in which Quot is a locally free $\mathcal{O}_S$ module on $[z] \cong S$. We twist by (d) with $d \geq d_1$, and apply the long exact cohomology sequence of the R^i to get short exact sequences of R^0's, in which the third term $Quot$ is a locally free $\mathcal{O}_S$ module of finite rank, to build up to arbitrary a, b, c in the allowed range. Once all the R^0 are locally free, then we have, for each $d \geq d_1$, a situation of an S-flat coherent sheaf on a proper X_S/S for which all the R^i are locally free, in which case base change is automatic (e.g., from [Mum-AV, page 46, Theorem] and universal coefficients).

The proof of (2) is entirely analogous, with the universal case now taking place over $S_{univ} = X \times X - Diag$. □

Lemma 2.2. *There exists an integer $d_2 = d_2(X, \mathbb{P}) \geq 1$ such that for any $d \geq d_2$, the restriction map*

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \to H^0(X, \mathcal{O}_X(d))$$

is surjective.

Proof. This is Serre vanishing on $\mathbb{P}$ for the ideal sheaf $I_{\mathbb{P}}(X)$. □

We now define the integer $d_0 = d_0(Z, X, \mathbb{P})$ by

$$d_0 := Max(d_1(Z, X, 3), d_1(\emptyset, X, 3), d_2(X, \mathbb{P})).$$

When Z is nonempty, then for any affine k-scheme $S = Spec(A)$, any pair of everywhere disjoint sections $x_1, x_2 \in (X - Z)(S)$, any connected component Z_i of Z and any section $z \in Z_i(S)$, we denote by

$$Hyp_d(Z_S + a[x_1] + b[x_2] + c[z]) \subset H^0(\mathbb{P}_S, \mathcal{O}_{\mathbb{P}_S}(d) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \otimes_k A$$

the kernel of the composite restriction map

$$\begin{array}{c} H^0(\mathbb{P}_S, \mathcal{O}_{\mathbb{P}_S}(d)) \longrightarrow H^0(X_S, \mathcal{O}_{X_S}(d)) \\ \downarrow \\ H^0(X_S, (\mathcal{O}_{X_S}/I_{X_S}(Z_S + a[x_1] + b[x_2] + c[z]))(d)). \end{array}$$

Thus for $d \geq d_0$ and a, b, c all ≤ 3, the various $Hyp_d(Z_S + a[x_1] + b[x_2] + c[z])$ are locally free A-modules of finite rank. If $0 \leq a \leq 2$, then the quotient

$$Hyp_d(Z_S + a[x_1] + b[x_2] + c[z])/Hyp_d(Z_S + (a+1)[x_1] + b[x_2] + c[z])$$

is the locally free A-module of rank $Binom(n + a - 1, a)$ given by $Sym^a(I/I^2)$, for I the ideal $I_{X_S}([x_1])$.

If $Spec(A)$ is connected, then the point $z \in Z(S)$ lies entirely in one connected component, say $z \in Z_i(S)$. If also $0 \leq c \leq 2$, then the quotient

$$Hyp_d(Z_S + a[x_1] + b[x_2] + c[z])/Hyp_d(Z_S + a[x_1] + b[x_2] + (c+1)[z])$$

is the locally free A-module of rank $Binom(codim_X(Z_i) + c - 1, c)$ given by the pullback to $[z] \subset Z_{i,S}$ of the locally free sheaf of that rank on $Z_{i,S}$ given by $Sym^c(I/I^2)$, for I the ideal $I_{X_S}(Z_{i,S})$.

When Z is empty, then for any affine k-scheme $S = Spec(A)$, any pair of everywhere disjoint sections $x_1, x_2 \in X(S)$, we denote by

$$Hyp_d(a[x_1] + b[x_2]) \subset H^0(\mathbb{P}_S, \mathcal{O}_{\mathbb{P}_S}(d) \cong H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d)) \otimes_k A$$

the kernel of the composite restriction map

$$H^0(\mathbb{P}_S, \mathcal{O}_{\mathbb{P}_S}(d)) \longrightarrow H^0(X_S, \mathcal{O}_{X_S}(d))$$
$$\downarrow$$
$$H^0(X_S, (\mathcal{O}_{X_S}/I_{X_S}(a[x_1] + b[x_2]))(d)).$$

Again in this case, for $d \geq d_0$ and a, b both ≤ 3, the various $Hyp_d(a[x_1]+b[x_2])$ are locally free A-modules of finite rank. If $0 \leq a \leq 2$, then the quotient

$$Hyp_d(a[x_1] + b[x_2])/Hyp_d((a + 1)[x_1] + b[x_2])$$

is the locally free A-module of rank $Binom(n + a - 1, a)$ given by $Sym^a(I/I^2)$, for I the ideal $I_{X_S}([x_1])$.

For $d \geq d_0$ and a, b, c all ≤ 3, we denote by

$$PHyp_d(Z_S + a[x_1] + b[x_2] + c[z]) \subset PHyp_{d,S}$$

and

$$PHyp_d(a[x_1] + b[x_2]) \subset PHyp_{d,S}$$

the projective bundles over S of lines in the vector bundles $Hyp_d(Z_S + a[x_1] + b[x_2] + c[z])$ and $Hyp_d(a[x_1] + b[x_2])$ respectively.

Fix $d \geq 1$. The incidence variety $Inc_d \subset PHyp_d \times_k X$ is the closed subscheme consisting of pairs (H, x) with H a degree d hypersurface such that $H(x) = 0$ and such that the scheme-theoretic intersection $X \cap H$ is singular at x. If we view Inc_d as mapping to X by the second projection, it is a $\mathbb{P}^M$ bundle, for $M = dim(Hyp_d) - 1 - n$; the fibre over a point $x \in X$ is the projective space $PHyp_d(2[x])$, which is a linear subspace of $PHyp_d$ of codimension $1 + n$. Thus Inc_d is itself proper, smooth, and geometrically connected of dimension $dim(Hyp_d) - 1$, being the total space of a $\mathbb{P}^M$ bundle over the n-dimensional X. The image of Inc_d in $PHyp_d$ under the first projection, with its induced reduced structure, is called the dual variety $X_d^\vee \subset PHyp_d$. It is thus a geometrically irreducible subvariety of $PHyp_d$ of codimension at least one.

Lemma 2.3. *Suppose that either $d \geq 3$ or that $d = 2$ and $n = dim(X)$ is odd. Then $X_d^\vee \subset Hyp_d$ has codimension one, i.e., it is a hypersurface in $PHyp_d$.*

Proof. The statement is geometric, so we may extend scalars to reduce to the case when k is algebraically closed. We argue by contradiction. If our geometrically irreducible $X_d^\vee \subset Hyp_d$ has codimension ≥ 2, then after extending scalars to any infinite overfield of k, we can find a line L (which we identify to $\mathbb{P}^1$) in $PHyp_d$ which is disjoint from $X_d^\vee$, i.e., we can find a Lefschetz pencil of degree d hypersurface sections of X which has no singular fibres. Denote by $\Delta \subset X$ the base of the pencil, i.e., the intersection of any two of the fibres. For general L, $\Delta \subset X$ is smooth of codimension 2. Denote by $\tilde{X} := Blow_\Delta(X)$ the blow up of X along Δ, and by

$$\rho : \tilde{X} \to \mathbb{P}^1$$

the corresponding fibration. This morphism is proper and smooth, and hence for any prime number ℓ invertible in k, the $\mathbb{Q}_\ell$-sheaves $R^i\rho_\star\mathbb{Q}_\ell$ on the base $\mathbb{P}^1$ are everywhere lisse, hence constant. So for every i we have $H^1(\mathbb{P}^1, R^i\rho_\star\mathbb{Q}_\ell) = 0$. Thus the only possibly nonvanishing E_2 terms in the Leray spectral sequence are

$$E_2^{0,i} = H^0(\mathbb{P}^1, R^i\rho_\star\mathbb{Q}_\ell)$$

and

$$E_2^{2,i} = H^2(\mathbb{P}^1, R^i\rho_\star\mathbb{Q}_\ell).$$

One knows that this spectral sequence degenerates at E_2, either by combining Deligne's general degeneration theorem [De-Degen, 2.4] with the Hard Lefschetz Theorem [De-Weil II, 4.1.1], or by doing a "spreading out" argument to reduce to the case of a finite field, and using Grothendieck's original weight argument, cf. [Ka-MMP, 7.5.2]. Thus we have a short exact sequence, for every i,

$$0 \to H^2(\mathbb{P}^1, R^{i-2}\rho_\star\mathbb{Q}_\ell) \to H^i(\tilde{X}, \mathbb{Q}_\ell) \to H^0(\mathbb{P}^1, R^i\rho_\star\mathbb{Q}_\ell).$$

Now let $X \cap H$ be one of the fibres of the pencil. By the constancy of $R^i\rho_\star\mathbb{Q}_\ell$, we have

$$H^0(\mathbb{P}^1, R^i\rho_\star\mathbb{Q}_\ell) \cong H^i(X \cap H, \mathbb{Q}_\ell),$$

and thus we find that the restriction map

$$H^i(\tilde{X}, \mathbb{Q}_\ell) \to H^i(X \cap H, \mathbb{Q}_\ell)$$

is surjective. On the other hand, one knows [SGA 7 II, XVIII, 5.1.6] that for $i \le n-1$, this restriction map has the same image in $H^i(X \cap H, \mathbb{Q}_\ell)$ as does the restriction map

$$H^i(X, \mathbb{Q}_\ell) \to H^i(X \cap H, \mathbb{Q}_\ell).$$

Taking $i = n-1$, we find a surjective restriction map

$$H^{n-1}(X, \mathbb{Q}_\ell) \twoheadrightarrow H^{n-1}(X \cap H, \mathbb{Q}_\ell).$$

But the dimension of the cokernel of this map, denoted

$$N_d := N_d(X, \text{given embedding in } \mathbb{P})$$

in [Ka-Pan], is strictly positive in the stated range of d, cf. [Ka-Pan, Theorem 1 and preceding two paragraphs]. This is the desired contradiction. $\square$

One knows [SGA 7 II, XVII, 3.2] that the points $H \in X_d^\vee$ such that $X \cap H$ has one and only one singular point, and such that this unique singular point is an ordinary double point, form an open set $Good(X_d^\vee) \subset X_d^\vee$. We define

$$Bad(X_d^\vee) := X_d^\vee \setminus Good(X_d^\vee).$$

Thus $Bad(X_d^\vee) \subset X_d^\vee$ is closed.

We can be more precise in the case when either n is even or $char(k) \neq 2$. Then for $d \geq 2$, we see from [SGA 7 II, XVII, 3.3 and 3.7.1] that the first projection, from the incidence variety Z_d to $PHyp_d$, is generically unramified. Hence $X_d^\vee$ is a hypersurface, and by [SGA 7 II, XVII, 3.5] the set $Good(X_d^\vee)$ is precisely its smooth locus.

For ease of future reference, we state explicitly the following slight sharpening of the previous lemma.

Lemma 2.4. *Suppose that $d \geq 2$. Then $X_d^\vee \subset Hyp_d$ is a hypersurface.*

Proof. The only case not covered by the previous lemma is when $d = 2$ and n is even, and that case is handled by the [SGA 7 II] results recalled just above. □

We will need the following existence results.

Lemma 2.5. *Suppose that $d \geq d_0$, and that k is infinite. Fix a point $x_0 \in X(k)$. Then there exists a degree d hypersurface H such that $X \cap H$ has an ordinary double point at x_0 and is smooth elsewhere.*

Proof. That $X \cap H$ be singular at x_0 means precisely that $H \in PHyp_d(2[x_0])$. Since $d \geq d_0$, the map

$$Hyp_d(2[x_0]) \to I_X([x_0])^2/I_X([x_0])^3)(d)$$

is surjective, i.e., we can achieve arbitrary quadratic terms at x_0. The condition that quadratic terms define an ordinary double point, i.e. that their vanishing define a smooth quadric in the projective space on $\mathcal{M}_{X,x_0}/\mathcal{M}_{X,x_0}^2$, is an open condition. So a dense open set, say U_1, of $PHyp_d(2[x_0])$ consists of degree d hypersurfaces H such that $X \cap H$ has ordinary double point at x_0. It remains to show that a second dense open set, say U_2, of $PHyp_d(2[x_0])$ consists of d hypersurfaces H such that $X \cap H$ is smooth outside of x_0, for then a point in $U_1 \cap U_2$ is the desired H. To construct U_2, we consider another incidence variety, call it

$$Inc_d(2[x_0]) \subset PHyp_d(2[x_0]) \times (X \setminus \{x_0\})$$

consisting of pairs (H, x) such that $X \cap H$ is singular at x. Viewed as lying over X $Inc_d(2[x_0])$ is a $\mathbb{P}^M$-bundle, for $M = dim(PHyp_d(2[x_0]))-$

$1-n$ (the fibre over $x \neq x_0$ is $PHyp_d(2[x_0]+2[x])$, which has codimension $n+1$ in $PHyp_d(2[x_0])$). Thus $Inc_d(2[x_0])$ is smooth and geometrically irreducible of dimension $dim(PHyp_d(2[x_0])) - 1$, and hence its image in $PHyp_d(2[x_0])$ under the first projection has a closure which is of codimension at least one. The complement of this closure in $PHyp_d(2[x_0])$ is the desired dense open set U_2. □

Corollary 2.6. *If $d \geq d_0$, the open set $Good(X_d^\vee) \subset X_d^\vee$ is nonempty, and its complement $Bad(X_d^\vee)$ has codimension ≥ 2 in $PHyp_d$.*

Proof. The assertion is geometric, so we may extend scalars and reduce to the case when$X(k)$ is nonempty. Then the first assertion is immediate from the lemma above, and the second then follows from the fact that $X_d^\vee$ is an irreducible hypersurface in $PHyp_d$. □

Now we put Z into the picture.

Lemma 2.7. *Suppose $d \geq d_0$, and that k is infinite. Then we have the following results.*

1. *There exists a degree d hypersurface $H \in PHyp_d(Z)$ such that $X \cap H$ is smooth.*
2. *The intersection $X_d^\vee \cap PHyp_d(Z)$ is an irreducible hypersurface in $PHyp_d(Z)$.*
3. *Fix a point $x_0 \in (X - Z)(k)$. There exists a degree d hypersurface $H \in PHyp_d(Z)$ such that $X \cap H$ has an ordinary double point at x and is smooth elsewhere.*
4. *The intersection $Good(X_d^\vee) \cap PHyp_d(Z)$ is nonempty. The intersection $Bad(X_d^\vee) \cap PHyp_d(Z)$ has codimension ≥ 2 in $PHyp_d(Z)$.*

Proof. Clearly (2) $\Rightarrow$ (1). Once we have proven (2) and (3), then (4) follows exactly as in the preceding corollary. To prove (2), we argue as follows. Let us denote by $Inc_d(Z) \subset PHyp_d(Z) \times X$ the incidence variety consisting of pairs (H, x) with $H \in PHyp_d(Z)$ and $x \in X$ such that $X \cap H$ is singular at x. We first view $Inc_d(Z)$ as mapping to X. Over $X \setminus Z$, $Inc_d(Z)$ is a $\mathbb{P}^M$-bundle, now for $M = dim(PHyp_d(Z)) - 1 - n$(the fibre over $x \in X \setminus S$ is $PHyp_d(Z + 2[x])$, which has codimension $n+1$ in $PHyp_d(Z)$). Over a point $z \in Z_i$, the fibre is $PHyp_d(Z + [z])$, which has codimension $codim_X(Z_i)$ in $PHyp_d(Z)$. So $Inc_d(Z)$ is the union of an open set which is smooth and geometrically connected of dimension $dim(PHyp_d(Z))-1$, namely the total space of a $\mathbb{P}^M$-bundle over $X \setminus Z$, and of a finite union of closed sets, namely the total spaces of projective bundles over the Z_i of fibre dimensions $dim(PHyp_d(Z)) - codim_X(Z_i)$. Because of the hypothesis

$$dim(Z_i) < codim_X(Z_i) - 1,$$

we see that each of these total spaces has dimension at most $dim(PHyp_d(Z)) - codim_X(Z_i) + dim(Z_i) \leq dim(PHyp_d(Z)) - 2$. The image of $Inc_d(Z)$ in $PHyp_d(Z)$, which is precisely the intersection $X_d^\vee \cap PHyp_d(Z)$, is therefore the union of a geometrically irreducible variety of codimension ≥ 1 and of finitely many geometrically irreducible varieties of codimension ≥ 2. So certainly $X_d^\vee \cap PHyp_d(Z)$ has codimension ≥ 1 in $PHyp_d(Z)$. On the other hand, $X_d^\vee$ is a hypersurface in $PHyp_d$, so the intersection $X_d^\vee \cap PHyp_d(Z)$ is either all of $PHyp_d(Z)$, or it is a hypersurface in $PHyp_d(Z)$. The first case being impossible, we conclude that $X_d^\vee \cap PHyp_d(Z)$ is a hypersurface in $PHyp_d(Z)$. From its description as the image of $Inc_d(Z)$, we conclude it is the union of a geometrically irreducible variety of codimension precisely 1 and of finitely many geometrically irreducible varieties of codimension ≥ 2. Looking at the maximal points of the hypersurface $X_d^\vee \cap PHyp_d(Z)$, we see that there is only one. This proves (2).

It remains to prove (3). Here we proceed exactly as we did in proving the double point lemma above. Again, it is an open dense condition on $PHyp_d(Z + 2[x_0])$ that $X \cap H$ have an ordinary double point at x_0. We next consider the incidence variety

$$Inc_d(Z + 2[x_0]) \subset PHyp_d(Z + 2[x_0]) \times (X \setminus x_0)$$

consisting of points (H, x) such that $X \cap H$ is singular at x. The complement of its image in $PHyp_d(Z + 2[x_0])$ is the set of those hypersurfaces $H \in PHyp_d(Z + 2[x_0])$ such that $X \cap H$ is smooth outside of x_0. It remains to show that this complement contains an open dense set, i.e., that the closure of the image of $Inc_d(Z + 2[x_0])$ has codimension ≥ 1 in $PHyp_d(Z + 2[x_0])$. For this, it suffices to show that

$$dim(Inc_d(Z + 2[x_0])) = dim(PHyp_d(Z + 2[x_0])) - 1.$$

Indeed over $X \setminus Z$, $Inc_d(Z+2[x_0])$ is a $\mathbb{P}^M$-bundle with $M = dim(PHyp_d(Z+2[x_0])) - n - 1$. Over a point $z \in Z_i$, its fibre is the space $PHyp_d(Z + 2[x_0] + [z])$, of dimension $dim(PHyp_d(Z + 2[x_0])) - codim_X(Z_i)$. Thus over Z_i, $Inc_d(Z + 2[x_0])$ is the total space of a $\mathbb{P}^M$-bundle with $M = dim(PHyp_d(Z + 2[x_0])) - codim_X(Z_i)$, hence has dimension at most $dim(PHyp_d(Z + 2[x_0])) - codim_X(Z_i) + dim(Z_i) \leq dim(PHyp_d(Z + 2[x_0])) - 2$. Thus $Inc_d(Z + 2[x_0])$ is the union of an open set which is smooth and geometrically connected of dimension $dim(PHyp_d(Z + 2[x_0])) - 1$, and of a closed set of strictly lower dimension. □

3. Lefschetz pencils, and proof of the theorem

A pencil of degree d hypersurfaces in $\mathbb{P}$ is a line $L \subset PHyp_d$, say $\mathbb{P}^1 \ni t = (\lambda, \mu) \mapsto H_t := \lambda F + \mu G$, for F and G two linearly independent elements of Hyp_d. Its axis $\Delta \subset X$ is the closed subscheme $X \cap F \cap G$ obtained by intersecting X with any two distinct hypersurfaces in the pencil. One says that $L \subset PHyp_d$ is a Lefschetz pencil of degree d hypersurface sections of X if the following three conditions are satisfied:

1. L is not entirely contained in the dual variety $X_d^\vee$.
2. L is disjoint from $Bad(X_d^\vee)$.
3. $\Delta \subset X$ is smooth of codimension 2 in X.

Suppose now that we are in the situation which the theorem purports to treat. Thus k is an infinite field, $n = dim(X) \geq 2$, and $Z \subset X$ is a closed subscheme which is smooth, each of whose connected components $Z_i \subset X$ satisfies the inequality

$$dim(Z_i) < codim_X(Z_i) - 1.$$

We have shown that $X_d^\vee \cap PHyp_d(Z)$ is an irreducible hypersurface in $PHyp_d(Z)$ and the intersection $Bad(X_d^\vee) \cap PHyp_d(Z)$ has codimension ≥ 2 in $PHyp_d(Z)$.

Because the intersection $X_d^\vee \cap PHyp_d(Z)$ is an irreducible hypersurface in $PHyp_d(Z)$, the lines $L \in Gr(1, PHyp_d(Z))$ not contained in it form a dense open set, say U_1. The fact that $Bad(X_d^\vee) \cap PHyp_d(Z)$ has codimension ≥ 2 in $PHyp_d(Z)$ insures that the lines $L \in Gr(1, PHyp_d(Z))$ which are disjoint from $Bad(X_d^\vee)$ form a second dense open set, say U_2. In the Grassmannian $Gr(1, PHyp_d(Z))$ of lines in $PHyp_d(Z)$, the condition that the axis $\Delta \subset X$ be smooth of codimension 2 in X defines a third open set, say U_3. We will show that the set U_1 is nonempty (and hence dense). Then the intersection $U_1 \cap U_2 \cap U_3$ is a dense open set in the rational variety $Gr(1, PHyp_d(Z))$, so has k-points so long as k is infinite.

To show that U_1 is nonempty, it suffices to show that in the product space $Hyp_d(Z) \times Hyp_d(Z)$, the open set U_0 consisting of pairs (F, G) such that $X \cap F \cap G$ is smooth of codimension 2 in X is nonempty. The pairs (F, G) which lie outside of U_0 are those for which there exists a geometric point $x \in X$ at which $F(x) = 0$, $G(x) = 0$, and such that at x, the linear terms of F and G fail to be linearly independent. Thus we are led to consider the incidence variety

$$Inc_{d,d}(Z) \subset Hyp_d(Z) \times Hyp_d(Z) \times X \times \mathbb{P}^1,$$

consisting of quadruples $(F, G, x, (\lambda, \mu))$ for which $F(x) = 0$, $G(x) = 0$, and $\lambda F - \mu G$ has no linear term at x. This third condition means explicitly that

(1) If $x \in X \setminus Z$, then $\lambda F - \mu G \in Hyp_d(Z + 2[x])$,
(1) If $x = z \in Z$, then $\lambda F - \mu G \in Hyp_d(Z + [z])$.

The image of $Inc_{d,d}(Z)$ in $Hyp_d(Z) \times Hyp_d(Z)$ is then the complement of U_0. So it suffices to show that

$$dim(Inc_{d,d}(Z)) \leq 2dim(Hyp_d(Z)) - 1.$$

For this, we view $Inc_{d,d}(Z)$ as lying over $X \times \mathbb{P}^1$. Over a point $(x, (\lambda, \mu)) \in (X \setminus Z) \times \mathbb{P}^1$, the fibre is the linear subspace of $Hyp_d(Z+[x]) \times Hyp_d(Z+[x])$ consisting of those pairs (F, G) for which $\lambda F - \mu G \in Hyp_d(Z+2[x])$. For a fixed representative $(\lambda_0, \mu_0) \in \mathbb{A}^2 \setminus (0,0)$ of $(\lambda, \mu) \in \mathbb{P}^1$, and a fixed element $H \in Hyp_d(Z + 2[x])$, the equation for variable $(F, G) \in Hyp_d(Z + [x]) \times Hyp_d(Z + [x])$

$$\lambda_0 F - \mu_0 G = H$$

certainly has solutions (e.g. $F = (1/\lambda_0)H, G = 0$ if $\lambda_0 \neq 0$), and the set of all solutions is principal homogeneous under the space of pairs (F, G) with

$$\lambda_0 F = \mu_0 G.$$

This space is isomorphic to $Hyp_d(Z + [x])$ (e.g., by $(F, G) \mapsto G$ if $\lambda_0 \neq 0$). Thus the fibre of $Inc_{d,d}(Z)$ over a point $(x, (\lambda, \mu)) \in (X \setminus Z) \times \mathbb{P}^1$ is itself a $Hyp_d(Z + [x])$-bundle over $Hyp_d(Z + 2[x])$, so has dimension

$$dim(Hyp_d(Z)) - 1 + dim(Hyp_d(Z)) - 1 - n = 2dim() - 2 - n.$$

Thus the part of $Inc_{d,d}(Z)$ lying over$(X \setminus Z) \times \mathbb{P}^1$ has dimension at most $2dim(Hyp_d(Z)) - 1$.

Over a point $(z, (\lambda, \mu)) \in Z_i \times \mathbb{P}^1$, the fibre is analyzed in the same way. We pick a representative $(\lambda_0, \mu_0) \in \mathbb{A}^2 \setminus (0,0)$ of $(\lambda, \mu) \in \mathbb{P}^1$. The fibre is then the linear subspace of $Hyp_d(Z) \times Hyp_d(Z)$ consisting of those pairs (F, G) for which $\lambda_0 F - \mu_0 G \in Hyp_d(Z+[z])$. Just as above, this fibre is a $Hyp_d(Z)$-bundle over $Hyp_d(Z + [z])$, so has dimension

$$dim(Hyp_d(Z)) + dim(Hyp_d(Z)) - codim_X(Z_i).$$

Thus the part of $Inc_{d,d}(Z)$ lying over$Z_i \times \mathbb{P}^1$ has dimension at most

$$2dim(Hyp_d(Z)) - codim_X(Z_i) + dim(Z_i) + 1 \leq 2dim(Hyp_d(Z)) - 1.$$

Putting together these pieces of $Inc_{d,d}(Z)$, we get the asserted estimate on its dimension.

References

[De-Degen] Deligne, P. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. Publ. Math. IHES 35 (1968), 259-278.

[De-Weil II] Deligne, P., La conjecture de Weil II. Publ. Math. IHES 52 (1981), 313-428.

[Ha] Hartshorne, R., Algebraic Geometry, Springer, 1977.

[Ka-MMP] Katz, N., Moments, monodromy, and perversity: a Diophantine perspective. Annals of Math. Study 159. Princeton University Press, 2005.

[Ka-Pan] Katz, N.; Pandharipande, R., Inequalities related to Lefschetz pencils and integrals of Chern classes. Geometric aspects of Dwork theory. Vol. I, II, 805-818, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.

[Mum-AV] Mumford, D., Abelian Varieties, Oxford, 1970.

[SGA 7 II] Groupes de monodromie en géométrie algébrique. II. Séminaire de Géométrie Algébrique du Bois-Marie 1967-1969 (SGA 7 II). Dirig par P. Deligne et N. Katz. Lecture Notes in Mathematics, Vol. 340. Springer-Verlag, 1973.

Princeton University, Mathematics, Fine Hall, NJ 08544-1000, USA
E-mail address: nmk@math.princeton.edu