A note on classical groups and moments

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We work over an algebraically closed field \( \mathbb{C} \) of characteristic zero. Let \( V \) be a \( \mathbb{C} \)-vector space of dimension \( N \geq 2 \). We fix a (not necessarily connected) Zariski closed reductive subgroup \( G \subseteq \text{GL}(V) \).

For each pair \( (a, b) \) of non-negative integers, we denote by \( M_{a,b}(G, V) \) the dimension of the space of \( G \)-invariant vectors in \( V^{\otimes a} \otimes (V^\vee)^{\otimes b} \):

\[
M_{a,b}(G, V) := \text{dim}_{\mathbb{C}} (V^{\otimes a} \otimes (V^\vee)^{\otimes b})^G.
\]

We call \( M_{a,b}(G, V) \) the \((a, b)\)'th moment of \((G, V)\). For each even integer \( 2n \geq 2 \), we denote by \( M_{2n}(G, V) \) the \( 2n \)'th absolute moment, defined by

\[
M_{2n}(G, V) := M_{n,n}(G, V).
\]

[Recall that if \( \mathbb{C} \) is the field of complex numbers, and if \( K \subseteq G(\mathbb{C}) \) is a maximal compact subgroup of \( G(\mathbb{C})^{\text{an}} \), then \( K \) is Zariski dense in \( G \) (Weyl's unitarian trick). If we denote by \( dk \) the Haar measure on \( K \) of total mass one, and by \( \chi : G(\mathbb{C}) \to \mathbb{C} \)

\[
\chi(g) := \text{Trace}(g|V),
\]

the character of \( V \) as \( G \)-module, then we have the formulas

\[
M_{a,b}(G, V) = \int_K \chi(k)^a \overline{\chi}(k)^b dk,
\]

\[
M_{2n}(G, V) = \int_K |\chi(k)|^{2n} dk.
\]

Thus the terminology "moments" and "absolute moments".

Theorem Let \( V \) be a \( \mathbb{C} \)-vector space of dimension \( N \geq 2 \), \( G \subseteq \text{GL}(V) \) a (not necessarily connected) Zariski closed reductive subgroup of \( \text{GL}(V) \).

1) If \( M_4(G, V) = 2 \), then either \( G \supseteq \text{SL}(V) \), or \( G/(G/\text{ scalars}) \) is finite. If in addition \( G \) is semisimple, then either \( G_0 = \text{SL}(V) \), or \( G \) is finite.

2) Suppose \( <,> \) is a nondegenerate symmetric bilinear form on \( V \), and suppose \( G \) lies in the orthogonal group \( O(V) := \text{Aut}(V, <,>) \). If \( M_4(G, V) = 3 \), then either \( G = O(V) \), or \( G = \text{SO}(V) \), or \( G \) is finite. If \( \dim(V) \) is 2 or 4, then \( G \) is not contained in \( \text{SO}(V) \).

3) Suppose \( <,> \) is a nondegenerate alternating bilinear form on \( V \) (such a form exists only if \( \dim(V) \) is even), suppose \( G \) lies in the symplectic group \( \text{Sp}(V) := \text{Aut}(V, <,>) \), and suppose \( \dim(V) > 2 \). If \( M_4(G, V) = 3 \), then either \( G = \text{Sp}(V) \), or \( G \) is finite.

Proof We first give two elementary interpretations of the fourth absolute moment. Decompose the \( G \)-module \( V \otimes V^\vee = \text{End}(V) \) as a sum of irreducibles with multiplicities:

\[
\text{End}(V) \cong \bigoplus_i n_i W_i.
\]

Then
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\[ M_4(G, V) = \sum_i (n_i)^2. \]

Or decompose \( V^{\otimes 2} \) as a sum of irreducibles with multiplicities:

\[ V^{\otimes 2} \cong \bigoplus_i m_i W_i. \]

Then

\[ M_4(G, V) = \sum_i (m_i)^2. \]

We also remark that if \( M_4(G, V) \leq 3 \), then \( V \) is \( G \)-irreducible. For if \( V = V_1 \oplus V_2 \) is the direct sum of two non-zero \( G \)-modules, then \( \operatorname{End}(V) \) is the sum of the four non-zero \( G \)-modules \( V_1 \otimes (V_j)^V \), and this in turn forces \( M_4(G, V) \geq 4 \).

To prove 1), we use the first interpretation. If \( M_4(G, V) = 2 \), then \( \operatorname{End}(V) \) is the sum of two distinct irreducible representations of \( G \). But under the bigger group \( \operatorname{GL}(V) \), \( \operatorname{End}(V) \) is the sum of two representations of \( \operatorname{GL}(V) \), namely

\[ \operatorname{End}(V) = \operatorname{End}^0(V) \oplus \mathbb{I} = \operatorname{Lie}(\operatorname{SL}(V)) \oplus \mathbb{I}. \]

The two summands are inequivalent irreducible representations of \( \operatorname{GL}(V) \), but we will not use this fact. Because \( M_4(G, V) = 2 \), this must be the decomposition of \( \operatorname{End}(V) \) as the sum of two distinct irreducible representations of \( G \). In particular, \( \operatorname{Lie}(\operatorname{SL}(V)) \) is \( G \)-irreducible.

The derived group \( G^\text{der} \) lies in \( \operatorname{SL}(V) \), so \( \operatorname{Lie}(G^\text{der}) \) lies in \( \operatorname{Lie}(\operatorname{SL}(V)) \). As \( G^\text{der} \) is a normal subgroup of \( G \), \( \operatorname{Lie}(G^\text{der}) \) is a \( G \)-stable submodule of \( \operatorname{Lie}(\operatorname{SL}(V)) \). So by the \( G \)-irreducibility of \( \operatorname{Lie}(\operatorname{SL}(V)) \), either \( \operatorname{Lie}(G^\text{der}) = \operatorname{Lie}(\operatorname{SL}(V)) \), or \( \operatorname{Lie}(G^\text{der}) = 0 \). In the first case, \( (G^\text{der})^0 = \operatorname{SL}(V) \), and so \( G \supset \operatorname{SL}(V) \). Thus if in addition \( G \) is semisimple, \( G^0 \) is \( \operatorname{SL}(V) \).

In the second case, \( G^\text{der} \) is finite. For any fixed element \( \gamma \) in \( G(C) \), the morphism from \( G^0 \) to \( G^\text{der} \) defined by \( g \mapsto gyg^{-1} \gamma^{-1} \) is therefore the constant map \( g \mapsto e \). Therefore \( G^0 \) lies in \( Z(G) \). As \( G \) acts irreducibly on \( V \), its center \( Z(G) \) lies in the \( G_m \) of scalars. But \( G^0 \subset Z(G) \), so \( G^0 \) lies in the \( G_m \) of scalars. Therefore \( G^0 \subset G \cap \text{scalars} \), whence \( G/(G \cap \text{scalars}) \) is finite. If in addition \( G \) is semisimple, then \( Z(G) \) is finite, and so \( G \cap \text{scalars} \) is finite, hence \( G \) itself is finite.

To prove 2), we use the second interpretation. If \( M_4(G, V) = 3 \), then \( V^{\otimes 2} \) is the sum of three distinct irreducible representations of \( G \). Under \( \operatorname{GL}(V) \), we first decompose

\[ V^{\otimes 2} = \operatorname{Sym}^2(V) \oplus \Lambda^2(V). \]

As \( \operatorname{O}(V) \)-modules, we have an isomorphism

\[ \Lambda^2(V) \cong \operatorname{Lie}(\operatorname{SO}(V)) \]

and the further decomposition

\[ \operatorname{Sym}^2(V) = \operatorname{SphHarm}^2(V) \oplus \mathbb{I}. \]

Thus as \( \operatorname{O}(V) \)-module, we have the three term decomposition

\[ V^{\otimes 2} = \operatorname{SphHarm}^2(V) \oplus \mathbb{I} \oplus \operatorname{Lie}(\operatorname{SO}(V)). \]

[For \( \dim(V) \geq 2 \), the three summands are distinct irreducible representations of \( \operatorname{O}(V) \); if in addition
dim(V) is neither 2 nor 4, they are distinct irreducible representations of SO(V). We will not use these facts.]

If $M_4(G, V) = 3$, then

$$V^\otimes 2 = \text{SphHarm}^2(V) \oplus \mathbb{I} \oplus \text{Lie}(\text{SO}(V))$$

must be the decomposition of $V^\otimes 2$ as the sum of three distinct irreducible representations of G.

We now exploit the fact that Lie(\text{SO}(V)) is G–irreducible. Since $G \subset \text{O}(V)$, $G^0 \subset \text{SO}(V)$, so Lie($G^0$) is a G–stable submodule of Lie(\text{SO}(V)). By G–irreducibility, Lie($G^0$) is either Lie(\text{SO}(V)) or is zero. If Lie($G^0$) = Lie(\text{SO}(V)), then $G^0$ is SO(V) and G, being caught between SO(V) and O(V), is either SO(V) or O(V). If Lie($G^0$) is zero, then G is finite.

If dim(V) is 2 or 4, we claim G cannot lie in SO(V). Indeed, for dim(V) = 2, SO(V) is a, Lie(SO(V)) is \text{\textit{\sigma}} as SO(V)-module, and SphHarm^2(V) is SO(V)-reducible, so if G \subset SO(V) then $M_4(G, V) \geq 6$. If dim(V) = 4, then SO(4) is (SL(2)xSL(2))/±(1,1), hence Lie(SO(4)) is SO(4)-reducible: so if G \subset SO(V) then $M_4(G, V) \geq 4$.

To prove 3), we begin with the GL(V)–decomposition

$$V^\otimes 2 = \text{Sym}^2(V) \oplus \Lambda^2(V).$$

As Sp(V)–modules, we have an isomorphism

$$\text{Lie}(\text{Sp}(V)) \cong \text{Sym}^2(V),$$

and the further (because dim(V) > 2) decomposition

$$\Lambda^2(V) = (\Lambda^2(V)/\mathbb{I}) \oplus \mathbb{I}.$$

Thus as Sp(V)–module we have a three term decomposition

$$V^\otimes 2 = \text{Lie}(\text{Sp}(V)) \oplus (\Lambda^2(V)/\mathbb{I}) \oplus \mathbb{I}.$$

[The three summands are distinct irreducible representations of Sp(V), but we will not use this fact.] Exactly as in the SO case above, we infer that Lie(\text{Sp}(V)) is G–irreducible. But G \subset Sp(V), so Lie($G^0$) is a G–stable submodule of Lie(\text{Sp}(V)), and so either Lie($G^0$) = Lie(\text{Sp}(V)), or Lie($G^0$) is zero. In the first case, G is Sp(V), and in the second case G is finite. QED

**Remarks** We should call attention to a striking result of Beukers, Brownawell and Heckmann, [BBH, Theorems A5 and A7 together], which is similar in spirit to our result (though more difficult): if G is a Zariski closed subgroup of GL(V) which acts irreducibly on Sym^2(V), then either G/(G\cap\text{scalars}) is finite, or G contains SL(V), or dim(V) is even and Sp(V) \subset G \subset \text{GSp}(V).

There are connected semisimple subgroups G \subset GL(V) with $M_4(G, V) = 3$ other than SO(V) (for dim(V) \geq 3, but \neq 4) and Sp(V) (for dim(V) \geq 4). The simplest examples are these. Take a C–vector space W of dimension $\ell+1$. Then for V either Sym^2(W), if $\ell \geq 2$, or $\Lambda^2(W)$, if $\ell \geq 4$, the image G of SL(W) in GL(V) has $M_4(G, V) = 3$, but V is not self–dual as a representation of G (not self dual because we excluded the case $\ell=3, V = \Lambda^2(W)$). Here is a bad proof. In the
Bourbaki [Bour–L8] notation, \( \text{Sym}^2(W) \) is the highest weight module \( E(2\omega_1) \), and \( \Lambda^2(W) \) is the highest weight module \( E(\omega_2) \). We use the first interpretation of the fourth absolute moment. We have

\[
\text{End}(E(2\omega_1)) = E(2\omega_1) \otimes E(2\omega_1)^\vee = E(2\omega_1) \otimes E(2\omega_f)
\]

and

\[
\text{End}(E(\omega_2)) = E(\omega_2) \otimes (\omega_2)^\vee = E(\omega_2) \otimes E(\omega_{f-1}).
\]

Now \( \text{End}(\text{any nontrivial rep'n of } SL(W)) \) contain both the trivial representation \( \mathbb{I} \) of \( SL(W) \) and its adjoint representation \( E(\omega_1 + \omega_f) \). From looking at highest weights, we see that \( \text{End}(E(2\omega_1)) \) contains \( E(2\omega_1 + 2\omega_f) \), and we see that \( \text{End}(E(\omega_2)) \) contains \( E(\omega_2 + \omega_{f-1}) \).

Thus we have a priori decompositions

\[
\begin{align*}
\text{End}(E(2\omega_1)) &= \mathbb{I} \oplus E(\omega_1 + \omega_f) \oplus E(2\omega_1 + 2\omega_f) \oplus (?), \\
\text{End}(E(\omega_2)) &= \mathbb{I} \oplus E(\omega_1 + \omega_f) \oplus E(\omega_2 + \omega_{f-1}) \oplus (?).
\end{align*}
\]

To see that in both cases there is no (?) term, it suffices to check that the dimensions add up, an exercise in the Weyl dimension formula we leave to the reader.

Other examples are (the image of) \( E_6 \) in either of its 27–dimensional irreducible representations, or \( \text{Spin}(10) \) in either of its 16–dimensional spin representations: according to simpLie, these all have fourth absolute moment 3.

What about finite groups \( G \subset GL(V) \) with \( M_4(G, V) = 2 \), or finite groups \( G \) in \( O(V) \) or \( \text{Sp}(V) \) with \( M_4(G, V) = 3 \)?

We begin with some examples of finite groups \( G \subset GL(V) \) with \( M_4(G, V) = 2 \), pointed out to me by Deligne. Let \( q \) be a power of an odd prime \( p \), i.e., \( q \) is the cardinality of a finite field \( \mathbb{F}_q \) of odd characteristic \( p \). Fix an integer \( n \geq 1 \), and a \( 2n \)–dimensional \( \mathbb{F}_q \)–vector space \( F \), endowed with a nondegenerate symplectic form \( <,> \). The Heisenberg group \( \text{Heis}_{2n}(\mathbb{F}_q) \) is the central extension of \( F \) by \( \mathbb{F}_q \) defined as the set of pairs \( (\lambda \in \mathbb{F}_q^*, f \in F) \), with group operation

\[
(\lambda, f)(\mu, g) := (\lambda + \mu + <f, g>, f + g).
\]

The symplectic group \( \text{Sp}(F) \) acts on \( \text{Heis}_{2n}(\mathbb{F}_q) \), \( \gamma \) in \( \text{Sp}(F) \) acting by

\[
\gamma(\lambda, f) := (\lambda, \gamma(f)).
\]

The irreducible \( \mathbb{C} \)–representations of the group \( \text{Heis}_{2n}(\mathbb{F}_q) \) are well–known. There are \( q^{2n} \) one–dimensional representations, those trivial on the center. For each of the \( q–1 \) nontrivial \( \mathbb{C}^\times \)–valued characters \( \psi \) of the center, there is precisely one irreducible representation with central character \( \psi \), say \( V_\psi \), which has dimension \( q^n \). Because the action of \( \text{Sp}(F) \) on \( \text{Heis}_{2n}(\mathbb{F}_q) \) is trivial on the center, the action of \( \text{Heis}_{2n}(\mathbb{F}_q) \) on \( V_\psi \) extends to a projective representation of the semidirect product group \( \text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F) \) on \( V_\psi \). Because we are over a finite field, this projective representation in turn extends to a linear representation of \( \text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F) \) on \( V_\psi \), the Weil–Shale representation.
We claim that for any nontrivial character \( \psi \) of the center, we have
\[
M_4(\text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F), V_\psi) = 2.
\]
To see this, it suffices to work over the complex numbers. We fix a choice of the nontrivial character \( \psi \), and denote by
\[
\chi : \text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F) \to \mathbb{C}
\]
the character of \( V_\psi \):
\[
\chi((\lambda, f, \gamma)) := \text{Trace}((\lambda, f, \gamma)|_{V_\psi}).
\]
According to Howe \cite{Howe}, \( \chi \) is supported on those conjugacy classes which meet (the center \( Z \) of \( \text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F) \), where it is given by
\[
|\chi((\lambda, 0, \gamma))|^2 = q^{\dim(\text{Ker}(\gamma^{-1}) \text{ in } F)}.
\]
Moreover, an element \((\lambda, f, \gamma)\) in \( \text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F) \) is conjugate to an element of \( Z \gamma \text{Sp}(F) \) if and only if it is conjugate to \((\lambda, 0, \gamma)\), and this happens if and only if \( f \) lies in \( \text{Image}(\gamma^{-1}) \), cf. \cite[page 294, first paragraph]{Howe}. Thus we have
\[
|\chi((\lambda, f, \gamma))|^2 = q^{\dim(\text{Ker}(\gamma^{-1})), \text{ if } f \in \text{Image}(\gamma^{-1}),}
\]
\[
|\chi((\lambda, f, \gamma))|^2 = 0, \text{ if not.}
\]
Using this explicit formula, we find a striking relation between the absolute moments of \( \text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F) \) on \( V_\psi \) and the absolute moments of its subgroup \( \text{Sp}(F) \) on \( V_\psi \). For any integer \( k \geq 1 \), we have
\[
M_{2k+2}(\text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F), V_\psi) = M_{2k}(\text{Sp}(F), V_\psi).
\]
To see this, we use the fact that \( \dim(\text{Ker}(\gamma^{-1})) + \dim(\text{Image}(\gamma^{-1})) = \dim(F) \), and simply compute:
\[
\begin{align*}
\#(\text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F)) \times \text{M}_{2k+2}(\text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F), V_\psi) \\
= \sum_{(\lambda, f, \gamma)} |\chi((\lambda, f, \gamma))|^{2k+2} \\
= \sum_{(\lambda, 0, \gamma)} \sum_{f \in \text{Image}(\gamma^{-1})} |\chi((\lambda, f, \gamma))|^{2k+2} \\
= \sum_{(\lambda, 0, \gamma)} q^{\dim(\text{Image}(\gamma^{-1})) \times q^{\dim(\text{Ker}(\gamma^{-1}))}} \\
= \sum_{\gamma \in \text{Sp}(F)} q^{1+\text{dim}(F) \times q^{\dim(\text{Ker}(\gamma^{-1}))}} \\
= \sum_{\gamma \in \text{Sp}(F)} q^{1+\text{dim}(F) \times |\chi((0, 0, \gamma))|^{2k}} \\
= q^{1+\text{dim}(F) \times \#(\text{Sp}(F)) \times \text{M}_{2k}(\text{Sp}(F), V_\psi)} \\
= \#(\text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F)) \times \text{M}_{2k}(\text{Sp}(F), V_\psi).
\end{align*}
\]
So in particular we have
\[
M_4(\text{Heis}_{2n}(\mathbb{F}_q) \gamma \text{Sp}(F), V_\psi) = M_2(\text{Sp}(F), V_\psi).
\]
The formula
\[
|\chi((0, 0, \gamma))|^2 = q^{\dim(\text{Ker}(\gamma^{-1}))} = \#(\text{fixed points of Sp}(F) \text{ on } F)
\]
means precisely that \( \text{End}(V_\psi) \) as \( \text{Sp}(F) \)-module is isomorphic to the natural permutation representation of \( \text{Sp}(F) \) on the space of \( \mathbb{C} \)-valued functions on \( F \). So
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\[ M_2(\text{Sp}(F), V\psi) = M_{1,0}(\text{Sp}(F), \text{Fct}(F, \mathbb{C})) \]

is the dimension of the space of \(\text{Sp}(F)\)-invariant functions on \(F\), which is in turn equal to the number of \(\text{Sp}(F)\)-orbits in \(F\). But \(\text{Sp}(F)\) acts transitively on \(F - \{0\}\), so there are just two orbits. Thus

\[ M_4(\text{Heis}_{2n}(\mathbb{F}_q)\gamma\text{Sp}(F), V\psi) = M_2(\text{Sp}(F), V\psi) = M_{1,0}(\text{Sp}(F), \text{Fct}(F, \mathbb{C})) = 2, \]

as asserted.

A perusal of the Atlas [CCNPW] gives some finite simple groups \(G\) with a low dimensional irreducible representation \(V\) for which we have \(M_4(G, V) = 2\). Here are some of them. In the table below, we give (in Atlas notation) the simple group \(G\), the character \(\chi\) of the lowest dimensional such \(V\), the dimension of \(V\), and the expression of \(|\chi|^2\) as the sum of two distinct irreducible characters.

| \(G\)     | character \(\chi\) of \(V\) | \(\text{dim}(V)\) | \(|\chi|^2 = ?\) |
|-----------|-----------------------------|-------------------|----------------|
| \(L_3(2) = L_2(7)\) | \(\chi_2, \chi_3\) | 3                 | \(\mathbb{I} + \chi_6\) |
| \(U_4(2) = S_4(3)\) | \(\chi_2, \chi_3\) | 5                 | \(\mathbb{I} + \chi_{10}\) |
| \(U_5(2)\) | \(\chi_3, \chi_4\) | 11                | \(\mathbb{I} + \chi_{16}\) |
| \(2F_4(2)'\) | \(\chi_2, \chi_3\) | 26                | \(\mathbb{I} + \chi_{15}\) |
| \(M_{23}\) | \(\chi_3, \chi_4\) | 45                | \(\mathbb{I} + \chi_{17}\) |
| \(M_{24}\) | \(\chi_3, \chi_4\) | 45                | \(\mathbb{I} + \chi_{19}\) |
| \(J_4\) | \(\chi_2, \chi_3\) | 1333              | \(\mathbb{I} + \chi_{11}\) |

What about finite subgroups of \(O(V)\) with \(M_4(G, V) = 3\)? Again the Atlas gives some examples of finite simple groups \(G\) with a low dimensional irreducible orthogonal representation \(V\) for which we have \(M_4(G, V) = 3\). Here are some of them.

<table>
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<tr>
<th>(G)</th>
<th>character (\chi) of (V)</th>
<th>(\text{dim}(V))</th>
<th>(\chi^2 = ?)</th>
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<td>(U_4(2))</td>
<td>(\chi_4)</td>
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<td>(\mathbb{I} + \chi_7 + \chi_9)</td>
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<tr>
<td>(S_6(2))</td>
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<td>(\mathbb{I} + \chi_4 + \chi_6)</td>
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<td>Dimension</td>
<td>( \chi^2 )</td>
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<td>( \mathbb{I} + \chi_5 + \chi_6 )</td>
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**Questions**

1. Given a connected algebraic group G over \( \mathbb{C} \) with \( \text{Lie}(G) \) simple, what if any are the finite subgroups of G which act irreducibly on \( \text{Lie}(G) \)?

2. Given a finite set of irreducible representations \( \{ V_i \} \) of such a G, what if any are the finite subgroups \( \Gamma \) of G which act irreducibly on every \( V_i \)? From the data \( (G, \{ V_i \}) \), how can one tell if any such \( \Gamma \) will exist? For example, if G is simple and simply connected, can we find such a \( \Gamma \) if we take for \( \{ V_i \} \) all the fundamental representations of G. (For SL(N), pick any even \( m \geq 4 \): then the subgroup \( \Gamma_m \subseteq \text{SL}(N) \) consisting of all permutation–shaped matrices of determinant one with entries in \( \mu_m \) is such a subgroup.) If we take for \( \{ V_i \} \) all the irreducible representations whose highest weight is the sum of at most two fundamental weights? (For SL(N), the groups \( \Gamma_m \) above fail here, already for \( \text{Sym}^2(\text{std}_N) = E(2\omega_1) \). Indeed, the \( \mathbb{C} \)-span of the squares \( (e_1)^2 \) of the standard basis elements \( e_i \) of \( \mathbb{C}^N \) is a \( \Gamma_m \)-stable subspace of \( \text{Sym}^2(\text{std}_N) \).)

3. Given a reductive, Zariski closed subgroup G of \( \text{GL}(V) \), can one classify the finite subgroups \( \Gamma \subseteq G \) for which \( M_4(\Gamma, V) = M_4(G, V) \)?

4. Given \( G \) as in 3) above, and an integer \( k \geq 1 \), let us say that a finite subgroup \( \Gamma \subseteq G \) "spoofs" \( G \) to order \( k \) if we have

\[
M_{2\ell}(\Gamma, V) = M_{2\ell}(G, V) \quad \text{for all} \quad 1 \leq \ell \leq k.
\]

For a given \( G \), what can we say about the set \( \text{Spoof}(G) \) of integers \( k \geq 1 \) for which there exists a finite subgroup \( \Gamma \subseteq G \) which spoofs \( G \) to order \( k \)? This set may consist of all \( k \geq 1 \). Take for \( G \) the diagonal subgroup of \( \text{GL}(N) \), and, for each integer \( m \geq 2 \), take \( \Gamma_m \) the finite subgroup of \( G \) consisting of diagonal matrices with entries in \( \mu_m \). Then \( \Gamma_m \) spoofs \( G \) to order \( m-1 \). Or take \( G \).
itself to be finite, then $\Gamma = G$ spoofs $G$ to any order. Is it true that if $G^0$ is semisimple and nontrivial, then the set $\text{Spoof}(G)$ is finite?

References


