(9.1) Weierstrass sheaves

(9.1.1) In this chapter, we will study various several parameter families of elliptic curves, which will serve as "inputs" in later chapters. What these families have in common is that all of them are given as families of Weierstrass equations.

(9.1.2) We begin with some general results.

Proposition 9.1.3 Let k be a field, and S/k a separated k-scheme of finite type which is locally a complete intersection, everywhere of some dimension d. Pick functions a_1, a_2, a_3, a_4, a_6 in $\Gamma(S, \mathcal{O}_S)$, and consider the locus $\mathcal{E}^{\mathrm{aff}}$ in $\mathbb{A}^2\times S$ defined by the generalized Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

View $\mathcal{E}^{\mathrm{aff}}$ as a relative affine curve $\pi : \mathcal{E}^{\mathrm{aff}} \to S$. Then for any prime number ℓ invertible in k, $R^1\pi_!\overline{\mathbb{Q}}_\ell[d]$ is a perverse sheaf on S.

proof Because $\mathcal{E}^{\mathrm{aff}}$ is defined in $\mathbb{A}^2\times S$ by one equation, and S itself is locally a complete intersection, we see that $\mathcal{E}^{\mathrm{aff}}$ itself is locally a complete intersection, everywhere of dimension d+1. Therefore [Ka-SMD, Corollary 6] $\overline{\mathbb{Q}}_\ell[d+1]$ is perverse on $\mathcal{E}^{\mathrm{aff}}$. Now $\mathcal{E}^{\mathrm{aff}}$ is a finite flat double cover of the x-line, by the map

$$\rho : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^1\times S,$$
$$(x, y, s) \mapsto (x, s).$$

Since ρ is finite, $\rho_*\overline{\mathbb{Q}}_\ell[d+1] = \rho_!\mathbb{Q}_\ell[d+1]$ is perverse on $\mathbb{A}^1\times S$. Because ρ is finite and flat of degree 2, we have the Trace map

$$\mathrm{Trace} : \rho_*\overline{\mathbb{Q}}_\ell = \rho_*\rho^*\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell.$$

We also have the natural adjunction map

$$\overline{\mathbb{Q}}_\ell \to \rho_*\rho^*\overline{\mathbb{Q}}_\ell.$$

The composite

$$\overline{\mathbb{Q}}_\ell \to \rho_*\rho^*\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$$

is multiplication by 2, cf. [SGA 4, Expose XVIII, Thm. 2.9, Var4, I].
So we have a direct sum decomposition

$$\rho_*\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell \oplus (\mathcal{K} := \mathrm{Ker}(\mathrm{Trace} : \rho_*\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell)).$$

Thus $\mathcal{K}[d+1]$ is a direct summand of $\rho_*\overline{\mathbb{Q}}_\ell[d+1]$, hence is itself perverse on $\mathbb{A}^1\times S$.

We make use of this perversity as follows. We factor π as

$$\mathcal{E}^{\mathrm{aff}} \xrightarrow{\rho} \mathbb{A}^1\times S \xrightarrow{\mathrm{pr}_2} S.$$

We will show that

$$R^1(\mathrm{pr}_2)_!\mathcal{K} \cong R^1\pi_!\overline{\mathbb{Q}}_\ell,$$

$$R^i(\mathrm{pr}_2)_!\mathcal{K} = 0 \text{ for } i \neq 1.$$

Once we have shown this, then we have

$$R(\mathrm{pr}_2)_!\mathcal{K}[d+1] = R^1(\mathrm{pr}_2)_!\mathcal{K}[d] = R^1\pi_!\overline{\mathbb{Q}}_\ell[d].$$

This shows that $R(\mathrm{pr}_2)_!\mathcal{K}[d+1]$ is semiperverse on S, since it is a single sheaf, concentrated in dimension d = dim(S). To show that $R(\mathrm{pr}_2)_!\mathcal{K}[d+1]$ is perverse, we must show that its dual $R(\mathrm{pr}_2)_*(D(\mathcal{K}[d+1]))$ is semiperverse. But $D(\mathcal{K}[d+1])$ is perverse, being the dual of the perverse object $\mathcal{K}[d+1]$, and pr_2 is an affine morphism, so the semiperversity results from Artin's theorem, cf. [SGA 4, Expose XIV, 3.1] and [BBD, 4.1.1].

We next calculate the $R^i(\mathrm{pr}_2)_!\mathcal{K}$. Because ρ is finite, the Leray spectral sequence gives

$$R^i\pi_!\overline{\mathbb{Q}}_\ell = R^i(\mathrm{pr}_2)_!(\rho_*\overline{\mathbb{Q}}_\ell) = R^i(\mathrm{pr}_2)_!\overline{\mathbb{Q}}_\ell \oplus R^i(\mathrm{pr}_2)_!\mathcal{K}.$$

The cohomology of $\mathbb{A}^1$ is known: we have

$$R^2(\mathrm{pr}_2)_!\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell(-1),$$

$$R^i(\mathrm{pr}_2)_!\overline{\mathbb{Q}}_\ell = 0 \text{ for } i \neq 2.$$

The vanishing of $R^1(\mathrm{pr}_2)_!\overline{\mathbb{Q}}_\ell$ shows that $R^1(\mathrm{pr}_2)_!\mathcal{K} \cong R^1\pi_!\overline{\mathbb{Q}}_\ell$. The only possibly nonvanishing $R^i(\mathrm{pr}_2)_!\mathcal{K}$ have i in {0, 1, 2}. To show that $R^i(\mathrm{pr}_2)_!\mathcal{K} = 0$ for i = 0 and 2, we argue as follows. The geometric fibres of $\mathcal{E}^{\mathrm{aff}}/S$ are geometrically irreducible affine plane curves, so $R^0\pi_!\overline{\mathbb{Q}}_\ell = 0$ [Ka-Sar-RMFEM, 10.1.5, applied fibre by fibre] and $R^2\pi_!\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell(-1)$. Looking fibre by fibre and comparing dimensions, we get the vanishing of $R^i(\mathrm{pr}_2)_!\mathcal{K}$ for i = 0 and 2. QED

Proposition 9.1.4 Hypotheses and notations as in Proposition 9.1.3

above, suppose in addition that k is a finite field, and that S/k is smooth and geometrically connected. Suppose further that the discriminant Δ of our Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

is nonzero in $\Gamma(S, \mathcal{O}_S)$, and that its j invariant is nonconstant. Then we have the following results.

1) The perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[d]((d+1)/2)$ on S is mixed of weight ≤ 0.

2) $\mathrm{Gr}^0(R^1\pi_!\overline{\mathbb{Q}}_\ell[d]((d+1)/2))$ is geometrically irreducible, geometrically nonconstant, and autodual. Its restriction to $S[1/\Delta]$ is $\mathcal{W}(d/2)[d]$, for $\mathcal{W}$ the lisse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell(1/2)|S[1/\Delta]$, which is pure of weight zero and symplectically self dual, with $G_{geom} = SL(2)$.

proof That $R^1\pi_!\overline{\mathbb{Q}}_\ell[d]((d+1)/2)$ is mixed of weight ≤ 0, or equivalently that $R^1\pi_!\overline{\mathbb{Q}}_\ell$ on S is mixed of weight ≤ 1, is a special case of Deligne's theorem [De-Weil II, 3.3.1]. In the case at hand, it is due to Hasse. On the open set $S[1/\Delta]$ where Δ is invertible, $R^1\pi_!\overline{\mathbb{Q}}_\ell$ is lisse of rank 2 and (by [Hasse]) pure of weight one. At finite field-valued points of S where Δ vanishes, the stalk of $R^1\pi_!\overline{\mathbb{Q}}_\ell$ is the H_c^1 of a singular Weierstrass cubic, which either vanishes ("additive case") or is one-dimensional ("multiplicative case"); in the latter case, the Frobenius eigenvalue is ± 1. The hypothesis that j is nonconstant implies [De-Weil II, 3.5.5] that G_{geom} for $R^1\pi_!\overline{\mathbb{Q}}_\ell \mid S[1/\Delta]$ is SL(2). This in turn implies that on $S[1/\Delta]$, the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[d]((d+1)/2)$ is geometrically irreducible, and is its own Gr^0.

To show that $\mathrm{Gr}^0(R^1\pi_!\overline{\mathbb{Q}}_\ell[d]((d+1)/2))$ is geometrically irreducible on S, we apply the criterion of Second Corollary 1.8.3, part 3). Putting $M := R^1\pi_!\overline{\mathbb{Q}}_\ell[d]((d+1)/d)$, we must show that there exists a real $\varepsilon > 0$ such that for variable finite extensions E/k, we have

$$\Sigma_{x \text{ in } S(E)} |M(E, x)|^2 = 1 + O((\#E)^{-\varepsilon/2}).$$

Thanks to the Othogonality Theorem 1.7.2, part 3), the geometric irreducibility of $R^1\pi_!\overline{\mathbb{Q}}_\ell \mid S[1/\Delta]$ tells that we have

$$\Sigma_{x \text{ in } S[1/\Delta](E)} |M(E, x)|^2 = 1 + O((\#E)^{-1/2}).$$

The terms where $\Delta = 0$ make a negligible contribution. Indeed, each term $|M(E, x)|^2$ for x an E-valued point of the locus $\Delta = 0$ is, as noted above, either 0 or $(\#E)^{-d-1}$, and the number of terms over which we sum is $O((\#E)^{d-1})$.

That the geometrically irreducible object M is self dual results (by Third Corollary 1.8.4, part 3)) from the fact that via any ι, its trace function takes real values.

That $\mathrm{Gr}^0(R^1\pi_!\overline{\mathbb{Q}}_\ell[d]((d+1)/2))$ is not geometrically constant is obvious from the fact that already on the dense open set $S[1/\Delta]$ it is not geometrically constant, indeed its G_{geom} is $SL(2)$. QED

Vanishing Proposition 9.1.5 Let k be a field, $d \geq 1$ an integer, $\mathbb{A}^d/k$ the affine space $\mathrm{Spec}(k[t_1,\ldots, t_d])$. Choose a_1, a_2, a_3, and a_4 arbitrarily in the subring $k[t_1,\ldots, t_{d-1}]$ of polynomials which do not involve t_d. Choose $a_6 := t_d$. Over $\mathbb{A}^d/k$, consider the Weierstrass cubic $\mathcal{E}^{aff}$

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Then for any prime ℓ invertible in k, we have

$$H_c{}^*(\mathbb{A}^d\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}_\ell) = 0,$$

$$H_c{}^*(\mathbb{A}^d\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}_\ell[d]) = 0.$$

proof The two statements are trivially equivalent. We will prove the first. Consider the Leray spectral sequence for π,

$$E_2{}^{a,b} := H_c{}^a(\mathbb{A}^d\otimes_k\overline{k}, R^b\pi_!\overline{\mathbb{Q}}_\ell) \Rightarrow H_c{}^{a+b}(\mathcal{E}^{aff}\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell).$$

On the one hand, we have

$$R^i\pi_!\overline{\mathbb{Q}}_\ell = 0 \text{ for } i \text{ not in } \{1, 2\},$$

$$R^2\pi_!\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell(-1).$$

So $E_2{}^{a,b}$ vanishes unless b is 1 or 2. The only nonvanishing $E_2{}^{a,2}$ term is $E_2{}^{2d,2}$. The terms $E_2{}^{a,1}$ vanish for a outside $[0, 2d]$. From this pattern of vanishing

$$\begin{array}{cccccc} & & & & & * \\ * & * & * & * & \dots & * \end{array}$$

we see that the spectral sequence degenerates at E_2, and that

$$H_c{}^a(\mathbb{A}^d\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}_\ell) \cong H_c{}^{a+1}(\mathcal{E}^{aff}\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell), \text{ for } 0 \le a \le 2d.$$

Because a_6 is t_d, and the other a_i do not involve t_d, we can use the Weierstrass equation to "solve for a_6", i.e., we have

$$\mathcal{E}^{aff} \cong \mathbb{A}^{d+1}, \text{ coordinates } x,\ y,\ t_1,\dots,\ t_{d-1}.$$

Therefore we have

$$H_c{}^i(\mathcal{E}^{aff}\otimes_k\overline{k}, \overline{\mathbb{Q}}_\ell) = 0 \text{ for } i \neq 2d+2.$$

Hence we have $H_c{}^*(\mathbb{A}^d\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}_\ell) = 0$. QED

(9.2) The situation when 2 is invertible

(9.2.1) In this section, we work over a $\mathbb{Z}[1/2]$-scheme S, i.e., a scheme S on which 2 is invertible. This allows us to complete the square in a Weierstrass cubic, and to reduce to the case $a_1 = a_3 = 0$. More precisely, pick functions a_1, a_2, a_3, a_4, a_6 in $\Gamma(S, \mathcal{O}_S)$, and consider the Weierstrass cubic $\mathcal{E}^{aff}$ defined by

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The quantity $\tilde{y} := y + (a_1x + a_3)/2$ satisfies the equation

$$\tilde{y}^2 = x^3 + \tilde{a}_2x^2 + \tilde{a}_4x + \tilde{a}_6,$$

where

$$\tilde{a}_2 := a_2 + (a_1)^2/4,$$
$$\tilde{a}_4 := a_4 + (a_1a_3)/2,$$
$$\tilde{a}_6 := a_6 + (a_3)^2/4.$$

Twisting Lemma 9.2.2 Let S be a $\mathbb{Z}[1/2]$-scheme, a_2, a_4, a_6 functions in $\Gamma(S, \mathcal{O}_S)$. Consider the locus $\mathcal{E}^{aff}$ in $\mathbb{A}^2\times S$ defined by the Weierstrass cubic

$$y^2 = x^3 + a_2x^2 + a_4x + a_6.$$

View $\mathcal{E}^{aff}$ as a relative affine curve $\pi : \mathcal{E}^{aff} \to S$.

Fix an invertible function f on S, and denote by $\mathcal{E}_f{}^{aff}$ the locus in $\mathbb{A}^2 \times S$ defined by the Weierstrass equation

$$y^2 = x^3 + fa_2x^2 + f^2a_4x + f^3a_6.$$

View $\mathcal{E}_f{}^{aff}$ as a relative affine curve $\pi_f : \mathcal{E}^{aff} \to S$.

For any prime number ℓ invertible on S, we have a canonical isomorphism of sheaves on S,

$$R^1(\pi_f)_!\overline{\mathbb{Q}}_\ell \cong \mathcal{L}_{\chi_2(f)} \otimes R^1\pi_!\overline{\mathbb{Q}}_\ell,$$

where χ_2 denotes the quadratic character, and $\mathcal{L}_{\chi_2}$ the corresponding Kummer sheaf.

proof Denote by $\rho : \mathcal{E}^{aff} \to \mathbb{A}^1 \times S$ the projection onto the x-line. Because 2 is invertible, and $\mathcal{E}^{aff}$ has equation of the form $y^2 = f(x)$, we have

$$\rho_*\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell \oplus \mathcal{L}_{\chi_2(f(x))}$$

$$= \overline{\mathbb{Q}}_\ell \oplus \mathcal{L}_{\chi_2(x^3 + a_2x^2 + a_4x + a_6)}.$$

Just as in the proof of Proposition 9.1.3, the vanishing of $R^1(pr_2)_!\overline{\mathbb{Q}}_\ell$ gives

$$R^1\pi_!\overline{\mathbb{Q}}_\ell \cong R^1(pr_2)_!\mathcal{L}_{\chi_2(x^3 + a_2x^2 + a_4x + a_6)}.$$

Applying this same formula to π_f, we get

$$R^1(\pi_f)_!\overline{\mathbb{Q}}_\ell \cong R^1(pr_2)_!\mathcal{L}_{\chi_2(x^3 + fa_2x^2 + f^2a_4x + f^3a_6)}.$$

Now apply the S-automorphism $(x, s) \mapsto (xf, s)$ of $\mathbb{A}^1 \times S$ to rewrite the left hand side as

$$R^1(pr_2)_!\mathcal{L}_{\chi_2(x^3 + fa_2x^2 + f^2a_4x + f^3a_6)}$$

$$\cong R^1(pr_2)_!\mathcal{L}_{\chi_2((f^3)(x^3 + a_2x^2 + a_4x + a_6))}$$

$$\cong R^1(pr_2)_!\mathcal{L}_{\chi_2((f)(x^3 + a_2x^2 + a_4x + a_6))}$$

(because χ_2 is quadratic)

$$\cong \mathcal{L}_{\chi_2(f)} \otimes R^1(pr_2)_!\mathcal{L}_{\chi_2(x^3 + a_2x^2 + a_4x + a_6)}$$

(by the projection formula)

$$\cong \mathcal{L}_{\chi_2(f)} \otimes R^1\pi_!\overline{\mathbb{Q}}_\ell.$$

QED

(9.3) Theorems of geometric irreducibility in odd characteristic

(9.3.1) In this section, we work over a field k in which 2 is invertible. We will give a number of families of Weierstrass cubics over affine spaces, $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^d$, for which the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[d]$ on $\mathbb{A}^d$ is geometrically irreducible.

(9.3.2) There are three simple principles we will apply, given as 9.3.3-5 below. The first two are standard. We include proofs for ease of reference.

External tensor product Lemma 9.3.3 Let k be a field in which a prime ℓ is invertible. Let S and T be geometrically irreducible nonempty separated k-schemes of finite type. Suppose K and L are $\overline{\mathbb{Q}}_\ell$-perverse sheaves on S and T respectively. Suppose that K is geometrically irreducible on S, and that L is geometrically irreducible on T. Then their external tensor product

$$K\boxtimes L := (pr_1{}^*K)\otimes(pr_2{}^*L)$$

is geometrically irreducible on $S\times_k T$. If K and L are both self dual, then $K\boxtimes L$ is self dual.

proof We first reduce to the case when k is algebraically closed, S is the support of K, T is the support of L, and S and T are both reduced. Then [BBD, 4.3.2 and 4.3.3] we can find affine open sets $U \subset S$ and $V \subset T$, inclusions denoted $j : U \to S$, $k : V \to T$, with U/k and V/k smooth and connected, and irreducible lisse sheaves $\mathcal{G}$ and $\mathcal{H}$ on U and V respectively, such that $K = j_{!*}\mathcal{G}[\dim U]$, $L = k_{!*}\mathcal{H}[\dim V]$. Their external tensor product $K\boxtimes L$ is the middle extension $(j\times k)_{!*}((\mathcal{G}\boxtimes\mathcal{H})[\dim(U\times V)])$. So it suffices [BBD, 4.3.1 and 4.3.3] to see that $\mathcal{G}\boxtimes\mathcal{H}$ is geometrically irreducible on $U\times V$, i.e., that if we pick base points u in U and v in V, the action ρ of $\pi_1(U\times V, u\times v)$ on $\mathcal{G}\boxtimes\mathcal{H}$ is irreducible. But there is a canonical homomorphism

$$\alpha : \pi_1(U, u)\times\pi_1(V, v) \to \pi_1(U\times V, u\times v).$$

The composite $\rho\circ\alpha$ is the external tensor product of the representations $\mathcal{G}$ of $\pi_1(U, u)$ and $\mathcal{H}$ of $\pi_1(V, v)$. Since the external tersor product of finite-dimensional irreducible $\overline{\mathbb{Q}}_\ell$-representations of two groups G and H is an irreducible representation of $G\times H$, $\rho\circ\alpha$ is irreducible, and a fortiori ρ itself is irreducible.

That the external tensor product of self dual perverse sheaves is self dual results from [BBD, 4.2.7 (b)]. QED

Local Nature Lemma 9.3.4 Let k be a field in which a prime ℓ is invertible. Let S be a geometrically irreducible separated k-scheme of finite type, which is the union of finitely many Zariski open sets U_i. Suppose K is a $\overline{\mathbb{Q}}_\ell$-perverse sheaf on S, whose support is S. Then the following conditions are equivalent:
1) K is geometrically irreducible on S,
2) $K|U_i$ is geometrically irreducible on U_i, for each i.

proof We reduce immediately to the case when k is algebraically closed. Since the support of K is S, there exists an affine open set $V \subset S$ such that $K|V$ is $\mathcal{G}[d]$, for a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{G}$ on V. At the expense of shrinking V, we may further assume that $V \subset U_i$ for each i. Now K is geometrically irreducible on S if and only if $\mathcal{G}$ is geometrically irreducible as lisse sheaf on V and if, denoting by $j : V \to S$ the inclusion, the canonical adjunction map of perverse sheaves on S

$$K \to Rj_\star j^\star K = Rj_\star(\mathcal{G}[d])$$

is injective, and maps K isomorphically to the perverse subsheaf $j_{!\star}(\mathcal{G}[d])$ of $Rj_\star(\mathcal{G}[d])$.

On the other hand, $K|U_i$ is geometrically irreducible on U_i if and only if $\mathcal{G}$ is geometrically irreducible as lisse sheaf on V and if, denoting by $j_i : V \to U_i$ the inclusion, the canonical adjunction map of perverse sheaves on U_i

$$K|U_i \to Rj_{i\star}j^\star K = Rj_{i\star}(\mathcal{G}[d])$$

is injective, and maps $K|U_i$ isomorphically to the perverse subsheaf $j_{i!\star}(\mathcal{G}[d])$ of $Rj_{i\star}(\mathcal{G}[d])$. By transitivity [BBD, 2.1.7], if we denote by $k_i : U_i \to S$ the inclusion, we have

$$j_{!\star}(\mathcal{G}[d]) = k_{i!\star}j_{i!\star}(\mathcal{G}[d]).$$

Restricting to U_i, this gives

$$(j_{!\star}(\mathcal{G}[d]))|\ U_i = j_{i!\star}(\mathcal{G}[d]).$$

If either 1) or 2) holds, then $\mathcal{G}$ is geometrically irreducible as lisse sheaf on V. If 1) holds, the above equality says that $K|U_i$ is the middle extension $j_{i!\star}(\mathcal{G}[d])$, and hence 2) holds. If 2) holds, both K and $j_{!*}(\mathcal{G}[d])$ are perverse subsheaves of the perverse sheaf $Rj_*(\mathcal{G}[d])$ which agree on each U_i, and hence coincide. Thus 1) holds. QED

Missing Points Lemma 9.3.5 Let k be an algebraically closed field in which a prime ℓ is invertible. Let S/k be an affine, smooth, connected k-scheme, of dimension $d \geq 1$. Let $\mathcal{G}$ be a constructible $\overline{\mathbb{Q}}_\ell$-sheaf on S. Suppose that the following three conditions hold.

1) $\mathcal{G}[d]$ is a perverse sheaf on S.

2) $H_c{}^*(S, \mathcal{G}[d]) = 0$.

3) There exists a finite subset $Z \subset S(k)$ such that $\mathcal{G}|Z = 0$ and such that $\mathcal{G}[d]|S - Z$ is irreducible and self dual as perverse sheaf on S - Z. Then $\mathcal{G}[d]$ is irreducible and self dual as a perverse sheaf on S.

proof Denote by $j : S - Z \to S$ the inclusion. By 3), $j^*\mathcal{G}[d]$ is perverse irreducible and self dual on S - Z. So its middle extension $j_{!*}(j^*\mathcal{G}[d])$ is perverse irreducible and self dual on S. So it suffices to show that $\mathcal{G}[d] \cong j_{!*}(j^*\mathcal{G}[d])$. But $\mathcal{G}|Z = 0$ by 3), i.e., $\mathcal{G} \cong j_!(j^*\mathcal{G})$. So it suffices to show that $j_!(j^*\mathcal{G}[d]) \cong j_{!*}(j^*\mathcal{G}[d])$. Since $j^*\mathcal{G}[d]$ is self dual, it suffices by duality to show that $j_{!*}(j^*\mathcal{G}[d]) \cong Rj_*(j^*\mathcal{G}[d])$. Since Z is a finite set of k-points, we obtain $j_{!*}(j^*\mathcal{G}[d])$ by first forming $Rj_*(j^*\mathcal{G}[d])$ and then by throwing away all cohomology sheaves in degree ≥ 0. So we have $j_{!*}(j^*\mathcal{G}[d]) \cong Rj_*(j^*\mathcal{G}[d])$ if and only if the object $Rj_*(j^*\mathcal{G}[d])$ has no nonzero cohomology sheaves in degree ≥ 0, if and only if the object $Rj_*j^*\mathcal{G}$ has no nonzero cohomology sheaves in degree $\geq d$. Now above degree zero, $Rj_*j^*\mathcal{G}$ is concentrated on Z. So we must show that $(Rj_*j^*\mathcal{G})|Z$ is concentrated in degree $\leq d$. As Z is finite, it is the same to show that $H^i(Z, (Rj_*j^*\mathcal{G})|Z) = 0$ for $i \geq d$.

To see this, we argue as follows. We have a distinguished triangle

$$\ldots\ j_!j^*\mathcal{G} \to Rj_*j^*\mathcal{G} \to (Rj_*j^*\mathcal{G})|Z \to \ldots .$$

By 2), we have $H_c{}^*(S, \mathcal{G}[d]) = 0$, a vanishing we rewrite as

$$H_c{}^*(S, j_!(j^*\mathcal{G}[d])) = 0.$$

Since $j^*\mathcal{G}[d]$ is self dual, by 3), the dual of this vanishing is

$$H^*(S, Rj_*(j^*\mathcal{G}[d])) = 0,$$

i.e.,

$$H^*(S, Rj_*j^*\mathcal{G}) = 0.$$

Looking back at the distinguished triangle, we see that the coboundary map is an isomorphism

$$H^i(Z, (Rj_*j^*\mathcal{G})|Z) \cong H^{i+1}(S, j_!j^*\mathcal{G}).$$

Because S is affine of dimension d, and $j_!j^*\mathcal{G}$ is a single constructible sheaf, we have

$$H^{i+1}(S, j_!j^*\mathcal{G}) = 0 \text{ for } i \geq d.$$

Therefore we find

$$H^i(Z, (Rj_*j^*\mathcal{G})|Z) = 0 \text{ for } i \geq d,$$

as required. QED

Theorem 9.3.6 Let k be a field in which 2 is invertible, $\mathbb{A}^2 = \mathrm{Spec}(k[s, t])$, and $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^2$ the relative affine curve defined by the Weierstrass equation

$$y^2 = x^3 + sx^2 + s^3t.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

proof Immediate reduction to the case when k is algebraically closed. Over each point of the locus $s = 0$, we have fibre $y^2 = x^3$. Looking fibre by fibre, we see that $R^1\pi_!\overline{\mathbb{Q}}_\ell \mid (s = 0)$ vanishes. On the locus where s is invertible, our curve is the s-twist of the curve

$$\mathcal{E}_1{}^{\mathrm{aff}} : y^2 = x^3 + x^2 + t,$$

whose structural morphism we denote $\pi_1 : \mathcal{E}_1{}^{\mathrm{aff}} \to \mathbb{A}^1$.

By Twisting Lemma 9.2.2, $R^1\pi_!\overline{\mathbb{Q}}_\ell$ on $\mathbb{A}^2[1/s] \cong \mathbb{G}_m\times\mathbb{A}^1$ is the external tensor product

$$R^1\pi_!\overline{\mathbb{Q}}_\ell \cong \mathcal{L}_{\chi_2(s)}\boxtimes R^1\pi_{1!}\overline{\mathbb{Q}}_\ell.$$

Denote by $j : \mathbb{G}_m \to \mathbb{A}^1$ the inclusion. Then $j_*\mathcal{L}_{\chi_2(s)}$ vanishes at s=0. So if we extend by zero from $\mathbb{A}^2[1/s]$ to $\mathbb{A}^2$, we get an isomorphism on $\mathbb{A}^2$,

$$R^1\pi_!\overline{\mathbb{Q}}_\ell \cong j_*\mathcal{L}_{\chi_2(s)} \boxtimes R^1\pi_{1!}\overline{\mathbb{Q}}_\ell,$$

i.e.,

$$R^1\pi_!\overline{\mathbb{Q}}_\ell[2] \cong j_{!*}(\mathcal{L}_{\chi_2(s)}[1]) \boxtimes R^1\pi_{1!}\overline{\mathbb{Q}}_\ell[1].$$

The asserted vanishing $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$ is obvious from this isomorphism, the Kunneth formula, and the vanishing of $H_c{}^*(\mathbb{A}^1\otimes_k\overline{k}, j_*\mathcal{L}_{\chi_2})$.

It remains to establish the irreducibility and self duality of $R^1\pi_!\overline{\mathbb{Q}}_\ell[2] \cong j_{!*}(\mathcal{L}_{\chi_2(s)}[1]) \boxtimes R^1\pi_{1!}\overline{\mathbb{Q}}_\ell[1]$. The first factor $j_{!*}(\mathcal{L}_{\chi_2(s)}[1])$ is perverse irreducible and self dual. So it suffices, by Lemma 9.3.3, to show that $R^1\pi_{1!}\overline{\mathbb{Q}}_\ell[1]$ is perverse irreducible and self dual. The curve $y^2 = x^3 + x^2 + t$ has nonconstant j, and its $\Delta = -16(4t + 27t^2)$ has only simple zeroes in any characteristic not two. So $R^1\pi_{1!}\overline{\mathbb{Q}}_\ell$ is a middle extension, by Corollary 7.5.5. Its restriction to $\mathbb{A}^1[1/\Delta]$ is geometrically irreducible, because the j invariant is not constant. Therefore $R^1\pi_{1!}\overline{\mathbb{Q}}_\ell[1]$ is the middle extension from $\mathbb{A}^1[1/\Delta]$ of a perverse irreducible which is self dual, so is itself perverse irreducible and self dual. QED

Corollary 9.3.7 Hypotheses as in Theorem 9.3.6 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.

proof Immediate from Proposition 9.1.4, part 2). QED

Theorem 9.3.8 Let k be a field in which 2 is invertible. Over $\mathbb{A}^2 = \mathrm{Spec}(k[a_2, a_6])$, consider the relative affine curve

$$\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^2$$

defined by the Weierstrass equation

$$y^2 = x^3 + a_2x^2 + a_6.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c^*(\mathbb{A}^2 \otimes_k \bar{k}, R^1\pi_!\bar{\mathbb{Q}}) = 0$.

proof Immediate reduction to the case when k is algebraically closed. The asserted vanishing $H_c^*(\mathbb{A}^2 \otimes_k \bar{k}, R^1\pi_!\bar{\mathbb{Q}}) = 0$ is a special case of Proposition 9.1.5.

We next show that over the open set $\mathbb{A}^2[1/a_2]$ where a_2 is invertible, $R^1\pi_!\bar{\mathbb{Q}}_\ell[2] \mid \mathbb{A}^2[1/a_2]$ is geometrically irreducible. Indeed, the change of variables $s = a_2$, $t = a_6/(a_2)^3$ gives us the family

$$y^2 = x^3 + sx^2 + s^3t$$

over the open set of the (s, t) plane where s is invertible. So by Theorem 9.3.6, we find that $R^1\pi_!\bar{\mathbb{Q}}_\ell[2] \mid \mathbb{A}^2[1/a_2]$ is geometrically irreducible and self dual. Since the j invariant is nonconstant, $R^1\pi_!\bar{\mathbb{Q}}_\ell[2] \mid \mathbb{A}^2[1/\Delta]$ is geometrically irreducible and self dual. The union of the the two open sets $\mathbb{A}^2[1/a_2]$ and $\mathbb{A}^2[1/\Delta]$ is $\mathbb{A}^2 - (0,0)$. Indeed, Δ is

$$\Delta = -16a_6(4(a_2)^3 + 27a_6),$$

so if Δ and a_2 both vanish, then a_6 also vanishes. Thus $R^1\pi_!\bar{\mathbb{Q}}_\ell[2]$ is irreducible and self dual as perverse sheaf on $\mathbb{A}^2 - (0,0)$.

We now observe that $R^1\pi_!\bar{\mathbb{Q}}_\ell$ vanishes at (0,0) (by inspection). By Proposition 9.1.3, $R^1\pi_!\bar{\mathbb{Q}}_\ell[2]$ is perverse on $\mathbb{A}^2$. By Proposition 9.1.5, we have $H_c^*(\mathbb{A}^2, R^1\pi_!\bar{\mathbb{Q}}_\ell) = 0$. So we may invoke the Missing Points Lemma 9.3.5, with $S = \mathbb{A}^2$, $Z = (0, 0)$, and $\mathcal{G} = R^1\pi_!\bar{\mathbb{Q}}_\ell$. QED

Corollary 9.3.9 Hypotheses as in Theorem 9.3.8 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\bar{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.

proof Immediate from Proposition 9.1.4, part 2). QED

Theorem 9.3.10 Let k be a field in which 6 is invertible, $\mathbb{A}^2 = \mathrm{Spec}(k[s, t])$, and $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^2$ the relative affine curve defined by the Weierstrass equation

$$y^2 = 4x^3 - s^2tx - s^3t.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

proof The proof is very similar to that of Theorem 9.3.6 Just as we did there, we first reduce to the case when k is algebraically closed. Looking fibre by fibre, we see that $R^1\pi_!\overline{\mathbb{Q}}_\ell$ vanishes on the locus s=0. On the open set where s is invertible, our curve is the s-twist of the curve

$$\mathcal{E}_1{}^{\mathrm{aff}} : y^2 = 4x^3 - tx - t,$$

whose structural morphism we denote $\pi_1 : \mathcal{E}_1{}^{\mathrm{aff}} \to \mathbb{A}^1$.
We find

$$R^1\pi_!\overline{\mathbb{Q}}_\ell[2] \cong j_{!*}(\mathcal{L}_{\chi_2(s)}[1])\boxtimes R^1\pi_{1!}\overline{\mathbb{Q}}_\ell[1],$$

which makes clear the asserted vanishing $H_c^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

It remains to show that $R^1\pi_{1!}\overline{\mathbb{Q}}_\ell[1]$ is perverse irreducible and self dual on $\mathbb{A}^1$. Since the j invariant j = 1728t/(t - 27) of $\mathcal{E}_1{}^{\mathrm{aff}}$ is nonconstant, $R^1\pi_{1!}\overline{\mathbb{Q}}_\ell \mid \mathbb{A}^1[1/\Delta]$ is lisse, irreducible, and self dual. As $\Delta = t^2(t - 27)$, we see from Corollary 7.5.5 that $R^1\pi_{1!}\overline{\mathbb{Q}}_\ell$ is a middle extension, and hence that $R^1\pi_{1!}\overline{\mathbb{Q}}_\ell[1]$ is perverse irreducible and self dual on $\mathbb{A}^1$. QED

Corollary 9.3.11 Hypotheses as in Theorem 9.3.10 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.

proof Immediate from Proposition 9.1.4, part 2). QED

Theorem 9.3.12 Let k be a field in which 6 is invertible. Over $\mathbb{A}^2 = \mathrm{Spec}(k[g_2, g_3])$, consider the relative affine curve $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^2$ defined by the Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

proof The vanishing $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$ is a special case of Proposition 9.1.5. On the open set where g_2g_3 is invertible, the change of variables

$$s = g_3/g_2,\ t = (g_2)^3/(g_3)^2,$$

i.e.,

$$g_2 = s^2t,\ g_3 = s^3t,$$

makes the open set $\mathbb{A}^2[1/g_2g_3]$ of the (g_2, g_3)-plane isomorphic to the open set $\mathbb{A}^2[1/st]$ of the (s, t)-plane, and carries our family

$$y^2 = 4x^3 - g_2x - g_3$$

to the twist family

$$y^2 = 4x^3 - s^2tx - s^3t.$$

By the previous Theorem 9.3.10, $R^1\pi_!\overline{\mathbb{Q}}_\ell[2] \mid \mathbb{A}^2[1/g_2g_3]$ is geometrically irreducible and self dual.

Since the j invariant $j = 1728(g_2)^3/((g_2)^3 - 27(g_3)^2)$ is nonconstant, $R^1\pi_!\overline{\mathbb{Q}}_\ell[2] \mid \mathbb{A}^2[1/\Delta]$ is also geometrically irreducible and self dual. The union of the two open sets $\mathbb{A}^2[1/g_2g_3]$ and $\mathbb{A}^2[1/\Delta]$ is $\mathbb{A}^2 - (0,0)$. Indeed, as $\Delta = (g_2)^3 - 27(g_3)^2$, if both Δ and g_2g_3 vanish, then both g_2 and g_3 vanish. Therefore the restriction of $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ to $\mathbb{A}^2 - (0,0)$ is geometrically irreducible and geometrically self dual. By inspection, $R^1\pi_!\overline{\mathbb{Q}}_\ell$ vanishes at (0,0). As already proven, we have $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}_\ell) = 0$. So we have only to apply the Missing Points Lemma 9.3.5 to the situation $S = \mathbb{A}^2$, $Z = (0,0)$, $\mathcal{G} = R^1\pi_!\overline{\mathbb{Q}}_\ell$. QED

Corollary 9.3.13 Hypotheses as in Theorem 9.3.12 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.

proof Immediate from Proposition 9.1.4, part 2). QED

(9.3.14) Strangely enough, the last two results 9.3.12-13 remain true in characteristic 3, although their proofs are necessarily quite different, since these are now families of curves whose j invariant $j = 1728 = 0$ is constant.

Theorem 9.3.15 Let k be a field of characteristic 3, $\mathbb{A}^1 = \mathrm{Spec}(k[t])$, and $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^1$ the relative affine curve defined by the Weierstrass equation

$$y^2 = 4x^3 - tx - t.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[1]$ on $\mathbb{A}^1$ is geometrically irreducible and geometrically self dual.

proof Immediate reduction to the case when k is algebraically closed. Since k has characteristic 3, $\Delta = t^3$. So $R^1\pi_!\overline{\mathbb{Q}}_\ell|\mathbb{G}_m$ is a lisse sheaf of rank two which is self dual.

By Corollary 7.5.5, $R^1\pi_!\overline{\mathbb{Q}}_\ell$ is a middle extension from $\mathbb{G}_m$, hence is self dual. What we must show is that the lisse sheaf $\mathcal{G} := R^1\pi_!\overline{\mathbb{Q}}_\ell|\mathbb{G}_m$ is geometrically irreducible on $\mathbb{G}_m$. For this, we argue as follows. Because our family has constant j invariant $j = 1728 = 0$, $\mathcal{G}$ has finite geometric monodromy and hence is a semisimple representation of $\pi_1(\mathbb{G}_m)$. Denote by

$$[2]: \mathbb{G}_m \to \mathbb{G}_m$$
$$t \mapsto t^2$$

the squaring map. It makes $\mathbb{G}_m$ a Galois finite etale covering of itself with Galois group ± 1. We will show that $[2]^*\mathcal{G}$ is the direct sum of two distinct characters α and β of $\pi_1(\mathbb{G}_m)$, which are different but which are interchanged by the nontrivial galois automorphism $[-1]$ of the upper $\mathbb{G}_m$. Once we have shown this, then Mackey theory tells us that $[2]_*\alpha = [2]_*\beta$ is irreducible. So we can compute the inner product (in the realm of semisimple finite-dimensional $\overline{\mathbb{Q}}_\ell$-representations of $\pi_1(\mathbb{G}_m)$)

$$\begin{aligned}\langle \mathcal{G}, [2]_*\alpha\rangle &= (1/2)(\langle \mathcal{G}, [2]_*\alpha\rangle + \langle \mathcal{G}, [2]_*\beta\rangle)\\ &= (1/2)\langle \mathcal{G}, [2]_*[2]^*\mathcal{G}\rangle\\ &= (1/2)\langle [2]^*\mathcal{G}, [2]^*\mathcal{G}\rangle\\ &= (1/2)\langle \alpha + \beta, \alpha + \beta\rangle\\ &= (1/2)(1 + 1) = 1.\end{aligned}$$

Since $\mathcal{G}$ has rank 2, we must have $\mathcal{G} = [2]_*\alpha$, which as noted above is irreducible.

To analyze $[2]^*\mathcal{G}$, we view it as $R^1\pi_{[2]!}\overline{\mathbb{Q}}_\ell$ for the curve $\pi_{[2]} : \mathcal{E}_{[2]}{}^{\mathrm{aff}} \to \mathbb{G}_m$ defined by

$$y^2 = 4x^3 - t^2x - t^2.$$

This curve over $\mathbb{G}_m$ is the t-twist of the curve $\pi_{[2t]} : \mathcal{E}_{[2t]}{}^{\mathrm{aff}} \to \mathbb{G}_m$ defined by

$$y^2 = 4x^3 - x - 1/t.$$

We recall that $4 = 1$, and write this last equation as

$$x^3 - x = y^2 + 1/t.$$

Let us denote by ψ and $\overline{\psi}$ the two nontrivial additive characters of $\mathbb{F}_3$, and by $\mathcal{L}_\psi$ and $\mathcal{L}_{\overline{\psi}}$ the corresponding Artin-Schreier sheaves on $\mathbb{A}^1$. Denote by ρ the projection of $\mathcal{E}_{[2t]}{}^{\mathrm{aff}}$ onto the y-line over $\mathbb{G}_m$. Then we have

$$\begin{aligned}\rho_*\overline{\mathbb{Q}}_\ell &= \overline{\mathbb{Q}}_\ell \oplus \mathcal{L}_{\psi(y^2 + 1/t)} \oplus \mathcal{L}_{\overline{\psi}(y^2 + 1/t)}\\ &= \overline{\mathbb{Q}}_\ell \oplus \mathcal{L}_{\psi(y^2 + 1/t)} \oplus \mathcal{L}_{\psi(-y^2 - 1/t)}\\ &\cong \overline{\mathbb{Q}}_\ell \oplus \mathcal{L}_{\psi(1/t)}\otimes\mathcal{L}_{\psi(y^2)} \oplus \mathcal{L}_{\psi(-1/t)}\otimes\mathcal{L}_{\psi(-y^2)}.\end{aligned}$$

Applying $pr_2 : \mathbb{A}^1\times\mathbb{G}_m \to \mathbb{G}_m$, we get

$$\begin{aligned}R^1\pi_{[2t]!}\overline{\mathbb{Q}}_\ell &= R^1pr_{2!}(\rho_*\overline{\mathbb{Q}}_\ell)\\ &\cong R^1pr_{2!}(\mathcal{L}_{\psi(1/t)}\otimes\mathcal{L}_{\psi(y^2)}) \oplus R^1pr_{2!}(\mathcal{L}_{\psi(-1/t)}\otimes\mathcal{L}_{\psi(-y^2)})\\ &\cong \mathcal{L}_{\psi(1/t)}\otimes R^1pr_{2!}(\mathcal{L}_{\psi(y^2)}) \oplus \mathcal{L}_{\psi(-1/t)})\otimes R^1pr_{2!}(\mathcal{L}_{\psi(-y^2)}).\end{aligned}$$

The sheaves $R^1pr_{2!}(\mathcal{L}_{\psi(y^2)})$ and $R^1pr_{2!}(\mathcal{L}_{\psi(-y^2)})$ on $\mathbb{G}_m$ are isomorphic (by $y \mapsto iy$) and constant (Kunneth formula), of rank one, so geometrically they are isomorphic to the constant sheaf. Thus we find

$$R^1\pi_{[2t]!}\overline{\mathbb{Q}}_\ell \cong \mathcal{L}_{\psi(1/t)} \oplus \mathcal{L}_{\psi(-1/t)}.$$

So by the twisting lemma, we have

$$\begin{aligned}[2]^*\mathcal{G} &\cong \mathcal{L}_{\chi_2(t)}\otimes R^1\pi_{[2t]!}\overline{\mathbb{Q}}_\ell\\ &\cong \mathcal{L}_{\chi_2(t)}\otimes\mathcal{L}_{\psi(1/t)} \oplus \mathcal{L}_{\chi_2(t)}\otimes\mathcal{L}_{\psi(-1/t)}.\end{aligned}$$

The sheaves $\mathcal{L}_{\chi_2(t)}$ and $\mathcal{L}_{\chi_2(-t)}$ are geometrically isomorphic. On the other hand, the sheaves $\mathcal{L}_{\psi(1/t)}$ and $\mathcal{L}_{\psi(-1/t)}$ are not geometrically isomorphic, as they are inverses and both nontrivial of order 3. So putting

$$\alpha := \mathcal{L}_{\chi_2(t)} \otimes \mathcal{L}_{\psi(1/t)},$$

we have

$$[2]^*\mathcal{G} = \alpha + \beta,\ \alpha \neq \beta,\ \beta = [-1]^*\alpha,$$

as required. QED

Corollary 9.3.16 Hypotheses as in Theorem 9.3.15 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[1](1)$ on $\mathbb{A}^1$ is self dual.

proof Immediate from Proposition 9.1.4, part 2). QED

Theorem 9.3.17 Let k be a field of characteristic 3, $\mathbb{A}^2 = \mathrm{Spec}(k[s, t])$, and $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^2$ the relative affine curve defined by the Weierstrass equation

$$y^2 = 4x^3 - s^2tx - s^3t.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

proof Repeat the first ten lines of the proof of Theorem 9.3.10, and then invoke Theorem 9.3.15 above. QED

Corollary 9.3.18 Hypotheses as in Theorem 9.3.17 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.

proof Immediate from Proposition 9.1.4, part 2). QED

Theorem 9.3.19 Let k be a field of characteristic 3, $\mathbb{A}^2 = \mathrm{Spec}(k[g_2, g_3])$. Consider the relative affine curve $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^2$ defined by the Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

proof Immediate reduction to the case when k is algebraically closed. The vanishing $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$ is a special case of Proposition 9.1.5.

Here Δ is $(g_2)^3$. On the open set $\mathbb{A}^2[1/g_2]$, $R^1\pi_!\overline{\mathbb{Q}}_\ell$ is lisse of rank two and self dual. Pull back by the map

$$\varphi : \mathbb{G}_m \to \mathbb{A}^2[1/g_2],$$
$$t \mapsto (t, t).$$

By Theorem 9.3.15, $\varphi^*(R^1\pi_!\overline{\mathbb{Q}}_\ell)$ is irreducible, and hence $R^1\pi_!\overline{\mathbb{Q}}_\ell \mid \mathbb{A}^2[1/g_2]$ is irreducible.

Denote by

$$j : \mathbb{A}^2[1/g_2] \to \mathbb{A}^2$$

the inclusion. It remains only to see that $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ is its own middle extension $j_{!*}j^*R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ from $\mathbb{A}^2[1/g_2]$. Because we are in characteristic 3, over the locus $g_2 = 0$, every geometric fibre is isomorphic to $y^2 = x^3$, and so $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ vanishes on this locus, i.e., we have

$$R^1\pi_!\overline{\mathbb{Q}}_\ell[2] = j_!j^*R^1\pi_!\overline{\mathbb{Q}}_\ell[2].$$

We will show that this holds outside a finite set $Z \subset \mathbb{A}^2(k)$ of the form $0\times Z_3$. Once we have shown this, then we have only to apply Missing Points Lemma 9.3.5.

For this, it suffices to show that

$$j_!(j^*R^1\pi_!\overline{\mathbb{Q}}_\ell[2]) \mid \mathbb{A}^2 - Z \cong Rj_*(j^*R^1\pi_!\overline{\mathbb{Q}}_\ell[2]) \mid \mathbb{A}^2 - Z.$$

This will be a consequence of Deligne's generic base change theorem [De-Fin, 1.9 (ii)]. We view $j : \mathbb{A}^2[1/g_2] \to \mathbb{A}^2$ as a morphism of $\mathbb{A}^1 := \mathrm{Spec}(k[g_3])$ schemes. Over a dense open set

$$U = \mathbb{A}^1 - Z_3 \subset \mathbb{A}^1 := \mathrm{Spec}(k[g_3]),$$

the formation of $Rj_*(j^*R^1\pi_!\overline{\mathbb{Q}}_\ell)$ commutes with arbitrary change of base on U.

Let us spell out what this means. We may shrink U, and assume 0 lies in Z_3. For any b in $U(k) \subset k^\times$, the restriction of $Rj_*(j^*R^1\pi_!\overline{\mathbb{Q}}_\ell)$ to the line $g_2 \mapsto (g_2, b)$ in $\mathbb{A}^2$ is the following. We consider the one-parameter family

$$\mathcal{E}_b{}^{\mathrm{aff}} : y^2 = x^3 + g_2x^2 + b$$

over the g_2-line, with structural morphism $\pi_b : \mathcal{E}_b{}^{\mathrm{aff}} \to \mathbb{A}^1$.
In characteristic 3, the discriminant is $(g_2)^3$. Let us denote by

$$j_b : \mathbb{A}^1 - \{0\} \to \mathbb{A}^1$$

the inclusion. Then $j_b{}^*R^1\pi_{b!}\overline{\mathbb{Q}}_\ell$ is lisse on $\mathbb{G}_m$, and we have

$$Rj_{b*}(j_b{}^*R^1\pi_{b!}\overline{\mathbb{Q}}_\ell) \cong Rj_*(j^*R^1\pi_!\overline{\mathbb{Q}}_\ell) \mid (\text{the line } g_2 \mapsto (g_2, b)).$$

How do we make use of this? Because $b \neq 0$ and k is algebraically closed, we can write b as β^3. Then our b-frozen family, written in terms of $\tilde{x} := x+\beta$ and y, has equation

$$y^2 = (\tilde{x})^3 + g_2(\tilde{x} - \beta) = (\tilde{x})^3 + g_2\tilde{x} - \beta g_2.$$

Here we may apply Corollary 7.5.5 to conclude that $R^1\pi_{b!}\overline{\mathbb{Q}}_\ell$ on $\mathbb{A}^1$ is a middle extension across $g_2 = 0$. Looking at the fibre over 0, we see that $R^1\pi_{b!}\overline{\mathbb{Q}}_\ell$ vanishes at 0. Since it is the middle extension across zero of the lisse sheaf $j_b{}^*R^1\pi_{b!}\overline{\mathbb{Q}}_\ell$, we have

$$j_{b!}(j_b{}^*R^1\pi_{b!}\overline{\mathbb{Q}}_\ell) \cong Rj_{b*}(j_b{}^*R^1\pi_{b!}\overline{\mathbb{Q}}_\ell).$$

Therefore the object

$$Rj_*(j^*R^1\pi_!\overline{\mathbb{Q}}_\ell) \mid (\text{the line } g_2 \mapsto (g_2, b))$$

vanishes at the point (0, b), so long as b does not lie in Z_3. Thus if we take for Z the finite set $0\times Z_3$, we have

$$j_!(j^*R^1\pi_!\overline{\mathbb{Q}}_\ell[2]) \mid \mathbb{A}^2 - Z \cong Rj_*(j^*R^1\pi_!\overline{\mathbb{Q}}_\ell[2]) \mid \mathbb{A}^2 - Z,$$

as required. QED

Corollary 9.3.20 Hypotheses as in Theorem 9.3.19 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.
proof Immediate from Proposition 9.1.4, part 2). QED

Remark 9.3.21 Here is another method of showing that for k a field of characteristic $p > 2$, and $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^2 = \mathrm{Spec}(k[g_2, g_3])$ defined by the Weierstrass equation

$$y^2 = 4x^3 - g_2x - g_3,$$

has $R^1\pi_!\overline{\mathbb{Q}}_\ell|\mathbb{A}^2[1/\Delta]$ geometrically irreducible. Rather than reduce to the case when k is algebraically closed, we will instead reduce to the case when k is $\mathbb{F}_p$. This is legitimate because for any field K (here taken to be $\mathbb{F}_p$), for any geometrically connected K-scheme X (here taken to be $\mathbb{A}^2[1/\Delta]$), for any algebraically closed overfield L/K, and for any geometric point x of $X\otimes_K L$, with image point x' in $X\otimes_K \overline{K}$, the natural homomorphism

$$\pi_1(X\otimes_K L, x) \to \pi_1(X\otimes_K \overline{K}, x')$$

is surjective, cf. [Ka-LG, 1.2.2] and [EGA IV 4, 5.21]. Denote by $pr_2 : \mathbb{A}^1\times\mathbb{A}^2 \to \mathbb{A}^2$ the projection of the x-line. Because 2 is invertible, we have (cf. the proof of Lemma 9.2.2)

$$R^1\pi_!\overline{\mathbb{Q}}_\ell = R^1pr_{2!}\mathcal{L}_{\chi_2}(4x^3 - g_2x - g_3),$$

and

$$R^ipr_{2!}\mathcal{L}_{\chi_2}(4x^3 - g_2x - g_3) = 0, \text{ for } i \neq 1.$$

Thus we may view $\mathbb{A}^2$ as the space $\mathcal{F}$ of polynomial functions of degree at most one on $\mathbb{A}^1$, and the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell(1)[2]$ on $\mathbb{A}^2$ as the object M = Twist(L, K, $\mathcal{F}$, h) attached to the standard input (cf. 1.15.4) over k = $\mathbb{F}_p$ given by

the integer m = 1,

the perverse sheaf $K = j_\star\mathcal{L}_{\chi_2}(1/2)[1]$ on $\mathbb{A}^m/k$,

the affine k-scheme of finite type $V = \mathbb{A}^1/k$,

the k-morphism $h : V \to \mathbb{A}^m$ given by $h(x) = 4x^3$,

the perverse sheaf $L = \overline{\mathbb{Q}}_\ell(1/2)[1]$ on V/k,

the integer $d \geq 2$,

the space of functions ($\mathcal{F}$, τ) on V given by polynomials of degree at most one.

Because $H_c{}^*(\mathbb{A}^1\otimes_k\overline{k}, j_\star\mathcal{L}_{\chi_2}(1/2)[1]) = 0$, we see from Lemma 1.15.5, part 3), that $Gr^0(M(1/2)) = Gr^0(M(1/2))_{ncst}$. By the Standard Input Theorem 1.15.6, part 2), $Gr^0(M(1/2)) = Gr^0(M(1/2))_{ncst}$ is geometrically irreducible on $\mathbb{A}^2$. Therefore $Gr^0(M(1/2)) \mid \mathbb{A}^2[1/\Delta]$ is either geometrically irreducible, or it is zero. But the object $M(1/2) \mid \mathbb{A}^2[1/\Delta]$ is already pure of weight zero and nonzero: it is the lisse rank two sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell(3/2)|\mathbb{A}^2[1/\Delta]$, placed in degree -2. Thus

$$R^1\pi_!\overline{\mathbb{Q}}_\ell(3/2)|\mathbb{A}^2[1/\Delta] = \mathrm{Gr}^0(M(1/2)) \mid \mathbb{A}^2[1/\Delta].$$

Hence the lisse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell|\mathbb{A}^2[1/\Delta]$ is geometrically irreducible. QED

(9.4) Geometric Irreducibility in even characteristic

Twisting Lemma 9.4.1 Let S be an $\mathbb{F}_2$-scheme, a_2, a_4, a_6 functions in $\Gamma(S, \mathcal{O}_S)$. Consider the locus $\mathcal{E}^{\mathrm{aff}}$ in $\mathbb{A}^2\times S$ defined by the Weierstrass cubic

$$y^2 + xy = x^3 + a_2x^2 + a_4x + a_6.$$

View $\mathcal{E}^{\mathrm{aff}}$ as a relative affine curve $\pi : \mathcal{E}^{\mathrm{aff}} \to S$.

Fix a function f on S, and denote by $\mathcal{E}_f{}^{\mathrm{aff}}$ the locus in $\mathbb{A}^2\times S$ defined by the Weierstrass equation

$$y^2 + xy = x^3 + (a_2 + f)x^2 + a_4x + a_6.$$

View $\mathcal{E}_f{}^{\mathrm{aff}}$ as a relative affine curve $\pi_f : \mathcal{E}^{\mathrm{aff}} \to S$.
For any prime number ℓ invertible on S, we have a canonical isomorphism of sheaves on S,

$$R^1(\pi_f)_!\overline{\mathbb{Q}}_\ell \cong \mathcal{L}_{\psi(f)}\otimes R^1\pi_!\overline{\mathbb{Q}}_\ell,$$

where ψ denotes the nontrivial additive character of $\mathbb{F}_2$, and $\mathcal{L}_\psi$ the corresponding Artin-Schreier sheaf.

proof Denote by $\rho : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^1\times S$ the projection onto the x-line. This finite flat double covering is ramified over x = 0, so if we denote by

$$j : \mathbb{G}_m\times S \to \mathbb{A}^1 \times S$$

the inclusion, we have

$$\rho_*\overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell \oplus (\mathcal{K} := \mathrm{Ker}(\mathrm{Trace} : \rho_*\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell)),$$

with $\mathcal{K}$ supported in $\mathbb{G}_m\times S$. Thus $\mathcal{K} = j_!j^*\mathcal{K}$, and we can compute $j^*\mathcal{K}$ as follows. Over $\mathbb{G}_m\times S$, we take new coordinates $(x, \tilde{y} := y/x)$. In terms of these, our equation becomes an Artin-Schreier equation

$$(\tilde{y})^2 + \tilde{y} = x + a_2 + a_4/x + a_6/x^2.$$

So we find

$$j^*\mathcal{K} = \mathcal{L}_{\psi(x + a_2 + a_4/x + a_6/x^2)}.$$

So we find that, denoting now by $pr_{2,\mathbb{G}_m}$ the projection

$$pr_{2,\mathbb{G}_m} : \mathbb{G}_m \times S \to S,$$

we have

$$R^1(\pi)_!\overline{\mathbb{Q}}_\ell \cong R^1(pr_{2,\mathbb{G}_m})_!\mathcal{L}_{\psi(x + a_2 + a_4/x + a_6/x^2)}.$$

Repeating the same argument with π_f instead of π, we find

$$\begin{aligned} &R^1(\pi_f)_!\overline{\mathbb{Q}}_\ell \\ &\cong R^1(pr_{2,\mathbb{G}_m})_!\mathcal{L}_{\psi(x + a_2 + f + a_4/x + a_6/x^2)} \\ &\cong R^1(pr_{2,\mathbb{G}_m})_!(\mathcal{L}_{\psi(f)}\otimes\mathcal{L}_{\psi(x + a_2 + a_4/x + a_6/x^2)}) \\ &\cong \mathcal{L}_{\psi(f)}\otimes R^1(pr_{2,\mathbb{G}_m})_!\mathcal{L}_{\psi(x + a_2 + a_4/x + a_6/x^2)} \\ &\cong \mathcal{L}_{\psi(f)}\otimes R^1\pi_!\overline{\mathbb{Q}}_\ell. \qquad \text{QED} \end{aligned}$$

Theorem 9.4.2 Let k be a field of characteristic 2, $\mathbb{A}^2 := \mathrm{Spec}(k[a_2, a_6])$. Consider the relative affine curve

$$\pi : \mathcal{E}^{aff} \to \mathbb{A}^2$$

defined by the Weierstrass equation

$$y^2 + xy = x^3 + a_2x^2 + a_6.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

proof The vanishing $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$ is a special case of Propostion 9.1.5. Denote by $\pi_{0,6} : \mathcal{E}_{0,6}{}^{aff} \to \mathbb{A}^1 := \mathrm{Spec}(k[a_6])$ the relative affine curve defined by putting $a_2 = 0$:

$$y^2 + xy = x^3 + a_6.$$

In this family, $\Delta = a_6$ and $j = 1/a_6$.

By the previous Lemma 9.4.1, $R^1\pi_!\overline{\mathbb{Q}}_\ell$ on $\mathbb{A}^2$ is the external tensor

$$R^1\pi_!\overline{\mathbb{Q}}_\ell \cong \mathcal{L}_{\psi(a_2)}\boxtimes R^1\pi_{0,6!}\overline{\mathbb{Q}}_\ell.$$

The sheaf $\mathcal{L}_{\psi(a_2)}$ is lisse of rank one on $\mathbb{A}^1$. So it remains to show that $R^1\pi_{0,6!}\overline{\mathbb{Q}}_\ell$ is a geometrically irreducible and geometrically self dual middle extension. It is lisse on $\mathbb{G}_m$ (because $\Delta = a_6$). It is geometrically irreducible and geometrically self dual on $\mathbb{G}_m$ because the j invariant $j = 1/a_6$ is nonconstant. It is a middle extension across $a_6 = 0$ by Corollary 7.5.5. QED

Corollary 9.4.3 Hypotheses as in Theorem 9.4.2 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.

proof Immediate from Proposition 9.1.4, part 2). QED

Theorem 9.4.4 Let k be a field of characteristic 2, $\mathbb{A}^2 := \mathrm{Spec}(k[a_2, a_4])$. Consider the relative affine curve

$$\pi : \mathcal{E}^{aff} \to \mathbb{A}^2$$

defined by the Weierstrass equation

$$y^2 + xy = x^3 + a_2x^2 + a_4x.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

proof The vanishing $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$ is a special case of Proposition 9.1.5. Denote by $\pi_{0,4} : \mathcal{E}_{0,4}{}^{aff} \to \mathbb{A}^1 := \mathrm{Spec}(k[a_4])$ the relative affine curve defined by putting $a_2 = 0$:

$$y^2 + xy = x^3 + a_4x.$$

In this family, $\Delta = (a_4)^2$ and $j = 1/(a_4)^2$.

Just as in the preceding theorem, we have

$$R^1\pi_!\overline{\mathbb{Q}}_\ell \cong \mathcal{L}_{\psi(a_2)}\boxtimes R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell.$$

We must show that $R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell$ is a geometrically irreducible and geometrically self dual middle extension. It is lisse on $\mathbb{G}_m$ (because $\Delta = (a_4)^2$). It is geometrically irreducible and geometrically self dual on $\mathbb{G}_m$ because the j invariant $j = 1/(a_4)^2$ is nonconstant. It is a middle extension across $t = 0$ because it is the absolute Frobenius pullback of the middle extension sheaf $R^1\pi_{0,6!}\overline{\mathbb{Q}}_\ell$ on $\mathbb{A}^1$ which occurred in the previous theorem. Indeed, on $\mathbb{A}^1 := \mathrm{Spec}(k[t])$, we have $\mathrm{Frob}_2{}^*(R^1\pi_{0,6!}\overline{\mathbb{Q}}_\ell)$ is $R^1\pi_{0,6,\mathrm{Fr}!}\overline{\mathbb{Q}}_\ell$ for the relative affine curve $\pi_{0,6,\mathrm{Fr}} : \mathcal{E}_{0,6,\mathrm{Fr}}{}^{\mathrm{aff}} \to \mathbb{A}^1$ defined by the Weierstrass equation

$$y^2 + xy = x^3 + t^2.$$

So here we have

$$R^1(\pi_{0,6,\mathrm{Fr}})_!\overline{\mathbb{Q}}_\ell \cong R^1(\mathrm{pr}_{2,\mathbb{G}_m})_!\mathcal{L}_{\psi(x + (t/x)^2)}$$
$$\cong R^1(\mathrm{pr}_{2,\mathbb{G}_m})_!\mathcal{L}_{\psi(x + t/x)}$$
$$:= R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell.$$

Thus $R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell \cong \mathrm{Frob}_2{}^*(R^1\pi_{0,6!}\overline{\mathbb{Q}}_\ell)$ is itself a middle extension. QED

Remark 9.4.5 The middle extension sheaf $R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell$ on $\mathbb{A}^1$ is the middle extension of the Kloosterman sheaf $\mathrm{Kl}_2(\psi)$ [Ka-GKM, 11.0.1] on $\mathbb{G}_m$.

Corollary 9.4.6 Hypotheses as in Theorem 9.4.4 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.

proof Immediate from Proposition 9.1.5, part 2). QED

(9.4.7) To end this section, we given a case where j remains constant, j = 0 = 1728, in our family.

Theorem 9.4.8 Let k be a field of characteristic 2, $\mathbb{A}^2 := \mathrm{Spec}(k[a_4, a_6])$. Consider the relative affine curve $\pi : \mathcal{E}^{\mathrm{aff}} \to \mathbb{A}^2$ defined by the Weierstrass equation

$$y^2 + y = x^3 + a_4x + a_6.$$

Then the perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2]$ on $\mathbb{A}^2$ is geometrically irreducible and geometrically self dual, and $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$.

proof The vanishing $H_c{}^*(\mathbb{A}^2\otimes_k\overline{k}, R^1\pi_!\overline{\mathbb{Q}}) = 0$ is a special case of

Proposition 9.1.5. In this family, $\Delta = 1$. So $R^1\pi_!\overline{\mathbb{Q}}_\ell$ is lisse of rank two on all of $\mathbb{A}^2$, and geometrically self dual. To see that it is geometrically irreducible, pull back by the map

$$\varphi : \mathbb{A}^1 \to \mathbb{A}^2,$$
$$t \mapsto (t, 0),$$

obtaining the curve $\pi_{0,4} : \mathcal{E}_{0,4}{}^{\mathrm{aff}} \to \mathbb{A}^1 := \mathrm{Spec}(k[t])$ defined by

$$y^2 + y = x^3 + tx.$$

Then $\varphi^{*1}\pi_!\overline{\mathbb{Q}}_\ell$ is $R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell$, which in turn is

$$R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell = R^1(pr_2 : \mathbb{A}^1\times\mathbb{A}^1 \to \mathbb{A}^1)_!\mathcal{L}_{\psi(x^3 + tx)}.$$

It is geometrically irreducible on $\mathbb{A}^1$, because

$$R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell[1] = R^1(pr_2 : \mathbb{A}^1\times\mathbb{A}^1 \to \mathbb{A}^1)_!\mathcal{L}_{\psi(x^3 + a_4x)}[1]$$

is the Fourier Transform on $\mathbb{A}^1$ of the perverse, geometrically irreducible object $\mathcal{L}_{\psi(x^3)}$ on $\mathbb{A}^1$. QED

Remark 9.4.9 It is also useful to keep in mind the description (verification left to the reader) of $R^1\pi_!\overline{\mathbb{Q}}_\ell$ as the external tensor product

$$R^1\pi_!\overline{\mathbb{Q}}_\ell \cong (R^1\pi_{0,4!}\overline{\mathbb{Q}}_\ell)\boxtimes\mathcal{L}_{\psi(a_6)}.$$

Corollary 9.4.10 Hypotheses as inTheorem 9.4.8 above, suppose in addition that k is a finite field. Then the geometrically irreducible perverse sheaf $R^1\pi_!\overline{\mathbb{Q}}_\ell[2](3/2)$ on $\mathbb{A}^2$ is self dual.

proof Immediate from Proposition 9.1.5, part 2). QED