Chapter 13: Average analytic rank and large N limits

(13.1) **The basic setting**

(13.1.1) In this chapter, we will make explicit the application of our results to $L$-functions of elliptic curves over function fields. We first give applications to average rank, along the lines of [deJ-Ka, 9.7] and [Ka-TLFM, 10.3]. We then pass to the large $N$ limit, and give applications to the distribution of low lying zeroes, along the lines of [Ka-Sar-RMFEM, Chapters 12 and 13] and [Ka-TLFM, 10.5 and 10.6].

(13.1.2) To begin, we formulate a version, Theorem 13.1.5 below, of Uniform Output Theorem 12.4.18 in the case when $G$ is the orthogonal group $O(N)$, which pays attention to the “sign in the functional equation” $\varepsilon$. For $\varepsilon$ either choice of $\pm 1$, let us denote by

$$O_{\text{sign}\varepsilon}(N, \mathbb{R}) := \{ A \in O(N, \mathbb{R}) \text{ with } \det(-A) = \varepsilon \}.$$ 

Thus for $A$ in $O_{\text{sign}\varepsilon}(N, \mathbb{R})$, the polynomial

$$P(T) = \det(1 - TA)$$

has the functional equation

$$T^N P(1/T) = \varepsilon P(T).$$

(13.1.3) For $\varepsilon$ either choice of $\pm 1$, we denote by $\mu_{\text{sign}\varepsilon}$ the restriction to $O_{\text{sign}\varepsilon}(N, \mathbb{R})$ of twice (sic) the normalized Haar measure $\mu$ on $O(N, \mathbb{R})$, i.e., $\mu_{\text{sign}\varepsilon}$ is normalized to give $O_{\text{sign}\varepsilon}(N, \mathbb{R})$ total mass one. A function on $O_{\text{sign}\varepsilon}(N, \mathbb{R})$ is said to be “central” if it is invariant under $O(N, \mathbb{R})$-conjugation. We denote by $O_{\text{sign}\varepsilon}(N, \mathbb{R})^\#$ the quotient of $O_{\text{sign}\varepsilon}(N, \mathbb{R})$ by the equivalence relation of $O(N, \mathbb{R})$-conjugation, and we denote by $\mu_{\text{sign}\varepsilon}^\#$ the direct image of $\mu_{\text{sign}\varepsilon}$ on $O_{\text{sign}\varepsilon}(N, \mathbb{R})^\#$. A continuous $\mathbb{C}$-valued function $f$ on $O_{\text{sign}\varepsilon}(N, \mathbb{R})^\#$ is none other than a continuous central $\mathbb{C}$-valued function $\tilde{f}$ on $O_{\text{sign}\varepsilon}(N, \mathbb{R})$, and

$$\int_{O_{\text{sign}\varepsilon}(N, \mathbb{R})} f d\mu_{\text{sign}\varepsilon} = \int_{O_{\text{sign}\varepsilon}(N, \mathbb{R})^\#} \tilde{f} d\mu_{\text{sign}\varepsilon}^\#$$

$$= 2 \int_{O(N, \mathbb{R}) \{ \tilde{f} \text{ extended by } 0 \}} d\mu.$$

**Lemma 13.1.4** Suppose that we are in the situation of the Uniform Output Theorem 12.4.8, with constants $A$ and $C$. Suppose in addition that $G$ is the full orthogonal group $O(N)$. For each choice of $\varepsilon = \pm 1$, each choice of finite field $E$ with $\#E > \text{Max}(4A^2, 4C^2)$, and for each $s$ in $S(E)$, the set...
is nonempty. More precisely, for \( \#E > 4A^2 \), we have the estimate
\[
|\#U_{s,E,\text{sign}}(E)/\#U_{s,E}(E) - 1/2| \leq C/E^{1/2}.
\]

**Proof** Recall from 12.4.18 that for \( \#E > 4A^2 \), we have
\[
2(\#E)^n \geq \#U_{s,E}(E) \geq (1/2)(\#E)^n,
\]
and for any irreducible nontrivial representation \( \Lambda \) of \( O(N, \mathbb{R}) \), we have
\[
|\sum_{x \in U_{s,E}(E)} \text{Trace}(\Lambda(\theta(E, s, \alpha, x)))| \leq C\text{dim}(\Lambda)(\#E)^{n-1/2}.
\]

We apply this with \( \Lambda \) the one-dimensional representation \( \text{det} \). It is tautologous that we have
\[
2\#U_{s,E,\text{sign}}(E) = \sum_{x \in U_{s,E}(E)} (1 + \text{det}(-\theta(E, s, \alpha, x))) = \#U_{s,E}(E) + \varepsilon\sum_{x \in U_{s,E}(E)} \text{det}(-\theta(E, s, \alpha, x)).
\]

Thus we find
\[
|2\#U_{s,E,\text{sign}}(E) - \#U_{s,E}(E)| \leq C(\#E)^{n-1/2}.
\]

Dividing through by \( 2\#U_{s,E}(E) \geq (\#E)^n \), we find the desired estimate.

QED

**Theorem 13.1.5** Suppose that we are in the situation of the Uniform Output Theorem 12.4.8, with constants \( A \) and \( C \). Suppose in addition that \( G \) is the full orthogonal group \( O(N) \). For each choice of \( \varepsilon = \pm 1 \), each choice of finite field \( E \) with \( \#E > \text{Max}(4A^2, 4C^2) \), and for each \( s \) in \( S(E) \), form the probability measure \( \mu_{\text{sign}_\varepsilon}(E, s, \alpha) \) on
\[
O_{\text{sign}_\varepsilon}(N, \mathbb{R})_{\neq}
\]
defined by
\[
\mu_{\text{sign}_\varepsilon}(E, s, \alpha) := (1/\#U_{s,E,\text{sign}_\varepsilon}(E)) \sum_{x \in U_{s,E,\text{sign}_\varepsilon}(E)} \delta_{\theta(E, s, \alpha, x)}.
\]

For either choice of \( \varepsilon = \pm 1 \), and for any sequence of data \( (E_i, s_i, \alpha_i) \) with each \( \#E_i > \text{Max}(4A^2, 4C^2) \) and with \( \#E_i \) strictly increasing, the measures \( \mu_{\text{sign}_\varepsilon}(E_i, s_i, \alpha_i) \) on \( O_{\text{sign}_\varepsilon}(N, \mathbb{R})_{\neq} \) converge weak \( \star \) to \( \mu_{\text{sign}_\varepsilon} \), i.e., for any continuous \( \mathbb{C} \)-valued central function \( f \) on \( O_{\text{sign}_\varepsilon}(N, \mathbb{R}) \), we have
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\[ \int_{O(N, \mathbb{R})} f d\mu_{\text{sign}} = \lim_{i \to \infty} \frac{1}{\# U_{s_i, E_i, \text{sign}}(E_i)} \sum_{x \in U_{s_i, E_i, \text{sign}}(E_i)} f(\theta(E_i, s_i, \alpha_i, x)). \]

**proof** Thanks to Lemma 13.1.4, this results from applying 12.4.18 to continuous central functions on \( O(N, \mathbb{R}) \) which are supported in one of the sets \( O_{\text{sign}}(N, \mathbb{R}) \). QED

(13.1.6) We define three integer-valued functions on \( O(N, \mathbb{R}) \), as follows. Each has values in the closed interval \([0, N]\). The first is the "analytic rank",
\[ r_{\text{an}}(A) := \text{ord}_{T=1}(\det(1 - TA)). \]

The second is the "quadratic analytic rank", defined as
\[ r_{\text{quad}, \text{an}}(A) := r_{\text{an}}(A^2) = \sum_{\varepsilon \in \{\pm 1\}} \text{ord}_{T=\varepsilon}(\det(1 - TA)). \]

The third is the "geometric analytic rank"
\[ r_{\text{geom}, \text{an}}(A) := \sum_{\zeta \text{ in } \mu_{\text{f}}(\mathbb{C}) \text{ of deg } \leq N \text{ over } \mathbb{Q}} \text{ord}_{T=\zeta}(\det(1 - TA)). \]

[The definition of geometric analytic rank we have taken above is only the "correct" one when we are dealing with a situation where all the classes \( \theta(E, s, \alpha, x) \) have characteristic polynomials in \( \mathbb{Q}[T] \), as is the case, e.g., when we are dealing with \( L \)-functions of elliptic curves. In general, one must pick a finite extension \( E_{\lambda}/\mathbb{Q}_\ell \) such that all the characteristic polynomials lie in \( E_{\lambda}[T] \), and replace "N" by the maximum degree over \( \mathbb{Q} \) of any of the finitely many roots of unity in \( \overline{\mathbb{Q}}_\ell \) whose degree over \( E_{\lambda} \) is at most \( N \). With any such definition, the Analytic Rank Theorem 13.1.7 below remains valid.]

**Analytic Rank Theorem 13.1.7** Suppose that we are in the situation of the Uniform Output Theorem 12.4.8, with constants \( A \) and \( C \). Suppose in addition that \( G \) is the full orthogonal group \( O(N) \). For each choice of \( \varepsilon = \pm 1 \), each choice of finite field \( E \) with \( \neq E \rangle \) \( \text{Max}(4A^2, 4C^2) \), and for each \( s \) in \( S(E) \), we denote by
\[ U_{s, E, \text{sign}}(E) := \{ x \in U_{s, E}(E) \text{ with } \theta(E, s, \alpha, x) \in O_{\text{sign}}(N, \mathbb{R}) \}. \]

For any sequence of data \((E_i, s_i, \alpha_i)\) with each \( \neq E_i \rangle \) \( \text{Max}(4A^2, 4C^2) \) and with \( \neq E_i \rangle \) strictly increasing, we have the following tables of limit formulas for various average analytic ranks. In the tables, the first column is the quantity being averaged, the second is the set
over which it is being averaged, and the third column is the limit, as \( i \to \infty \), of the average.

\[
\begin{align*}
\quad r_{\text{an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i}(E_i) & 1/2, \\
r_{\text{an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}+}(E_i) & 0, \\
r_{\text{an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}-}(E_i) & 1, \\
r_{\text{quad,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i}(E_i) & 1, \\
r_{\text{geom,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i}(E_i) & 1,
\end{align*}
\]

Supplementary table, when \( N \) is odd
\[
\begin{align*}
r_{\text{quad,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}+}(E_i) & 1, \\
r_{\text{quad,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}-}(E_i) & 1, \\
r_{\text{geom,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}+}(E_i) & 1, \\
r_{\text{geom,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}-}(E_i) & 1,
\end{align*}
\]

Supplementary table, when \( N \) is even
\[
\begin{align*}
r_{\text{quad,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}+}(E_i) & 0, \\
r_{\text{quad,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}-}(E_i) & 2, \\
r_{\text{geom,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}+}(E_i) & 0, \\
r_{\text{geom,an}}(\theta(E_i, s_i, \alpha_i, x)) & \quad U_{s_i,E_i,\text{sign}-}(E_i) & 2,
\end{align*}
\]

**proof** Straightforward application of equidistribution as incarnated in Theorem 13.1.5, cf. the proof of [deJ-Ka, 6.11], or of [Ka-TLMF, 8.3.3]. QED

(13.1.8) Let us now recall the definition of the eigenvalue location measures \( \nu(c, \Omega_{\text{sign}+}(N, \mathbb{R})) \) of [Ka-Sar-RMFEM, 6.20.1]. Here \( r \geq 1 \) is an integer, \( c = (c(1),\ldots,c(r)) \) in \( \mathbb{Z}^r \) is an “offset vector”, i.e.,
\[
0 < c(1) < c(2) < \ldots < c(r),
\]
and
\[
N > 2c(r) + 2.
\]
We must proceed through cases, depending on the parity of \( N \).
and the value of $\varepsilon = \pm 1$.

(13.1.9) For $N$ even $= 2d$, the eigenvalues of an element $A$ in $O_{\text{sign}+}(N, \mathbb{R}) = SO(2d, \mathbb{R})$ are of the form 
\[ e^{\pm i\varphi(1)}, e^{i\varphi(2)}, \ldots, e^{\pm i\varphi(d)} , \]
for a unique sequence of angles 
\[ 0 \leq \varphi(1) \leq \varphi(2) \leq \ldots \leq \varphi(d) \leq \pi. \]
Formation of any given $\varphi(i)$ defines a continuous central function on $O_{\text{sign}+}(N, \mathbb{R}) = SO(2d, \mathbb{R})$,
\[ A \mapsto \varphi(i)(A). \]
We rescale this function, and call it $\vartheta(i)$:
\[ \vartheta(i)(A) := N\varphi(i)(A)/2\pi. \]
Given the offset vector $c$, we define the continuous central function 
\[ F_c : O_{\text{sign}+}(N, \mathbb{R}) \to \mathbb{R}^r, \]
\[ F_c(A) := (\vartheta(c(1))(A), \ldots, \vartheta(c(r))(A)). \]
We then define the probability measure $\nu(c, O_{\text{sign}+}(N, \mathbb{R}))$ on $\mathbb{R}^r$ to be
\[ \nu(c, O_{\text{sign}+}(N, \mathbb{R})) := (F_c)_\ast (\text{normalized Haar measure } \mu_{\text{sign}+} \text{ on } O_{\text{sign}+}(N, \mathbb{R})). \]

(13.1.10) For $N$ even $= 2d$, the eigenvalues of an element $A$ in $O_{\text{sign}-}(N, \mathbb{R}) = O_{\text{sign}}(2d, \mathbb{R})$ are of the form 
\[ \pm 1, e^{\pm i\varphi(1)}, e^{i\varphi(2)}, \ldots, e^{\pm i\varphi(d-1)} , \]
for a unique sequence of angles 
\[ 0 \leq \varphi(1) \leq \varphi(2) \leq \ldots \leq \varphi(d-1) \leq \pi. \]
Formation of any given $\varphi(i)$ defines a continuous central function on $O_{\text{sign}-}(N, \mathbb{R}) = O_{\text{sign}}(2d, \mathbb{R})$,
\[ A \mapsto \varphi(i)(A). \]
We rescale this function, and call it $\theta(i)$:
\[ \theta(i)(A) := N\varphi(i)(A)/2\pi. \]
Given the offset vector $c$, we define the continuous central function 
\[ F_c : O_{\text{sign}-}(N, \mathbb{R}) \to \mathbb{R}^r, \]
\[ F_c(A) := (\theta(c(1))(A), \ldots, \theta(c(r))(A)). \]
We then define the probability measure $\nu(c, O_{\text{sign}-}(N, \mathbb{R}))$ on $\mathbb{R}^r$ to be
\[ \nu(c, O_{\text{sign}-}(N, \mathbb{R})) \]
(13.1.11) For $N$ odd $= 2d+1$, the eigenvalues of an element $A$ in $O_{signε}(N, \mathbb{R})$ are of the form
\[-ε, e^{±iφ(1)}, e^{iφ(2)}, ..., e^{iφ(d)},\]
for a unique sequence of angles
\[0 ≤ φ(1) ≤ φ(2) ≤ ... ≤ φ(d) ≤ π.\]
Formation of any given $φ(i)$ defines a continuous central function on $O_{signε}(N, \mathbb{R})$,
\[A \mapsto φ(i)(A).\]
We rescale this function, and call it $θ(i)$:
\[θ(i)(A) := Nφ(i)(A)/2π.\]
Given the offset vector $c$, we define the continuous central function
\[F_c : O_{signε}(N, \mathbb{R}) \to \mathbb{R}^r,\]
\[F_c(A) := (θ(c(1))(A), ..., θ(c(r))(A)).\]
We then define the probability measure $ν(c, O_{signε}(N, \mathbb{R}))$ on $\mathbb{R}^r$ to be
\[ν(c, O_{signε}(N, \mathbb{R})) := (F_c)_*(normalized \ Haar \ measure \ μ_{signε} \ on \ O_{signε}(N, \mathbb{R})).\]

**Eigenvalue Location Theorem 13.1.12** Fix an integer $r ≥ 1$, and an offset vector $c$ in $\mathbb{Z}^r$. Suppose that we are in the situation of the Uniform Output Theorem 12.4.8, with constants $A$ and $C$. Suppose in addition that $G$ is the full orthogonal group $O(N)$, with $N > 2c(r) + 2$. For each choice of $ε = ±1$, each choice of finite field $E$ with $ζE > \text{Max}(4A^2, 4C^2)$, and for each $s$ in $S(E)$, the $ζU_{s,E,signε}(E)$ points $F_c(θ(E, s, α, x))$ in $\mathbb{R}^r$ are approximately equidistributed for the measure $ν(c, O_{signε}(N, \mathbb{R}))$. More precisely, define a probability measure $ν_{signε}(c, E, s, α)$ on $\mathbb{R}^r$ by
\[ν_{signε}(c, E, s, α) := (F_c)_*(μ_{signε}(E, s, α)) := (1/ζU_{s,E,signε}(E)) \sum_{x \in U_{s,E,signε}(E)} δ_{F_c(θ(E, s, α, x))}.\]
In any sequence of data $(E_i, s_i, α_i)$ with each $ζE_i > \text{Max}(4A^2, 4C^2)$ and with $ζE_i$ strictly increasing, the measures $ν_{signε}(c, E, s, α)$ tend weak $*$ to the measure $ν(c, O_{signε}(N, \mathbb{R}))$ on $\mathbb{R}^r$. In fact, for any
continuous \( \mathbb{C} \)-valued function \( g \) on \( \mathbb{R}^r \) of polynomial growth (and not just of compact support!), we have the limit formula
\[
\int_{\mathbb{R}^r} g d\nu(c, O_{\text{sign}}(N, \mathbb{R})) = \lim_{i \to \infty} \int_{\mathbb{R}^r} g d\nu_{\text{sign}}(c, E_i, s_i, \alpha_i).
\]

**Proof** That the measures \( \nu_{\text{sign}}(c, E, s, \alpha) := (F_c)^* \mu_{\text{sign}}(E, s, \alpha) \) converge weak * to \( (F_c)^* \mu_{\text{sign}}_{\text{on O_{\text{sign}}}(N, \mathbb{R})} \) is the direct image by \( F_c \) of Theorem 13.1.5. That we have convergence for continuous functions of polynomial growth results from the tail estimates of [Ka-Sar-RMFEM, 7.11.2]. QED

(13.2) **Passage to the large \( N \) limit: general results**

(13.2.1) Fix a prime \( \ell \). Suppose we are given a sequence, indexed by integers \( j \geq 1 \), of data as in 12.4.10, i.e.,

- \( S_j \), a normal connected affine \( \mathbb{Z}[1/\ell] \)-scheme \( \text{Spec}(A_j) \) of finite type,
- \( a_j \) and \( b_j \), integers,
- \( m_j > 1 \) an integer,
- \( V_j/S_j \) an affine \( S_j \)-scheme of finite type,
- an \( S_j \)-morphism \( h_j : V_j \to \mathbb{A}^{m_j} \),
- an object \( L_j \) in \( D^b_c(V_j, \mathbb{Q}_\ell) \), which is \( \nu \)-mixed of weight \( \leq b_j \) and which is fibrewise perverse on \( V_j/S_j \),
- an integer \( d_j \geq 2 \),
- a space of functions \( (\mathcal{F}_j, \tau_j) \) on \( V_j \), i.e., a locally free \( A_j \)-module of finite rank \( \mathcal{F}_j \) and an \( A_j \)-linear map
  \[
  \tau_j : \mathcal{F}_j \to \text{Hom}_{S_j\text{-schemes}}(V_j, \mathbb{A}^{m_j}).
  \]

(13.2.2) We suppose that for each \( j \), this data is strong standard input of type \( (a_j, b_j) \), in the sense of 12.4.11 We further suppose that for each \( j \), we are given a dense open set \( U_j \subset \mathcal{F}_j \) which meets every geometric fibre of \( \mathcal{F}_j/S_j \), and an integer \( N_j \geq 1 \), such that for every \( j \), our strong standard input data produces uniform output of type \( (U_j, N_j, O(N_j)) \) relative to \( S_j \), in the sense of 12.4.14. Assume further that the sequence of integers \( N_j \) is strictly increasing. Denote by \( (A_j, C_j) \) the constants \( (A, C) \) occurring in Uniform Output Theorem 12.4.18 for the \( j \)th input data.
Theorem 13.2.3 Suppose the hypotheses of 13.2.1 and 13.2.2 hold. Fix an integer \( r \geq 1 \), and an offset vector \( c \) in \( \mathbb{Z}^r \). Then we have the following double limit formulas for the Katz-Sarnak measures \( \nu(\epsilon, c) \) on \( \mathbb{R}^r \). Fix a continuous \( \mathbb{C} \)-valued function \( g \) of polynomial growth on \( \mathbb{R}^r \). For each \( j \) large enough that \( N_j > 2c(r) + 2 \), pick a sequence of pairs \((E_{i,j}, s_{i,j})\) consisting of

- a finite field \( E_{i,j} \) with \( \#E_{i,j} > \text{Max}(4A_j^2, 4C_j^2) \),
- a point \( s_{i,j} \) in \( U_{j}(E_{i,j}) \),

such that \( \#E_{i,j} \) is a strictly increasing function of \( i \). Form the measures

\[
\mu_{\text{sign}}(E_{i,j}, s_{i,j}, \alpha_{i,j}) \text{ on } \Omega_{\text{sign}}(N, \mathbb{R})^\#,
\]

as in 13.1.5, and their direct images

\[
\nu_{\text{sign}}(c, E, s, \alpha) := (F_c)_{*} \mu_{\text{sign}}(E, s, \alpha) \text{ on } \mathbb{R}^r,
\]

as in 13.1.9-11. Then we have the double limit formula

\[
\int_{\mathbb{R}^r} gd\nu(\epsilon, c) = \lim_{j \to \infty} \lim_{i \to \infty} \int_{\mathbb{R}^r} gd\nu_{\text{sign}}(c, E_{i,j}, s_{i,j}, \alpha_{i,j}).
\]

Remark 13.2.4 In the next section, we will regard this double limit formula as a statement about the totality of conjugacy classes \( \Theta(E_{i,j}, s_{i,j}, \alpha_{i,j}, x) \).

(13.3) Application to generalized Weierstrass families of elliptic curves

(13.3.1) Here we are given a sequence, indexed by integers \( j \geq 1 \), of data \((S_j/\mathbb{Z}[1/\ell], C_j/S_j \) of genus \( g_j \), \( D_j \) on \( C_j \) of degree \( d_j \geq 2g_j + 3 \), as in 12.7.1-5. We define the integer \( N_j \) by

\[
N_j := 4g_j - 4 + 12\deg(D_j).
\]

We suppose that \( N_j \) is a strictly increasing function of \( j \). For each \( j \), we have the smooth \( S_j \)-scheme \( GW_{\text{Ifd}}(C_j, D_j) \), which carries the corresponding family of elliptic curves in generalized Weierstrass form

\[
y^2 + f_1 x y + f_3 y = x^3 + f_2 x^2 + f_4 x + f_6,
\]

with \( f_\psi \) in \( L(vD_j) \). For each finite field \( k_j \) with \( \#k_j > \text{Max}(A_j^2, C_j^2) \), for each point \( s_j \) in \( S_j(k_j) \), and for each point \( f \)'s in \( GW_{\text{Ifd}}(C_j, D_j)_{s_j} \),
we have an elliptic curve \( E_{f's,s_j,k_j} \) over the function field of \( C_{j,s_j}/k_j \), whose unitarized \( L \)-function \( L(E_{f's,s_j,k_j}(C_{j,s_j}), T) \) is given as follows, cf. 12.7.8. There is a unique conjugacy class \( \theta(k_j, s_j, \alpha_j, f's) \) in \( O(N, \mathbb{R})^* \) such that
\[
L(E_{f's,s_j,k_j}/k_j(C_{j,s_j}), T) = \det(1 - T\theta(k_j, s_j, \alpha_j, f's)).
\]
[Here \( \alpha_j = (\#k_j)^2 \). In 12.7.8, \( \alpha_j \) is omitted in the notation.]

(13.3.2) For each \( j \), the input data gives uniform output of type \( (U, N, G) = (GW_{f'd}(C_j, D_j), N_j, O(N_j)) \), cf. 12.7.5. So we obtain the following two theorems. [Notice that not only the divisors \( D_j \), but also the bases \( S_j \) and the curves \( C_j/S_j \) are allowed to vary with \( j \). We expect that in most applications, \( S \) and \( C \) will be fixed, and that only \( D_j \) on \( C/S \) will vary. But by prudence we state the general version.]

**Theorem 13.3.3** Hypotheses and notations as in 13.3.1 above, fix any \( j \geq 1 \). Consider all the conjugacy classes \( \theta(k_j, s_j, \alpha_j, f's) \), which define the \( L \)-functions of the elliptic curves in the corresponding generalized Weierstrass family. The Analytic Rank Theorem 13.1.7 and the Eigenvalue Location Theorem 13.1.12 apply to these conjugacy classes.

**Theorem 13.3.4** Fix an integer \( r \geq 1 \), and an offset vector \( c \) in \( \mathbb{Z}^r \). Hypotheses and notations as in 13.3.1 above, for \( j \) with \( N_j > 2c(r) + 2 \), consider all the conjugacy classes \( \theta(k_j, s_j, \alpha_j, f's) \). Then the double limit formulas of Theorem 13.2.3 hold for them, cf. 13.2.4.

**13.4 Application to usual Weierstrass families of elliptic curves**

(13.4.1) Here we are given a sequence, indexed by integers \( j \geq 1 \), of data \( (S_j/\mathbb{Z}[1/6\delta], C_j/S_j \) of genus \( g_j \), pairwise disjoint sections \( \{P_{i,j}\}_i \) in \( C_j(S_j) \), effective divisors \( D_{2,j} = \Sigma a_{i,j}P_{i,j} \) and \( D_{3,j} = \Sigma b_{i,j}P_{i,j} \) on \( C_j \) of degrees \( d_{2,j} \) and \( d_{3,j} \), both \( \geq 2g_j + 3 \), as in 12.8.1-5. [In particular, all the hypotheses of Theorem 12.8.5 are assumed to hold.] We define the integers \( c_{i,j} \) by
\[
c_{i,j} := \text{Max}(3a_{i,j}, 2b_{i,j}),
\]
and the divisor $D_{\text{max},j}$ by
\[ D_{\text{max},j} := \sum_i c_{i,j} P_{i,j}. \]

We then define the integer $N_j$ by
\[ N_j := 4g_j - 4 + \text{deg}(D_{\text{max},j}) + 2\#(i \text{ with } c_{i,j} \neq 0 \mod 12). \]

We suppose that $N_j$ is a strictly increasing function of $j$. For each $j$, we have the smooth $S_j$-scheme $W_{\text{ld}}(C_j, D_{2,j}, D_{3,j})$, which carries the corresponding family of elliptic curves in Weierstrass form
\[ y^2 = 4x^3 - f_2x - f_3, \]
with $f_2$ in $L(D_{2,j})$ and $f_3$ in $L(D_{3,j})$. For each finite field $k_j$ with $\#k_j > \max(A_j^2, C_j^2)$, for each point $s_j$ in $S_j(k_j)$, and for each point $(f_2, f_3)$ in $W_{\text{ld}}(C_j, D_j)s_j(k_j)$, we have an elliptic curve $E_{f's, s_j, k_j}$ over the function field of $C_j/s_j/k_j$, whose unitarized $L$-function
\[ L(E_{f's, s_j, k_j}/k_j(C_j, s_j), T) \]
and $L(E_{f's, s_j, k_j}/k_j(C_j, s_j), T)$ is given as follows, cf. 12.8.7. There is a unique conjugacy class $\theta(k_j, s_j, \alpha_j, f's)$ in $O(N_j, \mathbb{R})^\times$ such that
\[ L(E_{f's, s_j, k_j}/k_j(C_j, s_j), T) = \det(1 - T\theta(k_j, s_j; \alpha_j, f's)). \]

[Here $\alpha_j = 1$. In 12.8.7, $\alpha_j$ is omitted in the notation.]

For each $j$, the input data gives uniform output of type
\[(U, N, G) = (W_{\text{ld}}(C_j, D_j), N_j, O(N_j)), \]
cf. 12.8.5. So we obtain the following two theorems. (Once again, notice that not only the divisors $D_{2,j}$ and $D_{3,j}$, but also the bases $S_j$ and the curves $C_j/S_j$ are allowed to vary with $j$.)

**Theorem 13.4.3** Hypotheses and notations as in 13.4.1 above, fix any $j \geq 1$. Consider all the conjugacy classes $\theta(k_j, s_j, \alpha_j, f's)$, which define the $L$-functions of the elliptic curves in the corresponding usual Weierstrass family. The Analytic Rank Theorem 13.1.7 and the Eigenvalue Location Theorem 13.1.12 apply to these conjugacy classes.

**Theorem 13.4.4** Fix an integer $r \geq 1$, and an offset vector $c$ in $\mathbb{Z}^r$. Hypotheses and notations as in 13.4.1 above, for $j$ with $N_j > 2c(r) + 2$, consider all the conjugacy classes $\theta(k_j, s_j, \alpha_j, f's)$. Then the double
Applications to FJTwist families of elliptic curves

(13.5.1) Here we are given a sequence, indexed by integers \( j \geq 1 \), of data \((S_j / \mathbb{Z}[1/6\ell], C_j / S_j)\) of genus \( g_j \), pairwise disjoint sections \( \{P_{i,j}\}_i \) in \( C_j(S_j) \), effective divisors \( D_{0,j} = \sum_i a_{i,j} P_{i,j} \) and \( D_{1,j} = \sum_i b_{i,j} P_{i,j} \) on \( C_j \) of degrees \( d_{0,j} \) and \( d_{1,j} \), both \( \geq 2g_j + 3 \), as in 12.9.1-5. We define the integers \( c_{i,j} \) by

\[
c_{i,j} := 2a_{i,j} + b_{i,j},
\]
and the integer \( N_j \) by

\[
N_j := 4g_j - 4 + 2\deg(D_{0,j}) + 3\deg(D_{1,j}) + 2\# \{i \mid c_{i,j} \not\equiv 0 \mod 4\}.
\]

We suppose that \( N_j \) is a strictly increasing function of \( j \). For each \( j \), we have the smooth \( S_j \)-scheme \( \text{FJTwist}(D_{0,j}, D_{1,j}) \), which carries the corresponding family of elliptic curves in FJTwist form

\[
y^2 = 4x^3 - 3f^2g x - f^3g,
\]
with \( f \) in \( L(D_{0,j}) \) and \( g \) in \( L(D_{1,j}) \). For each finite field \( k_j \) with

\[ \#k_j > \max(A_j^2, C_j^2), \]
for each point \( s_j \) in \( S_j(k_j) \), and for each point \((f, g)\) in \( \text{FJTwist}(D_{0,j}, D_{1,j})_{s_j}(k_j) \), we have an elliptic curve \( E_{f,g,s_j,k_j} \)

over the function field of \( C_j, s_j / k_j \), whose unitarized \( L \)-function

\[
L(E_{f,g,s_j,k_j}/k_j(C_j,s_j), T) = \det(1 - T\sigma(k_j, s_j, \alpha_j, f,g)).
\]

[Here \( \alpha_j = 1 \). In 12.9.7, \( \alpha_j \) is omitted in the notation.]

(13.5.2) For each \( j \), the input data gives uniform output of type \((U, N, G) = (\text{FJTwist}(D_{0,j}, D_{1,j}), N_j, O(N_j))\), cf. 12.9.5. So we obtain the following two theorems. [Once again, notice that not only the

divisors \( D_{0,j} \) and \( D_{1,j} \), but also the bases \( S_j \) and the curves \( C_j/S_j \) are allowed to vary with \( j \).]

Theorem 13.5.3 Hypotheses and notations as in 13.5.1 above, fix any \( j \geq 1 \). Consider all the conjugacy classes \( \sigma(k_j, s_j, \alpha_j, f,g) \), which define the \( L \)-functions of the elliptic curves in the corresponding FJTwist family. The Analytic Rank Theorem 13.1.7 and the
Eigenvalue Location Theorem 13.1.12 apply to these conjugacy classes.

**Theorem 13.5.4** Fix an integer \( r \geq 1 \), and an offset vector \( c \) in \( \mathbb{Z}^r \). Hypotheses and notations as in 13.5.1 above, for \( j \) with \( N_j > 2c(r) + 2 \), consider all the conjugacy classes \( \theta(k_j, s_j, \alpha_j, f, g) \). Then the double limit formulas of Theorem 13.2.3 hold for them, cf. 13.2.4.

**Applications to pullback families of elliptic curves**

(13.6.1) Here we fix a normal connected affine \( \mathbb{Z}[1/6\ell] \)-scheme \( S = \text{Spec}(A) \) of finite type, an integer \( t \geq 1 \), and a Cartier divisor \( T \) in \( \mathbb{A}^1_S \) which is finite étale over \( S \) of degree \( t \). We suppose further that \( T \) is defined by the vanishing of a monic polynomial \( T(x) \) in \( A[x] \) which is monic of degree \( t \), and whose discriminant is a unit in \( A \).

We suppose given a relative elliptic curve
\[
\pi : \mathcal{E} \to \mathbb{A}^1_S - T,
\]
which, on each geometric fibre of \( \mathbb{A}^1_S / S \), has multiplicative reduction at some point of \( T \).

(13.6.2) Having fixed this initial input data as in 12.10.1, we suppose given a sequence, indexed by integers \( j \geq 1 \), of data \( (C_j/S \) of genus \( g_j \), pairwise disjoint sections \( \{P_{i,j}\}_{i} \) in \( C_j(S) \), an effective divisors \( D_j = \sum a_{i,j} P_{i,j} \) in \( C_j \), as in 12.10.2. For each \( j \), we form the integers \( N_{\text{down}, j} \), \( N_{\text{up}, j} \), and \( N_j := N_{\text{up}, j} - N_{\text{down}, j} \), as in 12.10.5. We suppose that each \( N_j \geq 9 \), and that \( N_j \) is a strictly increasing function of \( j \). For each \( j \), we have the smooth \( S \)-scheme \( U_{D_j,T} \) of 12.10.3, which carries the family of pullback elliptic curves
\[
f^* \mathcal{E} \quad \text{over} \quad C_j - D_j - f^{-1} T,
\]
f in \( U_{D_j,T} \). For each finite field \( k_j \) with \( k_j > \text{Max}(A_j^2, C_j^2) \), for each point \( s_j \) in \( S(k_j) \), and for each point \( f \) in \( (U_{D_j,T})_{s_j}(k_j) \), we have an elliptic curve \( f^* \mathcal{E}_{s_j,k_j} \) over the function field of \( C_{j,s_j}/k_j \). The unitarized \( L \)-function \( L(f^* \mathcal{E}_{s_j,k_j}/k_j(C_{j,s_j}, T)) \) is a polynomial of degree \( N_{\text{up}, j} \), which is always divisible as a polynomial by the
unitarized L-function $L(\xi_{s_j,k_j}/k_j(\mathbb{P}^1), T)$ of the original elliptic curve $\xi_{s_j,k_j}/k_j(\mathbb{P}^1)$. This latter L-function is a polynomial of degree $N_{\text{down}}$. The quotient of these L-functions, the "new part", is given as follows, cf. 12.10.9. There is a unique conjugacy class $\theta(k_j, s_j, \alpha_j, f)$ in $O(N_j, \mathbb{R})^*$ such that

$$L(f^{*} \xi_{s_j,k_j}/k_j(C_{j,s_j}), T)/L(\xi_{s_j,k_j}/k_j(\mathbb{P}^1), T) = \det(1 - T\theta(k_j, s_j, \alpha_j, f)).$$

[Here $\alpha_j = 1$. In 12.10.9, $\alpha_j$ is omitted in the notation.]

(13.6.3) For each $j$, the input data gives uniform output of type $(U, N, G) = (U_{D_j,T}, N_j, O(N_j))$, cf. 12.10.7. So we obtain the following two theorems.

**Theorem 13.6.4** Hypotheses and notations as in 13.6.1 and 13.6.2 above, fix any $j \geq 1$. Consider the conjugacy classes $\theta(k_j, s_j, \alpha_j, f)$, which define the "new parts" of the L-functions of the elliptic curves in the corresponding pullback families. The Analytic Rank Theorem 13.1.7 and the Eigenvalue Location Theorem 13.1.12 apply to these conjugacy classes.

**Theorem 13.6.5** Fix an integer $r \geq 1$, and an offset vector $c$ in $\mathbb{Z}^r$. Hypotheses and notations as in 13.6.1 and 13.6.2 above, for $j$ with $N_j > 2c(r) + 2$, consider all the conjugacy classes $\theta(k_j, s_j, \alpha_j, f)$. Then the double limit formulas of Theorem 13.2.3 hold for them, cf. 13.2.4.

**Applications to quadratic twist families of elliptic curves**

(13.7.1) We work over an affine $\mathbb{Z}[1/2\ell]$-scheme $S = \text{Spec}(A)$ of finite type. We fix a genus $g \geq 0$ and a projective smooth curve $C/S$ with geometrically connected fibres of genus $g$. We suppose given a sequence, indexed by integers $j \geq 1$, of data (pairwise disjoint sections $\{P_{i,j}\}$ in $C(S)$, disjoint effective Cartier divisors $T_j$ and $D_j = \sum_i a_{i,j}P_{i,j}$ in $C$, with $T_j/S$ finite etale of degree $t_j$, $D_j/S$ finite flat of degree $d_j \geq 2g+3$, an elliptic curve $\xi_j/C - T_j - D_j$ which, on every
geometric fibre of \((C - T_j - D_j)/S\) has multiplicative reduction at some point of \(T_j\), and is everywhere tamely ramified), as in 12.11.1 and 12.11.2. We form the integer \(N_j\) as in 12.11.5. We suppose that each \(N_j \geq 9\), and that \(N_j\) is a strictly increasing function of \(j\). For each \(j\), we have the smooth \(S\)-scheme \(\text{Fct}(C d_j, D_j, T_j)\), cf. 12.11.4, which carries the quadratic twist family 

\[ E_j \otimes \chi_2(f), \]

\(f\) in \(\text{Fct}(C d_j, D_j, T_j)\). For each finite field \(k_j\) with 

\[ k_j > \text{Max}(A_j^2, C_j^2), \]

for each point \(s_j\) in \(S(k_j)\), and for each point \(f\) in \(\text{Fct}(C d_j, D_j, T_j)\), we have an elliptic curve \(E_j, s_j, k_j \otimes \chi_2(f)\) over the function field of \(C_{s_j}/k_j\). The unitarized \(L\)-function

\[ L(E_j, s_j, k_j \otimes \chi_2(f)/k_j(C_{s_j}), T) \]

is given as follows, cf. 12.11.7. There is a unique conjugacy class \(\theta(k_j, s_j, \alpha_j, f)\) in \(O(N_j, R)^\#\) such that

\[ L(E_j, s_j, k_j \otimes \chi_2(f)/k_j(C_{s_j}), T) = \det(1 - T\theta(k_j, s_j, \alpha_j, f, g)). \]

[Here \(\alpha_j\) is a choice of \(\text{Sqrt}(k_j)\). In 12.11.7, \(\alpha_j\) is omitted in the notation.]

(13.7.2) For each \(j\), the input data gives uniform output of type 

\((U, N, G) = (\text{Fct}(C d_j, D_j, T_j), N_j, O(N_j))\), cf. 12.11.6. So we obtain the following two theorems.

**Theorem 13.7.3** Hypotheses and notations as in 13.1.1 above, fix any \(j \geq 1\). Consider the conjugacy classes \(\theta(k_j, s_j, \alpha_j, f)\), which define the \(L\)-functions of the elliptic curves in the corresponding quadratic twist families. The Analytic Rank Theorem 13.1.7 and the Eigenvalue Location Theorem 13.1.12 apply to these conjugacy classes.

**Theorem 13.7.4** Fix an integer \(r \geq 1\), and an offset vector \(c\) in \(\mathbb{Z}^r\). Hypotheses and notations as in 13.7 above, for \(j\) with \(N_j > 2c(r) + 2\), consider all the conjugacy classes \(\theta(k_j, s_j, \alpha_j, f)\). Then the double limit formulas of Theorem 13.2.3 hold for them, cf. 13.2.4.