ELLIPTIC CONVOLUTION, G_2, AND ELLIPTIC SURFACES

NICHOLAS M. KATZ

Abstract. This is (a slightly more detailed version of) our talk at the conference in honor of Laumon's sixtieth birthday. We report here on some unexpected occurrences of G_2, first stumbled upon experimentally, later proven, but still not understood. Proofs will appear elsewhere.

1. elliptic sums

Let k be a finite field, E/k an elliptic curve, and $f : E(k) \to \mathbb{C}$ a function on the finite abelian group $E(k)$. Given f, we define a function $S(f)$ of characters $\Lambda \in \mathrm{Hom}_{group}(E(k), \mathbb{C}^\times)$ by

$$S(f)(\Lambda) := \sum_{P \in E(k)} f(P)\Lambda(P).$$

This function $S(f)$ is the "Fourier transform" of f in the sense of finite abelian groups. Given two functions f, g on $E(k)$, their convolution is the function on $E(k)$ defined by

$$(f \star g)(P) := \sum_{R+S=P} f(R)g(S).$$

Their Fourier transforms are related by the usual identity $S(f \star g) = S(f)S(g)$, i.e., for each Λ we have

$$S(f \star g)(\Lambda) = S(f)(\Lambda)S(g)(\Lambda).$$

For a given function f, the moments of its Fourier transform $S(f)$, defined by

$$M_n(S(f)) := (1/\#E(k)) \sum_{\Lambda} S(f)(\Lambda)^n$$

are thus given in terms of the multiple self-convolutions $f^{\star n}$ of f with itself by

$$(1/\#E(k)) \sum_{\Lambda} S(f^{\star n})(\Lambda) = f^{\star n}(0).$$

For any writing of n as $a+b$ with a,b strictly positive integers, we thus have

$$M_n(S(f)) = (f^{\star n})(0) = \sum_P f^{\star a}(P) f^{\star b}(-P).$$

2. ELLIPTIC EQUIDISTRIBUTION

Fix a prime number ℓ invertible in k, and an embedding ι of $\overline{\mathbb{Q}_\ell}$ into $\mathbb{C}$. There is an obvious notion of convolution of objects in $D^c_b(E, \overline{\mathbb{Q}_\ell})$, defined in terms of the addition map $sum : E \times_k E \to E$, by $(A,B) \mapsto A \star B := Rsum_\star(A \boxtimes B)$. If we attach to $A \in D^c_b(E, \overline{\mathbb{Q}_\ell})$ its trace function on $E(k)$, given by $f_{A,k}(P) := \mathrm{Trace}(Frob_{k,P}|A)$, then by the Lefschetz Trace Formula we have the identity $f_{A,k} \star f_{B,k} = f_{A\star B,k}$ of functions on $E(k)$.

In general, if A and B are each perverse sheaves on E, their convolution need not be perverse. To remedy that, we work first on $E_{\overline{k}}$, the extension of scalars of E to $\overline{k}$. We say that an object $A \in D^c_b(E_{\overline{k}}, \overline{\mathbb{Q}_\ell})$ has property $\mathcal{P}$ if, for all lisse rank one sheaves $\mathcal{L}$ on $E_{\overline{k}}$, we have

$$H^i(E_{\overline{k}}, A \otimes \mathcal{L}) = 0 \text{ for i} \neq 0.$$

We have the following lemma.

Lemma 2.1. *Let $A \in D^c_b(E_{\overline{k}}, \overline{\mathbb{Q}_\ell})$ have property $\mathcal{P}$. Then A is perverse.*

Because lisse rank one $\mathcal{L}$'s on $E_{\overline{k}}$ are primitive in the sense that $sum^\star(\mathcal{L}) \cong \mathcal{L} \boxtimes \mathcal{L}$, the A's with property $\mathcal{P}$ are stable by convolution. Thus perverse sheaves with property $\mathcal{P}$ are stable by convolution. An irreducible perverse sheaf on $E_{\overline{k}}$ has property $\mathcal{P}$ unless it is an $\mathcal{L}[1]$.

Corollary 2.2. *The perverse sheaves on $E_{\overline{k}}$ with property $\mathcal{P}$ form a neutral Tannakian category, with convolution as the tensor operation, δ_0 as the identity, $N \mapsto N^\vee := [P \mapsto -P]^\star DN$ as the dual, and "dim"$(N) := \chi(E_{\overline{k}}, N) = h^0(E_{\overline{k}}, N)$. For any lisse rank one $\mathcal{L}$ on $E_{\overline{k}}$, $N \mapsto H^0(E_{\overline{k}}, N \otimes \mathcal{L})$ is a fibre functor.*

Remark 2.3. Just as in Gabber-Loeser [Ga-Loe], the abelian category structure on the above Tannakian category is the one induced by viewing it **not** as a full subcategory of the category $Perv$ of all perverse sheaves on $E_{\overline{k}}$, but rather as the quotient cateory $Perv/Neg$ of $Perv$ by the subcategory Neg consisting of those perverse sheaves which are of Euler characteristic zero, or (equivalently) of the form $\mathcal{F}[1]$ for $\mathcal{F}$ a lisse sheaf on $E_{\overline{k}}$, or (equivalently) successive extensions of objects $\mathcal{L}[1]$. The irreducible (resp. semisimple) objects in $Perv/Neg$ are just the irreducible (resp. semisimple) perverse sheaves with property $\mathcal{P}$. The semisimple perverse sheaves with property $\mathcal{P}$ themselves form a

Tannakian category; its structure of abelian category is equal to the naive one.

We now return to working on E/k. Recall that for a character Λ of $E(k)$, the Lang torsor construction [De-ST, 1.4] gives a lisse rank one sheaf $\mathcal{L}_\Lambda$ on E, whose trace function on $E(k)$ is Λ. The perverse sheaves on E which, pulled back to $E_{\overline{k}}$, have property $\mathcal{P}$, themselves form a neutral Tannakian category. For each character Λ of $E(k)$, $N \mapsto H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)$ is a fibre functor. The action of $Frob_k$ on $H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)$ is an automorphism of this fibre functor, so gives a conjugacy class $Frob_{k,\Lambda}$ in the Tannakian group $G_{arith,N}$ attached to N. Notice in passing that, by the Lefschetz trace formula,

$$\mathrm{Trace}(Frob_k | H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)) = \sum_{P \in E(k)} \mathrm{Trace}(Frob_{k,P}|N)\Lambda(P)$$

is the value at Λ of the elliptic sum $S(f_{N,k})$ attached to the trace function $f_{N,k}$ on N on $E(k)$.

Suppose N is perverse on E, has property $\mathcal{P}$, is arithmetically semisimple, is ι-pure of weight zero, and has dimension $n :=$ "dim"(N). Denote by $G_{arith,N}$, respectively $G_{geom,N}$, the Tannakian groups attached to N on E , respectively on $E_{\overline{k}}$. In general we have inclusions of reductive $\overline{\mathbb{Q}_\ell}$-algebraic groups

$$G_{geom,N} \lhd G_{arith,N} \subset GL(\text{“}dim\text{”}(N)).$$

Pick a maximal compact subgroup K of $G_{arith,N}(\mathbb{C})$. The semisimplification (in the sense of Jordan decomposition) $Frob^{ss}_{k,\Lambda}$ of the conjugacy class $Frob_{k,\Lambda}$ intersects K in a single conjugacy class $\theta_{k,\Lambda}$ of K. Via the inclusion of $K \subset G_{arith,N}(\mathbb{C})$ into $GL(n)$, we have

$$\det(1 - T\theta_{k,\Lambda}) = \det(1 - TFrob_k | H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)),$$

so in particular

$$\begin{aligned}\mathrm{Trace}(\theta_{k,\Lambda}) &= \mathrm{Trace}(Frob_k | H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)) \\ &= \sum_{P \in E(k)} \mathrm{Trace}(Frob_{k,P}|N)\Lambda(P).\end{aligned}$$

Exactly as in [Ka-CE, 1.1, 7.3], Deligne's Weil II results [De-Weil II, 3.3.1] and the Tannakian formalism give the following theorem.

Theorem 2.4. *In the above situation, suppose* $G_{geom,N} = G_{arith,N}$. *Then as* L/k *runs over larger and larger finite extension fields of* k, *the conjugacy classes* $\{\theta_{L,\Lambda}\}_{\Lambda \text{ char. of } \mathrm{E(L)}}$ *become equidistributed in the space* $K^\#$ *of conjugacy classes of* K, *for its "Haar measure" of total mass one.*

3. The search for G_2

We work over $\mathbb{C}$. Recall that G_2, the automorphism group of the octonions, is the fixer in $SO(7)$ of an alternating trilinear form. It is a connected irreducible subgroup of $SO(7)$. According to a theorem of Gabber [Ka-ESDE, 1.6], the only connected irreducible subgroups of $SO(7)$ are $SO(7)$ itself, G_2, and the image of $SL(2)$ in $\mathrm{Sym}^6(std_2)$, which we shall denote "$\mathrm{Sym}^6(SL(2))$". For each of these three groups G, its normalizer in the full orthogonal group $O(7) = \{\pm 1\} \times SO(7)$ is the group $\pm G := \{\pm 1\} \times G$. Among these six groups, we can distinguish G_2 by its moments (for the given seven dimensional representation, call it V). For an integer $n \geq 1$ and H any of these six groups, we define

$$M_n(H) := \dim((V^{\otimes n})^H).$$

For K a maximal compact subgroup of H, we have

$$M_n(H) = \int_K \mathrm{Trace}(k|V)^n.$$

The third and fourth moments are given by the following table.

	M_3	M_4
$\mathrm{Sym}^6(SL(2))$	1	7
$\pm\mathrm{Sym}^6(SL(2))$	0	7
G_2	1	4
$\pm G_2$	0	4
$SO(7)$	0	3
$O(7)$	0	3

So if M_3 is nonzero, we have either G_2 or $\mathrm{Sym}^6(SL(2))$. We can distinguish these two cases by their M_4. But there is another, computationally easier, way to distinguish the two. Take maximal compact subgroups UG_2 and $\mathrm{Sym}^6(SU(2))$ of these two groups. For UG_2, its traces in the given seven dimensional representation lie in the interval $[-2, 7]$, while the traces of $\mathrm{Sym}^6(SU(2))$ (namely the values of the function $\sin(7\theta)/\sin(\theta)$) lie in the interval $[-1.64, 7]$.

4. Beauville families of elliptic curves

Starting with an elliptic curve E/k, how can we produce geometrically irreducible perverse sheaves N which have $\mathcal{P}$, are ι-pure of weight zero, and which, in the Tannakian sense, are self dual of dimension seven? Start with a "seven point sheaf" on E, by which we mean a geometrically irreducible lisse sheaf $\mathcal{F}$ of rank two on a dense open set $j : U \subset E$ of E which is ι-pure of weight zero, whose determinant is trivial, and such that $(E \setminus U)(\overline{k})$ consists of seven points, at each of which the local monodromy of $\mathcal{F}$ is unipotent and nontrivial. Then

$$N := j_\star \mathcal{F}(1/2)[1]$$

is perverse, ι-pure of weight zero, and geometrically irreducible of "dimension" $\chi(E_{\overline{k}}, N) = 7$. If in addition $\mathcal{F}$ is isomorphic to its pullback by $P \mapsto -P$, then N is self dual. Because N is geometrically irreducible, the autoduality has a sign. Because N has odd "dimension", the autoduality must be orthogonal.

One way to get such an N on E, at least if 2 is invertible in k, is to view E as a double covering of $\mathbb{P}^1$. Concretely, write E as a Weierstrass equation $y^2 = g(x)$, $g \in k[x]$ a cubic with distinct roots in $\overline{k}$, so that $x : E \to \mathbb{P}^1$ is the double covering. If we start with a "four point sheaf" $\mathcal{G}$ on $\mathbb{P}^1$, one of whose bad points is ∞ but none of whose bad points is a zero of the cubic $g(x)$, then its pullback to E by x to $E \setminus x^{-1}(\{\text{the bad points}\})$ is a "seven point sheaf" on E, providing an N of the desired type.

The simplest way to produce a four point sheaf $\mathcal{G}$ on $\mathbb{P}^1$ is to take the $R^1\pi_\star\overline{\mathbb{Q}_\ell}(1/2)$ for an elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$ with precisely four bad fibres, each of which is semistable. Over $\mathbb{C}$, these are precisely the elliptic surfaces classified by Beauville [Beau] thirty years ago, of which there are six. Up to isogeny there are only four, to wit

$$y^2 = -x(x-1)(x-\lambda^2),\ \lambda \neq 0, 1, -1, \infty,$$

$$y^2 = 4x^3 + ((a+2)x + a)^2,\ a \neq 0, 1, -8, \infty,$$

$$y^2 = 4x^3 + (b^2+6b-11)x^2 + (10-10b)x + 4b-3,\ b \neq 0, \infty,\ \text{root of } b^2+11b-1,$$

and

$$y^2 = 4x^3 + (3cx+1)^2,\ c \neq \infty, c^3 \neq 1.$$

Attached to each of these four families is the monic cubic polynomial $f(x)$ whose roots are its three finite bad points, namely the cubics

$$x^3 - x, \quad x(x-1)(x+8), \quad x(x^2+11x-1), \quad x^3 - 1,$$

and its four point sheaf $\mathcal{G}(x)$ on the projective x-line.

Theorem 4.1. *For each of the four families, with associated cubic $f(x)$ and four point sheaf $\mathcal{G}(x)$, there is an explicit nonzero integer polynomial $P(T) \in \mathbb{Z}[T]$ with the following property. For each finite field k in which ℓ is invertible, and for each $t \in k$ at which $P(t) \neq 0$ in k, the equation*

$$E_t : y^2 = tf(x) + t^2$$

defines an elliptic curve over k, and the N on this E_t gotten by pulling back $\mathcal{G}(x)$ has

$$G_{geom,N} = G_{arith,N} = G_2.$$

The proof, sadly, is essentially a computer verification. We have a priori inclusions

$$G_{geom,N} \lhd G_{arith,N} \subset O(7).$$

One first shows, conceptually, that $G_{geom,N}$ is Lie-irreducible, i.e., that $(G_{geom,N})^0$ is an irreducible subgroup of $SO(7)$. One then shows, again conceptually, that the moments M_3 and M_4 for the data (k,t) are each independent of (k,t), provided that $P(t)$ is nonzero in k. And one shows, again conceptually, that if $M_3 = 0$, then we would have an explicit upper bound (something like $294/\sqrt{\#k}$) for the absolute value of the empirical M_3 computed over k, as in section 1. One then finds numerically a single good data point $(\mathbb{F}_p, t)$, with p around 10^5, for which the empirical M_3 exceeds 1.0. This shows that M_3 is nonzero, so must be 1, at this data point and hence at every good data point. This in turn forces $G_{geom,N}$ to be either G_2 or $\mathrm{Sym}^6(SL(2))$. In either of these cases, $G_{arith,N}$ will be either the same group, or $\pm$ that group. In the latter case, it will be $-\theta_{\mathbb{F}_p,\Lambda}$ rather than $\theta_{\mathbb{F}_p,\Lambda}$ which lies in G_2 or in $\mathrm{Sym}^6(SL(2))$ accordingly. One then finds a single good data point $(\mathbb{F}_p, t)$ at which there are traces both more negative than -1.64 and strictly greater than 2. At this point we must have $G_{geom,N} = G_{arith,N} = G_2$. Because M_4 is constant, we must have $M_4 = 4$ at every good data point, hence we must have $G_{geom,N} = G_2$ at every good data point.

It remains to show that $G_{arith,N}$ is always G_2, never $\pm G_2$, at any good data point (k,t). For this, we argue as follows. We have $G_{arith,N} = \pm G_2$, if and only if every $\theta_{k,\Lambda}$, Λ a character of $E_t(k)$, lies in $-G_2$, i.e., has determinant -1.Thus we have $G_{arith,N} = G_2$ precisely when $\theta_{k,\mathbb{1}}$, $\mathbb{1}$ the trivial character of $E_t(k)$, has determinant 1. Unscrewing these definitions, we must show that for any good data point (k,t), we have

$$\det(Frob_k | H^1(E_t/\overline{k}, \mathcal{F}(1/2))) = 1.$$

We now use the Leray spectral sequence for the x double covering $E_t \to \mathbb{P}^1$. For the four point sheaf $\mathcal{G}$, the cohomology groups $H^i(\mathbb{P}^1/\overline{k}, \mathcal{G})$ all vanish, so we find that

$$H^1(E_t/\overline{k}, \mathcal{F}(1/2)) = H^1(\mathbb{P}^1/\overline{k}, \mathcal{G}(1/2) \otimes \mathcal{L}_{\chi_2(tf(x)+t^2)}),$$

for $\mathcal{L}_{\chi_2}$ the Kummer sheaf attached to the quadratic character χ_2 of $k^\times$. In other words, at time t we are looking at the "interesting part" of $H^2(1)$ of the Beauville elliptic surface over the x line, quadratically twisted by $tf(x) + t^2$. The entire H^2 has Hodge numbers $(2, 32, 2)$. There are 29 "trivial" algebraic classes over $\overline{k}$, given by the zero section and classes of components of fibres. The orthogonal of this 29 dimensional subspace is the "interesting part" we are looking at. Its Hodge numbers are $(2, 3, 2)$.

We now analyze $t \mapsto H^1(\mathbb{P}^1/\overline{k}, \mathcal{G}(1/2) \otimes \mathcal{L}_{\chi_2(tf(x)+t^2)})$ as a sheaf on the t-line over $\mathbb{Z}$. We need to invert 2, t, the discriminant of $f(x)$, and the discriminant of $f(x) + t$. In the four families, this amounts to inverting the integer polynomial $D(t)$ given respectively by

$$2t(4-27t^2),\ \ 6t(5184-2380t-27t^2),\ \ 10t(125-5522t-27t^2),\ \ 6t(t-1).$$

To insure that these polynomials have zeroes which stay disjoint from each other and from ∞, we invert the integer d given by $6, 6 \times 73, 30 \times 31, 6$ in the four cases. Then over Spec $(\mathbb{Z}[1/d])$ we have the punctured affine t line $S := \mathbb{A}^1[1/dD(t)]/\mathbb{Z}[1/d]$, and over S we have the projective x line $(\mathbb{P}^1)_S$, with structural map denoted $\rho : (\mathbb{P}^1)_S \to S$. This $(\mathbb{P}^1)_S$ carries the four point sheaf $\mathcal{G}$, which is lisse outside ∞ and the three roots of $f(x)$, and it carries the twisting sheaf $\mathcal{L}_{\chi_2(tf(x)+t^2)}$. The sheaf

$$\mathcal{H} := R^1\rho_\star(\mathcal{G}(1/2) \otimes \mathcal{L}_{\chi_2(tf(x)+t^2)})$$

is lisse (use Deligne's semicontinuity theorem, cf. [Lau-SCCS, Cor. 2.1.2]) of rank seven, ι-pure of weight zero, and orthogonally self dual on $S := \mathbb{A}^1[1/dD(t)]/\mathbb{Z}[1/d]$. It is automatically tamely ramified along ∞ and the zeroes of $dD(t)$, and so by the tame specialization theorem [Ka-ESDE, 8.17.13] it has the "same" G_{geom} on each geometric fibre of $S/\mathbb{Z}[1/d]$. Factoring out the $\mathcal{L}_{\chi_2(t)}$, we can write $\mathcal{H}$ as the tensor product of $\mathcal{L}_{\chi_2(t)}$ with the sheaf

$$\mathcal{K} := R^1\rho_\star(\mathcal{G}(1/2) \otimes \mathcal{L}_{\chi_2(f(x)+t)}).$$

This last sheaf $\mathcal{K}$ is, on each geometric fibre, the middle additive convolution [Ka-RLS, 2.6.2] of $\mathcal{L}_{\chi_2}$ with the direct image sheaf $[-f]_\star\mathcal{G}(1/2)$. Since we know the local monodromies of $\mathcal{G}(1/2)$, we can first compute the local monodromies of $[-f]_\star\mathcal{G}(1/2)$, then those of $\mathcal{K}$ (using [Ka-RLS, 3.3.6]), then those of $\mathcal{H}$. The upshot is that the (Jordan block structures of the) local monodromies of $\mathcal{H}$ are given by

$$3\mathbb{1} \oplus 4\chi_2 \text{ at } 0,$$

$$Unip(3) \oplus \chi_3 Unip(2) \oplus \overline{\chi_3} Unip(2) \text{ at } \infty,$$

and, for the first three Beauville families

$$2Unip(2) \oplus 3\mathbb{1} \text{ at the two invertible zeroes of} D(t),$$

while for the last Beauville family we get

$$2\chi_6 \oplus 2\overline{\chi_6} \oplus 3\mathbb{1} \text{ at the unique invertible zero of } D(t).$$

Since all the local monodromies have trivial determinant, we see that $\det(\mathcal{H})$ is geometrically trivial on each geometric fibre of $S/\mathbb{Z}[1/d]$. Therefore (use the homotopy sequence) $\det(\mathcal{H})$ is the pullback from $\mathrm{Spec}\,(\mathbb{Z}[1/d])$ of a ± 1-valued character, i.e., a quadratic Dirichlet character whose conductor divides a power of d. In the four cases, this forces the conductor to divide, respectively $24, 24\times 73, 24\times 5\times 31, 24$. In each of the four cases, we then test numerically enough primes to show that this Dirichlet character is in fact trivial.Thus $\det(\mathcal{H})$ is arithmetically trivial on S.

5. G_2 as a "usual" monodromy group

Theorem 5.1. *For the first three Beauvile families (but* **not** *the fourth), the sheaf $\mathcal{H}$ has*

$$G_{geom} = G_{arith} = G_2.$$

The proof is, once again, essentially a computational verification. The first step is to show that the sheaf $[-f]_\star\mathcal{G}(1/2)$ is geometrically irreducible on each geometric fibre. [It is this step which fails for the fourth family.] This geometric irreducibility either holds on all geometric fibres, or on none, and one uses a numerical calculation to show that it holds in some low characteristic. Then the sheaf $\mathcal{K}$, and hence also the sheaf $\mathcal{H}$, is geometrically irreducible. If it were not Lie-irreducible, because its rank is the prime seven, it would either have finite global monodromy or be induced from a rank one sheaf. In either case all its local monodromies would be semisimple. But one of its local monodromies is unipotent, so in fact $\mathcal{H}$ is Lie-irreducible. So on each fibre of $S/\mathbb{Z}[1/d]$, its groups G_{geom} and G_{arith} sit in

$$G_{geom} \lhd G_{arith} \subset SO(7).$$

By the same trick as before, we show that $G_{geom,N} = G_2$ by showing it in one low characteristic p, by first computing the empirical M_3 over $\mathbb{F}_p$ to show that $M_3 \neq 0$, then finding $\mathbb{F}_p$ points where traces are both < -1.64 and > 2.0 to show that we have we cannot have $\mathrm{Sym}^6(SL(2))$ in this characteristic. So we have $G_{geom} = G_2$ in this characteristic, and hence in every characteristic. Since G_2 is its own normalizer in $SO(7)$, we have $G_{geom} = G_{arith} = G_2$.

Remark 5.2. Thus we have, in each good characteristic and hence over $\mathbb{C}$ as well, a family of quadratic twists of each of the first three

Beauville surfaces in which a 7 dimensional piece of H^2 has monodromy group G_2. What is the conceptual explanation for this? Can one "see" an alternating trilinear form on this piece of H^2?

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Princeton University, Mathematics, Fine Hall, NJ 08544-1000, USA
E-mail address: nmk@math.princeton.edu