ELLIPTIC CONVOLUTION, G_2, AND ELLIPTIC SURFACES

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Abstract. This is (a slightly more detailed version of) our talk at the conference in honor of Laumon's sixtieth birthday. We report here on some unexpected occurrences of G_2, first stumbled upon experimentally, later proven, but still not understood. Proofs will appear elsewhere.

1. elliptic sums

Let k be a finite field, E/k an elliptic curve, and $f : E(k) \to \mathbb{C}$ a function on the finite abelian group $E(k)$. Given f, we define a function $S(f)$ of characters $\Lambda \in \mathrm{Hom}_{group}(E(k), \mathbb{C}^\times)$ by

$$S(f)(\Lambda) := \sum_{p \in E(k)} f(P)\Lambda(P).$$

This function $S(f)$ is the "Fourier transform" of f in the sense of finite abelian groups. Given two functions f, g on $E(k)$, their convolution is the function on $E(k)$ defined by

$$(f \star g)(P) := \sum_{R+S=P} f(R)g(S).$$

Their Fourier transforms are related by the usual identity $S(f \star g) = S(f)S(g)$, i.e., for each Λ we have

$$S(f \star g)(\Lambda) = S(f)(\Lambda)S(g)(\Lambda).$$

For a given function f, the moments of its Fourier transform $S(f)$, defined by

$$M_n(S(f)) := (1/\#E(k)) \sum_{\Lambda} S(f)(\Lambda)^n$$

are thus given in terms of the multiple self-convolutions $f^{\star n}$ of f with itself by

$$(1/\#E(k)) \sum_{\Lambda} S(f^{\star n})(\Lambda) = f^{\star n}(0).$$

For any writing of n as $a+b$ with a, b strictly positive integers, we thus have

$$M_n(S(f)) = (f^{\star n})(0) = \sum_P f^{\star a}(P) f^{\star b}(-P).$$

2. ELLIPTIC EQUIDISTRIBUTION

Fix a prime number ℓ invertible in k, and an embedding ι of $\overline{\mathbb{Q}_\ell}$ into $\mathbb{C}$. There is an obvious notion of convolution of objects in $D^c_b(E, \overline{\mathbb{Q}_\ell})$, defined in terms of the addition map $sum : E \times_k E \to E$, by $(A, B) \mapsto A \star B := Rsum_\star(A \boxtimes B)$. If we attach to $A \in D^c_b(E, \overline{\mathbb{Q}_\ell})$ its trace function on $E(k)$, given by $f_{A,k}(P) := \mathrm{Trace}(Frob_{k,P}|A)$, then by the Lefschetz Trace Formula we have the identity $f_{A,k} \star f_{B,k} = f_{A\star B,k}$ of functions on $E(k)$.

In general, if A and B are each perverse sheaves on E, their convolution need not be perverse. To remedy that, we work first on $E_{\overline{k}}$, the extension of scalars of E to $\overline{k}$. We say that an object $A \in D^c_b(E_{\overline{k}}, \overline{\mathbb{Q}_\ell})$ has property $\mathcal{P}$ if, for all lisse rank one sheaves $\mathcal{L}$ on $E_{\overline{k}}$, we have

$$H^i(E_{\overline{k}}, A \otimes \mathcal{L}) = 0 \text{ for i} \neq 0.$$

We have the following lemma.

Lemma 2.1. *Let $A \in D^c_b(E_{\overline{k}}, \overline{\mathbb{Q}_\ell})$ have property $\mathcal{P}$. Then A is perverse.*

Because lisse rank one $\mathcal{L}$'s on $E_{\overline{k}}$ are primitive in the sense that $sum^\star(\mathcal{L}) \cong \mathcal{L} \boxtimes \mathcal{L}$, the A's with property $\mathcal{P}$ are stable by convolution. Thus perverse sheaves with property $\mathcal{P}$ are stable by convolution. An irreducible perverse sheaf on $E_{\overline{k}}$ has property $\mathcal{P}$ unless it is an $\mathcal{L}[1]$.

Corollary 2.2. *The perverse sheaves on $E_{\overline{k}}$ with property $\mathcal{P}$ form a neutral Tannakian category, with convolution as the tensor operation, δ_0 as the identity, $N \mapsto N^\vee := [P \mapsto -P]^\star DN$ as the dual, and "dim"$(N) := \chi(E_{\overline{k}}, N) = h^0(E_{\overline{k}}, N)$. For any lisse rank one $\mathcal{L}$ on $E_{\overline{k}}$, $N \mapsto H^0(E_{\overline{k}}, N \otimes \mathcal{L})$ is a fibre functor.*

Remark 2.3. Just as in Gabber-Loeser [Ga-Loe], the abelian category structure on the above Tannakian category is the one induced by viewing it **not** as a full subcategory of the category $Perv$ of all perverse sheaves on $E_{\overline{k}}$, but rather as the quotient cateory $Perv/Neg$ of $Perv$ by the subcategory Neg consisting of those perverse sheaves which are of Euler characteristic zero, or (equivalently) of the form $\mathcal{F}[1]$ for $\mathcal{F}$ a lisse sheaf on $E_{\overline{k}}$, or (equivalently) successive extensions of objects $\mathcal{L}[1]$. The irreducible (resp. semisimple) objects in $Perv/Neg$ are just the irreducible (resp. semisimple) perverse sheaves with property $\mathcal{P}$. The semisimple perverse sheaves with property $\mathcal{P}$ themselves form a

Tannakian category; its structure of abelian category is equal to the naive one.

We now return to working on E/k. Recall that for a character Λ of $E(k)$, the Lang torsor construction [De-ST, 1.4] gives a lisse rank one sheaf $\mathcal{L}_\Lambda$ on E, whose trace function on $E(k)$ is Λ. The perverse sheaves on E which, pulled back to $E_{\overline{k}}$, have property $\mathcal{P}$, themselves form a neutral Tannakian category. For each character Λ of $E(k)$, $N \mapsto H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)$ is a fibre functor. The action of $Frob_k$ on $H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)$ is an automorphism of this fibre functor, so gives a conjugacy class $Frob_{k,\Lambda}$ in the Tannakian group $G_{arith,N}$ attached to N. Notice in passing that, by the Lefschetz trace formula,

$$\mathrm{Trace}(Frob_k | H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)) = \sum_{P \in E(k)} \mathrm{Trace}(Frob_{k,P}|N)\Lambda(P)$$

is the value at Λ of the elliptic sum $S(f_{N,k})$ attached to the trace function $f_{N,k}$ on N on $E(k)$.

Suppose N is perverse on E, has property $\mathcal{P}$, is arithmetically semisimple, is ι-pure of weight zero, and has dimension $n :=$ "dim"(N). Denote by $G_{arith,N}$, respectively $G_{geom,N}$, the Tannakian groups attached to N on E , respectively on $E_{\overline{k}}$. In general we have inclusions of reductive $\overline{\mathbb{Q}_\ell}$-algebraic groups

$$G_{geom,N} \lhd G_{arith,N} \subset GL(\text{“}dim\text{”}(N)).$$

Pick a maximal compact subgroup K of $G_{arith,N}(\mathbb{C})$. The semisimplification (in the sense of Jordan decomposition) $Frob^{ss}_{k,\Lambda}$ of the conjugacy class $Frob_{k,\Lambda}$ intersects K in a single conjugacy class $\theta_{k,\Lambda}$ of K. Via the inclusion of $K \subset G_{arith,N}(\mathbb{C})$ into $GL(n)$, we have

$$\det(1 - T\theta_{k,\Lambda}) = \det(1 - TFrob_k | H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)),$$

so in particular

$$\begin{aligned}\mathrm{Trace}(\theta_{k,\Lambda}) &= \mathrm{Trace}(Frob_k | H^0(E_{\overline{k}}, N \otimes \mathcal{L}_\Lambda)) \\ &= \sum_{P \in E(k)} \mathrm{Trace}(Frob_{k,P}|N)\Lambda(P).\end{aligned}$$

Exactly as in [Ka-CE, 1.1, 7.3], Deligne's Weil II results [De-Weil II, 3.3.1] and the Tannakian formalism give the following theorem.

Theorem 2.4. *In the above situation, suppose $G_{geom,N} = G_{arith,N}$. Then as L/k runs over larger and larger finite extension fields of k, the conjugacy classes $\{\theta_{L,\Lambda}\}_{\Lambda \text{ char. of E(L)}}$ become equidistributed in the space $K^\#$ of conjugacy classes of K, for its "Haar measure" of total mass one.*

3. The search for G_2

We work over $\mathbb{C}$. Recall that G_2, the automorphism group of the octonions, is the fixer in $SO(7)$ of an alternating trilinear form. It is a connected irreducible subgroup of $SO(7)$. According to a theorem of Gabber [Ka-ESDE, 1.6], the only connected irreducible subgroups of $SO(7)$ are $SO(7)$ itself, G_2, and the image of $SL(2)$ in $Sym^6(std_2)$, which we shall denote "$Sym^6(SL(2))$". For each of these three groups G, its normalizer in the full orthogonal group $O(7) = \{\pm 1\} \times SO(7)$ is the group $\pm G := \{\pm 1\} \times G$. Among these six groups, we can distinguish G_2 by its moments (for the given seven dimensional representation, call it V). For an integer $n \geq 1$ and H any of these six groups, we define

$$M_n(H) := \dim((V^{\otimes n})^H).$$

For K a maximal compact subgroup of H, we have

$$M_n(H) = \int_K \mathrm{Trace}(k|V)^n.$$

The third and fourth moments are given by the following table.

	M_3	M_4
$\mathrm{Sym}^6(SL(2))$	1	7
$\pm Sym^6(SL(2))$	0	7
G_2	1	4
$\pm G_2$	0	4
SO(7)	0	3
O(7)	0	3

So if M_3 is nonzero, we have either G_2 or $Sym^6(SL(2))$. We can distinguish these two cases by their M_4. But there is another, computationally easier, way to distinguish the two. Take maximal compact subgroups UG_2 and $Sym^6(SU(2))$ of these two groups. For UG_2, its traces in the given seven dimensional representation lie in the interval $[-2, 7]$, while the traces of $Sym^6(SU(2))$ (namely the values of the function $\sin(7\theta)/\sin(\theta)$) lie in the interval $[-1.64, 7]$.

4. Beauville families of elliptic curves

Starting with an elliptic curve E/k, how can we produce geometrically irreducible perverse sheaves N which have $\mathcal{P}$, are ι-pure of weight zero, and which, in the Tannakian sense, are self dual of dimension seven? Start with a "seven point sheaf" on E, by which we mean a geometrically irreducible lisse sheaf $\mathcal{F}$ of rank two on a dense open set $j : U \subset E$ of E which is ι-pure of weight zero, whose determinant is trivial, and such that $(E \setminus U)(\overline{k})$ consists of seven points, at each of which the local monodromy of $\mathcal{F}$ is unipotent and nontrivial. Then

$$N := j_\star \mathcal{F}(1/2)[1]$$

is perverse, ι-pure of weight zero, and geometrically irreducible of "dimension" $\chi(E_{\overline{k}}, N) = 7$. If in addition $\mathcal{F}$ is isomorphic to its pullback by $P \mapsto -P$, then N is self dual. Because N is geometrically irreducible, the autoduality has a sign. Because N has odd "dimension", the autoduality must be orthogonal.

One way to get such an N on E, at least if 2 is invertible in k, is to view E as a double covering of $\mathbb{P}^1$. Concretely, write E as a Weierstrass equation $y^2 = g(x)$, $g \in k[x]$ a cubic with distinct roots in $\overline{k}$, so that $x : E \to \mathbb{P}^1$ is the double covering. If we start with a "four point sheaf" $\mathcal{G}$ on $\mathbb{P}^1$, one of whose bad points is ∞ but none of whose bad points is a zero of the cubic $g(x)$, then its pullback to E by the x to $E \setminus x^{-1}(\{\text{the bad points}\})$ is a "seven point sheaf" on E, providing an N of the desired type.

The simplest way to produce a four point sheaf $\mathcal{G}$ on $\mathbb{P}^1$ is to take the $R^1\pi_\star\overline{\mathbb{Q}_\ell}(1/2)$ for an elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$ with precisely four bad fibres, each of which is semistable. Over $\mathbb{C}$, these are precisely the elliptic surfaces classified by Beauville [Beau] thirty years ago, of which there are six. Up to isogeny there are only four, to wit

$$y^2 = -x(x-1)(x-\lambda^2),\ \lambda \neq 0, 1, -1, \infty,$$
$$y^2 = 4x^3 + ((a+2)x + a)^2,\ a \neq 0, 1, -8, \infty,$$
$$y^2 = 4x^3 + (b^2+6b-11)x^2 + (10-10b)x + 4b-3,\ b \neq 0, \infty,\ \text{root of } b^2+11b-1,$$

and

$$y^2 = 4x^3 + (3cx+1)^2,\ c \neq \infty, c^3 \neq 1.$$

Attached to each of these four families is the monic cubic polynomial $f(x)$ whose roots are its three finite bad points, namely the cubics

$$x^3 - x, \quad x(x-1)(x+8), \quad x(x^2+11x-1), \quad x^3 - 1,$$

and its four point sheaf $\mathcal{G}(x)$ on the projective x-line.

Theorem 4.1. *For each of the four families, with associated cubic $f(x)$ and four point sheaf $\mathcal{G}(x)$, there is an explicit nonzero integer polynomial $P(T) \in \mathbb{Z}[T]$ with the following property. For each finite field k in which ℓ is invertible, and for each $t \in k$ at which $P(t) \neq 0$ in k, the equation*

$$E_t : y^2 = tf(x) + t^2$$

defines an elliptic curve over k, and the N on this E_t gotten by pulling back $\mathcal{G}(x)$ has

$$G_{geom,N} = G_{arith,N} = G_2.$$

The proof, sadly, is essentially a computer verification. We have a priori inclusions

$$G_{geom,N} \lhd G_{arith,N} \subset O(7).$$

One first shows, conceptually, that $G_{geom,N}$ is Lie-irreducible, l.e., that $(G_{geom,N})^0$ is an irreducible subgroup of $SO(7)$. So One then shows, again conceptually, that the moments M_3 and M_4 for the data (k,t) are each independent of (k,t), provided that $P(t)$ is nonzero in k. And one shows, again conceptually, that if $M_3 = 0$, then we would have an explicit upper bound (something like $294/\sqrt{\#k}$) for the absolute value of the empirical M_3 computed over k, as in section 1. One then finds numerically a single good data point $(\mathbb{F}_p, t)$, with p around 10^5, for which the empirical M_3 exceeds 1.0. This shows that M_3 is nonzero, so must be 1, at this data point and hence at every good data point. This in turn forces $G_{geom,N}$ to be either G_2 or $Sym^6(SL(2))$. In either of these cases, $G_{arith,N}$ will be either the same group, or $\pm$ that group. In the latter case, it will be $-\theta_{\mathbb{F}_p,\Lambda}$ rather than $\theta_{\mathbb{F}_p,\Lambda}$ which lies in G_2 or in $Sym^6(SL(2))$ accordingly. One then finds a single good data point $(\mathbb{F}_p, t)$ at which there are traces both more negative than -1.64 and strictly greater than 2. At this point we must have $G_{geom,N} = G_{arith,N} = G_2$. Because M_4 is constant, we must have $M_4 = 4$ at every good data point, hence we must have $G_{geom,N} = G_2$ at every good data point.

It remains to show that $G_{arith,N}$ is always G_2, never $\pm G_2$, at any good data point (k,t). For this, we argue as follows. We have $G_{arith,N} = \pm G_2$, if and only if every $\theta_{k,\Lambda}$, Λ a character of $E_t(k)$, lies in $-G_2$, i.e., has determinant -1.Thus we have $G_{arith,N} = G_2$ precisely when $\theta_{k,\mathbb{1}}$, $\mathbb{1}$ the trivial character of $E_t(k)$, has determinant 1. Unscrewing these definitions, we must show that for any good data point (k,t), we have

$$\det(Frob_k | H^1(E_t/\overline{k}, \mathcal{F}(1/2))) = 1.$$

We now use the Leray spectral sequence for the x double covering $E_t \to \mathbb{P}^1$. For the four point sheaf $\mathcal{G}$, the cohomology groups $H^i(\mathbb{P}^1/\overline{k}, \mathcal{G})$ all vanish, so we find that

$$H^1(E_t/\overline{k}, \mathcal{F}(1/2)) = H^1(\mathbb{P}^1/\overline{k}, \mathcal{G}(1/2) \otimes \mathcal{L}_{\chi_2(tf(x)+t^2)}),$$

for $\mathcal{L}_{\chi_2}$ the Kummer sheaf attached to the quadratic character χ_2 of $k^\times$. In other words, at time t we are looking at the "interesting part" of $H^2(1)$ of the Beauville elliptic surface over the x line, quadratically twisted by $tf(x) + t^2$. The entire H^2 has Hodge numbers $(2, 32, 2)$. There are 29 "trivial" algebraic classes over $\overline{k}$, given by the zero section and classes of components of fibres. The orthogonal of this 29 dimensional subspace is the "interesting part" we are looking at. Its Hodge numbers are $(2, 3, 2)$.

We now analyze $t \mapsto H^1(\mathbb{P}^1/\overline{k}, \mathcal{G}(1/2) \otimes \mathcal{L}_{\chi_2(tf(x)+t^2)})$ as a sheaf on the t-line over $\mathbb{Z}$. We need to invert 2, t, the discriminant of $f(x)$, and the discriminant of $f(x) + t$. In the four families, this amounts to inverting the integer polynomial $D(t)$ given respectively by

$$2t(4-27t^2),\ \ 6t(5184-2380t-27t^2),\ \ 10t(125-5522t-27t^2),\ \ 6t(t-1).$$

To insure that these polynomials have zeroes which stay disjoint from each other and from ∞, we invert the integer d given by $6, 6 \times 73, 30 \times 31, 6$ in the four cases. Then over $\mathrm{Spec}\,(\mathbb{Z}[1/d])$ we have the punctured affine t line $S := \mathbb{A}^1[1/dD(t)]/\mathbb{Z}[1/d]$, and over S we have the projective x line $(\mathbb{P}^1)_S$, with structural map denoted $\rho : (\mathbb{P}^1)_S \to S$. This $(\mathbb{P}_1)_S$ carries the four point sheaf $\mathcal{G}$, which is lisse outside ∞ and the three roots of $f(x)$, and it carries the twisting sheaf $\mathcal{L}_{\chi_2(tf(x)+t^2)}$. The sheaf

$$\mathcal{H} := R^1\rho_\star(\mathcal{G}(1/2) \otimes \mathcal{L}_{\chi_2(tf(x)+t^2)})$$

is lisse (use Deligne's semicontinuity theorem, cf. [Lau-SCCS, Cor. 2.1.2]) of rank seven, ι-pure of weight zero, and orthogonally self dual on $S := \mathbb{A}^1[1/dD(t)]/\mathbb{Z}[1/d]$. It is automatically tamely ramified along ∞ and the zeroes of $dD(t)$, and so by the tame specialization theorem [Ka-ESDE, 8.17.13] it has the "same" G_{geom} on each geometric fibre of $S/\mathbb{Z}[1/d]$. Factoring out the $\mathcal{L}_{\chi_2(t)}$, we can write $\mathcal{H}$ as the tensor product of $\mathcal{L}_{\chi_2(t)}$ with the sheaf

$$\mathcal{K} := R^1\rho_\star(\mathcal{G}(1/2) \otimes \mathcal{L}_{\chi_2(f(x)+t)}).$$

This last sheaf $\mathcal{K}$ is, on each geometric fibre, the middle additive convolution [Ka-RLS, 2.6.2] of $\mathcal{L}_{\chi_2}$ with the direct image sheaf $[-f]_\star\mathcal{G}(1/2)$. Since we know the local monodromies of $\mathcal{G}(1/2)$, we can first compute the local monodromies of $[-f]_\star\mathcal{G}(1/2)$, then those of $\mathcal{K}$ (using [Ka-RLS, 3.3.6]), then those of $\mathcal{H}$. The upshot is that the (Jordan block structures of the) local monodromies of $\mathcal{H}$ are given by

$$3\mathbb{1} \oplus 4\chi_2 \text{ at } 0,$$

$$Unip(3) \oplus \chi_3 Unip(2) \oplus \overline{\chi_3} Unip(2) \text{ at } \infty,$$

and, for the first three Beauville families

$$2Unip(2) \oplus 3\mathbb{1} \text{ at the two invertible zeroes of} D(t),$$

while for the last Beauville family we get

$$2\chi_6 \oplus 2\overline{\chi_6} \oplus 3\mathbb{1} \text{ at the unique invertible zero of } D(t).$$

Since all the local monodromies have trivial determinant, we see that $\det(\mathcal{H})$ is geometrically trivial on each geometric fibre of $S/\mathbb{Z}[1/d]$. Therefore (use the homotopy sequence) $\det(\mathcal{H})$ is the pullback from $\mathrm{Spec}\,(\mathbb{Z}[1/d])$ of a ± 1-valued character, i.e., a quadratic Dirichlet character whose conductor divides a power of d. In the four cases, this forces the conductor to divide, respectively $24, 24\times 73, 24\times 5\times 31, 24$. In each of the four cases, we then test numerically enough primes to show that this Dirichlet character is in fact trivial.Thus $\det(\mathcal{H})$ is arithmetically trivial on S.

5. G_2 as a "usual" monodromy group

Theorem 5.1. *For the first three Beauvile families (but* **not** *the fourth), the sheaf $\mathcal{H}$ has*

$$G_{geom} = G_{arith} = G_2.$$

The proof is, once again, essentially a computational verification. The first step is to show that the sheaf $[-f]_\star\mathcal{G}(1/2)$ is geometrically irreducible on each geometric fibre. [It is this step which fails for the fourth family.] This geometrically irreducibility either holds on all geometric fibres, or on none, and one uses a numerical calculation to show that it holds in some low characteristic. Then the sheaf $\mathcal{K}$, and hence also the sheaf $\mathcal{H}$, is geometrically irreducible. If it were not Lie-irreducible, because its rank is the prime seven, it would either have finite global monodromy or be induced from a rank one sheaf. In either case all its local monodromies would be semisimple. But one of its local monodromies is unipotent, so in fact $\mathcal{H}$ is Lie-irreducible. So on each fibre of $S/\mathbb{Z}[1/d]$, its groups G_{geom} and G_{arith} sit in

$$G_{geom} \lhd G_{arith} \subset SO(7).$$

By the same trick as before, we show that $G_{geom,N} = G_2$ by showing it in one low characteristic p, by first computing the empirical M_3 over $\mathbb{F}_p$ to show that $M_3 \neq 0$, then finding $\mathbb{F}_p$ points where traces are both < -1.64 and > 2.0 to show that we have we cannot have $Sym^6(SL(2))$ in this characteristic. So we have $G_{geom} = G_2$ in this characteristic, and hence in every characteristic. Since G_2 is its own normalizer in $SO(7)$, we have $G_{geom} = G_{arith} = G_2$.

Remark 5.2. Thus we have, in each good characteristic and hence over $\mathbb{C}$ as well, a family of quadratic twists of each of the first three

Beauville surfaces in which a 7 dimensional piece of H^2 has monodromy group G_2. What is the conceptual explanation for this? Can one "see" an alternating trilinear form on this piece of H^2?

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