Moments of Weil representations of finite special unitary groups

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We prove an “\(n\)th moment = 1” result for irreducible Weil representations of degree \((q^n + 1)/(q + 1)\) of special unitary groups \(SU_n(q)\) for any odd \(n \geq 3\) and any prime power \(q\).

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Dedicated to the memory of Kay Magaard

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1. Introduction

For an odd integer \( n \geq 3 \), and a prime power \( q \geq 2 \), the irreducible representations (over \( \mathbb{C} \)) of lowest degree after the trivial representation of the group \( \text{SU}_n(q) \) are a symplectic representation of dimension \( \frac{q^n+1}{q+1} - 1 = \frac{q^n-q}{q+1} \), and \( q \) representations of dimension \( \frac{q^n+1}{q+1} \). When \( q \) is odd, exactly one of these \( q \) representations is orthogonal, otherwise none is. The direct sum of these \( q+1 \) representations is called the (big, or reducible) Weil representation of \( \text{SU}_n(q) \), and the \( q+1 \) individual representations are referred to as (irreducible) Weil representations, see e.g. [14, Theorem 4.1] and [15, §4].

In the paper [7], we wrote down \( q+1 \) rigid local systems on the affine line \( \mathbb{A}^1/\mathbb{F}_p \) whose geometric monodromy groups we conjectured to be the images of \( \text{SU}_n(q) \) in these \( q+1 \) representations. We were able to prove this only in the case when \( n=3 \) and \( \gcd(n,q+1)=1 \). In the sequel [9], we used a completely different method, which starts with results of Gross [4] and relies on [8], to prove these conjectures for any odd \( n \geq 3 \) and for any odd prime power \( q \).

In the course of thinking about these questions, we stumbled upon a striking representation-theoretic fact about the \( q \) Weil representations of \( \text{SU}_n(q) \) \((n \geq 3 \) odd\) of dimension \( \frac{q^n+1}{q+1} \). For each of them, their \( n \)th moment (i.e. the dimension of the space of invariants in the \( n \)th tensor power of the representation in question) is exactly one. For the irreducible representation of dimension \( \frac{q^n+1}{q+1} - 1 \), the \( n \)th moment vanishes. At present we do not have a conceptual explanation for this phenomenon.

**Theorem 1.** Let \( q \) be a prime power, \( n \geq 3 \) any odd integer, and let \( G = \text{SU}_n(q) \). Suppose in addition that \((n,q) \neq (3,2)\). Let \( V \) be one of the \( q+1 \) complex irreducible Weil modules of \( G \), of dimension \( (q^n+1)/(q+1) \) or \((q^n-q)/(q+1)\). Then the subspace of \( G \)-invariants on \( V^{\otimes n} \) has dimension 1 if \( \dim(V) = (q^n+1)/(q+1) \), and 0 if \( \dim(V) = (q^n-q)/(q+1) \).

As stated in Theorem 1, each of the Weil modules of \( \text{SU}_n(q) \) of dimension \( (q^n+1)/(q+1) \) has a unique (up to scalar) polynomial invariant of degree \( n \). It would be interesting to know what is the geometric significance of this polynomial invariant, and to find an explicit construction of it.

Given this result about \( n \)th moments for \( \text{SU}_n(q) \) when \( n \) is odd, it is natural to wonder about the situation for \( n \)th moments when \( n \) is even. [For \( n \) even and \( q \geq 3 \) a prime power, the irreducible representations (over \( \mathbb{C} \)) of lowest degree after the trivial representation of the group \( \text{SU}_n(q) \) are an orthogonal representation of dimension \( \frac{q^n-1}{q+1} + 1 = \frac{q^n+q}{q+1} \), and \( q \) representations of dimension \( \frac{q^n-1}{q+1} \)] Already for \( n = 4 \), the result is not so nice, cf. Theorem 4.1.

For the Weil representations of finite special linear groups \( \text{SL}_n(q) \) and symplectic groups \( \text{Sp}_{2n}(q) \), the latter with \( q \) odd, one also does not expect any nice regularity about the \( n \)th moments. We record however a curious fact about the 4th moments of Weil representations of \( \text{Sp}_{2n}(3) \), see Proposition 4.2.
2. Preliminaries

Let \( q = p^f \) be any prime power and \( n \geq 2 \). It is well known, see e.g. [15, §4], that the function

\[
\zeta_{i,n} = \zeta_n : g \mapsto (-1)^n (-q)^{\text{dim}_q \ker(g-1w)}
\]

defines a complex character, called the (reducible) Weil character, of the general unitary group \( \text{GU}_n(q) = \text{GU}(W) \), where \( W = F_q^\times \) is a non-degenerate Hermitian space with Hermitian product \( \circ \). Note that the \( F_q \)-bilinear form

\[
(u|v) = \text{Trace}_{F_q^2/F_q}(\theta u \circ u)
\]
on \( W \), for a fixed \( \theta \in F_q^\times \) with \( \theta^{q-1} = -1 \), is non-degenerate symplectic. This leads to an embedding

\[
\tilde{G} := \text{GU}_n(q) \hookrightarrow \text{Sp}_{2n}(q).
\]

Moreover, if \( q \) is odd then the restriction of any of the two big Weil characters (of degree \( q^n \), and denoted \( \text{Weil}_{1,2} \) in [8]) of \( \text{Sp}_{2n}(q) \) to \( \text{GU}_n(q) \) is exactly the big Weil character \( \zeta_n \) multiplied by the unique linear character of order 2 of \( \tilde{G} \), cf. [15, §4]. We will also denote by \( \zeta_n \) the restriction of this character to the special unitary group \( G := \text{SU}_n(q) \).

Fix a generator \( \sigma \) of \( F_q^\times \) and set \( \rho := \sigma^{q-1} \). We also fix a primitive \((q^2 - 1)\)th root of unity \( \sigma \in \mathbb{C}^\times \) and let \( \rho = \sigma^{q-1} \). Then

\[
\zeta_n = \sum_{i=0}^q \tilde{\zeta}_{i,n} \tag{2.0.1}
\]
decomposes as the sum of \( q + 1 \) characters of \( \tilde{G} \), where

\[
\tilde{\zeta}_{i,n}(g) = \frac{(-1)^n}{q+1} \sum_{i=0}^q \rho^i (-q)^{\text{dim}_q \ker(g-\rho^i-1w)}; \tag{2.0.2}
\]

see [15, Lemma 4.1]. In particular, \( \tilde{\zeta}_{i,n} \) has degree \((q^n - (-1)^n)/(q+1)\) if \( i > 0 \) and \((q^n + (-1)^n q)/(q+1)\) if \( i = 0 \).

We will let \( \zeta_{i,n} \) denote the restriction of \( \tilde{\zeta}_{i,n} \) to \( G = \text{SU}_n(q) \), for \( 0 \leq i \leq q \). If \( n \geq 3 \), then these \( q + 1 \) characters are all irreducible and distinct. If \( n = 2 \), then \( \zeta_{i,n} \) is irreducible, unless \( q \) is odd and \( i = (q+1)/2 \), in which case it is a sum of two irreducible characters of degree \((q-1)/2\), see [15, Lemma 4.7]. Formula \((2.0.2)\) implies that Weil characters \( \zeta_{i,n} \) enjoy the following branching rule while restricting to the natural subgroup \( H := \text{Stab}_G(w) \cong \text{SU}_{n-1}(q) \) (\( w \in W \) any anisotropic vector):
Furthermore, the complex conjugation fixes $\tilde{\zeta}_{0,n}$ and sends $\tilde{\zeta}_{j,n}$ to $\tilde{\zeta}_{q+1-j,n}$ when $1 \leq j \leq q$. As $n \geq 3$ is odd, it is also known that $\tilde{\zeta}_{0,n}$ is of symplectic type; let $\Psi_0 : \tilde{G} \to \text{Sp}(V)$ be a complex representation affording this character. If $2 \mid q$, then $\tilde{\zeta}_{(q+1)/2,n}$ is of orthogonal type; let $\Psi_{(q+1)/2} : \tilde{G} \to \text{O}(V)$ be a complex representation affording this character. In the remaining cases, let $\Psi_i : \tilde{G} \to \text{GL}(V)$ be a complex representation affording the character $\tilde{\zeta}_{i,n}$.

3. Odd-dimensional unitary groups

In this section, we will consider special unitary groups $G := \text{SU}_n(q) = \text{SU}(W)$ where $q$ is any prime power and $n \geq 3$ is odd. In fact, up until Theorem 3.11 we will assume that $n = 2k + 1 \geq 5$, and fix a basis $(e_1, \ldots, e_k, f_1, \ldots, f_k, w)$ of the Hermitian space $W = \mathbb{F}_{q^2}^n$, in which the Hermitian form $\diamond$ takes values

$$e_i \diamond e_j = f_i \diamond f_j = e_i \diamond w = f_i \diamond w = 0, \quad e_i \diamond f_j = \delta_{i,j}, \quad w \diamond w = 1.$$  

(3.0.1)

We also fix the notation

$$P_1 := \text{Stab}_G(\langle e_1 \rangle_{\mathbb{F}_{q^2}}) = Q_1L_1, \quad P_k := \text{Stab}_G(\langle e_1, \ldots, e_k \rangle_{\mathbb{F}_{q^2}}) = Q_kL_k,$$

where $Q_1 = \text{O}_p(P_1)$, $Q_k = \text{O}_p(P_k)$, $L_k \cong \text{GL}_k(q^2)$. The action of any $X \in L_k = \text{GL}_k(q^2)$ in the indicated basis of $W$ is given by $\text{diag}(X, t^X - q, \det(X)^{q-1})$, see [13, §5.1].

As shown in [5, Lemmas 12.5, 12.6], the Levi subgroup $L$ has a unique orbit $\mathcal{O}$ on $\text{Irr}(Z(Q_k)) \setminus \{1_{Z(Q_k)}\}$ of smallest length $q^{2k} - 1)/(q + 1)$, which then occurs in the restriction of any Weil character $\zeta_{i,n}$. Moreover, any $\lambda \in \mathcal{O}$ can only lie under an irreducible character of degree $q$ of $Q_k$. In particular, this shows that

Lemma 3.1. Suppose $n = 2k + 1 \geq 5$. Then $\zeta_{0,n}$ is irreducible over $P_k$. If $1 \leq i \leq q$, then $\zeta_{i,n}|_{P_k} = \nu_i + \theta_i$, where $\theta_i \in \text{Irr}(P_k)$ affords the orbit $\mathcal{O}$, and $\nu_i$ is a linear character of $P_k$ trivial at $Z(Q_k)$.

Lemma 3.2. In the notation of Lemma 3.1, assume that $1 \leq i \leq q$. Then $\text{Ker}(\nu_i) \geq Q_k$, and if $X \in L_k$ has determinant $\sigma^t$ as an element in $\text{GL}_k(q^2)$ with $t \in \mathbb{Z}$, then $\nu_i(X) = \sigma^{(q-1)t}$.

Proof. As noted in Lemma 3.1, $\nu_i$ is trivial at $Z(Q_k)$, and it is $P_k$-invariant. But $L_k$ acts transitively on the $q^{2k} - 1$ nontrivial linear characters of $Q_k/Z(Q_k)$, so $\text{Ker}(\nu_i) \geq Q_k$. Next, $[L_k, L_k] \cong \text{SL}_k(q^2)$ is perfect, so $\nu_i$ is trivial at $[L_k, L_k]$. Thus there is some $0 \leq s \leq q^2 - 2$ such that $\nu_i(X) = \sigma^{ts}$ for the listed $X \in L_k$. To find $s$, it suffices to
evaluate $\nu_i(X)$ for some $X_0$ that generates $L_k$ modulo $[L_k, L_k]$. Let $\gamma$ be a generator of $\mathbb{F}_{q^{2k}}^\times$ such that $\gamma(q^{2k-1})/(q^2 - 1) = \sigma$, and choose $X_0 \in L_k$ conjugate to

$$\text{diag}(\gamma, \gamma q^2, \ldots, \gamma q^{2k-2})$$

over $\mathbb{F}_q$, so that $\det(X_0) = \sigma$. Since no eigenvalue of $X_0$ belongs to $\mathbb{F}_{q^2}$, $X_0$ cannot fix any $\lambda \in \mathcal{O}$, see formula (20) of [13]), and so $\theta_i(X_0) = 0$ and $\nu_i(X_0) = \zeta_{i,n}(X_0)$. The absence of eigenvalues in $\mathbb{F}_{q^2}$ and the equality $\det(X_0)^{q-1} = \rho$ imply by (2.0.2) that $\zeta_{i,n}(X_0) = \rho^i = \sigma^{(q-1)i}$, i.e. $s = (q-1)i$ as stated. \hfill \Box

**Proposition 3.3.** Suppose $n = 2k+1 \geq 5$. Then $(\zeta_n)^{n-1}$ contains $\zeta_{i,n}$ with multiplicity one if $i > 0$, and zero if $i = 0$.

**Proof.** Note that $(\zeta_n)^2$ is just the permutation character of $G$ acting on the point set of $W$. Hence $(\zeta_n)^{n-1}$ is the permutation character of $G$ acting on the set $\Omega$ of ordered $k$-tuples $\omega = (v_1, \ldots, v_k)$, $v_i \in W$. Let $\pi_\omega = \text{Ind}_{G_\omega}^{G}(1_{G_\omega})$ denote the permutation character of $G$ acting on the $G$-orbit of $\omega = (v_1, \ldots, v_k)$, where $G_\omega = \text{Stab}_G(\omega)$, and suppose that $\zeta_{i,n}$ is an irreducible constituent of $\pi_\omega$. Then

$$0 < [\pi_\omega, \zeta_{i,n}]_G = [1_{G_\omega}, \zeta_{i,n}|_{G_\omega}]_{G_\omega}; \quad (3.3.1)$$

in particular, $1_{G_\omega}$ is an irreducible constituent of $\zeta_{i,n}|_{G_\omega}$.

(i) First we consider the case where $X := \langle v_1, \ldots, v_k \rangle_{\mathbb{F}_{q^2}}$ is contained in a non-degenerate subspace $Y$ of $W$ of codimension $\geq 2$. Without loss we may assume that $e_1, f_1 \in Y^\perp$. Then $G_\omega$ contains a natural subgroup $M := \text{SU}((e_1, f_1)_{\mathbb{F}_{q^2}}) \cong \text{SU}_2(q)$ (that acts trivially on $Y$). The branching rule (2.0.3) then shows that $\zeta_{i,n}|_{M}$ is a sum of Weil characters $\zeta_{j,2}$ of $M$. As mentioned above, an irreducible constituent $\lambda$ of $\zeta_{j,2}$ can have degree 1 only when $(q, j) = (2, \neq 0)$ or $(q, j) = (3, (q+1)/2)$. In the former case, one can check that $\lambda$ is actually the sign character of $M = \text{SU}_2(2) \cong \text{Sym}_4$. In the latter case, $\lambda(z) \neq 1$ for some element $z$ of $M \cong \text{SU}_2(3)$ of order 3. Thus $\lambda$ can never be equal to $1_M$, contradicting (3.3.1).

In particular, we have shown that $X$ cannot be non-degenerate.

(ii) Suppose now that $0 \neq X \cap X^\perp$ has dimension $j \leq k - 1$. By Witt’s lemma, we may then assume that $X = \langle e_1, \ldots, e_j, w_1, \ldots, w_{k-j} \rangle_{\mathbb{F}_{q^2}}$, where $\langle w_1, \ldots, w_{k-j} \rangle_{\mathbb{F}_{q^2}}$ is a non-degenerate subspace of

$$\langle e_{j+1}, \ldots, e_k, f_{j+1}, \ldots, f_k \rangle_{\mathbb{F}_{q^2}}.$$

But then $X$ is contained in the non-degenerate subspace

$$Y := \langle e_1, \ldots, e_j, f_1, \ldots, f_j, w_1, \ldots, w_{k-j} \rangle_{\mathbb{F}_{q^2}}$$

of codimension $n - (k + j) \geq 2$, contradicting (i).
(iii) We have shown that \( \dim(\mathcal{X} \cap \mathcal{X}^\perp) = k \), i.e. \( \mathcal{X} \) is totally singular of dimension \( k \). There is only one \( G \)-orbit of such \( \omega \), and we may assume that \( \omega = (e_1, \ldots, e_k) \). The description of \( P_k \) given in \([13, \S 5.1]\) shows that \( G_\omega = Q_k \). Now Lemmas 3.1, 3.2, and (3.3.1) show that \([\pi_\omega, \zeta_{i,n}]_G = 1 - \delta_{0,i} \), as stated. \( \square \)

Next we define the following linear characters \( \lambda_i \) of the parabolic subgroup \( P_i = \text{Stab}_G((e_1)_{F_{q^2}}) \) for \( 1 \leq i \leq q \): if \( g \in P_i \) sends \( e_1 \) to \( \sigma^t \) for \( 0 \leq t \leq q^2 - 2 \), then \( \lambda_i(g) = \sigma^{-(q-1)it} \), and set
\[
\Lambda_i := \text{Ind}^G_{P_i}(\lambda_i).
\]

**Proposition 3.4.** Suppose \( n = 2k + 1 \geq 5 \), \((n,q) \neq (5,2)\), and \( 1 \leq i \leq q \). Then \( \Lambda_i \) enters the character \( (\zeta_n)^2 \), and \([\zeta_{i,n}]^2, \Lambda_i \) \( \geq 1 \).

**Proof.** (i) As discussed in \([5, \S 11]\), \( P'_i := \text{Stab}_G(e_1) = Q_1 \rtimes L'_i \), where \( L'_1 = \text{Stab}_G(e_1) \cap \text{Stab}_G(f_1) \cong SU_{n-2}(q) \). Note that \( \Lambda_i \) enters the character \( \text{Ind}^G_{P'_i}(\chi) \), which in turn enters the character \( (\zeta_n)^2 \). Furthermore, \( L_1 \) acts transitively on the \( q-1 \) nontrivial linear characters of \( \mathbf{Z}(Q_1) \) (which has order \( q \)), and for each such character \( \alpha \) there is a unique irreducible character of \( Q_1 \) of degree \( q^{n-2} \), which then extends to a unique character \( M_\alpha \) of \( P'_1 \). We fix some nontrivial \( \alpha \in \text{Irr}(\mathbf{Z}(Q_1)) \) and let \( K := \text{Stab}_{P_1}(\alpha) = P'_1 \cdot C_{q+1} \). By its uniqueness, \( M_\alpha \) extends to \( K \). Note that
\[
\zeta_{i,n}(1) = (q^n + 1)/(q + 1) < 2q^{n-2}(q - 1) = 2(q - 1)M_\alpha(1).
\]

It follows by Clifford’s theorem that
\[
\zeta_{i,n}|_{P_i} = \beta_i + \text{Ind}^K_{P_i}(M_\alpha), \tag{3.4.1}
\]
for some extension to \( K \) of \( M_\alpha \) which we also denote by \( M_\alpha \), and for some character \( \beta_i \) of \( P_1 \) of degree \( (q^{n-2}+1)/(q+1) \), with \( \mathbf{Z}(Q_1) \leq \text{Ker}(\beta_i) \). Next, \( M_\alpha|_{L'_1} = \zeta_{n-2} \). Applying (2.0.3) to the standard subgroup \( L'_1 \) and using (3.4.1), we get
\[
\beta_i|_{L'_1} = \zeta_{i,n}|_{L'_1} - (q - 1)\zeta_{n-2} = \sum_{j \neq i, j \neq i} \zeta_{n-2,j'} - (q - 1) \sum_{j' = 0}^{q} \zeta_{n-2,i'} = \zeta_{n-2,i}.
\]

In particular, \( \beta_i \in \text{Irr}(P_1) \).

(ii) As usual, \( \bar{\chi} \) denotes the complex conjugate of any character \( \chi \). Note that \( \text{Stab}_{P_1}(\bar{\alpha}) = K \). Hence, (3.4.1) implies that
\[
\bar{\zeta}_{i,n}|_{P_1} = \bar{\beta}_i + \text{Ind}^K_{P_1}(\bar{M}_\alpha). \tag{3.4.2}
\]

Observe that \( \bar{M}_\alpha \) affords the \( \mathbf{Z}(Q_1) \)-character \( q^{n-2}\bar{\alpha} \) and is irreducible over \( P'_1 \). By the aforementioned uniqueness, \( \bar{M}_\alpha \) agrees with \( M_\bar{\alpha} \) on \( P'_1 \), where \( M_\bar{\alpha} \) is the \( K \)-character
of the \(\tilde{\alpha}\)-isotypic component in \(\zeta_{i,n}|_{P_1}\). As \(K/P_1 \cong C_{q+1}\), these two characters differ from each other by a linear character of \(K/P'_1\), which extends to a linear character \(\delta\) of \(P_1/P'_1 \cong C_{q^2-1}\). We have shown that

\[
\text{Ind}_{K}^{P_1}(\overline{M}_\alpha) = \text{Ind}_{K}^{P_1}(M_\alpha \cdot \delta|_K) = \text{Ind}_{K}^{P_1}(M_\alpha) \cdot \delta, \tag{3.4.3}
\]

and

\[
\zeta_{i,n}|_{P_1} = \beta_i + \text{Ind}_{K}^{P_1}(M_\alpha). \tag{3.4.4}
\]

(iii) We aim to show that one can take \(\delta = \overline{\lambda}_i\) in (3.4.3). Let \(\tau\) be an element of \(\mathbb{F}_{q^{4k-2}}^\times\) of order \(q^{2k-1}+1\) chosen such that \(\tau^{(q^{2k-1}+1)/(q+1)} = \rho\). Then we can find an element \(h \in K\) such that \(h(e_1) = \rho e_1\) and \(h\) is conjugate to

\[
\text{diag}(\rho, \rho, \tau^{-2}, \tau^{2q}, \tau^{-2q^2}, \ldots, \tau^{-2(-q)^{2k-2}})
\]

over \(\mathbb{F}_{q^2}\). Since \(k \geq 2\) and \((k, q) \neq (2, 2)\), by [16] there is a prime divisor \(\ell\) of \(q^{4k-2}-1\) that does not divide \(\prod_{j=1}^{4k-3}(q^j-1)\). In particular, \(\ell\) divides \((q^{2k-1}+1)\), and moreover the \(\ell\)-part of \(|P_1|\) is equal to the \(\ell\)-part of \(\beta_i(1)\), whence \(\beta_i\) is an irreducible character of \(P_1\) of \(\ell\)-defect zero. On the other hand, for any \(1 \leq t \leq q\), \(\ell\) divides \(|h^t|\), whence \(\beta_i(t) = 0\), and so we obtain by using (2.0.2), (3.4.2), (3.4.4) that

\[
\text{Ind}_{K}^{P_1}(M_\alpha)(h^t) = \zeta_{i,n}(h^t) = -(q-1)\rho^{it},
\]

\[
\text{Ind}_{K}^{P_1}(M_\alpha)(h^t) = \overline{\zeta}_{i,n}(h^t) = -(q-1)\rho^{-it}.
\]

It now follows from (3.4.3) that

\[
\delta(h^t) = \rho^{-2it} = \rho^{(q-1)it} = \overline{\lambda}_i(h^t),
\]

whence \(\delta(g) = \overline{\lambda}_i(g)\) for all \(g \in K\), since the choice of \(h\) ensures that \(h\) generates \(K\) modulo \(P'_1\). Together with (3.4.3), we have shown that

\[
(\text{Ind}_{K}^{P_1}(M_\alpha) \cdot \delta)(g) = (\text{Ind}_{K}^{P_1}(M_\alpha) \cdot \overline{\lambda}_i)(g) \tag{3.4.5}
\]

for all \(g \in K\). If \(g \in P_1 \setminus K\) then \(\text{Ind}_{K}^{P_1}(M_\alpha)(g) = 0\) since \(K \triangleleft P_1\), and so (3.4.5) holds for \(g\) as well. Consequently,

\[
\text{Ind}_{K}^{P_1}(M_\alpha) = \text{Ind}_{K}^{P_1}(M_\alpha) \cdot \overline{\lambda}_i.
\]

This identity, together with (3.4.2) and (3.4.4), implies by Frobenius’ reciprocity that
\[(\zeta_{i,n})^2, \Lambda_i]_G = [\zeta_{i,n} \overline{\lambda_i}, \zeta_{i,n}]_G = [\zeta_{i,n} \cdot \text{Ind}_{P_i}^G(\overline{\lambda_i}), \zeta_{i,n}]_G
= [\text{Ind}_{P_i}^G(\zeta_{i,n}|_{P_i} \cdot \overline{\lambda_i}), \zeta_{i,n}]_G = [\zeta_{i,n}|_{P_i} \cdot \overline{\lambda_i}, \zeta_{i,n}]_{P_i}
\geq [\text{Ind}_{K}^{P_i}(\overline{\lambda_i}) \cdot \overline{\lambda_i}, \text{Ind}_{K}^{P_i}(\overline{\lambda_i})]_{P_i} = 1,
\]

as stated. \(\square\)

**Proposition 3.5.** Suppose \(n = 2k + 1 \geq 5\) and \(0 < i \leq q\). Then \([(\Lambda_i)^k, \zeta_{i,n}] = 1\).

**Proof.** Recall \(G\) acts transitively on the set \(\Xi\) of isotropic 1-spaces in \(W = \mathbb{F}_{q^2}^n\), with \(P_i = \text{Stab}_G(\pi_i)\), where we set \(\pi_j := \langle e_j \rangle_{\mathbb{F}_{q^2}}\) for \(1 \leq j \leq k\). Hence the character \(\Lambda_i\) is afforded by a \(\mathbb{C}G\)-module

\[V = \text{Ind}_{P_i}^G(\text{V}_{\pi_i}) = \oplus_{g \in G/P_i} V_{g(\pi_i)},\]

where \(V_{\pi_i} = \langle v_{\pi_i} \rangle_{\mathbb{C}}\) is a one-dimensional \(P_i\)-module with character \(\lambda_i\), and \(G\) permutes the summands via \(h(V_{g(\pi_i)}) = V_{h(g(\pi_i))}\). It follows that \((\Lambda_i)^k\) is afforded by the \(G\)-module

\[V^\otimes k = \langle v_\xi \mid \xi \in \Xi^k \rangle_{\mathbb{C}},\]

where \(v_\xi = v_{\xi_1} \otimes v_{\xi_2} \otimes \ldots \otimes v_{\xi_k}\) for \(\xi = (\xi_1, \xi_2, \ldots, \xi_k)\).

Consider the \(G\)-orbit \(\Pi\) of the \(k\)-tuple \(\pi := (\pi_1, \pi_2, \ldots, \pi_k) \in \Xi^k\). Then the \(G\)-submodule

\[V(\Pi) := \langle v_\xi \mid \xi \in \Pi \rangle_{\mathbb{C}}\]

of \(V^\otimes k\) affords the character \(\text{Ind}_{R}^{G}(\mu)\), where \(R := \cap_{j=1}^k \text{Stab}_G(\langle e_j \rangle_{\mathbb{F}_{q^2}})\), and

\[\mu(h) = \sigma^{-(q-1)i\sum_{j=1}^k t_j}\]

if \(h(e_j) = \sigma^{t_j}\) for \(0 \leq t_j \leq q^2 - 2\) and \(1 \leq j \leq k\).

Note that \(\mathbb{Q}_k < R < P_k\) and \(\mathbb{Q}_k \leq \text{Ker}(\mu)\). Furthermore, if \(h \in L_k\) belongs to \(R\) and \(h(e_j) = \sigma^{t_j}\), then \(\det(h)\) (as an element in \(\text{GL}_k(q^2)\)) is \(\sigma^{\sum_{j=1}^k t_j}\), and so

\[\nu_i(h) = \sigma^{-(q-1)i\sum_{j=1}^k t_j} = \mu(h)\]

for the character \(\nu_i\) considered in Lemma 3.2, i.e. \(\nu_i|_R = \mu\). By Lemma 3.1, we have therefore shown that

\[0 < [\mu, \zeta_{i,n}]_R = [\text{Ind}_{R}^{G}(\mu), \zeta_{i,n}]_G \leq [(\Lambda_i)^k, \zeta_{i,n}]_G.\]

On the other hand, \((\Lambda_i)^k\) enters the character \((\zeta_n)^{n-1}\) by Proposition 3.4, whence the upper bound \([(\Lambda_i)^k, \zeta_{i,n}] \leq 1\) follows from Proposition 3.3. \(\square\)
Next we will study some see-saw dual pairs (cf. [10]) to determine various branching rules. Our consideration is based on the following well-known formula [11, Lemma 5.5]:

**Lemma 3.6.** Let \( \omega \) be a character of the direct product \( S \times G \) of finite groups \( S \) and \( G \). Then

\[
\omega = \sum_{\alpha \in \text{Irr}(S)} D_{\alpha} \otimes \alpha,
\]

where

\[
D_{\alpha} : g \mapsto \frac{1}{|S|} \sum_{x \in S} \alpha(x) \omega(xg)
\]

is either zero, or a character of \( G \).

We will work with a finite group \( \Gamma \) that contains two dual pairs \( S_1 \times G_1 \) and \( S_2 \times G_2 \), where \( G_1 \geq G_2 \) and \( S_2 \geq S_1 \).

**Lemma 3.7.** Let \( \omega \) be a character of \( \Gamma \), and decompose

\[
\omega|_{G_1 \times S_1} = \sum_{\alpha \in \text{Irr}(S_1)} D_{\alpha} \otimes \alpha, \quad \omega|_{G_2 \times S_2} = \sum_{\gamma \in \text{Irr}(G_2)} \gamma \otimes E_{\gamma}
\]

as in Lemma 3.6. Then, for any \( \alpha \in \text{Irr}(S_1) \) and any \( \gamma \in \text{Irr}(G_2) \) we have that

\[
[D_{\alpha}|_{G_2}, \gamma]|_{G_2} = [\alpha, E_{\gamma}|_{S_1}]_{S_1},
\]

and hence

\[
D_{\alpha}|_{G_2} = \sum_{\gamma \in \text{Irr}(G_2)} [E_{\gamma}|_{S_1}, \alpha]|_{S_1} \cdot \gamma.
\]

**Proof.** Write \( a_{\alpha, \gamma} := [D_{\alpha}|_{G_2}, \gamma]|_{G_2} \), so that

\[
D_{\alpha}|_{G_2} = \sum_{\gamma \in \text{Irr}(G_2)} a_{\alpha, \gamma} \gamma.
\]

Then

\[
\omega|_{G_2 \times S_1} = \sum_{\alpha \in \text{Irr}(S_1), \ \gamma \in \text{Irr}(G_2)} a_{\alpha, \gamma} \otimes \alpha
\]

\[
= \sum_{\gamma \in \text{Irr}(G_2)} \gamma \otimes \sum_{\alpha \in \text{Irr}(S_1)} a_{\alpha, \gamma} \alpha.
\]

Thus \( E_{\gamma}|_{S_1} = \sum_{\alpha \in \text{Irr}(S_1)} a_{\alpha, \gamma} \alpha \), and the statements follow. \( \square \)
First we consider the dual pair

\[ G_2 \times S_2 \]  

(3.7.1)

inside \( \Gamma := \text{GU}_2(q) \), where \( S_2 = \text{GU}_2(q) \) and \( G_2 = \text{SU}_n(q) \), and \( \omega = \zeta_{2n} = \zeta_{2n,q} \). More precisely, we view \( S_2 \) as \( \text{GU}(U) \), where \( U = \langle v_1, v_2 \rangle_{F_q^2} \) is endowed with the Hermitian form \( \circ \), with an orthonormal basis \( (v_1, v_2) \). Next, \( G_2 = \text{SU}_n(q) \) is \( \text{SU}(W) \), where \( W = \mathbb{F}_{q^2}^n \) is endowed with the Hermitian form \( \circ \) defined in (3.0.1). Now we consider \( V = U \otimes_{F_q^2} W \) with the Hermitian form \( \circ \) defined via

\[(u \otimes w) \circ (u' \otimes w') = (u \circ u')(w \circ w')\]

for \( u \in U \) and \( w \in W \). The action of \( G_2 \times S_2 \) on \( V \) induces a homomorphism \( G_2 \times S_2 \to \Gamma := \text{GU}(V) \).

Now \( V \) is the orthogonal sum \( V_1 \oplus V_2 \), where \( V_i := v_i \otimes W \). This gives us a subgroup \( G_1 := \text{SU}(V_1) \times \text{SU}(V_2) \cong \text{SU}_n(q) \times \text{SU}_n(q) \) of \( \Gamma \) that contains (the image of) \( G_2 \). In fact, \( G_2 \) embeds diagonally in \( G_1 \): \( g \mapsto \text{diag}(g, g) \).

Next,

\[ S_1 := \text{GU}((v_1)_{F_q^2}) \times \text{GU}((v_2)_{F_q^2}) \cong \text{GU}_1(q) \times \text{GU}_1(q) \]

is just the non-split diagonal torus of \( S_2 \).

In the above basis \( (v_1, v_2) \) of \( U \) and for \( 0 \leq i, j \leq q \), we consider the character

\[ \lambda_{i,j} : \text{diag}(\rho^a, \rho^b) \mapsto \rho^{ia+jb} \]

of \( S_1 \). Then, as explained in [15, §4], \( \zeta_{i,n} \) corresponds to the \( \rho^i \)-eigenspace of the generator \( \rho \cdot 1_W \) of \( \mathbb{Z}(\text{GU}_n(q)) \), so that

\[ D_{\lambda_{ij}} = \zeta_{i,n} \otimes \zeta_{j,n} \quad (3.7.2) \]

for the dual pair \( G_1 \times S_1 \).

We use the notation of [1] for the irreducible characters of \( S_2 = \text{GU}_2(q) \) (with the parameter \( q + 1 \) in the superscripts of characters changed to 0). For instance

\[ \chi_1^{(t)}|_{S_1} = \lambda_{t,t}. \]

The decomposition

\[ \omega|_{S_2 \times G_2} = \sum_{\alpha \in \text{Irr}(S_2)} \alpha \otimes C_\alpha \quad (3.7.3) \]
was described in [11, Proposition 6.3]. In particular, the $G_2$-characters
\[ C_\alpha^0 := C_\alpha - k_\alpha \cdot 1_{G_2}, \] (3.7.4)
where $\alpha \in \text{Irr}(S_2)$, are irreducible and pairwise distinct, and $k_\alpha \in \{0,1\}$ is listed in Table I.

This implies

**Corollary 3.8.** For the decomposition
\[ \omega|_{G_2 \times S_2} = \sum_{\gamma \in \text{Irr}(G_2)} \gamma \otimes E_\gamma, \]
we have that
\[ E_\gamma = \begin{cases} \alpha, & \gamma = C_\alpha^0 \text{ for some } \alpha \in \text{Irr}(S_2), \\ \chi_1^{(0)} + \chi_q^{(0)}, & \gamma = 1_{G_2}, \\ 0, & \text{otherwise}. \end{cases} \]

**Proposition 3.9.** Suppose $n = 2k + 1 \geq 5$ and $(n,q) \neq (5,2)$. For $0 < i \leq q$, and in the notation of (3.7.3)–(3.7.4) we have
\[ \Lambda_i = C_{\chi_i^{(i)}} + C_{\chi_q^{(i)}}. \]

Among these two irreducible constituents, only $C_{\chi_i^{(i)}}$ enters $(\zeta_{i,n})^2$.

**Proof.** (i) First, an application of Mackey’s formula reveals that $\Lambda_i$ is the sum of two distinct irreducible characters of $G_2 = \text{SU}_n(q)$. Clearly, $[\Lambda_i, 1_{G_2}] = 0$. By Proposition 3.5, $\Lambda_i$ enters $(\zeta_n)^2 = \omega|_{G_2}$, so
\[ \Lambda_i = C_{\beta_1}^0 + C_{\beta_2}^0 \]
for some $\beta_1 \neq \beta_2 \in \text{Irr}(S_2)$. Next,
\[ \Lambda_i(1) = (q^n - (-1)^n)(q^{n-1} + (-1)^n)/(q^2 - 1), \]

so \( \beta_1, \beta_2 \neq \chi_{q+1}^{(t)} \), see Table I.

By Proposition 3.4, at least one of \( \gamma_j := C^\circ_{\beta_j} \), \( j = 1, 2 \), is an irreducible constituent of

\[ (\zeta_{i,n})^2 = D_{\lambda, i}|G_2, \]

see (3.7.2). As \( \gamma_j \neq 1_{G_2} \), by Lemma 3.6 and Corollary 3.8 we have

\[ [D_{\lambda, i}|G_2, \gamma_j]|G_2 = [\lambda_{i, i}, E_{\gamma_j}|S_1]|_{S_1} = [\lambda_{i, i}, \beta_j]|_{S_1}|_{S_1}. \]

We have shown that \( C^\circ_{\beta_j} \), is an irreducible constituent of \((\zeta_{i,n})^2\) precisely when \( \lambda_{i, i} \) is an irreducible constituent of \( \beta_j|_{S_1} \).

(ii) As in the proof of Proposition 3.4, let \( \tau \) be an element of \( \mathbb{F}^\times_{q^{2k-2}} \) of order \( q^{2k-1} + 1 \) chosen such that \( \tau(q^{2k-1} + 1)/(q+1) = \rho \). Then we fix an element \( g \in L_1 \) such that \( g(e_1) = \sigma e_1, g(f_1) = \sigma^{-q} f_1 \), and \( g \) is conjugate to

\[ \text{diag}(\sigma, \sigma^{-q}, \tau, \tau^{-q}, \tau^{q^2}, \ldots, \tau^{(-q)^{2k-2}}) \]

over \( \mathbb{F}_{q^2} \). By [16] there is a prime divisor \( \ell \) of \( q^{2k-2} - 1 \) that does not divide \( \prod_{j=1}^{4k-3} (q^j - 1) \). In particular, \( \ell \) divides \( |\tau| \). It follows that \( \sigma \) and \( \sigma^{-q} \) are the only eigenvalues of \( g \) that belong to \( \mathbb{F}_{q^2} \).

Assume in addition that \( q > 2 \); in particular, \( \sigma \neq \sigma^{-q} \). Then, \( \langle e_1 \rangle_{\mathbb{F}_{q^2}} \) and \( \langle f_1 \rangle_{\mathbb{F}_{q^2}} \) are the only two \( g \)-invariant isotropic 1-spaces in \( W \), and so

\[ \Lambda_i(g) = 2\rho^{-i}. \quad (3.9.1) \]

Next, for any \( x \in S_2 = GU_2(q) \), \( \omega(gx) = 1 \), unless \( x \) has, at least one, and therefore both, of \( \sigma^{-1} \) and \( \sigma^q \) as its eigenvalues. In this exceptional case, \( x \) belongs to class \( C_{4}^{(-1)} \) in the notation of [1], and \( \omega(gx) = q^2 \). It follows from Lemma 3.6 that

\[ C_i^{\circ}(g) = \begin{cases} \rho^{-t}, \quad \alpha = \chi_{1}^{(t)}, & 0 < t \leq q, \\ 2, \quad \alpha = \chi_{1}^{(0)}, \\ \rho^{-t}, \quad \alpha = \chi_{q}^{(t)}, & 0 < t \leq q, \\ 0, \quad \alpha = \chi_{q}^{(0)}, \\ 0, \quad \alpha = \chi_{q-1}^{(t,u)}, & 0 \leq t, u \leq q. \end{cases} \]

Together with \((3.9.1)\), this readily implies that \( \{\beta_1, \beta_2\} = \{\chi_{1}^{(i)}, \chi_{q}^{(i)}\} \). Note that \( \chi_{1}^{(i)}|_{S_1} = \lambda_{i, i} \), but \( \chi_{q}^{(i)}|_{S_1} \) does not contain \( \lambda_{i, i} \), so we are done.

(iii) Now we consider the case \( q = 2 \). As shown in (i), we may assume that \( \beta_1|_{S_1} \) contains \( \lambda_{i, i} \). It follows that \( \beta_1 \in \{\chi_{1}^{(i)}, \chi_{q-1}^{(2i,0)}\} \). However degree consideration using
Table I rules out $\chi_{q-1}^{(2i,0)}$ and shows that $\beta_1 = \chi_1^{(i)}$. Again by degree consideration we now see that $\beta_2 = \chi_q^{(t)}$ for some $t \in \{1, 2\}$. Furthermore, $g$ fixes exactly three isotropic 1-spaces in $W$ (namely, the ones spanned by $e_1$, $f_1$, and $e_1 + f_1$), so $\Lambda_i(g) = 3\rho^{-i}$.

Arguing as in (ii), we see that

$$C_{\alpha}^\circ(g) = \begin{cases} \rho^{-t}, & \alpha = \chi_1^{(t)}, \quad 0 < t \leq q; \\ 2, & \alpha = \chi_1^{(0)}; \\ 2\rho^{-t}, & \alpha = \chi_q^{(t)}, \quad 0 < t \leq q; \\ 0, & \alpha = \chi_q^{(0)}. \end{cases}$$

Hence $\beta_2 = \chi_q^{(i)}$, and we are done since $\chi_q^{(i)}|_{S_1}$ does not contain $\lambda_{i,i}$. □

We will now work with three new dual pairs. First, we consider the dual pair $G_3 \times S_3$ inside $\Gamma := \text{GU}_{2kn}(q)$, where $S_3 = \text{GU}_{2k}(q)$ and $G_3 = \text{SU}_n(q)$, and $\omega = \zeta_{2nk} = \zeta_{2nk,q}$. More precisely, we view $S_3$ as $\text{GU}(U)$, where $U = \langle v_1, \ldots, v_{2k} \rangle_{F^{2k}}$ is endowed with the Hermitian form $\circ$, with an orthonormal basis $(v_1, \ldots, v_{2k})$. Next, $G_3 = \text{SU}_n(q)$ is $\text{SU}(W)$, where $W = \mathbb{F}_{q^2}$ is endowed with the Hermitian form $\circ$ defined in (3.0.1). Now we consider $V = U \otimes_{F^{q^2}} W$ with the Hermitian form $\cdot$ defined via

$$(u \otimes w) \circ (u' \otimes w') = (u \circ u')(w \circ w')$$

for $u \in U$ and $w \in W$. The action of $G_3 \times S_3$ on $V$ induces a homomorphism $G_3 \times S_3 \to \Gamma := \text{GU}(V)$.

Now $V$ is the orthogonal sum $\oplus_{i=1}^{2k} V_i$, where $V_i := v_i \otimes W$. This gives us a subgroup

$$G_1 := \text{SU}(V_1) \times \text{SU}(V_2) \times \ldots \times \text{SU}(V_{2k}) \cong \text{SU}_n(q)^{2k}$$

of $\Gamma$ that contains (the image of) $G_3$. In fact, $G_3$ embeds diagonally in $G_1$: $g \mapsto \text{diag}(g, g, \ldots, g)$. Next,

$$S_1 := \text{GU}(\langle v_1 \rangle_{F^{q^2}}) \times \text{GU}(\langle v_2 \rangle_{F^{q^2}}) \times \ldots \times \text{GU}(\langle v_{2k} \rangle_{F^{q^2}}) \cong \text{GU}_1(q)^{2k}$$

is just the non-split diagonal torus of $S_3$. In the above basis $(v_1, v_2, \ldots, v_{2k})$ of $U$ and for $1 \leq i \leq q$, we consider the character

$$\mu_i : \text{diag}(\rho^{a_1}, \rho^{a_2}, \ldots, \rho^{a_{2k}}) \mapsto \rho^{i(\sum_{j=1}^{2k} a_j)} \quad (3.9.2)$$

of $S_1$.

Next, for each $1 \leq j \leq k$ we embed one copy of $\text{SU}(W)$ in

$$\text{SU}(\langle v_{2j-1}, v_{2j} \rangle_{F^{q^2}} \otimes W)$$
(by letting it act only on $W$). This gives an embedding of $G_2 := SU_n(q)^k$ in $G_1$ via
\[ \text{diag}(g_1, g_2, \ldots, g_k) \mapsto \text{diag}(g_1, g_1, g_2, g_2, \ldots, g_k, g_k). \]

At the same time, $G_3$ embeds diagonally in $G_2$ via $g \mapsto \text{diag}(g, g, \ldots, g)$. The action of $G_2$ is centralized by $S_2 := GU_2(k)$.

Recall the characters $C_\alpha$ of $SU_n(q)$ introduced in (3.7.3).

**Proposition 3.10.** Suppose $n = 2k + 1 \geq 5$, $(n, q) \neq (5, 2)$, and $0 < i \leq q$. Then both $(C_{\chi_i}^k)^k$ and $(\zeta_{i, n})^n$ contain $\zeta_{i, n}$.

**Proof.** (i) First we decompose
\[ \omega|_{G_3 \times S_3} = \sum_{\gamma \in \text{Irr}(G_3)} \gamma \otimes E_\gamma \]
for the dual pair $G_3 \times S_3$. By Proposition 3.3, $\omega|_{G_3} = (\zeta_n)^{n-1}$ contains $\zeta_{i, n}$ with multiplicity one. It follows that the $G_3$-character $E_{\zeta_{i, n}}$ has degree 1, so there is some $0 \leq m = m_i \leq q$ such that
\[ E_{\zeta_{i, n}}(X) = \rho^{mt} \]
whenever $X \in GU_{2k}(q)$ has determinant equal to $\rho^t$.

(ii) Next we decompose
\[ \omega|_{S_2 \times G_2} = \sum_{\beta \in \text{Irr}(S_2)} \beta \otimes F_\beta \]
for the dual pair $S_2 \times G_2$. Note by (3.7.3) that if
\[ \beta = \beta_1 \otimes \beta_2 \otimes \ldots \otimes \beta_k, \]
then
\[ F_\beta = C_{\beta_1} \otimes C_{\beta_2} \otimes \ldots \otimes C_{\beta_k}. \]  \hfill (3.10.1)

By Lemma 3.7,
\[ [F_\beta|_{G_3}, \zeta_{i, n}]_{G_3} = [\beta, E_{\zeta_{i, n}}]|_{S_2}. \]
Since $E_{\bar{\zeta}_{i,n}}$ has degree 1, we see that $\bar{\zeta}_{i,n}$ is an irreducible constituent of $F_{\beta}|G_3$ precisely when $\beta = E_{\bar{\zeta}_{i,n}}|_{S_2}$, that is when

$$
\beta(X_1, X_2, \ldots, X_k) = \rho^m \sum_{j=1}^k t_j
$$

whenever $X_j \in GU_2(q)$ has determinant equal to $\rho^{t_j}$ for $1 \leq j \leq k$. In the notation of [1] we then have

$$
\beta = \chi_i^{(m)} \otimes \chi_j^{(m)} \otimes \cdots \otimes \chi_k^{(m)}.
$$

(3.10.2)

(iii) Recall by Proposition 3.4 that $\Lambda_i$ enters $(\zeta_n)^2$. It follows that $\Lambda_i^{\otimes k} = \bigotimes_{1 \leq j \leq k} \Lambda_i$ enters $\omega|G_2$. Next, by Proposition 3.5, $\bar{\zeta}_{i,n}$ is an irreducible constituent of $(\Lambda_i)^k = \Lambda_i^{\otimes k}|G_3$. Furthermore, by Proposition 3.9, $\Lambda_i = C_{\chi_i^{(i)}} + C_{\chi_i^{(j)}}$. Hence, using (3.10.1) we see that

$$
\Lambda_i^{\otimes k} = \sum_{1 \leq j \leq k, \beta_j \in \{\chi_i^{(i)}, \chi_i^{(j)}\}} C_{\beta_1} \otimes C_{\beta_2} \otimes \cdots \otimes C_{\beta_k}
$$

$$
= \sum_{1 \leq j \leq k, \beta_j \in \{\chi_i^{(i)}, \chi_i^{(j)}\}} F_{\beta_1 \otimes \beta_2 \otimes \cdots \otimes \beta_k}.
$$

Applying the result (3.10.2) of (ii), we conclude that $m = i$ and $\bar{\zeta}_{i,n}$ is an irreducible constituent of

$$
F_{\chi_i^{(m)} \otimes \chi_j^{(m)} \otimes \cdots \otimes \chi_k^{(m)}}|G_3 = (C_{\chi_i^{(i)}})^k.
$$

(iv) The same argument as in (ii), but applied to the decomposition

$$
\omega|S_1 \times G_1 = \sum_{\alpha \in \text{Irr}(S_1)} \alpha \otimes D_\alpha
$$

for the dual pair $S_1 \times G_1$ implies that $\bar{\zeta}_{i,n}$ is an irreducible constituent of $D_\alpha|G_3$ precisely when $\alpha = E_{\bar{\zeta}_{i,n}}|_{S_1}$, that is when $\alpha = \mu_m$ as introduced in (3.9.2). As $m$ was shown to be equal to $i$ in (iii), we now have that $\bar{\zeta}_{i,n}$ is an irreducible constituent of

$$
D_\alpha|G_3 = D_{\mu_i}|G_3 = (\zeta_{i,n})^{n-1}.
$$

We can now prove Theorem 1, which we restate:

**Theorem 3.11.** Let $q$ be a prime power and let $G = SU_n(q)$ with $n = 2k + 1 \geq 3$. Suppose in addition that $(n, q) \neq (3, 2)$. Then $(\zeta_{i,n})^n$ contains $1_G$ with multiplicity exactly one if $1 \leq i \leq q$ and zero if $i = 0$. 

Proof. For \( n = 3 \), the statement was checked by A. Schaeffer Fry using the package Chevie [3]. Likewise, the case \((n, q) = (5, 2)\) was checked using the package GAP [2]. So we may assume that \( n \geq 5 \) and \((n, q) \neq (5, 2)\). Now for \( i = 0 \) the statement follows from Proposition 3.3. For \( 1 \leq i \leq q \) we have

\[
[(\zeta_{i,n})^{n-1}, \overline{\zeta}_{i,n}]_G = [(\zeta_{i,n})^n, 1_G]
\]

is at most 1 by Proposition 3.3 and at least 1 by Proposition 3.10. \( \square \)

4. Moments of Weil representations of \( SU_4(q) \)

Theorem 1 naturally brings up the question: what are the \( n \)th moments of Weil representations of \( SU_n(q) \) when \( 2 \mid n \)? Preliminary analysis indicates that the even-dimensional case does not behave as nicely as in the odd-dimensional case (particularly because real-valued characters usually have large even moments). We restrict ourselves to record the following result:

Theorem 4.1. Consider the irreducible Weil characters \( \zeta_{i,n}, 0 \leq i \leq q \), of \( G := SU_n(q) \) as given in (2.0.2), and suppose \( n = 4 \). Then

\[
[(\zeta_{i,4})^4, 1_G] = \begin{cases} 
q + 1, & i = 0, \\
q + 2, & 2 \nmid q, 
\quad i = (q + 1)/2, \\
q - 1, & 4|(q + 1), 
\quad i = (q + 1)/4, 3(q + 1)/4, \\
1, & \text{otherwise.}
\end{cases}
\]

Proof. (i) We will use the dual pairs \( G_1 \times S_1 = SU_n(q)^2 \times GU_1(q)^2 \) and \( G_2 \times S_2 = SU_n(q) \times GU_2(q) \) as in (3.7.1). By [11, Proposition 6.3],

\[
\omega|_{G_2 \times S_2} = \sum_{\alpha \in \text{Irr}(S_2)} C_\alpha \otimes \alpha = \sum_{\gamma \in \text{Irr}(G_2)} \gamma \otimes E_\gamma = \sum_{\alpha \in \text{Irr}(S_2)} C_\alpha^0 \otimes \alpha + 1_{G_2} \otimes (\chi_1^{(0)} + \chi_q^{(0)}),
\]

where \( C_\alpha^0(1) \) are listed in Table I. The only new feature that arises in the case \( n = 4 \) is that, according to [11, Proposition 6.5],

(a) if \( \alpha \neq \beta \), then \( C_\alpha^0 = C_\beta^0 \) precisely when \( \{\alpha, \beta\} = \{\chi_1^{(t)}, \chi_q^{(q+1-t)}\} \) for some \( t \in \{1, 2, \ldots, q\} \setminus \{(q + 1)/2\} \); and

(b) all \( C_\alpha^0 \) are irreducible, except when \( 2 \nmid q \) and \( \alpha = \chi_1^{(q+1)/2} \), in which case \( C_\alpha^0 \) is a sum of two distinct irreducible characters (of degree \((q^2 + 1)(q^2 - q + 1)/2\)).
Hence, instead of Corollary 3.8 now we have

$$E_\gamma = \begin{cases} 
\alpha, & \text{if } \gamma \text{ is an irreducible constituent of } C_\alpha^0 \text{ for some } \alpha \in \text{Irr}(GU_2(q)), \\
\chi_1^{(0)} + \chi_4^{(0)}, & \text{if } \gamma = 1_{G_2}, \\
0, & \text{otherwise.}
\end{cases} \quad (4.1.1)$$

On the other hand,

$$\omega|_{G_1 \times S_1} = \sum_{\alpha \in \text{Irr}(S_1)} D_\alpha \otimes \alpha,$$

where $D_\alpha$ is given in (3.7.2) for $\alpha = \lambda_{i,j} \in \text{Irr}(GU_1(q)^2)$. Applying Lemma 3.7 we then get

$$(\zeta_{i,4})^2|_{SU_4(q)} = D_{\lambda_{i,i}}|_{G_2} = \sum_{\gamma \in \text{Irr}(G_2)} [E_\gamma|_{GU_1(q)^2}, \lambda_{i,i}]_{GU_1(q)^2} \cdot \gamma. \quad (4.1.2)$$

Direct computations show for $\alpha \in \text{Irr}(GU_2(q))$ that

$$[\alpha|_{GU_1(q)^2}, \lambda_{i,i}]_{GU_1(q)^2} = \begin{cases} 
\delta_{t,i}, & \alpha = \chi_1^{(t)}, \\
\delta_{t,2i}, & \alpha = \chi_{q+1}^{(t)}, \\
\delta_{t+u,2i}, & \alpha = \chi_{q-1}^{(t, u)}, \\
\delta_{t,i+(q+1)/2}, & \alpha = \chi_q^{(t)}, 2 \nmid q, \\
0, & \alpha = \chi_q^{(t)}, 2|q,
\end{cases} \quad (4.1.3)$$

and $\delta_{i,j}$ is defined to be 1 if $i \equiv j (\text{mod } q + 1)$ and 0 otherwise. Recall that in the notation for $\alpha \in \text{Irr}(GU_2(q))$, the superscripts are viewed as elements of $\mathbb{Z}/(q + 1)\mathbb{Z}$ if $\alpha(1) \leq q$, and as elements of $\mathbb{Z}/(q^2 - 1)\mathbb{Z}$ if $\alpha(1) = q + 1$. Moreover, $\chi_{q-1}^{(t, u)} = \chi_q^{(u, t)}$ and $\chi_{q+1}^{(-t q)}$.

(ii) Consider the case $2|q$. Then (4.1.1)–(4.1.3) imply that

$$(\zeta_{0,4})^2 = 1_G + C_{\chi_{1}^{(0)}}^{0} + \sum_{1 \leq t \leq q/2} C_{\chi_{q-1}^{(t, -t)}}^{0} + \sum_{1 \leq s \leq (q - 2)/2} C_{\chi_{q+1}^{(s(q+1))}}^{0}.$$ 

As $\zeta_{0,4}$ is real-valued, it follows that $[(\zeta_{0,4})^4, 1_G]_G = q + 1$.

Likewise, if $i \neq 0$, then the irreducible summands of $(\zeta_{i,4})^2$ are $C_{\chi_{1}^{(i)}}^{0}$, $C_{\chi_{q+1}^{(i, t)}}^{0}$ with $t \neq i$, and $C_{\chi_{q+1}^{(i)}}^{0}$ with $s \equiv 2i (\text{mod } q + 1)$ (and $s \neq 0 (\text{mod } q - 1)$); all with multiplicity one. It follows that the only common irreducible constituent of $(\zeta_{i,4})^2$ and $(\zeta_{i,4})^2 = (\zeta_{q+1-i,4})^2$ is $C_{\chi_{1}^{(i)}}^{0} = C_{\chi_{q+1-i}^{(i)}}^{0}$, cf. (a) above. Thus $[(\zeta_{i,4})^4, 1_G]_G = 1$. In fact, this argument also
applies to the case where $2 \nmid q$ and $(q + 1) \nmid 4i$, where there is an extra irreducible summand $C_{\chi_q^{(i-1)(q+1)/2}}$ (also with multiplicity 1) in $(\zeta_{i,4})^2$:

(iii) Assume now that $2 \nmid q$. Then (4.1.1)–(4.1.3) imply that

$$((\zeta_{0,4})^2 = 1_G + C_{\chi_1}^0 + \sum_{1 \leq t \leq \frac{q-1}{2}} C_{\chi_{q-1}}^0(t,-t) + C_{\chi_q}^{(2+1)} + \sum_{1 \leq s \leq \frac{q-3}{2}} C_{\chi_{q+1}}^0(s(q+1)),$$

yielding $[(\zeta_{0,4}), 1_G]_G = q + 1$. Likewise,

$$((\zeta_{q+1,4})^2 = 1_G + C_{\chi_1}^0(2+1) + \sum_{1 \leq t \leq \frac{q-1}{2}} C_{\chi_{q-1}}^0(t,-t) + C_{\chi_q}^0(0) + \sum_{1 \leq s \leq \frac{q-3}{2}} C_{\chi_{q+1}}^0(s(q+1)).$$

Since $\zeta_{q+1,4}$ is real-valued and $C_{\chi_1}^0(2+1)$ is the sum of two distinct irreducible summands, $[(\zeta_{q+1,4}), 1_G]_G = q + 2$.

Finally, the irreducible summands of $(\zeta_{q+1,4})^2$ are $C_{\chi_q}^{(q-1)/2}, C_{\chi_{q+1}}^{(2+1)}$, $C_{\chi_{q+1}}^{(q-1)/2}$ with $t \neq \pm(q + 1)/4$, and $C_{\chi_{q+1}}^{(2+1)(q+1)/2}$ all with multiplicity one. As mentioned in (a), $C_{\chi_1}^{(q-1)/2} = C_{\chi_{q+1}}^{(q-1)/2}$. Thus all of these characters, except for the first one, are common irreducible summands between $(\zeta_{q+1,4})^2$ and $(\bar{\zeta}_{q+1,4})^2 = (\zeta_{2(q+1),4})^2$. It follows that $[(\zeta_{q+1,4}), 1_G]_G = q - 1$. □

We also record a curious fact about 4th moments of Weil representations of $Sp_{2n}(q)$, which holds specifically in the case $q = 3$.

**Proposition 4.2.** Let $n \geq 2$ and let $\xi, \eta$ denote an irreducible Weil character of $G = Sp_{2n}(3)$ of degree $(3^n + 1)/2$ and $(3^n - 1)/2$, respectively. Then

$$[\xi^4, 1_G]_G = 1 = [\eta^4, 1_G]_G.$$  

**Proof.** It was shown in [12, Proposition 5.4] that if $\chi \in \{\xi, \eta\}$ then $\text{Sym}^2(\chi)$ and $\wedge^2(\chi)$ are irreducible, of distinct degrees. Furthermore, Lemma 3.3(ii) and formula (3.5) of [6] show that

$$\text{Sym}^2(\xi) = \text{Sym}^2(\bar{\xi}), \text{Sym}^2(\eta) \neq \text{Sym}^2(\bar{\eta}), \wedge^2(\xi) \neq \wedge^2(\bar{\xi}), \wedge^2(\eta) = \wedge^2(\bar{\eta}).$$

Since $\chi^2 = \text{Sym}^2(\chi) + \wedge^2(\chi)$, the statement follows. □

**References**


